Convolution operators on quasianalytic classes that admit a continuous linear right inverse

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On joint work with R. Meise (Düsseldorf, Germany)
**Weight function**

A function \( \omega : \mathbb{R} \to [0, \infty] \) is called a **weight function** if it is continuous, even, increasing on \([0, \infty]\), satisfies \( \omega(0) = 0 \), and also the following conditions:

(\(\alpha\)) There exists \( K \geq 1 \) such that \( \omega(2t) \leq K \omega(t) + K \) for all \( t \geq 0 \).

(\(\beta\)) \( \omega(t) = o(t) \) as \( t \) tends to infinity.

(\(\gamma\)) \( \log(t) = o(\omega(t)) \) as \( t \) tends to infinity.

(\(\delta\)) \( \varphi : t \mapsto \omega(e^t) \) is convex on \([0, \infty]\).

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**quasianalytic weight function**

\[
\int_1^\infty \frac{\omega(t)}{t^2} dt = \infty
\]

Otherwise it is called **non-quasianalytic**.

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- **radial extension of** \( \omega : \tilde{\omega} : \mathbb{C}^n \to [0, \infty] \), \( \tilde{\omega}(z) := \omega(|z|) \).

**Example.** The function \( \omega(t) := |t|(\log(e + |t|))^{-\alpha} \), \( \alpha > 0 \) is a weight function which is quasianalytic if and only if \( 0 < \alpha \leq 1 \).
Let \( \omega \) be a given weight function, let \( K \) be a compact and \( G \) be an open subset of \( \mathbb{R}^N \).

### \( \omega \)-ultradifferentiable functions of Beurling type on \( G \)

\[
\mathcal{E}_\omega(G) = \{ f \in C^\infty(G) : \text{for each } K \subset G \text{ compact and } m \in \mathbb{N} \\
p_{K,m}(f) := \sup_{x \in K} \sup_{\alpha \in \mathbb{N}_0^N} |f^{(\alpha)}(x)| \exp \left( -m \varphi^*(|\alpha|) \right) < \infty \}.
\]

\( \varphi^*(x) := \sup \{ xy - \varphi(y) : y > 0 \}, \ x \geq 0. \)

- \( \mathcal{E}_\omega(G) \) is a Fréchet space if we endow it with the locally convex topology given by the semi-norms \( p_{K,m} \).

\[
\mathcal{E}_\omega(K) := \{ f \in C^\infty(K) : p_{K,m}(f) < \infty \quad \forall \ m \in \mathbb{N} \}.
\]
The Convolution Operators

For $\mu \in \mathcal{E}(\omega)'$, $\mu \neq 0$, and $\varphi \in \mathcal{E}(\omega)(\mathbb{R})$ we define

$\tilde{\mu}(\varphi) := \mu(\tilde{\varphi})$, $\tilde{\varphi}(x) := \varphi(-x)$, $x \in \mathbb{R}$.

The convolution operator

$T_\mu : \mathcal{E}(\omega)(\mathbb{R}) \to \mathcal{E}(\omega)(\mathbb{R})$

$f \mapsto T_\mu(f) := \tilde{\mu} * f$, $$(\tilde{\mu} * f)(x) := \tilde{\mu}(f(x - .)), x \in \mathbb{R}.$$ It is a well-defined, linear, continuous operator.

Problem

When does operator $T_\mu$ admit a continuous linear right inverse?
The problem was solved by **Meise** and **Vogt** in 1987

- Their proof used non quasianalyticity in an essential way.

- However, the basic tool here and in their paper is the Fourier Laplace transform

\[
\mathcal{F}(u)(z) = \hat{u}(z) := u_x(e^{-i\langle x, z \rangle}).
\]

\[\mathcal{F} : \mathcal{E}(\omega)(\mathbb{R})' \to A(\omega)\] is a linear topological isomorphism.

- Moreover, \( \mathcal{F} \circ T^t_\mu = M_\hat{\mu} \circ \mathcal{F} \) for each \( \mu \in \mathcal{E}(\omega)(\mathbb{R})' \). Here \( M_\hat{\mu} \) is the multiplication operator.

\[
A(\omega) = \{ f \in H(\mathbb{C}) : \exists n \in \mathbb{N} : \| f \|_n := \sup_{z \in \mathbb{C}} |f(z)| \exp(-n(|\text{Im } z| + \omega(z))) < \infty \}
\]
Theorem 1. (Momm, 1992)

For each weight function $\omega$ the following conditions are equivalent for $\mu \in \mathcal{E}'(\omega)(\mathbb{R})$, $\mu \neq 0$:

1. $T_\mu : \mathcal{E}(\omega)(\mathbb{R}) \to \mathcal{E}(\omega)(\mathbb{R})$ is surjective.

2. The principal ideal $\hat{\mu}A(\omega)$ is closed in $A(\omega)$.

3. $\hat{\mu}$ is $(\omega)$-slowly decreasing in the sense of Ehrenpreis, i.e., there exist $k > 0$, $x_0 > 0$, such that for each $x \in \mathbb{R}$ with $|x| \geq x_0$ there exists $t \in \mathbb{R}$ with $|t - x| \leq k\omega(x)$ such that

$$|\hat{\mu}(t)| \geq \exp(-k\omega(t)).$$

4. $\hat{\mu}$ is $(\omega)$-slowly decreasing, i.e., there exists $C > 0$ such that for each $x \in \mathbb{R}$, $|x| \geq C$ there exists $\xi \in \mathbb{C}$ such that

$$|x - \xi| \leq C\omega(x) \text{ and } |\hat{\mu}(\xi)| \geq \exp(-C|\text{Im}\,\xi| - C\omega(\xi)).$$
Proposition 2.

Let $\omega$ be a weight function which satisfies $(\alpha_1)$ and let $\mu \in \mathcal{E}'(\omega)(\mathbb{R})$ be given. If the convolution operator

$$T_\mu : \mathcal{E}(\omega)(\mathbb{R}) \to \mathcal{E}(\omega)(\mathbb{R})$$

admits a continuous linear right inverse then the following two conditions are satisfied:

(a) $\hat{\mu}$ is $(\omega)$-slowly decreasing.

(b) There exists $C > 0$ such that

$$|\text{Im } a| \leq C(1 + \omega(a)), \quad a \in \mathbb{C}, \quad \hat{\mu}(a) = 0.$$
A weight function $\omega$ satisfies the condition $(\alpha_1)$ if

$$\sup_{\lambda \geq 1} \limsup_{t \to \infty} \frac{\omega(\lambda t)}{\lambda \omega(t)} < \infty.$$ 

This condition was introduced by Petzsche and Vogt in 1984.

$$\exists C_1 > 0 \text{ such that for each } W \geq 1 \text{ there exists } C_2 > 0 \text{ such that }$$

$$\omega(Wt + W) \leq WC_1 \omega(t) + C_2, \quad t \geq 0.$$
Idea of the proof of Proposition 2.

- The surjectivity of $T_\mu$ follows trivially. Assume that condition (b) does not hold.

- We can choose a sequence $(a_j)_{j \in \mathbb{N}}$ of complex numbers, such that $\hat{\mu}(a_j) = 0$ and a weight function $\sigma$ with $\omega = o(\sigma)$, which also satisfies the condition $(\alpha_1)$, with $\sigma(a_j) = O(|\text{Im } a_j|)$ as $j \to \infty$, and such that

$$F(z) := \prod_{j=1}^{\infty} (1 - \frac{z}{a_j}), \quad z \in \mathbb{C},$$

is an entire function with

$$\sup_{z \in \mathbb{C}} |F(z)| \exp(-n\sigma_0(z)) < \infty$$

for some $n$ and some weight $\sigma_0 = o(\omega)$. 

Moreover, $M_f : A(\omega) \to A(\omega)$ has closed range, and there is $g \in A(\omega)$ such that $\hat{\mu} = gF$.

Since $T_\mu$ admits a continuous linear right inverse, $M_\hat{\mu}$ has a continuous linear left inverse $L_\hat{\mu}$. This implies that $M_F$ has this property too, since

$$L_\hat{\mu} M_g M_F(h) = L_\hat{\mu}(gFh) = L_\hat{\mu}(\hat{\mu}h) = h.$$ 

To continue, we denote by $A\{\sigma\}$ the space of all the entire functions $g$ satisfying that there is $n$ such that for each $m$ we have

$$\sup_{z \in \mathbb{C}} |g(z)| \exp(-n|\text{Im } z| - \frac{1}{m}\sigma(z)) < \infty.$$ 

Observe that $A(\omega) \subset A\{\sigma\}$. 
Define $\varrho : H(\mathbb{C}) \rightarrow \mathbb{C}^\mathbb{N}$, $\varrho(f) := (f(a_j))_{j \in \mathbb{N}}$. By the properties of the sequence $(a_j)_j$ and results due to Meise in 1985 and 1989, we have the diagram

$$
0 \rightarrow A(\omega) \xrightarrow{M_F} A(\omega) \xrightarrow{\varrho_1} E \rightarrow 0
$$

$$
\cap \quad \cap
$$

$$
0 \rightarrow A\{\sigma\} \xrightarrow{M_F} A\{\sigma\} \xrightarrow{\varrho_2} G \rightarrow 0,
$$

where $\varrho_1$ and $\varrho_2$ are the restrictions of $\varrho$, and $E$ and $G$ are sequence spaces, which coincide algebraically and topologically with the power series space $\Lambda_\infty((|\text{Im} a_j|)_{j \in \mathbb{N}})'$ and are isomorphic to the corresponding quotients.
Since $\varrho_1$ admits a continuous linear right inverse, so does $\varrho_2$.

Therefore $\Lambda_\infty((|\text{Im } a_j|)_{j \in \mathbb{N}})'$ is isomorphic to a complemented subspace of $A_{\{\sigma\}}$, hence $\Lambda_\infty((|\text{Im } a_j|)_{j \in \mathbb{N}})$ is isomorphic to a quotient of the space $\mathcal{E}_{\{\sigma\}}(\mathbb{R}) = A'_{\{\sigma\}}$ of Roumieu type.

By a result of Vogt 2002 (see also Bonet, Domanski 2006), this cannot be the case if condition $(\alpha_1)$ holds. This contradiction completes the proof.

Remark

For $\omega(t) = t$ we get $\mathcal{E}(\omega)(\mathbb{R}) = H(\mathbb{C})$. Meise and Taylor proved that each convolution operator on $H(\mathbb{C})$ admits a continuous linear right inverse. This is the reason why we assume $\omega(t) = o(t)$ as $t$ tends to infinity.
Sufficient condition for the existence of right inverse

To prove a partial converse of Proposition 2 we need the following condition on the weight.

**(DN)-weight function**

For each \( C > 1 \) there exist \( R_0 > 0 \) and \( 0 < \delta < 1 \) such that for each \( R \geq R_0 \)

\[
\omega^{-1}(CR)\omega^{-1}(\delta R) \leq (\omega^{-1}(R))^2.
\]

- (DN)-weight functions were introduced by Meise and Taylor in 1987.
- It is equivalent to the property \( (DN) \) of Vogt for the Fréchet space \( A'_\omega \).

\[
A_\omega = \{ f \in H(\mathbb{C}) \mid \exists n \in \mathbb{N} : \sup_{z \in \mathbb{C}} |f(z)| \exp(-n\omega(z)) < \infty \}
\]
Let $\omega$ be a weight function for which there exists $A > 0$ such that

$$2\omega(t) \leq \omega(At) + A, \ t \geq 0.$$ 

Then $\omega$ is a (DN)-weight function.

- The following functions are quasianalytic (DN)-weight functions which also satisfy ($\alpha_1$):

  (1) $\omega(t) := \frac{|t|}{(\log(e + |t|))^{\alpha}}, \ 0 < \alpha \leq 1.$

  (2) $\omega(t) := \frac{|t|}{(\log(e + \log(e + |t|)))}.$
Proposition 3

Let \( \omega \) be a (DN)-weight function. Then for \( \mu \in \mathcal{E}'(\omega)(\mathbb{R}) \) the convolution operator

\[
T_\mu : \mathcal{E}(\omega)(\mathbb{R}) \to \mathcal{E}(\omega)(\mathbb{R})
\]

admits a continuous linear right inverse if the following conditions hold.

(a) \( \hat{\mu} \) is \((\omega)\)-slowly decreasing.

(b) There exists \( C > 0 \) such that

\[
|\text{Im } a| \leq C(1 + \omega(a)), \ a \in \mathbb{C}, \ \hat{\mu}(a) = 0.
\]
Idea of the Proof of Proposition 3.

- By duality and the Fourier Laplace transform, it suffices to show that the short exact sequence

$$
0 \longrightarrow A_{(\omega)} \xrightarrow{M_{\hat{\mu}}} A_{(\omega)} \xrightarrow{\varrho} A_{(\omega)}/\hat{\mu}A_{(\omega)} \longrightarrow 0
$$

splits.

- It is no restriction to assume that $\hat{\mu}$ has infinitely many zeros. In this case, Meise proved in 1985 that the quotient $A_{(\omega)}/\hat{\mu}A_{(\omega)}$ is isomorphic to a sequence space $\Lambda$. 

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In fact, after guessing what \( \Lambda \) is, one defines a linear map 
\[ \Phi : A(\omega) \longrightarrow \Lambda \text{ with } \ker(\Phi) = \hat{\mu} A(\omega). \]

Several steps are needed to show that \( \Phi \) is surjective. All but one can be done in a continuous linear way.

Since \( \omega \) is a \((DN)\)-weight function, we can use a result due to Meise and Taylor on the splitting of certain \( \partial \)-complex, and show that \( \Phi \), and hence \( \varrho \), admits a continuous linear right inverse.
We restrict the attention now to \((\omega)\)-ultradifferential operators, and we see that the situation changes.

**Proposition 4**

Let \(\omega\) be a \((\text{DN})\)-weight function. Assume that for \(\mu \in \mathcal{E}'(\omega)(\mathbb{R})\) its Fourier-Laplace transform \(\hat{\mu}\) is \((\omega)\)-slowly decreasing and satisfies 
\[|\hat{\mu}(z)| \leq C \exp(C\omega(z))\]
for some \(C > 0\) and all \(z \in \mathbb{C}\). Then for each \(a, b \in \mathbb{R}\) with \(a < b\) the sequence
\[
0 \longrightarrow \ker T_{\mu,[a,b]} \longrightarrow \mathcal{E}(\omega)[a,b] \xrightarrow{T_{\mu,[a,b]}} \mathcal{E}(\omega)[a,b] \longrightarrow 0
\]
is exact and splits.

**Remark**

Let \(\omega\) be a quasianalytic \((\text{DN})\)-weight function, which satisfies the condition \((\alpha_1)\). Then there exist \((\omega)\)-ultradifferential operators \(T_\mu\) which admit a continuous linear right inverse on \(\mathcal{E}(\omega)[a,b]\), but which do not admit a continuous linear right inverse on \(\mathcal{E}(\omega)(\mathbb{R})\).
Ingredients of the proof of Proposition 4.

(1) $E(\omega)[a, b]$ has $(DN)$, since $\omega$ is a $(DN)$-weight.

(2) $\ker T_{\mu,[a,b]}$ is isomorphic to a power series space of infinite type.

(3) $T_{\mu,[a,b]}$ is surjective.

Hence the result follows from the splitting theorem of Vogt and Wagner.
We obtain the following result which is due in the real analytic case to Langenbruch 1994 and to Domanski and Vogt 2001.

**Theorem 5.**

Let $\omega$ be a quasianalytic (DN)-weight function which satisfies the condition $\left(\alpha_1\right)$. Assume that for $\mu \in \mathcal{E}'(\omega)(\mathbb{R})$ its Fourier-Laplace transform $\hat{\mu}$ is $(\omega)$-slowly decreasing and satisfies

$$|\hat{\mu}(z)| \leq C \exp(C\omega(z))$$

for some $C > 0$ and all $z \in \mathbb{C}$. Then the following assertions are equivalent:

1. $T_\mu : \mathcal{E}(\omega)(\mathbb{R}) \to \mathcal{E}(\omega)(\mathbb{R})$ admits a continuous linear right inverse.

2. There exists $C > 0$ such that $|\text{Im} \, a| \leq C(\omega(a) + 1)$ for each $a \in V(\hat{\mu})$.

3. For each/some $a, b \in \mathbb{R}$ with $a < b$ and each $f \in \ker T_\mu, [a,b]$ there exists $g \in \ker T_\mu$ such that $f = g|_{[a,b]}$. 
Let $\omega$ be a given weight function. For a compact subset $K$ of $\mathbb{R}^N$ and $m \in \mathbb{N}$

$$
\|f\|_{K,m} := \sup_{x \in K} \sup_{\alpha \in \mathbb{N}_0^N} |f^{(\alpha)}(x)| \exp \left( -\frac{1}{m} \varphi^*(m|\alpha|) \right).
$$

For an open set $G$ in $\mathbb{R}^N$,

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$$
\mathcal{E}_{\{\omega\}}(G) = \{ f \in C^\infty(G) : \forall \ K \subset G \text{ compact} \ \exists m \in \mathbb{N} \ \|f\|_{K,m} < \infty \}.
$$
The Fourier Laplace transform $\mathcal{F}$ defines a linear topological isomorphism between $\mathcal{E}_{\{\omega\}}(\mathbb{R}^N)'$ and the $(LF)$-space

$$A_{\{\omega\}} := \text{ind}_K A(K), \quad \text{where} \quad A(K) := \text{proj}_m A(K, \frac{1}{m}),$$

and

$$A(K, \frac{1}{m}) = \{ f \in H(\mathbb{C}^N) : \sup_{z \in \mathbb{C}^N} |f(z)| \exp(-K|\text{Im } z| - \frac{1}{m} \omega(|z|)) < \infty \}.$$

- The convolution operators $T_\mu, \mu \in \mathcal{E}_{\{\omega\}}(\mathbb{R}^N)'$ are defined in a similar way as in the Beurling case.
$F \in A_{\{\omega\}}$ is called $\{\omega\}$-slowly decreasing, if for each $m \in \mathbb{N}$ there exists $R > 0$ such that for each $x \in \mathbb{R}^N$ with $|x| \geq R$ there exists $\xi \in \mathbb{C}^N$ satisfying $|x - \xi| \leq \omega(x)/m$ such that

$$|F(\xi)| \geq \exp(-\omega(\xi)/m).$$

**Proposition 6.**

Let $\omega$ be a weight function and let $F \in A_{\{\omega\}}$ be given. Then the following assertions are equivalent:

(a) $F$ is $\{\omega\}$-slowly decreasing.

(b) There exists a weight function $\sigma$ satisfying $\sigma = o(\omega)$ such that $F \in A_{(\sigma)}$ and such that $F$ is $(\sigma)$-slowly decreasing.

(c) The multiplication operator $M_F : A_{\{\omega\}} \to A_{\{\omega\}}$, $M_F(g) := Fg$, has closed range.

(d) $M_F^{-1} : FA_{\{\omega\}} \to A_{\{\omega\}}$ is sequentially continuous.
Theorem 7

Let $\omega$ be a weight function and let $\mu \in \mathcal{E}_\omega(\mathbb{R})'$, $\mu \neq 0$, be given. Then the following assertions are equivalent:

1. $T_\mu : \mathcal{E}_\omega(\mathbb{R}) \to \mathcal{E}_\omega(\mathbb{R})$ is surjective.
2. The following two conditions are satisfied:
   a. $\hat{\mu}$ is $\{\omega\}$-slowly decreasing.
   b. There exists $\delta > 0$ such that each limit point of the set
      \[
      \{|\text{Im } a|/\omega(a) : a \in V(\hat{\mu}), \omega(a) \neq 0\}
      \]  
      is contained in $\{0\} \cup [\delta, \infty[.$

This result for non quasianalytic classes was proved by Braun, Meise and Vogt in 1990.

For quasianalytic ultradifferential operators by Meyer in 1997.

Our proof depends on recent results on $(LF)$-spaces due to Vogt and to Wengenroth.
Concerning the existence of continuous linear right inverses, we only know a necessary condition. A full characterization in the non quasianalytic case was obtained by Meise and Vogt in 1987.

**Proposition 8**

Let $\omega$ be a quasianalytic weight function which satisfies the condition $(\alpha_1)$ and let $\mu \in \mathcal{E}_\omega'(\mathbb{R})$, $\mu \neq 0$ be given. If $T_\mu : \mathcal{E}_\omega(\mathbb{R}) \to \mathcal{E}_\omega(\mathbb{R})$ admits a continuous linear right inverse, then the following assertions hold:

(a) $T_\mu$ is surjective and

(b) \[ \lim_{\substack{a \in V(\hat{\mu}) \\ |a| \to \infty}} \frac{|\text{Im } a|}{\omega(a)} = 0. \]