Composition and superposition operators on weighted Banach spaces of holomorphic functions of type H^∞

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Modern Complex Analysis and Operator Theory and Applications, IV
El Escorial (Madrid), Spain, June 17–21, 2009

On joint work with K.D. Bierstedt, P. Domański, P. Galindo, M. Lindström, J. Taskinen and E. Wolf

Notation

Weights

Let v be a strictly positive continuous **weight** on the open unit disk $\mathbb{D}$ in the complex plane which is radial (that is, $v(z) = v(|z|)$ for every $z \in \mathbb{D}$), non increasing with respect to $|z|$ and $\lim_{r\to 1} v(r) = 0$.

Examples

- The **standard weights** are $v_p(z) = (1 - |z|)^p$, $p > 0$.
- $v(r) = \exp(-\frac{1}{(1-r)^q}), \; q > 0$.
- $v(r) = (1 - \log(1 - r))^q, \; q < 0$.

Notation

Spaces

The **weighted Banach spaces of type H^∞** are

$$H_v^\infty := \{f \in H(\mathbb{D});\ \|f\|_v = \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty\}$$

and

$$H_v^0 := \{f \in H_v^\infty,\ \lim_{|z|\to 1} v(z)|f(z)| = 0\}$$

endowed with the norm $\|.\|_v$.

Here $H(\mathbb{D})$ denotes the space of all analytic functions. It is a Fréchet space endowed with the compact open topology.

If we do not assume that $\lim_{r\to 1} v(r) = 0$, then $H_v^\infty = H^\infty$.

Properties of the spaces

- H_v^∞ is a Banach space and H_v^0 is a closed subspace containing the polynomials as a dense subspace. **Shields, Williams, 1978 and Bierstedt and Summers, 1993** proved that H_v^∞ is canonically the bidual of H_v^0.
- **Lusky, 2006,** has completed the isomorphic classification of the spaces H_v^∞. Either H_v^∞ is isomorphic to H^∞ or to ℓ_∞. The characterization is in terms of a technical condition on the weight v.
- **Lusky, 2006,** also proved that H_v^0 has a basis. The basis need not be the monomials $z^n, n \in \mathbb{N}$.
- **Bonet, Wolf, 2003,** proved that for every continuous weight v on an open connected subset G of $\mathbb{C}^d$ the space $H_v^0(G)$ is isomorphic to a subspace of c_0. **Kaballo, 1980** proved that H_v^0 is not isomorphic to c_0 for logarithmic weights.

Composition operators

Operators

We consider a non-constant self map $\varphi \in H(D)$ satisfying $\varphi(D) \subset D$ and a function $\psi \in H(D)$ which is not identically equal zero.

They induce the **weighted composition operator**

$$C_{\varphi,\psi} f := \psi(f \circ \varphi).$$

If $\psi = 1$, then as usual we denote the **composition operator** $C_{\varphi,1}$ by C_φ

$$C_\varphi f := f \circ \varphi.$$

Questions to be considered

- Continuity, compactness and weak compactness of C_φ.
- Essential norm of C_φ.
- The spectrum.
- Characterization of C_φ which are isometries.
- Compact differences of composition operators.
- Topological structure of the set of composition operators.

The associated weights

The associated weight. Bierstedt, Bonet and Taskinen, 1998.

$$\tilde{v}(z) := \frac{1}{\sup\{|f(z)|;\ f \in H_v^\infty, \|f\|_v \le 1\}} = \frac{1}{\|\delta_z\|_{H_v^\infty{}'}},\ z \in \mathbb{D},$$

- $\tilde{v}$ is continuous, radial, non increasing with respect to $|z|$, and satisfies $v(0) \ge \tilde{v} \ge v > 0$.
- For each $z \in \mathbb{D}$ we can find $f_z \in H_v^\infty$, $||f||_v \le 1$, such that $|f_z(z)| = \frac{1}{\tilde{v}(z)}$.
- $H_{\tilde{v}}^\infty$ is isometrically isomorphic to H_v^∞ and $H_{\tilde{v}}^0$ is isometrically isometric to H_v^0.

Composition operators on Bloch spaces

- Let B_p, $0 < p < \infty$, denote the **Bloch type spaces** of functions $f \in H(\mathbb{D})$ with $f(0) = 0$ satisfying

$$||f||_p := \sup_{z \in \mathbb{D}} (1 - |z|)^p |f'(z)| < \infty.$$

- $\|.\|_p$ is a norm and B_p is a Banach space.
- The map $S_p : B_p \to H^\infty_{v_p}, \ S_p(f) = f'$ is a surjective isometry.
- Since

$$C_\varphi = S_p^{-1} \circ C_{\varphi,\varphi'} \circ S_p,$$

studying the properties of $C_\varphi : B_p \to B_p$ is equivalent to investigating the operator $C_{\varphi,\varphi'} : H^\infty_{v_p} \to H^\infty_{v_p}$.

Composition operators. Continuity.

Theorem, BDLT, 1998

Let v and w be weights. The following conditions are equivalent for C_φ:

(1) $C_\varphi(H_v^\infty) \subset H_w^\infty$.

(2) $C_\varphi(H_v^0) \subset H_w^0$.

(3) $C_\varphi : H_v^\infty \to H_w^\infty$ is continuous.

(4) $C_\varphi : H_v^0 \to H_w^0$ is continuous.

(5) $\sup_{z \in \mathbb{D}} \frac{\tilde{w}(z)}{\tilde{v}(\varphi(z))} < \infty$.

In this case $\|C_\varphi\| = \sup_{z \in \mathbb{D}} \frac{\tilde{w}(z)}{\tilde{v}(\varphi(z))}$.

It was extended to weighted composition operators by **Contreras and Hernández Díaz, 2000**.

Composition operators. Continuity.

Examples

- If $\varphi(z) = \frac{z+1}{2}$ and $v(r) = \exp(-\frac{1}{1-r})$, then C_φ is not continuous on H_v^∞, since $\frac{\tilde{v}(r)}{\tilde{v}(\varphi(r))} = \exp(\frac{1}{1-r})$.
- If $\varphi(0) = 0$, then C_φ is continuous on every space H_v^∞.

Theorem, BDLT, 1998

A weight v satisfies that every composition operator C_φ is continuous on H_v^∞ if and only if the following condition introduced by **Lusky** in 1995 (also **Shields and Williams, 1978**) holds:

$$(L1) \qquad \sup_k \frac{v(1-2^{-k})}{v(1-2^{-k-1})} < \infty.$$

The standard weights $v_p(z) = (1-|z|)^p$, $p > 0$, are weights which have $(L1)$. The condition means that the weight does not tend to 0 too fast.

Composition operators. Compactness.

Theorem, BDLT, 1998, BDL, 1999

Let v and w be weights. The following conditions are equivalent for C_φ:

(1) $C_\varphi : H_v^\infty \to H_w^\infty$ is (weakly) compact.

(2) $C_\varphi : H_v^0 \to H_w^0$ is (weakly) compact.

(3) $\lim_{r\to 1} \frac{\tilde{w}(z)}{\tilde{v}(\varphi(z))} = 0$.

(4) $\lim_{n\to\infty} \frac{\|\varphi(z)^n\|_w}{\|z^n\|_v} = 0$.

- The case of Hardy spaces was investigated by **Shapiro** in 1987.
- Examples of non-nuclear compact composition operators are due to **Taskinen**.
- Composition operators on weighted spaces of **vector valued** holomorphic functions were investigated by **BDL, 2001 and B-Friz, 2002**, continuing work by **Liu, Saksman, Tylli, 1998**.

Composition operators. The essential norm.

Essential norm

If T is an operator on a Banach space X, the essential norm $\|T\|_{e,X}$ of T is the distance to the space of compact operators on X.

Theorem, Montes, 2000

$$||C_\varphi||_{e,H_v^\infty} = \lim_{r\to 1} \sup_{|\varphi(z)|>r} \frac{v(z)}{\tilde{v}(\varphi(z))}.$$

If C_φ is also bounded on H_v^0, then $||C_\varphi||_{e,H_v^0} = \lim_{r\to 1} \sup_{|z|>r} \frac{v(z)}{\tilde{v}(\varphi(z))}.$

Weaker versions of this results were obtained by **BDL, 1999** and **Contreras, Hernández-Díaz, 2000**. The theorem has been recently extended to more general classes of operators by **Galindo, Lindström**.

Composition operators. The spectrum.

Spectrum

$$\sigma_{H_v^\infty}(C_\varphi) := \{\lambda \in \mathbb{C} \mid \lambda I - C_\varphi \text{ is not invertible } \}$$

Theorem, BGL, 2008, extending Zheng, 2003, Aron, Lindström, 2004

Suppose φ is not an automorphism and has fixed point $0 \in \mathbb{D}$. Then

$$\sigma_{H_v^\infty}(C_\varphi) = \{\lambda \in \mathbb{C} : |\lambda| \leq r_{e,H_v^\infty}(C_\varphi)\} \cup \{\varphi'(0)^n\}_{n=0}^{\infty}.$$

Here the **essential spectral radius** is $r_{e,H_v^\infty}(C_\varphi) = \lim_{n\to\infty} \|C_\varphi^n\|_e^{1/n}$.

Problem.

Investigate the spectrum of C_φ on H_v^∞ without the assumption that φ has a fixed point in $\mathbb{D}$.

Composition operators. The spectrum.

In **BGL, 2008** we also studied how the essential spectral radius of C_φ on both H_v^∞ and H_v^0 determines whether the Koenigs eigenfunction σ of C_φ belongs to H_v^∞ and H_v^0 respectively.

Every holomorphic self map φ having non-zero derivative at its Denjoy-Wolf point $w \in \mathbb{D}$ has a unique Koenigs eigenfunction $\sigma \in H(\mathbb{D})$ determined by $\sigma \circ \varphi = \varphi'(w)\sigma, \ \sigma'(w) = 1$.

Problem. Bourdon, 1998.

Let φ be an analytic self map on $\mathbb{D}$ which has an attractive fixed point in $\mathbb{D}$. Let $\sigma \in H(\mathbb{D})$ be the Koenigs eigenfunction of φ. Characterize when $\sigma \in H_v^\infty$.

Composition operators. Isometries on H^∞.

Theorem, De Leeuw, Rudin and Wermer

If S is an isometry of H^∞ **onto** H^∞, then S has the form $Sf = \lambda C_\varphi f$ for every $f \in H^\infty$, where φ is a conformal automorphism of D and $\lambda \in \partial D$.

Theorem, Roan, 1978, Sarason

The following statements are equivalent:

(i) $C_\varphi : H^\infty \to H^\infty$ is an isometry.

(ii) $C_\varphi : H^\infty \to H^\infty$ has closed range.

(iii) $\partial D \subset \overline{\varphi(D)}$.

Example.

The function $\varphi : D \to D, \varphi(z) = \frac{-1+2(1+iz)}{1+iz+\sqrt{2(1-z^2)}}$ is not a surjection, but satisfies $\overline{\varphi(D)} = \overline{D}$ and therefore C_φ is an isometry on H^∞.

Theorem, Cima, Wogen, 1980 for $H^\infty_{v_1}$.

If $S : H^\infty_{v_p} \to H^\infty_{v_p}$, $0 < p < \infty$, is a surjective isometry, then there is a conformal automorphism φ of $\mathbb{D}$ and $\lambda \in \partial\mathbb{D}$ such that $Sf = C_{\varphi,\lambda(\varphi')^p} f$ for every $f \in H^\infty_{v_p}$.

Problem.

Even for the standard weights a characterization of all isometries on these spaces does not seem to be known.

In the case of the smaller space $H^0_{v_p}$ all isometries can be described in this way with similar arguments.

Composition operators. Isometries.

M.J. Martin, Vukotic, 2006 analyzed when composition operators C_φ on the Bloch space are isometric and showed that every thin Blaschke product induces an isometric composition operator on the Bloch space.

Proposition, BLW, 2008

Assume that $\tilde{v}$ is strictly decreasing on $[0,1[$. Let φ be an analytic self map on $\mathbb{D}$. If the composition operator C_φ is an isometry on $H_v^\infty(\mathbb{D})$, then $\varphi(0) = 0$.

Theorem, BLW, 2008

Let v be a radial weight such that $\lim_{r\to 1}\tilde{v}(r) = 0$. Let φ be an analytic self map on $\mathbb{D}$ such that $\varphi(0) = 0$. The composition operator C_φ on H_v^∞ is an isometry if and only if φ is a rotation.

In **BLW, 2008** we also investigated isometric weighted composition operators on H_v^∞. It is more interesting, but we will not state the results here.

Palmberg, 2007 investigated weighted composition operators with closed range, extending work **by Ghatage, Yan and Zheng and by Zorborska**.

Topological structure of the set of composition operators.

The pseudohyperbolic metric

For $z, a \in \mathbb{D}$,

$$\rho(z,a) := |\sigma_a(z)|, \text{ where } \sigma_a(z) := \frac{a-z}{1-\bar{a}z}$$

is the automorphism of $\mathbb{D}$ which changes 0 and a.

Lemma, Lindström, Wolf, 2007

Let v be a radial weight with $(L1)$ such that v is continuously differentiable with respect to $|z|$. Then there is $M > 0$ such that for $f \in H_v^\infty$ we have

$$|v(p)f(p) - v(q)f(q)| \leq M\|f\|_v \rho(p,q)$$

for all $p, q \in \mathbb{D}$.

Topological structure of the set of composition operators.

Theorem, BLW, 2008

Let v be a radial weight such that v is continuously differentiable with respect to $|z|$, $v = \tilde{v}$ and satisfies the Lusky condition $(L1)$. Let $\psi^{(1)}, \psi^{(2)} \in H_w^0$. If $\phi^{(1)}, \phi^{(2)} : \mathbb{D} \to \mathbb{D}$ are analytic maps such that and $\psi^{(1)} C_{\phi^{(1)}}, \psi^{(2)} C_{\phi^{(2)}} : H_v^\infty \to H_w^\infty$ are bounded, then the operator $\psi^{(1)} C_{\phi^{(1)}} - \psi^{(2)} C_{\phi^{(2)}} : H_v^\infty \to H_v^\infty$ is compact if and only if

(a) $\lim_{|z|\to 1} v(z) \frac{|\psi^{(1)}(z)|}{v(\phi^{(1)}(z))} \rho(\phi^{(1)}(z), \phi^{(2)}(z)) = 0,$

(b) $\lim_{|z|\to 1} v(z) \frac{|\psi^{(2)}(z)|}{v(\phi^{(2)}(z))} \rho(\phi^{(1)}(z), \phi^{(2)}(z)) = 0,$

(c) $\lim_{|z|\to 1} v(z) \left| \frac{\psi^{(1)}(z)}{v(\phi^{(1)}(z))} - \frac{\psi^{(2)}(z)}{v(\phi^{(2)}(z))} \right| = 0.$

Important results on compact differences of composition operators are due to **MacCluer, Ohno, Zhao, 2001, Moorhouse, 2005 and Nieminen, 2006**.

Topological structure of the set of composition operators.

The problem of the topological structure of the space of (weighted) composition operators has been considered on several spaces of analytic functions, starting with **MacCluer, Ohno and Zhao, 2001**. More results were obtained by **T. Hosokawa, K. Izuchi, S. Ohno, 2005-08**.

Theorem, BLW, 2009

Let v be a radial, typical weight satisfying $(L1)$ such that $v = \tilde{v}$ and v is continuously differentiable with respect to $|z|$ and $\psi^{(1)}, \psi^{(2)} \in H_v^0$.

- The set of compact weighted composition operators on H_v^∞ is path connected.
- Assume that $\psi^{(1)} C_{\phi^{(1)}}$, $\psi^{(2)} C_{\phi^{(2)}}$, $\psi^{(1)} C_{\phi^{(2)}}$ and $\psi^{(2)} C_{\phi^{(1)}}$ are bounded and $\psi^{(1)} C_{\phi^{(1)}}$ or $\psi^{(2)} C_{\phi^{(2)}}$ is not compact on H_v^∞. Furthermore, let the difference $\psi^{(1)} C_{\phi^{(1)}} - \psi^{(2)} C_{\phi^{(2)}} : H_v^\infty \to H_v^\infty$ be compact. Then the operators $\psi^{(1)} C_{\phi^{(1)}}$ and $\psi^{(2)} C_{\phi^{(2)}}$ belong to the same path component.

Topological structure of the set of composition operators.

Aim

Compare the topologies τ_∞, τ_v and τ_w induced by the Banach spaces $L(H^\infty)$, $L(H_v^\infty)$ and $L(H_w^\infty)$ respectively on the space of all composition operators $\mathcal{C}$, where v and w are weights satisfying condition (L1). Our results complement those of **Saksman and Sundberg, 2006**.

Theorem, BLW, 2009

- Let v be a typical weight satisfying condition $(L1)$. Then τ_∞ is finer than τ_v.
- If $v_p(z) = (1 - |z|)^p$, $p > 0$, then τ_∞ is strictly finer than τ_{v_p}.

Problem

Compare the topologies τ_v and τ_w on $\mathcal{C}$ for two weights v and w.

Superposition operators

We briefly report on research in progress with D. Vukotic.

- X and Y are linear spaces of holomorphic functions on the unit disc $\mathbb{D}$ of the complex plane and φ is **an entire function**.
- The **superposition operator** is

$$S_\varphi : X \to Y, \ S_\varphi(f) := \varphi \circ f.$$

- **Aim:** Characterize those symbols φ such that S_φ maps X into Y.
- In case X and Y are Banach spaces, it is also important to determine when S_φ is bounded, in the sense that it maps bounded subsets of X into bounded subsets of Y or when S_φ is continuous.

Theorem, Bonet, Vukotic, 20??

Let p and q be positive numbers. Let φ be an entire function. The following conditions are equivalent:

(1) φ is a polynomial of degree less or equal than the integer part $[q/p]$ of q/p.

(2) The superposition operator S_φ maps Hv_p into Hv_q.

(3) The superposition operator S_φ maps Hv_p into Hv_q and it is bounded.

(4) The superposition operator S_φ maps Hv_p into Hv_q and it is continuous.

Theorem, Bonet, Vukotic, 20??

Let $v(z) = (1 - |z|)^q, z \in \mathbb{D}$ and $w(z) = \exp(-\frac{1}{(1-|z|)^p}), p, q > 0$. Let φ be an entire function. The following conditions are equivalent:

(1) For all $0 < d < 1$ there is $C \geq 1$ such that $|\varphi(z)| \leq C \exp(d|z|^{p/q})$ for all $z \in \mathbb{D}$.

(2) The superposition operator S_φ maps Hv into Hw.

(3) The superposition operator S_φ maps Hv into Hw and it is bounded.

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