Surjective convolution operators on vector valued distributions

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STATEMENT OF THE PROBLEM

Let

$$P(D)u_\lambda = f_\lambda$$  \hspace{1cm} (1)

be a linear partial differential equation with constant coefficients, where each $f_\lambda$ is a distribution, ultradistribution, real analytic function or ultradifferentiable function, etc.

We consider the question whether the equation (1) is solvable in such a way that

If $f_\lambda$ depends “nicely” on the parameter $\lambda$ (e.g. holomorphically, smoothly, etc.),

\[ \downarrow \]

The solution $u_\lambda$ can be chosen depending on $\lambda$ in the same way
The problem of parameter dependence of solutions can be translated into the question of surjectivity of the tensorized operator

\[ P(D) \otimes \text{id} : X \hat{\otimes}_\pi E \longrightarrow X \hat{\otimes}_\pi E, \]

where \( X \) is the class of objects to which \( f \) and \( u \) belong and \( E \) corresponds to the parameter dependence.

For instance, if

\[ P(D) : \mathcal{D}'(\Omega) \longrightarrow \mathcal{D}'(\Omega), \quad \Omega \subseteq \mathbb{R}^d, \text{ open}, \]

is surjective, we ask when

\[ P(D) \otimes \text{id} : \mathcal{D}'(\Omega, E) \longrightarrow \mathcal{D}'(\Omega, E) \simeq \mathcal{D}'(\Omega) \hat{\otimes}_\pi E \]

is surjective for suitable function space \( E \).
In particular, we show that a linear partial differential operator with constant coefficients $P(D)$ is surjective on the space of $E$-valued ultradistributions over an arbitrary convex set if $E'$ is a nuclear Fréchet space with property (DN) of Vogt. In particular, this holds if $E$ is isomorphic to the space of tempered distributions $S'$ or to the space of germs of holomorphic functions over a one-point set $H(\{0\})$.

The spaces of vector valued distributions and operators between them were introduced by L. Schwartz, 1957-59.

$$\mathcal{D}'(\Omega, E) := L(\mathcal{D}(\Omega), E) = \mathcal{D}'(\Omega) \hat{\otimes} E$$

Problems similar to those treated here for spaces of vector valued infinitely differentiable functions were considered before.
Suppose that $P(D) : \mathcal{E}(\Omega) \rightarrow \mathcal{E}(\Omega)$, $\Omega \subseteq \mathbb{R}^d$, open, is surjective, and consider the operator $P(D) \otimes \text{id} : \mathcal{E}(\Omega, E) \rightarrow \mathcal{E}(\Omega, E)$.

(1) **Grothendieck, 1955**
If $E$ is Fréchet, then $P(D) \otimes \text{id}$ is surjective on $\mathcal{E}(\Omega, E)$.

(2) **Vogt, 1983**
If $E = \mathcal{D}(\Omega)$ or $E = \mathcal{E}'(\Omega)$ and $P(D)$ is elliptic, then $P(D) \otimes \text{id}$ is not surjective on $\mathcal{E}(\Omega, E)$.

(3) **Vogt, 1983**
If $E = F'_b$ is the strong dual of a Fréchet space with the property (DN) and $P(D)$ is hypoelliptic, then $P(D) \otimes \text{id}$ is surjective on $\mathcal{E}(\Omega, E)$.

(4) **Vogt, 1975**
If $P(D)$ is the Cauchy Riemann operator on $\mathbb{C} = \mathbb{R}^2$ and $E = F'_b$ is the strong dual of a Fréchet space, then $P(D) \otimes \text{id}$ is surjective on $\mathcal{E}(\mathbb{C}, E)$ if and only if $F$ has the property (DN).
We consider the case when $X$ is a PLN-space, like the space of distributions

$$\mathcal{D}'(\Omega)$$

or, more generally, the space of ultradistributions of Beurling type

$$\mathcal{D}'_\omega(\Omega)$$
in the sense of Bjorck or Braun, Meise and Taylor, or a space of real analytic functions

$$\mathcal{A}(\Omega)$$

or a space of ultradifferentiable function of Roumieu type

$$\mathcal{E}_{\{\omega\}}(\Omega)$$

both non-quasianalytic and quasianalytic.
A locally convex space $X$ is a **PLS-space** (resp. **PLN-space**) if it is a projective limit of a sequence of strong duals of Fréchet -Schwartz (resp. nuclear Fréchet) spaces. Roughly speaking, PLS-spaces are “regular” spaces of the form

\[
\bigcap \bigcup_{N \in \mathbb{N}} X_{N,n}, \quad X_{N,n} \text{ Banach spaces,}
\]

with the natural topology.

Every PLS-space $X$ satisfies

\[
X := \text{proj}_{N \in \mathbb{N}} \text{ ind}_{n \in \mathbb{N}} X_{N,n}
\]

where $X_{N,n}$ are Banach spaces,

\[
X_N := \text{ind}_{n \in \mathbb{N}} X_{N,n} \text{ denotes the locally convex inductive limit of a sequence } (X_{N,n})_{n \in \mathbb{N}} \text{ with compact linking maps, and}
\]

\[
X = \text{proj}_{N \in \mathbb{N}} X_N \text{ denotes the topological projective limit of a sequence } (X_N)_{N \in \mathbb{N}} \text{ of locally convex spaces.}
\]

The linking maps will be denoted by

\[
i^K_N : X_K \to X_N \text{ and } i_N : X \to X_N.
\]

We denote the closed unit ball of $X_{N,n}$ by $B_{N,n}$. 
In what follows $\Omega \subseteq \mathbb{R}^d$ is an open domain and $(K_N)_{N \in \mathbb{N}}$, $K_1 \subset K_2 \subset \cdots \subset \Omega$, is a compact exhaustion.

- The space of distributions $\mathcal{D}'(\Omega)$ is the strong dual of the space

$$\mathcal{D}(\Omega) = \text{ind}_{N \in \mathbb{N}} \mathcal{D}_{K_N}$$

of test functions. It is a PLN-space.

- The space of real analytic functions is denoted by

$$\mathcal{A}(\Omega) := \{ f : \Omega \rightarrow \mathbb{C} : f \text{ analytic} \}.$$  

$\mathcal{A}(\Omega)$ is equipped with the unique locally convex topology such that

- for any $U \subseteq \mathbb{C}^d$ open, $\mathbb{R}^d \cap U = \Omega$, the restriction map $R : H(U) \rightarrow \mathcal{A}(\Omega)$ is continuous, and

- for any compact set $K \subseteq \Omega$ the restriction map $r : \mathcal{A}(\Omega) \rightarrow H(K)$ is continuous.
• We endow the space $H(U)$ of holomorphic functions on $U$ with the compact-open topology and the space $H(K)$ of germs of holomorphic functions on $K$ with its natural topology:

$$H(K) = \text{ind}_{n \in \mathbb{N}} H^\infty(U_n),$$

where $(U_n)_{n \in \mathbb{N}}$ is a basis of $\mathbb{C}^d$-neighbourhoods of $K$.

Martineau proved that there is exactly one topology on $\mathcal{A}(\Omega)$ satisfying the condition above and endowed with this topology one has

$$\mathcal{A}(\Omega) = \text{proj}_{N \in \mathbb{N}} H(K_N).$$

The space $H(K_N)$ is a DFN-space and $\mathcal{A}(\Omega)$ is a PLN-space.
We define the **Roumieu class of ultradifferentiable functions** $\mathcal{E}_{\{\omega\}}(\Omega)$.

Let $\omega : [0, \infty] \rightarrow [0, \infty]$ be a continuous increasing function (called **weight**) satisfying the following conditions:

\[(\alpha) \ \omega(2t) = O(\omega(t)) \quad (\beta) \ \omega(t) = O(t)\]

\[(\gamma) \ \log t = o(\omega(t)) \quad (\delta) \ \varphi \text{ is a convex function} \]

Let $\varphi^*(t) := \sup_{x \geq 0} (xt - \varphi(t))$ be the Young conjugate of $\varphi(t) := \omega(e^t)$.

\[
\mathcal{E}_{\{\omega\}}(\Omega) := \{ f \in C^\infty(\Omega) : \forall N \in \mathbb{N} \ \exists m \in \mathbb{N} : \|f\|_{N,m} < \infty \},
\]

where

\[
\|f\|_{N,m} := \sup_{x \in K_N} \sup_{\alpha \in \mathbb{N}^d} |f^{(\alpha)}(x)| \exp \left( -\frac{1}{m} \varphi^*(|\alpha|m) \right).
\]
These classes were systematically studied by Bjorck and Braun, Meise and Taylor.

The Gevrey classes (i.e., $\mathcal{E}_{\{\omega\}}$ with $\omega(t) = t^{1/p}$, $p \in (0, 1)$) are of that type.

For $\omega(t) = t$, we obtain the spaces of real analytic functions.

Clearly,

$$\mathcal{E}_{\{\omega\}}(\Omega) = \text{proj}_{N \in \mathbb{N}} \text{ind}_{n \in \mathbb{N}} \mathcal{E}_{\{\omega\},N,n}(K_N)$$

and

$$\mathcal{E}_{\{\omega\},N,n}(K_N) = \{ f \in C^\infty(K_N) : \|f\|_{N,n} < \infty \}$$

are Banach spaces with norms $\|f\|_{N,n}$.

- If

$$\int_0^\infty \frac{\omega(t)}{1 + t^2} dt = \infty$$

then the class (or the weight) is quasianalytic (i.e., there are no non-trivial elements with compact support in $\mathcal{E}_{\{\omega\}}(\Omega)$). Otherwise the class is non-quasianalytic.
The class of ultradistributions of Beurling type \( \mathcal{D}'(\omega)(\Omega) \), \( \omega \) a non-quasianalytic weight, is a PLN-space. It is defined to be the strong dual of

\[
\mathcal{D}(\omega)(\Omega) = \text{ind} \sum_{N \in \mathbb{N}} \mathcal{D}(\omega)(K_N),
\]

\[
\mathcal{D}(\omega)(K_N) := \{ f \in \mathcal{D}(\omega)(\Omega) : \text{supp} \ f \subseteq K_N \}.
\]

- The weight \( \omega(t) = \log(1 + |t|) \) does not satisfy condition \((\gamma)\). As it is well-known for this weight we have

\[
\mathcal{D}'(\omega)(\Omega) = \mathcal{D}'(\Omega).
\]
THE CASE OF BANACH VALUED ULTRADISTRIBUTIONS

**Theorem 1.** Let $X = \text{proj } X_N$ be an ultrabornological PLN-space, and let $Y = \text{proj } Y_N$ be a PLN-space. If $T : X \to Y$ is a surjective operator, then

$$T \otimes \text{id} : X \hat{\otimes}_\pi E \to Y \hat{\otimes}_\pi E$$

is always surjective for any Banach space $E$.

**Corollary 2.** Let $E$ be a Banach space, let $\omega$ be a weight, and let

$$T : \mathcal{D}'(\omega)(\Omega) \to \mathcal{D}'(\omega)(\Omega), \quad T : \mathcal{E}_{\{\omega\}}(\Omega) \to \mathcal{E}_{\{\omega\}}(\Omega)$$

be surjective operators. The operators

$$T \otimes \text{id} : \mathcal{D}'(\omega, E)(\Omega, E) \to \mathcal{D}'(\omega)(\Omega, E)$$

$$T \otimes \text{id} : \mathcal{E}_{\{\omega\}}(\Omega, E) \to \mathcal{E}_{\{\omega\}}(\Omega, E)$$

are surjective.
FUNCTIONAL ANALYTIC INGREDIENTS

The problem of parameter dependence of solutions is related to the splitting of short exact sequences and to the functor $\text{Proj}^1$. This relation gives the only accessible way to obtain the consequences mentioned before. The so-called functor $\text{Proj}^1$ was introduced in the theory of locally convex spaces by Palamodov, and developed by Vogt.

We concentrate below in the case when $E$ is the strong dual of a Fréchet Schwartz space $F$. The following topological invariants were introduced by Vogt:

A Fréchet space $F$ satisfies the property $(\text{DN})$ if

$$\exists \nu \quad \forall \mu, \theta \in ]0,1[ \quad \exists \kappa, C_2 \quad \forall y \in F :$$

$$\|y\|_{\mu} \leq C_2 \|y\|_{\nu}^{\theta} \|y\|^{1-\theta}_{\kappa}.$$

$F$ has property $(\text{DN})$ if the quantifier at $\theta$ is “exists” instead of “for all”.

**Proposition 3.** Let $E$ be an LS-space, let $\omega$ be a non-quasianalytic weight, and let

$$T : D'(\omega)(\Omega) \rightarrow D'(\omega)(\Omega), \quad T : \mathcal{E}_{\{\omega\}}(\Omega) \rightarrow \mathcal{E}_{\{\omega\}}(\Omega)$$

be surjective operators.

$\quad T \otimes \text{id} : D'(\omega)(\Omega, E) \rightarrow D'(\omega)(\Omega, E),$ \quad are surjective

$\quad T \otimes \text{id} : \mathcal{E}_{\{\omega\}}(\Omega, E) \rightarrow \mathcal{E}_{\{\omega\}}(\Omega, E)$

$\uparrow$

$\quad \text{Proj}^{1}_{N \in \mathbb{N}} L(E', X_N) = 0$

where

$\quad \ker T \cong \text{proj}_{N \in \mathbb{N}} X_N, \quad X_N \quad \text{LS-spaces.}$
Theorem 4. Let $F$ be a Fréchet space, $F = \proj_{\nu \in \mathbb{N}} F_{\nu}$, let $X = \proj_{N \in \mathbb{N}} X_N$ be a PLS-space, and assume that both projective spectra are reduced. Let $X_N = \ind_{n \in \mathbb{N}} X_{N,n}$ be LS-spaces, $X_{N,n}$ Banach spaces. If

$$\Proj^{1}_{N \in \mathbb{N}} L(F, X_N) = 0$$

then the pair $(F, X)$ satisfies the condition (H), i.e.,

$$\forall N \exists M \geq N \forall K \geq M \exists \nu, n \forall \mu \geq \nu, m \geq n$$

$$\exists \kappa \geq \mu, k \geq m, S > 0 \forall y \in F \forall x' \in X'_N :$$

$$\|y\|_{\mu} \|x' \circ i_{N}^{M}\|_{*}^{M,m} \leq$$

$$\leq S \left( \|y\|_{\kappa} \|x' \circ i_{N}^{K}\|_{*}^{K,k} + \|y\|_{\nu} \|x'\|_{*}^{N,n} \right).$$

The condition is also sufficient if one of the following assumptions is satisfied:

(a) $F$ is nuclear;

(b) $F$ is a Fréchet-Schwartz Köthe sequence space of order $1$, i.e., $F \simeq \lambda^{1}(A)$;

(c) $X$ is a Köthe type PLS-space of order $\infty$, i.e., $X \simeq \Lambda^{\infty}(A)$. 
In order to evaluate condition (H), we introduce analogues for PLS-spaces of conditions (Ω) and (Ω̅) of Vogt, which are known for Fréchet spaces.

We call them \((PΩ)\) and \((PΩ̅)\) respectively.

They coincide with Vogt’s conditions for Fréchet spaces, and are satisfied by every LS-space.

We do not give here the precise formulation. We prefer to mention examples and a result which constitutes a generalization of the famous (DN)-(Ω) Vogt-Wagner Splitting Theorem, due to the relation between the vanishing of the functors \(\text{Proj}^1\) and \(\text{Ext}^1\).
EXAMPLES.

**Proposition 5.** If \( \omega \) is a non-quasianalytic weight and \( \Omega \subseteq \mathbb{R}^d \) is an open domain, then \( \mathcal{D}'(\Omega) \), \( \mathcal{D}'(\omega)(\Omega) \) and \( \mathcal{E}_{\{\omega\}}(\Omega) \) have \((P\Omega)\).

**Theorem 6.** If the weight \( \omega \) is subadditive, then the space \( \mathcal{E}_{\{\omega\}}(\Omega) \) satisfies \((P\Omega)\) for every open convex set \( \Omega \subseteq \mathbb{R}^d \).

**Corollary 7.** The space \( \mathcal{A}(\Omega) \) has \((P\Omega)\) for every open (non-necessarily convex) set \( \Omega \subseteq \mathbb{R}^d \).
Theorem 8.

a) If \( F \) is a Fréchet space with \((DN)\)-property and \( X \) is a PLS-space with \((P\Omega)\), then \((F, X)\) satisfies \((H)\).

b) If \( F \) is a Fréchet space with \((DN)\) and \( X \) is a PLS-space with \((P\overline{\Omega})\), then \((F, X)\) satisfies \((H)\).

Corollary 9. Let \( \alpha \) be a stable sequence, \( X \) a PLS-space with \( \text{Proj}^1 X = 0 \). Then

\[
(\Lambda_0(\alpha), X) \text{ has } (H) \iff X \text{ has } (P\overline{\Omega})
\]

\[
(\Lambda_{\infty}(\alpha), X) \text{ has } (H) \iff X \text{ has } (P\Omega)
\]
MAIN RESULTS

As before, $E$ is the strong dual of a Fréchet Schwartz space $F$. We state now the consequences of our abstract results for the problem we consider.

**Proposition 10.** Let $\Omega \subseteq \mathbb{R}^d$ be an arbitrary open set, let $F$ be either a nuclear Fréchet space or $F \simeq \lambda^1(A)$ be a Fréchet-Schwartz space. If $T$ is an elliptic partial differential operator, then

$$T \otimes \text{id} : \mathcal{A}(\Omega, F') \to \mathcal{A}(\Omega, F')$$

is surjective if $F$ has (DN). If $\Omega$ is convex then the condition is also necessary.

**Proposition 11.** Let $T : \mathcal{D}'(\omega)(\Omega) \to \mathcal{D}'(\omega)(\Omega)$ be a surjective operator. Then $\ker T$ has $(P\Omega)$ if and only if

$$T \otimes \text{id} : \mathcal{D}'(\omega)(\Omega \times \mathbb{R}) \to \mathcal{D}'(\omega)(\Omega \times \mathbb{R})$$

is surjective.

**Corollary 12.** If $\Omega \subseteq \mathbb{R}^d$ is a convex, open set and $P(D) : \mathcal{D}'(\omega)(\Omega) \to \mathcal{D}'(\omega)(\Omega)$ is surjective, then $\ker P(D)$ has $(P\Omega)$. 
Proposition 13. Let $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R}^d)$ be an ultradistribution of compact support and let $T_\mu : \mathcal{D}'_{(\omega)}(\mathbb{R}^d) \longrightarrow \mathcal{D}'_{(\omega)}(\mathbb{R}^d)$ be a surjective convolution operator. Then $\ker T_\mu$ has $(P\Omega)$.

Proposition 14. Let $\Omega \subseteq \mathbb{R}^d$ be a convex, open set. If $E$ is an LS-space, $E'$ has (DN) and either $E$ is an LN-space or $E \simeq k_\infty(v)$ then

$$P(D) : \mathcal{D}'_{(\omega)}(\Omega, E) \longrightarrow \mathcal{D}'_{(\omega)}(\Omega, E) \text{ is surjective.}$$

Proposition 15. Let $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R}^d)$ be an ultradistribution of compact support and let

$$T_\mu : \mathcal{D}'_{(\omega)}(\mathbb{R}^d) \longrightarrow \mathcal{D}'_{(\omega)}(\mathbb{R}^d)$$

be surjective convolution operator. If $E$ is an LS-space, $E'$ has (DN) and either $E$ is an LN-space or $E \simeq k_\infty(v)$ then

$$T_\mu : \mathcal{D}'_{(\omega)}(\mathbb{R}^d, E) \longrightarrow \mathcal{D}'_{(\omega)}(\mathbb{R}^d, E) \text{ is surjective.}$$
**Proposition 16.** If $E$ is an LS-space and

$$ T_\mu : \mathcal{D}'(\omega)(\mathbb{R}) \longrightarrow \mathcal{D}'(\omega)(\mathbb{R}) $$

is a surjective convolution operator then

$$ T_\mu \otimes \text{id} : \mathcal{D}'(\omega)(\mathbb{R}, E) \longrightarrow \mathcal{D}'(\omega)(\mathbb{R}, E) $$

is surjective whenever $E'$ has (DN).

If there exist a subsequence $(z_j)$ of zeros of the Fourier-Laplace transform $\hat{\mu}$ of $\mu$ and a constant $C$ such that

$$ \frac{|\text{Im } z_j|}{\omega(z_j)} \longrightarrow \infty \quad \text{and} \quad |\text{Im } z_{j+1}| \leq C |\text{Im } z_j| $$

then the converse holds as well.
Proposition 17. Let $\omega$ be a non-quasianalytic weight and let

$$T : \mathcal{E}_{\{\omega\}}(\Omega) \longrightarrow \mathcal{E}_{\{\omega\}}(\Omega)$$

be a surjective operator. The map

$$T \otimes \text{id} : \mathcal{E}_{\{\omega\}}(\Omega \times \mathbb{R}) \longrightarrow \mathcal{E}_{\{\omega\}}(\Omega \times \mathbb{R})$$

is surjective if and only if $\ker T$ has $(P\overline{\Omega})$.

Corollary 18. Let $\omega$ be a non-quasianalytic weight and let $P(D) : \mathcal{E}_{\{\omega\}}(\mathbb{R}^d) \longrightarrow \mathcal{E}_{\{\omega\}}(\mathbb{R}^d)$ be surjective and homogeneous, then

$$\ker P(D) \text{ has } (P\overline{\Omega}) \Leftrightarrow \left( P(D) \text{ has a continuous linear right inverse. } \right)$$