The Canonical Spectral Mesure and Köthe spaces

José Bonet
Departament de Matemàtica Aplicada & IMPA
Universitat Politècnica de València

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Boolean algebras of projections/spectral measures in Banach spaces were intensively studied by W. Bade, N. Dunford and others. This is an extension of the notion of the resolution of the identity of a normal operator in Hilbert space which consists entirely of selfadjoint projections.

We consider the problem of describing the connection between

geometric/analytic properties of a general Fréchet space X	$\leftrightsquigarrow$	the operator/measure theoretic properties of any Bade complete Boolean algebra of projections in X (necessarily equicontinuous) which possesses a cyclic vector

This problem is reduced to a consideration of the multiplication operators (by characteristic functions) on some Fréchet function or sequence spaces over a σ-finite measure space.

We first recall the **Köthe echelon spaces**.

Let Γ denote either $\mathbb{N}$ or $\mathbb{N} \times \mathbb{N}$ or any infinite subset of these.

- An increasing sequence $A = (a_n)_{n \in \mathbb{N}}$ of strictly positive functions

$$a_n : \Gamma \longrightarrow (0, \infty)$$

is called a ***Köthe matrix*** on Γ, here increasing means

$$0 < a_n(i) \leq a_{n+1}(i), \qquad i \in \Gamma,\ n \in \mathbb{N}.$$

- To each $p \in [1, \infty)$ we associate the linear space

$$\boxed{\lambda_p(A) := \{x \in \mathbb{C}^\Gamma : q_n^{(p)}(x) := \Big(\sum_{i \in \Gamma} a_n(i)|x_i|^p\Big)^{1/p} < \infty \quad \text{for all} \quad n \in \mathbb{N}}$$

- For $p = 0$, we set

$$\boxed{\lambda_0(A) := \{x \in \mathbb{C}^\Gamma : a_n x \in c_0(\Gamma) \quad \text{for all } n \in \mathbb{N}\}}$$

equipped with the sup-seminorms $q_n^{(0)}(x)$.

The spaces $\lambda_p(A)$, for $p \in \{0\} \cup [1, \infty)$ are called ***Köthe echelon spaces (of order*** $\mathbf{p}$***)***; they are all Fréchet sequence spaces relative to the increasing sequence of seminorms

$$q_1^{(p)} \leq q_2^{(p)} \leq \dots$$

• For each $p \in [1, \infty)$ define the vector space

$$\ell^{p+} := \bigcap_{q>p} \ell^q$$

It is a Fréchet sequence space for the seminorms

$$q_{k,p}(x) := \left(\sum_{n=1}^{\infty} |x_n|^{\beta_k} \right)^{1/\beta_k}, \quad \text{for } x \in \ell^{p+}, \quad \text{where } \beta_k := p + \frac{1}{k} \quad \text{for } k \in \mathbb{N}.$$

The space ℓ^{p+} was investigated by **J.C. Diaz** , **Metafune** and **Moscatelli**.

• It is not Montel and has no infinite dimensional Banach subspaces. In particular, it is non-normable and not isomorphic to any Köthe echelon space $\lambda_q(A)$, for $q \in \{0\} \cup [1, \infty)$.

• The space ℓ^{p+} contains an infinite dimensional, complemented, nuclear Fréchet subspace with a basis.

The canonical spectral measure

- Let λ be one of the sequence spaces defined above.
- $2^{\mathbb{N}}$ denotes the σ-algebra of all the subsets of $\mathbb{N}$.
- For $E \in 2^{\mathbb{N}}$ and $x \in \lambda$, we set $\mathbf{P}(\mathbf{E})\mathbf{x} := \mathbf{x}\ \chi_{\mathbf{E}}$ which is also an element of λ.

In fact $P(E) : \lambda \mapsto \lambda$ is continuous, and we write $P(E) \in L(\lambda)$.

The set function

$$P(E) : x \mapsto x\chi_{E}, \qquad x \in \lambda,\ E \in 2^{\mathbb{N}},$$

is called the ***canonical spectral measure*** in λ.

The set $\{P(E) \mid E \in 2^{\mathbb{N}}\}$ is called a **Boolean algebra of projections** on $L(X)$.

• For a locally convex space X,

$\mathbf{L_s(X)}$ and $\mathbf{L_b(X)}$ denote the space of all the continuous linear operators from X into X endowed with the topology of uniform convergence on the finite subsets of X and on the bounded subsets of X respectively.

• If X is a Banach space,

$\mathbf{L_s(X)}$ is the space of all operators from X into X endowed with the strong operator topology SOT, and

$\mathbf{L_b(X)}$ is the space of all operators endowed with the operator norm.

Elementary properties of the spectral measure P

(1) $\mathbf{P}(\mathbb{N}) = \mathbf{I}$, the identity on λ, and $\mathbf{P}(\emptyset) = \mathbf{0}$.

(2) $\mathbf{P}$ **is multiplicative**, i.e. $P(E \cap F) = P(E)P(F)$.

In particular each $P(E)$ is a continuous projection on λ.

(3) $\mathbf{P} : \mathbf{2}^{\mathbb{N}} \mapsto \mathbf{L_s}(\lambda)$ **is** σ**-additive**

If $(E_k)_k$ is a sequence of disjoint subsets of $\mathbb{N}$, then

$$P(\cup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} P(E_k),$$

and the series converges in $L_s(\lambda)$.

It is clear now how to define a ***spectral measure***

$$P : \Sigma \mapsto L_s(X)$$

for a locally convex space X.

• The theory of spectral measures and Boolean algebras of projections in Banach spaces was initiated by Bade and N. Dunford. It is well understood.

• If X is a Hilbert space, the resolution of the identity of a normal operator yields a spectral measure.

• The case of non-normable locally convex spaces X has been investigated by Walsh, Okada and Ricker among others.

• There was a lack of concrete, non-trivial examples in the Fréchet-(DF) setting. This was one of the main motivations for our work.

• We investigate several questions about the spectral measure

$$P : \Sigma \mapsto L_s(X)$$

for

$$\lambda = \lambda_p(A) \quad \text{or} \quad \lambda = \ell^{p+},$$

and **connect the properties of the spectral measure with the structure of the sequence space.**

QUESTION 1

What can be said about the range $\mathbf{P}(\mathbf{2}^{\mathbb{N}}) := \{\mathbf{P}(\mathbf{E}) \mid \mathbf{E} \in \mathbf{2}^{\mathbb{N}}\}$ as a subset of $\mathbf{L}_{\mathrm{s}}(\lambda)$?

Let Y be a locally convex space and $m : \Sigma \to Y$ be a vector measure. The Orlicz-Pettis theorem implies

$$\sigma\text{-additivity of } m \quad \Leftrightarrow \quad \begin{cases} \sigma\text{-additivity of each } \mathbb{C}\text{-valued set function} \\ \qquad \langle m, y' \rangle : E \mapsto \langle m(E), y' \rangle, \\ \text{for } E \in \Sigma \text{ and } y' \in Y' \end{cases}$$

(a) (Kluvánek, Knowles, 1976) If Y is *quasicomplete*, then the range

$$m(\Sigma) := \{m(E) : E \in \Sigma\}$$

is a relatively weakly compact subset of Y.

Since the spectral measure $P : 2^{\mathbb{N}} \mapsto L_s(\lambda)$ has countably many atoms, a result of Hoffmann–Jørgensen, 1971, implies that

(b) The range $\mathbf{P}(\mathbf{2}^{\mathbb{N}})$ is compact in $\mathbf{L}_{\mathbf{s}}(\lambda)$.

Proposition 1.

(i) Let $p \in [1, 2)$. Then every $\lambda_p(A)$-valued vector measure has relatively compact range.

(ii) Let $p \in [2, \infty)$. Then

$$\lambda_p(A) \text{ is a Montel space} \quad \Leftrightarrow \left\{ \begin{array}{l} \text{if every } \lambda_p(A) - \text{valued vector measure} \\ \text{has relatively compact range.} \end{array} \right\}$$

(iii) Let $p \in [1, \infty)$. Then ℓ^{p+} has the property that

$$\left\{ \begin{array}{l} \text{every } \ell^{p+}\text{-valued vector measure} \\ \text{has relatively compact range} \end{array} \right\} \quad \Leftrightarrow \quad p \in [1, 2)$$

QUESTION 2

When is $\mathbf{P} : \mathbf{2}^{\mathbb{N}} \mapsto \mathbf{L_b}(\lambda)$ σ**-additive?**

A spectral measure $P : \Sigma \mapsto L_s(X)$ is called ***boundedly*** $\boldsymbol{\sigma}$***-additive*** if $P : \Sigma \mapsto L_b(X)$ is σ-additive.

• If X is a Banach space, then boundedly σ-additive spectral measures are trivial.

This is due to the fact that $R = 0$ whenever a projection $R \in L(X)$ satisfies $\|R\| < 1$.

• On the other hand, if a Fréchet space X is Montel, then every spectral measure $P : \Sigma \mapsto L_s(X)$ is boundedly σ-additive.

Proposition 2.

(i) *For every* $p \in [1, \infty)$, *the canonical spectral measure*

$$P : 2^{\mathbb{N}} \longrightarrow L_s(\ell^{p+})$$

fails to be boundedly σ*-additive.*

(ii) *For some (all)* $p \in \{0\} \cup [1, \infty)$ *and any Köthe matrix* A *on* Γ, *the canonical spectral measure* $P : 2^{\Gamma} \longrightarrow L_s(\lambda_p(A))$ *is boundedly* σ*-additive if and only if* $\lambda_p(A)$ *is a Montel space (and if and only if* $\lambda_1(A)$ *is reflexive).*

QUESTION 3

When does $P : 2^{\mathbb{N}} \mapsto L_s(\lambda)$ have finite variation?

Let Y be a lcHs with topology determined by a family of continuous seminorms $\mathscr{N}$.

Let $Y/q^{-1}(\{0\})$ be the quotient normed space determined by $q \in \mathscr{N}$ and Y_q denote its Banach space completion.

The norm in Y_q is denoted by $\|\cdot\|_q$ and the canonical quotient map of Y onto $Y/q^{-1}(\{0\})$ is denoted by ρ_q.

Given any Y-valued vector measure defined on a measurable space (Ω, Σ), the continuity of ρ_q ensures that $m_q := \rho_q \circ m$ is a vector measure on Σ with values in $Y/q^{-1}(\{0\}) \hookrightarrow Y_q$, for each $q \in \mathscr{N}$.

• The ***variation measure*** $|m_q| : \Sigma \to [0, \infty]$ of the Banach-space-valued measure m_q is defined in the usual way.

• The variation $|m_q|$ is called ***finite*** if $|m_q|(\Omega) < \infty$.

• We say that m has ***finite variation*** if m_q has finite variation for $every$ $q \in \mathscr{N}$.

Proposition 3. *Let X be a nuclear Fréchet space. Every $L_s(X)$-valued measure is boundedly σ-additive and has finite variation in both $L_s(X)$ and $L_b(X)$.*

Proposition 4. *Let A be a Köthe matrix.*

(i) *Let $p \in \{0\} \cup (1, \infty)$.*

$$\left\{ \begin{array}{c} \textit{The canonical spectral measure} \\ P : 2^{\mathbb{N}} \to L_s(\lambda_p(A)) \\ \textit{has finite variation} \end{array} \right\} \quad \Leftrightarrow \quad \lambda_p(A) \textit{ is nuclear.}$$

(ii) *The spectral measure $P : 2^{\mathbb{N}} \to L_s(\lambda_1(A))$ always has finite variation.*

(iii) *The canonical spectral measure $P : 2^{\mathbb{N}} \to L_s(\ell^{p+})$ fails to have finite variation for every $p \in [1, \infty)$.*

Proposition 5. *Let A be a Köthe matrix with $\lambda_1(A)$ Montel.*

$$\left\{ \begin{array}{c} \textit{The canonical spectral measure} \\ P : 2^{\mathbb{N}} \to L_b(\lambda_1(A)) \\ \textit{has finite variation} \end{array} \right\} \quad \Leftrightarrow \quad \lambda_1(A) \textit{ is nuclear.}$$

This result depends on the characterization of bounded subsets of $\lambda_1(A)$ due to Bierstedt, Meise and Summers.

QUESTION 4

What are the $\mathbf{P}$-integrable functions for $\mathbf{P} : \mathbf{2}^{\mathbb{N}} \mapsto \mathbf{L_b}(\lambda)$?

• Associated with any $L_s(X)$-valued spectral measure Q (defined on some measurable space (Ω, Σ)) is the space

$\mathscr{L}^1(Q)$ of all Q-integrable functions $f : \Omega \to \mathbb{C}$

and the space

$\mathscr{L}^\infty(Q)$ of Q-essentially bounded functions.

• In the setting of Banach spaces X,

$$\mathscr{L}^1(Q) = \mathscr{L}^\infty(Q) \quad \text{as vector spaces,}$$

that is,

the *only* Q-integrable functions are the Q-essentially bounded ones.

This is a result due to Dunford. For non-normable spaces X, this is surely not the case in general.

• We investigate when is the containment $\mathscr{L}^\infty(P) \subset \mathscr{L}^1(P)$ strict.

Instead of recalling the definition of Q-integrable functions, we prefer to present the following characterization, due to Okada and Ricker, 1999, which is valid in the present setting.

Proposition 6. *Let $P : 2^{\mathbb{N}} \to L_s(\lambda)$ be the canonical spectral measure.*

(i) A function $f \in \mathbb{C}^{\mathbb{N}}$ *belongs to* $\mathscr{L}^1(P) \quad \Leftrightarrow \quad \lambda f \subseteq \lambda.$

Moreover, $\displaystyle\int_{\mathbb{N}} f \, dP$ *is the multiplication operator*

$$M_f : x \mapsto xf, \text{ for } x \in \lambda.$$

(ii) A function $f \in \mathbb{C}^{\mathbb{N}}$ *belongs to* $\mathscr{L}^\infty(P) \quad \Leftrightarrow \quad f \in \ell^\infty.$

Our problem is then to decide whether there are unbounded multipliers on the sequence space λ. We state first the result for ℓ^{p+}.

Proposition 7. *The canonical spectral measure* $P : 2^{\mathbb{N}} \to L_s(\ell^{p+})$ *satisfies*

$$\mathscr{L}^1(P) = \ell^\infty.$$

Lemma 8. *Let* $p \in \{0\} \cup [1, \infty)$, A *be a Köthe matrix and* $f \in \mathbb{C}^{\mathbb{N}}$.

$$\lambda_p(A)f \subseteq \lambda_p(A) \quad \Leftrightarrow \quad \forall\, n \in \mathbb{N} \;\; \exists\, m_n \geq n : \quad \frac{a_n f}{a_{m_n}} \in \ell^\infty.$$

Proposition 9. *Let $p \in \{0\} \cup [1, \infty)$ and A be a Köthe matrix. Then*

$\exists\ f \in \mathbb{C}^{\mathbb{N}} \setminus \ell^{\infty}$: $\lambda_p(A) f \subseteq \lambda_p(A)$	$\Leftrightarrow$	*there exists an infinite set $J \subseteq \mathbb{N}$ such that the sectional subspace $\lambda_p(J, A)$ is Schwartz.*

Corollary 10. *Let $p \in \{0\} \cup [1, \infty)$ and A be a Köthe matrix.*

(i) For the canonical spectral measure P,

$\mathscr{L}^{\infty}(P) \subseteq \mathscr{L}^{1}(P)$ *is proper*	$\Leftrightarrow$	*there exists an infinite set $J \subseteq \mathbb{N}$ such that the sectional subspace $\lambda_p(J, A)$ is Schwartz.*

(ii) If $\lambda_p(A)$ satisfies the density condition and is non-normable, then the inclusion $\mathscr{L}^{\infty}(P) \subseteq \mathscr{L}^{1}(P)$ is proper.

• A Fréchet space satisfies the ***density condition*** if the bounded sets of its strong dual space are metrizable.

This condition in Fréchet and Köthe echelon spaces was thoroughly investigated by Bierstedt and Bonet.

Fréchet $+$ **density condition** $\Rightarrow$ **distinguished**
(its strong dual is barrelled)

Quasinormable $\searrow$

density condition

Montel Fréchet $\nearrow$

The following result summarizes work by Bierstedt, Meise, Bonet; it was later complemented by Bastin and Vogt.

Proposition 11. *Let A be a Köthe matrix on Γ.*

(a) $\lambda_p(A)$ satisfies the density condition for some (all) $p \in \{0\} \cup [1, \infty)$.

(b) $\lambda_1(A)$ is distinguished.

(c) Condition (D) holds for A, that is, there exists an increasing sequence $(\Gamma_m)_{m \in \mathbb{N}}$ of subsets of Γ such that:

$$\forall\, m \quad \exists\, n(m) \quad \forall\, k > n(m) : \quad \inf_{i \in \Gamma_m} \frac{a_{n(m)}(i)}{a_k(i)} > 0 \tag{D1}$$

and

$$\forall\, n\, \forall\, \Gamma_0 \subseteq \Gamma \text{ with } \Gamma_0 \cap (\Gamma \setminus \Gamma_m) \neq \emptyset\, (\forall\, m \in \mathbb{N}), \exists\, n^* = n^*(n, \Gamma_0) > n : \tag{D2}$$

$$\inf_{i \in \Gamma_0} \frac{a_n(i)}{a_{n^*}(i)} = 0.$$

Proposition 12. *Suppose A is a Köthe matrix on $\Gamma = \mathbb{N}$ such that $\lambda_p(A)$ is a Montel space for some $p \in \{0\} \cup [1, \infty)$. Then there exists an infinite set $J \subseteq \mathbb{N}$ such that the sectional subspace $\lambda_p(J, A)$ is a Schwartz space.*

Corollary 13. *Let A be a Köthe matrix on $\Gamma = \mathbb{N}$ and $p \in \{0\} \cup [1, \infty)$. Suppose that $\lambda_p(A)$ satisfies the density condition and is non-normable. Then there exists an infinite set $J \subseteq \mathbb{N}$ such that the sectional subspace $\lambda_p(J, A)$ is Schwartz.*

It remains to treat the case of Köthe echelon spaces $\lambda_p(A)$ without the density condition. We prefer to restrict our attention to a class of examples.

We recall a particular class of Köthe matrices A, the so called ***Köthe-Grothendieck*** (briefly, KG) matrices. In this case,

$\Gamma = \mathbb{N} \times \mathbb{N}$ and $a_n : \mathbb{N} \times \mathbb{N} \longrightarrow (0, \infty)$ for $n \in \mathbb{N}$

- $a_n(i, j) = 1,$ for all $j, n \in \mathbb{N}$ and $i > n$. (KG-1)
- $\sup_{j \in \mathbb{N}} a_n(n, j) = \infty,$ for all $n \in \mathbb{N}$. (KG-2)
- $a_p(i, j) = a_q(i, j),$ for all $i, j \in \mathbb{N}$ and all $p, q \geq i$. (KG-3)

The original KG-matrix corresponds to

$$a_n(i, j) := \begin{cases} j & \text{for } i \leq n \text{ and } j \in \mathbb{N} \\ 1 & \text{for } i > n \text{ and } j \in \mathbb{N}, \end{cases} \qquad \text{for each } n \in \mathbb{N}.$$

Some known facts are as follows; part (iii) is due to Albanese.

Proposition 14. *Let A be any KG-matrix on $\Gamma = \mathbb{N} \times \mathbb{N}$.*

(i) $\lambda_1(A)$ *is not distinguished.*

(ii) *For each $p \in \{0\} \cup [1, \infty)$, the Fréchet space $\lambda_p(A)$ fails to satisfy the density condition.*

(iii) *For each $p \in \{0\} \cup [1, \infty)$, the Fréchet space $\lambda_p(A)$ has no complemented subspace which is Montel.*

Proposition 15. *Let* $p \in \{0\} \cup [1, \infty)$ *and* A *be any KG-matrix on* $\Gamma = \mathbb{N} \times \mathbb{N}$. *Then, for the canonical spectral measure* P *in* $\lambda_p(A)$, *we have* $\mathscr{L}^1(P) = \mathscr{L}^\infty(P) = \ell^\infty(\Gamma)$. *That is, the only multipliers for* $\lambda_p(A)$ *are those in* $\ell^\infty(\Gamma)$.

Proof. If $f \in \ell^\infty(\Gamma) = \mathscr{L}^\infty(P) \subseteq \mathscr{L}^1(P) \quad \Rightarrow \quad \lambda_p(A) f \subseteq \lambda_p(A)$.

Conversely, suppose that $f \in \mathbb{C}^\Gamma$ satisfies $\lambda_p(A) f \subseteq \lambda_p(A)$.

For each $n \in \mathbb{N}$ there exists $m_n \geq n$ and $C_n > 0$ such that

$$a_n |f| \leq C_n a_{m_n} \text{ on } \Gamma.$$

$$\left. \begin{array}{l} \text{(KG-1)} KG - 1 \\ (n = 1) \end{array} \right\} \quad \Rightarrow \quad |f(i,j)| \leq C_1 \frac{a_{m_1}(i,j)}{a_1(i,j)} \leq C_1, \qquad i > m_1, j \in \mathbb{N}.$$

for some $C_1 > 0$ and $m_1 \in \mathbb{N}$.

Now select $k > m_1$ and $D > 0$ such that

$$|f| \leq D \, \frac{a_k}{a_{m_1}} \text{ on } \Gamma.$$

For $i \leq m_1$ and $j \in \mathbb{N}$,

$$\text{(KG-3)} \;\Rightarrow\; \quad |f(i,j)| \leq D \, \frac{a_k(i,j)}{a_{m_1}(i,j)} \;=\; D.$$

Accordingly, $f \in \ell^\infty(\Gamma)$. $\square$

This proposition gives a large class of non-normable Fréchet spaces, namely $\lambda_p(A)$ for $p \in \{0\} \cup [1, \infty)$ and A any KG-matrix, and a non-trivial spectral measure, namely P, with the property that $\mathscr{L}^1(P) = \mathscr{L}^\infty(P)$

Spaces of measurable functions

One can consider also spectral measures defined on spaces of measurable functions.

- We recall the separable Fréchet spaces

$$L_{p-} := \bigcap_{1 \leq r < p} L^r([0,1]), \quad \text{for } p \in (1,\infty),$$

equipped with the seminorms

$$q_{p,m}(f) := \|f\|_{\beta(m)} = \left(\int_0^1 |f(t)|^{\beta(m)}\, dt \right)^{1/\beta(m)}$$

for every $f \in L_{p-}$ and any increasing sequence $1 \leq \beta(m) \uparrow p$ as $m \to \infty$.

• We have, with continuous inclusions, that

$$L^p([0,1]) \quad \hookrightarrow \quad L_{p-} \hookrightarrow \quad L^r([0,1]), \qquad 1 \leq r < p.$$

Each of the spaces L_{p-}, for $p \in (1, \infty)$, is reflexive and none of them is Montel.

• For each $p \in (1, \infty)$, the set function given by

$$\widetilde{P}(E) : f \mapsto f\chi_{_E}, \qquad f \in L_{p-}$$

for $E \in \mathcal{B}$ (the σ-algebra of Borel subsets of $[0,1]$), defines a spectral measure

$$\widetilde{P} : \mathcal{B} \to L_s(L_{p-}).$$

The spectral measure $\widetilde{P}$ has no atoms.

Proposition 16.

(i) The spectral measure $\widetilde{P} : \mathcal{B} \to L_s(L_{p-})$ *fails to have finite variation for every* $p \in (1, \infty)$.

(ii) For every $p \in (1, \infty)$, *the spectral measure* $\widetilde{P} : \mathcal{B} \to L_s(L_{p-})$ *is boundedly* σ*-additive in* $L_b(L_{p-})$.

Proposition 17. *Let* $p \in (1, \infty)$.

(i) *A Borel measurable function* $\varphi : [0,1] \to \mathbb{C}$ *belongs to* $\mathscr{L}^1(\widetilde{P})$ *if and only if* $D_p(M_\varphi) = L_{p-}$, *that is,* $\varphi L_{p-} \subseteq L_{p-}$. *In this case*

$$\int_{[0,1]} \varphi \, d\widetilde{P} = M_\varphi.$$

(ii) *As a vector space*

$$\mathscr{L}^1(\widetilde{P}) = \bigcap_{1 \le q < \infty} L^q([0,1]).$$

In particular, the inclusion $\mathscr{L}^\infty(\widetilde{P}) \subseteq \mathscr{L}^1(\widetilde{P})$ *is proper.*

To show that $\mathscr{L}^\infty(\widetilde{P}) \subseteq \mathscr{L}^1(\widetilde{P})$ is a proper inclusion, let $\{F(n)\}_{n=1}^\infty$ be any pairwise disjoint sequence of sets in $\mathcal{B}$ satisfying

$$\lambda(F(n)) = e^{-n}, \quad \text{for } n \in \mathbb{N}.$$

Then $\varphi := \sum_{n=1}^{\infty} n\chi_{F(n)}$ is surely not in $L^\infty([0,1]) = \mathscr{L}^\infty(\widetilde{P})$.

However, for any $q \in [1, \infty)$ we have

$$\|\varphi\|_q^q = \sum_{n=1}^{\infty} n^q e^{-n} < \infty,$$

then

$$\varphi \in L^q([0,1]).$$

Accordingly, $\varphi \in \mathscr{L}^1(\widetilde{P})$.

It is also possible to consider the following more general frame.

(Ω, Σ, μ) is a σ-finite measure space.

$\mathscr{M}^+$ is the set of non negative measurable functions.

$\rho : \mathscr{M}^+ \to [0, \infty]$ is a *function norm*.

• If $a \in \mathscr{M}^+$, $0 < a < \infty$ (μ-a.e. on Ω) is a measurable function then

$$L_\rho(a) := \{f \in \mathscr{M} : \ \rho(af) < \infty\}$$

is a Banach function space.

A *Köthe matrix* $A = (a_n)$ on Ω is a sequence of functions $a_n \in \mathscr{M}^+$, for $n \in \mathbb{N}$, which satisfy $0 < a_n \leq a_{n+1} < \infty$ (μ-a.e. on Ω). Then

$$L_\rho(a_{n+1}) \subseteq L_\rho(a_n) \quad \text{for all } n \in \mathbb{N}$$

and

$$\boxed{L_\rho(A) := \bigcap_{n=1}^{\infty} L_\rho(a_n)} \qquad \textbf{(\textit{Köthe function space})}$$

is a Fréchet space (and lattice for the μ-a.e. order).

For $E \in \Sigma$ we define the multiplication operator

$$\begin{array}{rcl} Q(E) : L_\rho(A) & \longrightarrow & L_\rho(A) \\ f & \mapsto & \chi_E f \end{array} \qquad \Rightarrow \quad Q(E) \in L(L_\rho(A))$$

The map $Q : \Sigma \to L_s(L_\rho(A))$ is called the ***canonical spectral measure*** in $L_\rho(A)$.

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