Surjective convolution operators on vector valued distributions

José Bonet

Departament de Matemàtica Aplicada
Universitat Politècnica de València

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STATEMENT OF THE PROBLEM

Let

$$P(D)u_\lambda = f_\lambda \tag{1}$$

be a linear partial differential equation with constant coefficients, where each f_λ is a distribution, ultradistribution, real analytic function or ultradifferentiable function, etc.

We consider the question whether the equation (1) is solvable in such a way that

If f_λ depends "nicely" on the parameter λ (e.g. holomorphically, smoothly, etc.),

$$\Downarrow$$

The solution u_λ can be chosen depending on λ in the same way

The problem of parameter dependence of solutions can be translated into the question of surjectivity of the tensorized operator

$$P(D) \otimes \mathrm{id} : X \hat{\otimes}_\pi E \longrightarrow X \hat{\otimes}_\pi E,$$

where X is the class of objects to which f and u belong and E corresponds to the parameter dependence.

For instance, if

$$P(D) : \mathcal{D}'(\Omega) \longrightarrow \mathcal{D}'(\Omega), \qquad \Omega \subseteq \mathbb{R}^d, \text{ open},$$

is surjective, we ask when

$$P(D) \otimes \mathrm{id} : \mathcal{D}'(\Omega, E) \longrightarrow \mathcal{D}'(\Omega, E) \simeq \mathcal{D}'(\Omega) \hat{\otimes}_\pi E$$

is surjective for suitable function space E.

In particular, we show that a linear partial differential operator with constant coefficients $P(D)$ is surjective on the space of E-valued ultradistributions over an arbitrary convex set if E' is a nuclear Fréchet space with property (DN) of Vogt. In particular, this holds if E is isomorphic to the space of tempered distributions $\mathcal{S}'$ or to the space of germs of holomorphic functions over a one-point set $H(\{0\})$.

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The spaces of vector valued distributions and operators between them were introduced by L. Schwartz, 1957-59.

$$\mathcal{D}'(\Omega, E) := L(\mathcal{D}(\Omega), E) = \mathcal{D}'(\Omega)\hat{\otimes}_\varepsilon E$$

Problems similar to those treated here for spaces of vector valued infinitely differentiable functions were considered before.

Suppose that $P(D) : \mathcal{E}(\Omega) \longrightarrow \mathcal{E}(\Omega)$, $\Omega \subseteq \mathbb{R}^d$, open, is surjective, and consider the operator

$$P(D) \otimes \mathrm{id} : \mathcal{E}(\Omega, E) \longrightarrow \mathcal{E}(\Omega, E).$$

(1) **Grothendieck, 1955**
If E is Fréchet, then $P(D) \otimes \mathrm{id}$ is surjective on $\mathcal{E}(\Omega, E)$.

(2) **Vogt, 1983**
If $E = \mathcal{D}(\Omega)$ or $E = \mathcal{E}'(\Omega)$ and $P(D)$ is elliptic, then $P(D) \otimes \mathrm{id}$ is not surjective on $\mathcal{E}(\Omega, E)$.

(3) **Vogt, 1983**
If $E = F_b'$ is the strong dual of a Fréchet space with the property (DN) and $P(D)$ is hypoelliptic, then $P(D) \otimes \mathrm{id}$ is surjective on $\mathcal{E}(\Omega, E)$.

(4) **Vogt, 1975**
If $P(D)$ is the Cauchy Riemann operator on $\mathbb{C} = \mathbb{R}^2$ and $E = F_b'$ is the strong dual of a Fréchet space, then $P(D) \otimes \mathrm{id}$ is surjective on $\mathcal{E}(\mathbb{C}, E)$ if and only if F has the property (DN).

CONTEXT

We consider the case when X is a PLN-space, like the space of distributions

$$\mathcal{D}'(\Omega)$$

or, more generally, the space of ultradistributions of Beurling type

$$\mathcal{D}'_{(\omega)}(\Omega)$$

in the sense of Bjorck or Braun, Meise and Taylor, or a space of real analytic functions

$$\mathcal{A}(\Omega)$$

or a space of ultradifferentiable function of Roumieu type

$$\mathcal{E}_{\{\omega\}}(\Omega)$$

both non-quasianalytic and quasianalytic.

A locally convex space X is a ***PLS-space*** (resp. ***PLN-space)*** if it is a projective limit of a sequence of strong duals of Fréchet -Schwartz (resp. nuclear Fréchet) spaces. Roughly speaking, PLS-spaces are “regular” spaces of the form

$$\bigcap_{N \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} X_{N,n}, \qquad X_{N,n} \text{ Banach spaces,}$$

with the natural topology.

Every PLS-space X satisfies

$$\boxed{X := \mathrm{proj}_{N \in \mathbb{N}} \ \mathrm{ind}_{n \in \mathbb{N}} X_{N,n}}$$

where $X_{N,n}$ are Banach spaces,

$X_N := \mathrm{ind}_{n \in \mathbb{N}} X_{N,n}$ denotes the locally convex inductive limit of a sequence $(X_{N,n})_{n \in \mathbb{N}}$ with compact linking maps, and

$X = \mathrm{proj}_{N \in \mathbb{N}} \ X_N$ denotes the topological projective limit of a sequence $(X_N)_{N \in \mathbb{N}}$ of locally convex spaces.

The linking maps will be denoted by

$$i_N^K : X_K \to X_N \ \text{ and } \ i_N : X \longrightarrow X_N.$$

We denote the closed unit ball of $X_{N,n}$ by $B_{N,n}$.

In what follows $\Omega \subseteq \mathbb{R}^d$ is an open domain and $(K_N)_{N\in\mathbb{N}}$, $K_1 \Subset K_2 \Subset \cdots \Subset \Omega$, is a compact exhaustion.

• The space of **distributions** $\mathcal{D}'(\Omega)$ is the strong dual of the space

$$\mathcal{D}(\Omega) = \mathrm{ind}_{N\in\mathbb{N}}\, \mathcal{D}_{K_N}$$

of test functions. It is a PLN-space.

• The space of **real analytic functions** is denoted by

$$\mathcal{A}(\Omega) \ := \{f : \Omega \longrightarrow \mathbb{C} : f \text{ analytic}\}.$$

$\mathcal{A}(\Omega)$ is equipped with the unique locally convex topology such that

- for any $U \subseteq \mathbb{C}^d$ open, $\mathbb{R}^d \cap U = \Omega$, the restriction map $R : H(U) \longrightarrow \mathcal{A}(\Omega)$ is continuous, and

- for any compact set $K \subseteq \Omega$ the restriction map $r : \mathcal{A}(\Omega) \ \longrightarrow H(K)$ is continuous.

• We endow the space $H(U)$ of **holomorphic functions on $\mathbf{U}$** with the compact-open topology and the space $H(K)$ of germs of holomorphic functions on K with its natural topology:

$$H(K) = \mathrm{ind}_{n\in\mathbb{N}}\, H^{\infty}(U_n),$$

where $(U_n)_{n\in\mathbb{N}}$ is a basis of $\mathbb{C}^d$-neighbourhoods of K.

Martineau proved that there is exactly one topology on $\mathcal{A}(\Omega)$ satisfying the condition above and endowed with this topology one has

$$\mathcal{A}(\Omega) \; = \mathrm{proj}_{N\in\mathbb{N}}\; H(K_N).$$

The space $H(K_N)$ is a DFN-space and $\mathcal{A}(\Omega)$ is a PLN-space.

We define the **Roumieu class of ultradifferentiable functions** $\mathcal{E}_{\{\omega\}}(\Omega)$.

Let $\omega : [0,\infty[\longrightarrow [0,\infty[$ be a continuous increasing function (called *weight*) satisfying the following conditions:

(α) $\omega(2t) = O(\omega(t))$ (β) $\omega(t) = O(t)$

(γ) $\log t = o(\omega(t))$ (δ) φ is a convex function

Let $\varphi^*(t) := \sup_{x\geq 0}(xt - \varphi(t))$ be the Young conjugate of $\varphi(t) := \omega(e^t)$.

$$\mathcal{E}_{\{\omega\}}(\Omega) := \{f \in C^\infty(\Omega) : \forall\ N \in \mathbb{N}\ \exists\ m \in \mathbb{N} : \|f\|_{N,m} < \infty\},$$

where

$$\|f\|_{N,m} := \sup_{x\in K_N} \sup_{\alpha\in\mathbb{N}^d} |f^{(\alpha)}(x)| \exp\left(-\frac{1}{m}\varphi^*\left(|\alpha| m\right)\right).$$

These classes were systematically studied by Bjorck and Braun, Meise and Taylor.

The Gevrey classes (i.e., $\mathcal{E}_{\{\omega\}}$ with $\omega(t) = t^{1/p}$, $p \in (0,1)$) are of that type.

For $\omega(t) = t$, we obtain the spaces of real analytic functions.

Clearly,

$$\mathcal{E}_{\{\omega\}}(\Omega) = \mathrm{proj}_{N\in\mathbb{N}}\ \mathrm{ind}_{n\in\mathbb{N}}\ \mathcal{E}_{\{\omega\},N,n}(K_N)$$

and

$$\mathcal{E}_{\{\omega\},N,n}(K_N) = \{f \in C^\infty(K_N) : \|f\|_{N,n} < \infty\}$$

are Banach spaces with norms $\|f\|_{N,n}$.

• If

$$\int_0^\infty \frac{\omega(t)}{1+t^2}dt = \infty$$

then the class (or the weight) is ***quasianalytic*** (i.e., there are no non-trivial elements with compact support in $\mathcal{E}_{\{\omega\}}(\Omega)$). Otherwise the class is ***non-quasianalytic***.

The class of **ultradistributions of Beurling type** $\mathcal{D}'_{(\omega)}(\Omega)$, ω a non-quasianalytic weight, is a PLN-space. It is defined to be the strong dual of

$$\mathcal{D}_{(\omega)}(\Omega) = \text{ind}_{N\in\mathbb{N}} \mathcal{D}_{(\omega)}(K_N),$$

$$\mathcal{D}_{(\omega)}(K_N) := \{f \in \mathcal{D}_{(\omega)}(\Omega) : \operatorname{supp} f \subseteq K_N\}.$$

• The weight $\omega(t) = \log(1 + |t|)$ does not satisfy condition (γ). As it is well-known for this weight we have

$$\mathcal{D}'_{(\omega)}(\Omega) = \mathcal{D}'(\Omega).$$

THE CASE OF BANACH VALUED ULTRADISTRIBUTIONS

Theorem 1. *Let* $X = \mathrm{proj}\ X_N$ *be an ultrabornological PLN-space, and let* $Y = \mathrm{proj}\ Y_N$ *be a PLN-space. If* $T : X \to Y$ *is a surjective operator, then*

$$T \otimes \mathrm{id}\ : X\hat{\otimes}_\pi E \longrightarrow Y\hat{\otimes}_\pi E$$

is always surjective for any Banach space E.

Corollary 2. *Let* E *be a Banach space, let* ω *be a weight, and let*

$$T : \mathcal{D}'_{(\omega)}(\Omega) \longrightarrow \mathcal{D}'_{(\omega)}(\Omega), \qquad T : \mathcal{E}_{\{\omega\}}(\Omega) \longrightarrow \mathcal{E}_{\{\omega\}}(\Omega)$$

be surjective operators. The operators

$$T \otimes \mathrm{id}\ : \mathcal{D}'_{(\omega)}(\Omega, E) \longrightarrow \mathcal{D}'_{(\omega)}(\Omega, E),$$

$$T \otimes \mathrm{id}\ : \mathcal{E}_{\{\omega\}}(\Omega, E) \longrightarrow \mathcal{E}_{\{\omega\}}(\Omega, E)$$

are surjective.

FUNCTIONAL ANALYTIC INGREDIENTS

The problem of parameter dependence of solutions is related to the splitting of short exact sequences and to the functor Proj^1 . This relation gives the only accessible way to obtain the consequences mentioned before. The so called functor Proj^1 was introduced in the theory of locally convex spaces by Palamodov, and developed by Vogt.

We concentrate below in the case when E is the strong dual of a Fréchet Schwartz space F. The following topological invariants were introduced by Vogt:

A Fréchet space F satisfies the **property (DN)** if

$$\exists\ \nu \quad \forall\ \mu, \theta \in]0,1[\quad \exists\ \kappa, C_2 \quad \forall\ y \in F\ :$$

$$\|y\|_\mu \ \leq\ C_2\ \|y\|_\nu^\theta\ \|y\|_\kappa^{1-\theta}.$$

F has property $(\underline{DN})$ if the quantifier at θ is "exists" instead of "for all".

Proposition 3. *Let E be an LS-space , let ω be a non-quasianalytic weight, and let*

$$T : \mathcal{D}'_{(\omega)}(\Omega) \longrightarrow \mathcal{D}'_{(\omega)}(\Omega)\ ,\ T : \mathcal{E}_{\{\omega\}}(\Omega) \longrightarrow \mathcal{E}_{\{\omega\}}(\Omega)$$

be surjective operators.

$$\left.\begin{array}{l} T \otimes \mathrm{id}\ : \mathcal{D}'_{(\omega)}(\Omega, E) \longrightarrow \mathcal{D}'_{(\omega)}(\Omega, E), \\ T \otimes \mathrm{id}\ : \mathcal{E}_{\{\omega\}}(\Omega, E) \longrightarrow \mathcal{E}_{\{\omega\}}(\Omega, E) \end{array}\right. \quad \textit{are surjective}$$

$$\Updownarrow$$

$$\mathrm{Proj}^1\ {}_{N\in\mathbb{N}}\ L(E', X_N) = 0$$

where

$$\ker T \simeq \mathrm{proj}_{N\in\mathbb{N}}\ X_N, \quad X_N \quad \textit{LS-spaces.}$$

Theorem 4. *Let F be a Fréchet space, $F = \text{proj}_{\ \nu\in\mathbb{N}} F_\nu$, let $X = \text{proj}_{N\in\mathbb{N}} X_N$ be a PLS-space, and assume that both projective spectra are reduced. Let $X_N = \text{ind}_{n\in\mathbb{N}} X_{N,n}$ be LS-spaces, $X_{N,n}$ Banach spaces. If*

$$\text{Proj}^1_{\ N\in\mathbb{N}} L(F, X_N) = 0$$

then the pair (F, X) satisfies the ***condition (H)****, i.e.,*

$$\forall\ N\ \ \exists\ M \geq N\ \ \forall\ K \geq M\ \ \exists\ \nu, n\ \ \forall\ \mu \geq \nu, m \geq n$$

$$\exists\ \kappa \geq \mu, k \geq m, S > 0\quad \forall\ y \in F\quad \forall\ x' \in X'_N :$$

$$\|y\|_\mu\ \|x' \circ i_N^M\|^*_{M,m}\ \leq$$

$$\leq\ S\left(\|y\|_\kappa\ \|x' \circ i_N^K\|^*_{K,k}\ +\ \|y\|_\nu \|x'\|^*_{N,n}\right).$$

The condition is also sufficient if one of the following assumptions is satisfied:

(a) F is nuclear;

(b) F is a Fréchet-Schwartz Köthe sequence space of order 1, i.e., $F \simeq \lambda^1(A)$;

(c) X is a Köthe type PLS-space of order ∞, i.e., $X \simeq \Lambda^\infty(A)$.

In order to evaluate condition (H), we introduce analogues for PLS-spaces of conditions (Ω) and $(\overline{\overline{\Omega}})$ of Vogt, which are known for Fréchet spaces.

We call them $(P\Omega)$ and $(P\overline{\overline{\Omega}})$ respectively.

They coincide with Vogt's conditions for Fréchet spaces, and are satisfied by every LS-space.

We do not give here the precise formulation. We prefer to mention examples and a result which constitutes a generalization of the famous (DN)-(Ω) Vogt-Wagner Splitting Theorem, due to the relation between the vanishing of the functors Proj^1 and Ext^1.

EXAMPLES.

Proposition 5. *If ω is a non-quasianalytic weight and $\Omega \subseteq \mathbb{R}^d$ is an open domain, then $\mathcal{D}'(\Omega)$, $\mathcal{D}'_{(\omega)}(\Omega)$ and $\mathcal{E}_{\{\omega\}}(\Omega)$ have $(P\overline{\overline{\Omega}})$.*

Theorem 6. *If the weight ω is subadditive, then the space $\mathcal{E}_{\{\omega\}}(\Omega)$ satisfies $(P\overline{\overline{\Omega}})$ for every open convex set $\Omega \subseteq \mathbb{R}^d$.*

Corollary 7. *The space $\mathcal{A}(\Omega)$ has $(P\overline{\overline{\Omega}})$ for every open (non-necessarily convex) set $\Omega \subseteq \mathbb{R}^d$.*

Theorem 8.

a) *If F is a Fréchet space with (DN)-property and X is a PLS-space with $(P\Omega)$* $\Rightarrow (F, X)$ *satisfies (H).*

$$\left.\begin{array}{l}\text{If } F \text{ is a Fréchet space}\\ \text{with (DN)-property and}\\ X \text{ is a PLS-space with } (P\Omega)\end{array}\right\} \Rightarrow (F,X) \text{ satisfies (H).}$$

b) *If F is a Fréchet space with ($\underline{DN}$) and X is a PLS-space with $(P\overline{\overline{\Omega}})$* $\Rightarrow (F, X)$ *satisfies (H).*

$$\left.\begin{array}{l}\text{If } F \text{ is a Fréchet space}\\ \text{with } (\underline{DN}) \text{ and}\\ X \text{ is a PLS-space with } (P\overline{\overline{\Omega}})\end{array}\right\} \Rightarrow (F,X) \text{ satisfies (H).}$$

Corollary 9. *Let α be a stable sequence, X a PLS-space with $\mathrm{Proj}^1 X = 0$. Then*

$$(\Lambda_0(\alpha), X) \;\; \text{has (H)} \qquad \Leftrightarrow \qquad X \;\; \text{has} \;\; (P\overline{\overline{\Omega}})$$

$$(\Lambda_\infty(\alpha), X) \;\; \text{has (H)} \qquad \Leftrightarrow \qquad X \;\; \text{has} \;\; (P\Omega)$$

MAIN RESULTS

As before, E is the strong dual of a Fréchet Schwartz space F. We state now the consequences of our abstract results for the problem we consider.

Proposition 10. *Let $\Omega \subseteq \mathbb{R}^d$ be an arbitrary open set, let F be either a nuclear Fréchet space or $F \simeq \lambda^1(A)$ be a Fréchet-Schwartz space. If T is an elliptic partial differential operator, then*

$$T \otimes \mathrm{id} \;: \mathcal{A}(\Omega, F') \to \mathcal{A}(\Omega, F')$$

is surjective if F has (DN). If Ω is convex then the condition is also necessary.

Proposition 11. *Let $T : \mathcal{D}'_{(\omega)}(\Omega) \longrightarrow \mathcal{D}'_{(\omega)}(\Omega)$ be a surjective operator. Then $\ker T$ has $(P\Omega)$ if and only if*

$$T \otimes \mathrm{id} \;: \mathcal{D}'_{(\omega)}(\Omega \times \mathbb{R}) \longrightarrow \mathcal{D}'_{(\omega)}(\Omega \times \mathbb{R})$$

is surjective.

Corollary 12. *If $\Omega \subseteq \mathbb{R}^d$ is a convex, open set and $P(D) : \mathcal{D}'_{(\omega)}(\Omega) \longrightarrow \mathcal{D}'_{(\omega)}(\Omega)$ is surjective, then $\ker P(D)$ has $(P\Omega)$.*

Proposition 13. *Let* $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R}^d)$ *be an ultradistibution of compact support and let* $T_\mu : \mathcal{D}'_{(\omega)}(\mathbb{R}^d) \longrightarrow \mathcal{D}'_{(\omega)}(\mathbb{R}^d)$ *be a surjective convolution operator. Then* $\ker T_\mu$ *has* $(P\Omega)$.

Proposition 14. *Let* $\Omega \subseteq \mathbb{R}^d$ *be a convex, open set. If* E *is an LS-space,* E' *has (DN) and either* E *is an LN-space or* $E \simeq k_\infty(v)$ *then*

$$P(D) : \mathcal{D}'_{(\omega)}(\Omega, E) \longrightarrow \mathcal{D}'_{(\omega)}(\Omega, E) \quad \textit{is surjective.}$$

Proposition 15. *Let* $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R}^d)$ *be an ultradistribution of compact support and let*

$$T_\mu : \mathcal{D}'_{(\omega)}(\mathbb{R}^d) \longrightarrow \mathcal{D}'_{(\omega)}(\mathbb{R}^d)$$

be surjective convolution operator. If E *is an LS-space,* E' *has (DN) and either* E *is an LN-space or* $E \simeq k_\infty(v)$ *then*

$$T_\mu : \mathcal{D}'_{(\omega)}(\mathbb{R}^d, E) \longrightarrow \mathcal{D}'_{(\omega)}(\mathbb{R}^d, E) \quad \textit{is surjective.}$$

Proposition 16. *If E is an LS-space and*

$$T_\mu : \mathcal{D}'_{(\omega)}(\mathbb{R}) \longrightarrow \mathcal{D}'_{(\omega)}(\mathbb{R})$$

is a surjective convolution operator then

$$T_\mu \otimes \mathrm{id} : \mathcal{D}'_{(\omega)}(\mathbb{R}, E) \longrightarrow \mathcal{D}'_{(\omega)}(\mathbb{R}, E)$$

is surjective whenever E' has (DN).

If there exist a subsequence (z_j) of zeros of the Fourier-Laplace transform $\hat{\mu}$ of μ and a constant C such that

$$\frac{|\operatorname{Im} z_j|}{\omega(z_j)} \to \infty \quad \textit{and} \quad |\operatorname{Im} z_{j+1}| \leq C |\operatorname{Im} z_j|$$

then the converse holds as well.

Proposition 17. *Let ω be a non-quasianalytic weight and let*

$$T : \mathcal{E}_{\{\omega\}}(\Omega) \longrightarrow \mathcal{E}_{\{\omega\}}(\Omega)$$

be a surjective operator. The map

$$T \otimes \mathrm{id} \; : \mathcal{E}_{\{\omega\}}(\Omega \times \mathbb{R}) \longrightarrow \mathcal{E}_{\{\omega\}}(\Omega \times \mathbb{R})$$

is surjective if and only if $\ker T$ *has* $(P\overline{\overline{\Omega}})$.

Corollary 18. *Let ω be a non-quasianalytic weight and let $P(D) : \mathcal{E}_{\{\omega\}}(\mathbb{R}^d) \longrightarrow \mathcal{E}_{\{\omega\}}(\mathbb{R}^d)$ be surjective and homogeneous, then*

$$\ker P(D) \text{ has } (P\overline{\overline{\Omega}}) \quad \Leftrightarrow \quad \left(\begin{array}{l} P(D) \text{ has a continuous} \\ \text{linear right inverse.} \end{array} \right)$$