The dual of the space of holomorphic functions on locally closed convex sets

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NOTATION AND STATEMENT OF THE PROBLEM

A subset $Q$ of $\mathbb{C}^N$ is called **locally closed** if for each $z \in Q$ there is a closed neighbourhood $U$ of $z$ in $\mathbb{C}^N$ such that $Q \cap U$ is closed.

- Every open subset and every closed subset of $\mathbb{C}^N$ is locally closed.
- Every convex open set in $\mathbb{R}^N$ is locally closed too.

For a convex set $Q \subset \mathbb{C}^N$ the symbols $\text{int}_r Q$ denote the relative interior and $\partial_r Q$ the relative boundary of $Q$ **with respect to the affine hull of $Q$**.

For example, if $0 \in Q$, the affine hull of $Q$ is the the real linear span of $Q$. We write $\omega := Q \cap \partial_r Q$. 
Proposition 1. The following assertions are equivalent for a convex subset $Q$ of $\mathbb{C}^N$:

- $Q$ is locally closed.

$\iff$

- $Q$ admits a countable fundamental sequence $(Q_n)_{n \in \mathbb{N}}$ of compact subsets

$\iff$

- $Q$ is the union of the relative interior $\text{int}_r Q$ of $Q$ and a subset $\omega$ of $\partial_r Q$ which is open in $\partial_r Q$
A locally closed convex set $Q$ is called (C-)\textbf{strictly convex at the relative boundary of } $\omega$ if the intersection of $Q$ with each supporting (complex) hyperplane to the closure $\overline{Q}$ of $Q$ is compact.

- If the interior of $Q$ is empty,

\begin{center}
\begin{tabular}{c}
$Q$ is strictly convex at the relative boundary of $\omega$ \iff $Q$ is compact
\end{tabular}
\end{center}

- If the interior of $Q$ is not empty,

\begin{center}
\begin{tabular}{c}
$Q$ is (C-)strictly convex at the relative boundary of $\omega$ \iff each line segment (of which the C-linear affine hull belongs to some supporting hyperplane of $\overline{Q}$) of $\omega = Q \cap \partial_r Q$ is relatively compact in $\omega$
\end{tabular}
\end{center}

\textbf{Proposition 2.} A locally closed convex set $Q$ is strictly convex at the relative boundary of $\omega$ if and only if $Q$ has a neighbourhood basis of convex domains.

For example $Q$ is strictly convex at the relative boundary of $\omega$ if $Q$ is open or compact.
Strictly convex

Not strictly convex
Strictly convex  

Not strictly convex
Let $Q \subset \mathbb{C}^N$ be a convex locally closed set such that
\[
\overline{Q} = \{ x \in \mathbb{C}^N \mid x_1 \geq f(x_2, \ldots, x_{2N}), \ (x_2, \ldots, x_{2N}) \in \mathbb{R}^{2N-1} \}
\]
for a convex function $f : \mathbb{R}^{2N-1} \to \mathbb{R}$.

- If the function $f$ is strictly convex $\Rightarrow \{ Q$ is strictly convex at the relative boundary of $\omega \}$

- If $Q$ is closed and there is a unbounded interval in $\mathbb{R}^{2N-1}$ on which $f$ is affine $\Rightarrow \{ Q$ is not strictly convex at the relative boundary of $\omega \}$
We denote by $H(Q)$ the vector space of all functions which are holomorphic on some open neighbourhood of the locally closed convex set $Q$.

Let $(Q_n)_{n \in \mathbb{N}}$ be an increasing fundamental sequence of compact convex sets in $Q$. Since the algebraic equality $H(Q) = \bigcap_{n \in \mathbb{N}} H(Q_n)$ holds, we endow $H(Q)$ with the projective topology of

$$H(Q) := \text{proj}_n H(Q_n)$$

This topology does not depend of the choice of the fundamental system $(Q_n)_{n \in \mathbb{N}}$. The space $H(Q)$ is a (PLN)-space.

• If $Q$ is an open convex subset of $\mathbb{R}^N$, then

$$H(Q) = A(Q)$$

where $A(Q)$ denotes the space of all real analytic functions on $Q$. 
1966, Martineau investigated the spaces $H(Q)$ of analytic functions on a locally closed convex set $Q$ in $\mathbb{C}^N$ and convolution operators on these spaces in 1967.

1990’s, Napalkov, Udakov, Korobeinik and Maltsev,

2000, Melikhov and Momm.

The Laplace transform

$$\mathcal{F} : H(Q)_b' \longrightarrow VH(\mathbb{C}^N) := \ind_{n \rightarrow} \proj_{\leftarrow k} H(v_{n,k}, \mathbb{C}^N)$$

is a linear topological isomorphism.

The Laplace transform: $\mathcal{F}(\varphi)(z) := \varphi(\exp\langle \cdot, z \rangle)$, $z \in \mathbb{C}^N$, 
The steps $H(v, \mathbb{C}^N)$ are defined, for a positive weight $v$ on $\mathbb{C}^N$, as

$$H(v, \mathbb{C}^N) := \{ f \in H(\mathbb{C}^N) | \sup_{z \in \mathbb{C}^N} v(z) |f(z)| < \infty \} \quad \text{(Banach space)}$$

and $v_{n,k}(z) := \exp(-H_n(z) - |z|/k)$, $n, k \in \mathbb{N}$, $z \in \mathbb{C}^N$.

For each set $D \subset \mathbb{C}^N$ we denote by $H_D$ the **support function of $D$**:

$$H_D(z) := \sup_{w \in D} \text{Re}\langle z, w \rangle, \quad z \in \mathbb{C}^N \quad \text{with} \quad \langle z, w \rangle := \sum_{j=1}^{N} z_j w_j.$$
Bierstedt, Meise, and Summers (1982)

System $\bar{V}$ of all those weights $\bar{v} : \mathbb{C}^N \to ]0, \infty[$ which are continuous and have the property that

for each $n$ there are $\alpha_n > 0$ and $k = k(n)$ such that $\bar{v} \leq \alpha_n v_{n,k}$ on $\mathbb{C}^N$.

The projective hull of the weighted inductive limit is defined by

$$H\bar{V}(\mathbb{C}^N) := \{ f \in H(\mathbb{C}^N) \mid ||f||_{\bar{v}} := \sup_{z \in \mathbb{C}^N} \bar{v}(z)|f(z)| < \infty \text{ for all } \bar{v} \in \bar{V} \},$$

endowed with the Hausdorff locally convex topology defined by the system of seminorms

$$\{ ||.||_{\bar{v}} \mid \bar{v} \in \bar{V} \}$$

The projective hull is a complete locally convex space and

$$VH(\mathbb{C}^N) \subset H\bar{V}(\mathbb{C}^N) \quad \text{with continuous inclusion.}$$
Characterize in terms of the locally closed convex set $Q$ when the inclusion

$$VH(\mathbb{C}^N) \subset H\overline{V}(\mathbb{C}^N)$$

is a topological isomorphism into and when it is surjective.
Spaces of holomorphic functions

**Theorem 3.** Let $Q \subset \mathbb{C}^N$ be a convex locally closed set. If $Q$ is strictly convex at the relative boundary of $\omega$, then

$$VH(\mathbb{C}^N) = H\overline{V}(\mathbb{C}^N) \quad (algebraically and topologically)$$

- **Maltsev (1994):** Permit us to construct, if $Q \subset \mathbb{C}$ is locally closed and not strictly convex at the relative boundary of $\omega$, an entire function $P(z)$ of order at most one and zero type such that the linear differential operator $P(D)$ associated with $P(z)$ is not surjective on $H(Q)$. A reduction argument for $N > 1$ yields

**Proposition 4.** Suppose that $Q \subset \mathbb{C}^N$ is convex, locally closed and has a neighbourhood basis of domains of holomorphy.

If each nonzero differential operator $P(D) : H(Q) \rightarrow H(Q)$ is surjective

$$\Rightarrow \quad Q \text{ is strictly convex at the relative boundary of } \omega$$
**Theorem 5.** Let $Q \subset \mathbb{C}^N$ be a convex locally closed set. Suppose that $Q$ has a neighbourhood basis of domains of holomorphy. If $VH(\mathbb{C}^N)$ is a topological subspace of $H\overline{V}(\mathbb{C}^N)$, then $Q$ is strictly convex at the relative boundary of $\omega$.

**Proof.** Suppose that $VH(\mathbb{C}^N)$ is a topological subspace of $H\overline{V}(\mathbb{C}^N)$. With a division argument one shows that, for each nonzero entire function of order at most one and zero type $P$, the multiplication operator $M_P = P(D)^t : VH(\mathbb{C}^N) \to VH(\mathbb{C}^N)$ is an injective topological homomorphism. Since the space $H(Q)$ is reflexive, an application of Hahn-Banach theorem gives that

$$P(D) : H(Q) \longrightarrow H(Q)$$

is surjective for each such $P$. By Proposition 4, $Q$ is strictly convex at the relative boundary of $\omega$.

**Corollary 6.** Let $Q$ be a convex subset of $\mathbb{R}^N$ which is locally closed.

$VH(\mathbb{C}^N)$ is a topological subspace of its projective hull $H\overline{V}(\mathbb{C}^N)$

$\iff$ $Q$ is compact.
If a convex and locally closed set \( Q \subset \mathbb{C}^N \) is \( \mathbb{C} \)-strictly convex at the relative boundary of \( \omega \) then \( Q \) has a neighbourhood basis of domains of holomorphy.

In fact, by Martineau 1966, if \( Q \) is \( \mathbb{C} \)-strictly convex at the relative boundary of \( \omega \), then \( Q \) has a neighbourhood basis of linearly convex open sets, hence a basis of domains of holomorphy.

An open convex set in \( \mathbb{C}^N \) is linearly convex if its complement is a union of complex hyperplanes.

If \( Q \) is a convex and locally closed subset of \( \mathbb{R}^N \), then also \( Q \) has a neighbourhood basis of domains of holomorphy by a lemma of Cartan.
Spaces of continuous functions

The weighted (LF)-space of continuous functions $VC(\mathbb{C}^N)$ and its projective hull $C\widehat{V}(\mathbb{C}^N)$ associated with the sequence $V = (v_{n,k})_{n,k \in \mathbb{N}}$ of Section 2 are defined by replacing entire functions by continuous ones.

**Theorem 7. (Bierstedt, Meise, Summers, 1982)** For every locally closed convex set $Q \subset \mathbb{C}^N$ the weighted (LF)-space $VC(\mathbb{C}^N)$ is a topological subspace of its projective hull $C\widehat{V}(\mathbb{C}^N)$.

**Theorem 8.** Let $Q \subset \mathbb{C}^N$ is convex and locally closed. The following are equivalent:

(i) The algebraic equality $VC(\mathbb{C}^N) = C\widehat{V}(\mathbb{C}^N)$ holds.

(i)' $VC(\mathbb{C}^N) = C\widehat{V}(\mathbb{C}^N)$ (algebraically and topologically)

(ii) $Q$ is strictly convex at the relative boundary of $\omega$. 

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A necessary and sufficient condition for the algebraic equality $VH(C) = H\overline{V}(C)$ in the case of a bounded convex locally closed set $Q$ in $\mathbb{C}$ was obtained in 2003.

**Theorem 9.** Let $Q$ be a bounded convex locally closed subset of $\mathbb{C}^N$.

(i) Assume that the following conditions (*) holds:

There is a supporting hyperplane $\Pi$ to $\overline{Q}$ such that $\Pi \cap Q \neq \emptyset$ and there exists a $z_0 \in (\Pi \cap \overline{Q}) \setminus Q$ which is a smooth point of $\partial Q$,

then $VH(\mathbb{C}^N) \neq H\overline{V}(\mathbb{C}^N)$.

(ii) $VH(\mathbb{C}) \neq H\overline{V}(\mathbb{C})$ if and only if the condition (*) holds.

The algebraic identity $VH(\mathbb{C}^N) = H\overline{V}(\mathbb{C}^N)$ also holds in case $Q$ is a convex open subset of $\mathbb{R}^N$ as it was proved in 2004. This is the case of the Fourier Laplace transform of the space of analytic functionals.