Non-commutative locally convex measures

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Joint work with J. D. Maitland Wright, Aberdeen, UK
Part of a program on **non-commutative measure theory**.

Started by **John D. Maitland Wright** in 1980.

Continued together with his coauthors **J.K. Brooks, A. Peralta, K. Saitô, I. Villanueva, K. Ylinen**

and Pepe Bonet (the locally convex touch)
Classical measure theory

In classical theory, measures and integrals correspond to functionals or vector valued operators defined on a function algebra, like $C(K)$, $K$ compact; or $L^\infty$.

Non-commutative measure theory

Replacing these commutative algebras by non-commutative $C^*$-algebras, like $L(H)$, $H$ Hilbert space, gave birth to non-commutative measure theory. But replacing $C^*$-algebras by more general classes of Banach spaces give rise to fruitful new insights.
Question 1
Let $K$ be a compact Hausdorff space and let $E$ be a complete locally convex space. Let $T \in L(C(K), E)$ a continuous operator. When does there exist an $E$-valued Baire measure $\mu$ on $K$ such that

$$T(f) = \int_K f(t)\mu(dt) \quad \text{for each } f \in C(K)?$$

Question 2
Let $B(K)$ be the algebra of bounded Borel measurable functions on $K$. When does there exist an operator $T^\infty \in L(C(K), E)$ extending $T$ and, whenever $(f_n)_n$ is a bounded monotone increasing sequence in $B(K)$ with pointwise limit $f$, it follows $T^\infty(f_n) \rightarrow T^\infty f$ in $E$?
When $E$ is one dimensional, the answer to Question 1 is “always”. This is the classical **Riesz Representation Theorem**.

When $E$ is a Banach space, the answer is: precisely when $T$ is a weakly compact operator. This was proved by **Grothendieck (1953) and Bartle, Dunford and Schwartz (1955)**.

**Theorem (Lewis, 1970)**

Let $K$ be a compact Hausdorff space, let $E$ be a complete locally convex space and let $T \in \mathcal{L}(C(K), E)$ a continuous operator. $T$ is weakly compact if and only if there is a regular measure $\mu : \Sigma \to E$ on the Borel subsets of $K$ such that

$$T(f) = \int_K f(t)\mu(dt) \quad \text{for each } f \in C(K).$$
Weakly compact operators

**Definition**

If $X$ is a Banach space and $E$ is a locally convex space an operator $T \in L(X, E)$ is weakly compact if $T(X_1)$ is relatively $\sigma(E, E')$-compact in $E$. Here $X_1$ stands for the closed unit ball of $X$.

**Grothendieck’s extension of Gantmacher’s Theorem**

Let $X$ be a Banach space and let $E$ be a complete locally convex space. The following are equivalent for $T \in L(X, E)$

1. $T$ is weakly compact.
2. $T''(X'') \subset E$.
3. For each $C \subset E'$ which is equicontinuous, $T'(C)$ is relatively $\sigma(X', X'')$-compact.
A is a $C^*$-algebra.

$A'$ is the dual of $A$.

The bidual $A''$ can be identified with the von Neumann envelope of $A$ in its universal representation.

$A'$ is the predual of a von Neumann algebra, hence $(A', \sigma(A', A''))$ is sequentially complete.
Weakly compact operators

**Theorem**

Let $A$ be a Banach space such that $(A', \sigma(A', A'''))$ is sequentially complete and let $E$ be a complete locally convex space. Let $(T_n)_n$ be a sequence of weakly compact operators from $A$ into $E$.

If $(T_n''z)_n$ is a Cauchy sequence in $E$ for each $z \in A''$, then there is a weakly compact operator $T : A \to E$ such that $T_n''z \to T''z$ in $E$ for each $z \in A''$.

**Theorem (Dieudonné, 1951)**

Let $K$ be a compact metric space, let $(\mu_n)_n$ be a sequence of Borel measures such that $\lim_{n \to \infty} \mu_n(U)$ exists for each open subset $U$ of $K$. Then there exists a Borel measure $\mu$ such that $\mu_n(f) \to \mu(f)$ for each bounded Borel measurable function $f \in B(K)$. 
Let $A$ be a $C^*$-algebra. A projection $p \in A''$ is said to be a range projection for $A$ if there exists $b \in A$, $0 \leq b \leq 1$, such that the monotone increasing sequence $(b^{1/n})_n$ converges to $p$ in the topology $\sigma(A'', A')$. We write $p = \text{RP}(b)$.

Theorem (Brooks, Saitô, Wright (2003), Bonet, Wright (2009))

Let $A$ be a $C^*$-algebra, let $E$ be a complete locally convex space and let $(T_n)_n$ be a sequence of weakly compact operators $T_n : A \to E$. Suppose that, whenever $p \in A''$ is a range projection, $\lim_{n \to \infty} T_n''(p)$ exists in $E$. Then there is a unique weakly compact operator $T : A \to E$ such that $T''(x) = \lim_{n \to \infty} T_n''(x)$ for each $x \in A''$. 

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Extending a result of Ryan on weakly compact operators

- $E$ is a Fréchet space or a complete (DF)-space.
- $c_0(E)$ and $c(E)$.
- $\ell_1(E)$ and $\ell_\infty(E)$.
- $c(E)'_b \simeq \ell_1(E'_b)$.
- $\ell_1(E'_b)'_b \simeq \ell_\infty(E''_b)$.
This is our extension of a result due to Ryan (1979).

**Theorem**

Let $A$ be a Banach space and let $E$ be a Fréchet space or a complete (DF)-space.

Let $T_n$ be the operators from $A$ into $E$ such that, for each $a \in A$, $\lim_n T_n a = 0$ in $E$.

Then the operator

$$\tilde{T} : A \to c_0(E), \quad \tilde{T} a := (T_n a)_n, \quad a \in A,$$

is weakly compact if and only if each $T_n$ is weakly compact and $\lim_n T_n'' z = 0$ for each $z \in A''$. 

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Theorem

Let $A$ be a Banach space such that $(A', \sigma(A', A'''))$ is sequentially complete and let $E$ be a Fréchet space or a complete (DF)-space. Let $(T_n)_n$ be a sequence of weakly compact operators from $A$ into $E$, such that $(T''_n z)_n$ is a Cauchy sequence in $E$ for each $z \in A'''$. Then $\tilde{T} : A \to c(E), \tilde{T} a := (T_n a)_n, a \in A$, is a weakly compact operator.

If the assumption that $A$ has a weakly sequentially complete dual is removed in this Theorem, the conclusion is no longer valid, as the example constructed by Ylinen in 2005 shows.
The Right topology $\rho(E)$ is the topology induced in $E$ by the Mackey topology $\mu(E'', E')$ of the dual pair $(E'', E')$. Recall that $\mu(E'', E')$ is the topology on $E''$ of the uniform convergence on the absolutely convex $\sigma(E', E'')$-compact subsets of $E'$.

**Theorem (Ruess (1982), Peralta, Villanueva, Ylinen, Wright (2007))**

Let $F$ and $E$ be complete, barrelled locally convex spaces. Let $T \in L(F, E)$ be a continuous linear operator. The following conditions are equivalent:

1. $T$ maps bounded subsets in $F$ into relatively weakly compact subsets of $E$.
2. $T : (F, \rho(F)) \to E$ is continuous.
3. There is an absolutely convex neighbourhood $V \in U_0(F)$ such that the restriction $T|_V$ of $T$ from $V$, equipped with the topology induced by the Right topology $\rho(F)$, into $E$ is continuous.
A classical theorem of Nikodym asserts that if a sequence of countably additive measures converges pointwise, then the limit is also a countably additive measure and, furthermore, countable additivity is uniform for the sequence of measures. As a consequence of our previous results, we get

**Theorem**

Let $A$ be a Banach space and let $E$ be a Fréchet space or a complete (DF)-space. Let $(T_n)_n$ be a sequence of weakly compact operators from $A$ into $E$ such $(T_n''z)_n$ converges to $0$ in $E$ for each $z \in A''$.

If $(a_j)_j \subset A$ converges to $0$ in the Right topology of $A$, then $\sup_{n \in \mathbb{N}} p(T_n(a_j))$ converges to $0$ as $j \to \infty$ for each continuous seminorm $p$ on $E$. 
Theorem

Let $A$ be a Banach space such that $(A', \sigma(A', A'''))$ is sequentially complete and let $E$ be a Fréchet space or a complete (DF)-space. Let $(T_n)_n$ be a sequence of weakly compact operators from $A$ into $E$ such that $(T''_n z)_n$ is a Cauchy sequence in $E$ for each $z \in A'''$. Let $S a := \lim T_n a$ for each $a \in A$. Then

1. If $(a_j)_j \subset A$ converges to 0 in the Right topology of $A$, then $\sup_{n \in \mathbb{N}} p((T_n - S)(a_j))$ converges to 0 as $j \to \infty$ for each continuous seminorm $p$ on $E$.

2. If $(z_j)_j$ is a sequence in $A''$ converging to 0 in the topology $\mu(A'', A')$, then $\sup_{n \in \mathbb{N}} q((T''_n - S'')(z_j))$ converges to 0 as $j \to \infty$ for each continuous seminorm $q$ on $(E'', \beta(E'', E'))$. 

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Let $\psi \in A'$ be a positive functional on a $C^*$-algebra $A$, and define

$$p_\psi(a) := \psi((aa^* + a^*a)^{1/2}), \quad a \in A.$$ 

Then $p_\psi$ is a seminorm on $A$.

A positive functional $\psi \in A'$ satisfying $\|\psi\| = 1$ is called a state.

The universal $\sigma$-strong* topology of a $C^*$-algebra $A$ is the topology induced by all seminorms $p_\psi$, where $\psi$ is a positive functional on $A$. 

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It follows from a fundamental result of Akemann, 1967, that the restriction of the $\sigma$-strong* topology to the unit ball $A_1$ of $A$ coincides with the restriction of the Right topology $\rho(A)$ to $A_1$.

An **orthogonal sequence** in the $C^*$-algebra $A$ is a sequence $(a_n)_n$ of self-adjoint elements of the closed unit ball of $A$ such that $a_na_m = 0$ whenever $n \neq m$. 
The omnibus theorem

Theorem

Let $A$ be a $C^*$-algebra, $E$ a complete locally convex space and $T : A \to E$ a continuous linear operator. The following conditions are equivalent:

1. $T$ is a weakly compact operator.
2. $T : (A, \rho(A)) \to E$ is sequentially continuous.
3. If $(a_n)_n$ is an orthogonal sequence in $A$, then $(Ta_n)_n$ converges to 0 in $E$.
4. For every bounded universal strong $*$-null net $(a_\lambda)_\lambda$ in $A$ we have $(T(a_\lambda))_\lambda$ converges to 0 in $E$.
5. If $(a_n)_n$ is a sequence in $A$ which is convergent in the universal $\sigma$-strong* topology, then $(Ta_n)_n$ converges in $E$. 
The omnibus theorem continued

(6) \( T : (A, \rho(A)) \to E \) is continuous.

(7) For each \( q \in \text{cs}(E) \) there exist a state \( \phi_q \in A' \) and \( N_q : [0, \infty) \to [0, \infty] \) such that

\[
q(Ta) \leq N_q(\varepsilon)p_{\phi_q}(a) + \varepsilon \|a\|.
\]

Lemma, Akemann (1967)

A subset \( K \subset A' \) is relatively \( \sigma(A', A'') \)-compact if and only if \( K \) is bounded and there exists a state \( \phi \in A' \) such that for all \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for all \( a \in A_1 \) for which \( p_{\phi}(a) < \delta \) then \( |\psi(a)| < \varepsilon \) for all \( \psi \in K \).
The characterization in the lemma is a deep theorem due to Akemann (1967). It is very important in the proof.

The implication (3) ⇒ (1) is based on a very deep theorem due to Pfitzner (1994).

Conditions of type (7) in this context were introduced by Jarchow (1986).

- $A_{sa}$ the set of self adjoint elements in $A$.

- $A^\sigma$ stands for the smallest subspace of $A''$ containing $A$ with the property that whenever $(b_n)_n$ is a monotonic sequence in $(A^\sigma)_{sa}$ with limit $b$ in the weak operator topology of $A''$ (or in the topology $\sigma(A'', A')$), then $b \in A^\sigma$.

- By a fundamental theorem of Pedersen 1979, $A^\sigma$ is a $C^*$-subalgebra of $A''$ and it is called the Baire $*$-envelope of $A$ or the Pedersen envelope of $A$. 
Definition

\[ \tilde{T} : A^\sigma \to E \] is a **weak $E$-valued integral for** $A$ if it is continuous and for all $(b_n)_n$ monotonic sequence of self adjoint elements in $A^\sigma$ such that $b_n$ tends to $b$ for the $\sigma(A^\sigma, A')$-topology then $\tilde{T}(b_n)$ tends to $\tilde{T}(b)$ for the $\sigma(E, E')$-topology.

Theorem. It extends a result of Wright (1980)

A continuous operator $T : A \to E$ is weakly compact if and only if there is a weak $E$-valued integral $\tilde{T} : A^\sigma \to E$ for $A$ whose restriction to $A$ coincides with $T$. 

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Corollary

If $A$ be a $C^*$-algebra and $E$ is a complete locally convex space which contains no copy of $c_0$, then every $T \in L(A, E)$ is weakly compact.

- The commutative case of this result is due to Panchapagesan, 1998. The non-commutative case for a Banach space $E$ is due to Akemann, Dodds and Gamlen (1972).

- It is based on a theorem of Tumarkin, 1970, extending a result of Bessaga and Pelczynski, on the unconditional convergence of weakly unconditionally Cauchy series in locally complete spaces not containing $c_0$. 


