Power bounded composition operators on spaces of real analytic functions

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Definitions

X is a Hausdorff locally convex space (lcs).

$\mathcal{L}(X)$ is the space of all continuous linear operators on X.

Power bounded operators

An operator $T \in \mathcal{L}(X)$ is said to be *power bounded* if $\{T^m\}_{m=1}^{\infty}$ is an equicontinuous subset of $\mathcal{L}(X)$.

If X is a Fréchet space, an operator T is power bounded if and only if the orbits $\{T^m(x)\}_{m=1}^{\infty}$ of all the elements $x \in X$ under T are bounded. This is a consequence of the uniform boundedness principle.

Definitions

Mean ergodic operators

An operator $T \in \mathcal{L}(X)$ is said to be *mean ergodic* if the limits

$$Px := \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} T^m x, \quad x \in X, \tag{1}$$

exist in X.

A power bounded operator T is mean ergodic precisely when

$$X = \mathrm{Ker}(I - T) \oplus \overline{\mathrm{Im}(I - T)}, \tag{2}$$

where I is the identity operator, $\mathrm{Im}(\mathrm{I} - \mathrm{T})$ denotes the range of $(I - T)$ and the bar denotes the "closure in X".

Motivation

Trivial observation

$$\lambda \in \mathbb{C}, \ |\lambda| \leq 1.$$

$$\lambda_{[n]} := \frac{1}{n}(\lambda + \lambda^2 + \ldots + \lambda^n)$$

$$\lambda = 1 \Rightarrow \lambda_{[n]} = 1, n = 1, 2, 3, \ldots$$

$$\lambda \neq 1 \Rightarrow |\lambda_{[n]}| \leq \frac{2}{n|1-\lambda|} \to 0$$

$$\text{as } n \to \infty.$$

Motivation

von Neumann, 1931

Let H be a Hilbert space and let $T \in \mathcal{L}(H)$ be unitary. Then there is a projection P on H such that $T_{[n]} := \frac{1}{n} \sum_{m=1}^{n} T^m$ converges to P in the strong operator topology.

Lorch, 1939

Let X be a reflexive Banach space. If $T \in \mathcal{L}(X)$ is a power bounded operator, then there is a projection P on X such that $T_{[n]}$ converges to P in the strong operator topology; i.e. T is mean ergodic.

Hille, 1945

There exist mean ergodic operators T on $L_1([0,1])$ which are not power bounded.

Motivation

Problem attributed to Sucheston, 1976

Let X be a Banach space such that every power bounded operator $T \in \mathcal{L}(X)$ is mean ergodic. Does it follow that X is reflexive?

- YES if X is a Banach lattice (**Emelyanov, 1997**).
- YES if X is a Banach space with basis (**Fonf, Lin, Wojtaszczyk, J. Funct. Anal. 2001**). This was a major breakthrough.

Yosida's mean ergodic Theorem

- **Barrelled locally convex spaces**
- $\mathcal{L}_s(X), \quad \mathcal{L}_b(X)$

Yosida, 1960

Let X be a barrelled lcs. The operator $T \in \mathcal{L}(X)$ is mean ergodic if and only if it $\lim_{n\to\infty} \frac{1}{n} T^n = 0$, in $\mathcal{L}_s(X)$ and

$$\{T_{[n]}x\}_{n=1}^{\infty} \text{ is relatively sequentially } \sigma(X, X')\text{–compact, } \forall x \in X. \quad (3)$$

Setting $P := \tau_s\text{-}\lim_{n\to\infty} T_{[n]}$, the operator P is a projection which commutes with T and satisfies $\mathrm{Im}(P) = \mathrm{Ker}(I - T)$ and $\mathrm{Ker}(P) = \overline{\mathrm{Im}(I - T)}$.

Yosida's mean ergodic Theorem

If $\{T_{[n]}\}_{n=1}^{\infty}$ happens to be convergent in $\mathcal{L}_b(X)$, then T is called **uniformly mean ergodic**.

Corollary

- Let X be a reflexive lcs in which every relatively $\sigma(X, X')$–compact set is relatively sequentially $\sigma(X, X')$–compact. Then every power bounded operator on X is mean ergodic.
- Let X be a Montel space in which every relatively $\sigma(X, X')$–compact set is relatively sequentially $\sigma(X, X')$–compact. Then every power bounded operator on X is uniformly mean ergodic.

The assumption on the weakly compact subsets of X is satisfied by Fréchet spaces, duals of Fréchet spaces and the space $\mathscr{A}(\Omega)$ of real analytic functions.

Main Results

- Let X be a complete barrelled lcs with a Schauder basis and in which every relatively $\sigma(X, X')$–compact subset of X is relatively sequentially $\sigma(X, X')$–compact. Then X is reflexive if and only if every power bounded operator on X is mean ergodic.
- Let X be a complete barrelled lcs with a Schauder basis and in which every relatively $\sigma(X, X')$–compact subset of X is relatively sequentially $\sigma(X, X')$–compact. Then X is Montel if and only if every power bounded operator on X is uniformly mean ergodic.
- Let X be a sequentially complete lcs which contains an isomorphic copy of the Banach space c_0. Then there exists a power bounded operator on X which is not mean ergodic.

Results of Albanese, Bonet, Ricker, 2008

Results about Schauder basis

- A complete, barrelled lcs with a basis is reflexive if and only if every basis is shrinking if and only if every basis is boundedly complete.

  This extends a result of Zippin for Banach spaces and answers positively a problem of Kalton from 1970.

- Every non-reflexive Fréchet space contains a non-reflexive, closed subspace with a basis.

  This is an extension of a result of A. Pelczynski for Banach spaces.

Bonet, de Pagter and Ricker (2009) have proved that a Fréchet lattice X is reflexive if and only if every power bounded operator on X is mean ergodic.
This is an extension of the result of Emelyanov.

Composition operators on spaces of analytic functions

- Let $U \subseteq \mathbb{C}^d$ be a connected open domain.
- Let $\varphi : U \to U$ be a holomorphic mapping.
- The **composition operator** $C_\varphi : H(U) \to H(U)$ is defined by $C_\varphi(f) := f \circ \varphi$.
- $H(U)$ endowed with the compact open topology is a Fréchet Montel space. Mean ergodic and uniformly mean ergodic operators on $H(U)$ coincide.
- Universal functions for composition operators have been investigated by **Bernal, Bonilla, Godefroy, Grosse-Erdmann, León, Luh, Montes, Mortini, Shapiro and others.**

Theorem

Let $U \subseteq \mathbb{C}^d$ be a connected domain of holomorphy (or even a Stein manifold). The following assertions are equivalent:

(a) $C_\varphi : H(U) \to H(U)$ is power bounded;

(b) $C_\varphi : H(U) \to H(U)$ is uniformly mean ergodic;

(c) $C_\varphi : H(U) \to H(U)$ is mean ergodic;

(d) $\forall\, K \Subset U \;\exists\, L \Subset U$ such that $\varphi^n(K) \subseteq L$ for every $n \in \mathbb{N}$;

(e) There is a fundamental family of (connected) compact sets (L_j) in U such that $\varphi(L_j) \subseteq L_j$ for every $j \in \mathbb{N}$.

The equivalence (a)⇔(b)⇔(c) is true for arbitrary open connected $U \subseteq \mathbb{C}^d$.

Theorem

If for every compact set $K \Subset U$ there is a compact set $L \Subset U$ such that $\varphi^n(K) \subseteq L$ for every $n \in \mathbb{N}$, then there are a holomorphic submanifold M of U and a holomorphic surjective retraction $\rho : U \to M$ such that $\psi := \varphi|_M$ is an automorphism of M. Moreover,

$$G := \overline{\{\varphi^n : n \in \mathbb{N}\}}^{H(M,M)}$$

is a compact abelian group of automorphisms on M such that every cluster point of (φ^n) in $H(U,U)$ is of the form $\gamma \circ \rho$ where $\gamma \in G$ and

$$P(f)(z) := \frac{1}{N}\sum_{n=1}^{N} C_{\varphi^n}(f)(z) = \int_G f(\gamma \circ \rho(z))dH(\gamma),$$

H being the Haar measure on G and

$$\text{im } P = \{f : f \text{ is constant on } \rho^{-1}(\{\gamma \circ \rho(z) : \gamma \in G\}) \quad \forall\, z \in U\}.$$

- $\Omega \subseteq \mathbb{R}^d$ open connected set.

- The space of real analytic functions $\mathscr{A}(\Omega)$ is equipped with the unique locally convex topology such that for any $U \subseteq \mathbb{C}^d$ open, $\mathbb{R}^d \cap U = \Omega$, the restriction map $R : H(U) \longrightarrow \mathscr{A}(\Omega)$ is continuous and for any compact set $K \subseteq \Omega$ the restriction map $r : \mathscr{A}(\Omega) \longrightarrow H(K)$ is continuous. In fact,

$$\mathscr{A}(\Omega) = \mathrm{proj}_{N\in\mathbb{N}} \; H(K_N) = \mathrm{proj}_{N\in\mathbb{N}} \; \mathrm{ind}_{n\in\mathbb{N}} \; H^{\infty}(U_{N,n}).$$

- $\mathscr{A}(\Omega)$ is complete, separable, barrelled and Montel.

- **Domański, Vogt, 2000.** The space $\mathscr{A}(\Omega)$ has no Schauder basis.

Theorem

Let $\varphi : \Omega \to \Omega$, $\Omega \subseteq \mathbb{R}^d$ open connected, be a real analytic map. Then the following assertions are equivalent:

(a) $C_\varphi : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ is power bounded;

(b) $C_\varphi : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ is uniformly mean ergodic;

(c) $C_\varphi : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ is mean ergodic;

(d) $\forall\, K \Subset \Omega\ \exists\, L \Subset \Omega\ \ \forall\, U$ complex neighbourhood of L $\exists\, V$ complex neighbourhood of K:

$$\forall\, n \in \mathbb{N} \quad \varphi^n \text{ is defined on } V \text{ and } \varphi^n(V) \subseteq U;$$

(e) For every complex neighbourhood U of Ω there is a complex (open!) neighbourhood $V \subseteq U$ of Ω such that φ extends as a holomorphic function to V and $\varphi(V) \subseteq V$.

Corollary

Let $\varphi : \Omega \to \Omega$ be a real analytic map, $\Omega \subseteq \mathbb{R}^d$ open connected set.

If $C_\varphi : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ is power bounded, then for every complex neighbourhood U of Ω there is a complex neighbourhood $V \subseteq U$ of Ω and there is a fundamental system of compact sets (L_j) in V such that $\varphi(L_j) \subseteq L_j$.

In particular, $\varphi(V) \subseteq V$ and $C_\varphi : H(V) \to H(V)$ is power bounded and (uniformly) mean ergodic.

Corollary

Let $\Omega \subseteq \mathbb{R}^d$ be an open connected set and let $\varphi : \Omega \to \Omega$ be a real analytic map with a fixed point $u \in \Omega$.

If $C_\varphi : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ is power bounded, then either $|\det \varphi'(u)| < 1$ or φ is a biholomorphic automorphism of a (fundamental) family of hyperbolic complex neighbourhoods of Ω.

In the latter case, if $\varphi'(u)$ is the identity map then φ is the identity.

All criteria for power bounded composition operators $C_\varphi : \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$ contain some condition on the behavior of φ outside Ω.

Remark. Even in the first case of the Corollary, orbits of φ need not converge to a fixed point (and there could exist many fixed points). Consider the map $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ defined by

- $\varphi_1(x, y, z) := (x \cos z - y \sin z) \cdot \left(0.5 + \frac{1}{2\sqrt{x^2+y^2}}\right)$,
- $\varphi_2(x, y, z) := (x \sin z + y \cos z) \cdot \left(0.5 + \frac{1}{2\sqrt{x^2+y^2}}\right)$,
- $\varphi_3(x, y, z) := z$,

defined on a cylinder with basis in the x-y plane being an annulus ($z \in (-1, 1)$).
The only fixed points are of the form $(x, y, 0)$ where $x^2 + y^2 = 1$ but orbits starting from (x, y, z) tend to the circle $\{(x, y, z) : x^2 + y^2 = 1\}$.

Example: Even if $\varphi :]-1,1[\to]-1,1[$ maps all bounded sets in $]-1,1[$ in one compact set it does not follow that $C_\varphi : \mathscr{A}(]-1,1[) \to \mathscr{A}(]-1,1[)$ is power bounded.

$$\varphi_\alpha(z) := \frac{2}{\alpha \cdot \pi} \ln \left(\frac{1 - iz}{1 + iz} \right).$$

- φ_α maps the unit disc onto the vertical strip with $\operatorname{Re} z \in \left(-\frac{1}{\alpha}, \frac{1}{\alpha}\right)$.
- $\varphi_\alpha(i) = \infty$, and φ_α maps the real line into the real line and the imaginary line into the imaginary line.
- If $\frac{\pi \cdot \alpha}{4} < 1$, the zero point is an attractive fixed point on the imaginary line for φ_α^{-1}. This implies that $\varphi_\alpha^{-n}(i) \to 0$ as $n \to \infty$.
- Since on this sequence φ_α^{n+1} is not defined, there is no common neighbourhood of zero such that all φ_α^n are defined for all $n \in \mathbb{N}$ (α, $\alpha < 1$, $\frac{\pi \cdot \alpha}{4} < 1$). Thus $C_\varphi : \mathscr{A}(]-1,1[) \to \mathscr{A}(]-1,1[)$ is not power bounded.

Theorem

Let $a, b \in \bar{\mathbb{R}}$ and let $\varphi :]a, b[\to]a, b[$ be real analytic. The following are equivalent:

(a) $C_\varphi : \mathscr{A}(]a, b[) \longrightarrow \mathscr{A}(]a, b[)$ is power bounded;

(b) there exists a complex neighbourhood U of $]a, b[$ such that $\varphi(U) \subseteq U$, $\mathbb{C} \setminus U$ contains at least two points, and φ has a (real) fixed point u, or equivalently, there is a fundamental family of such neighbourhoods of $]a, b[$;

(c) φ is one of the following forms:

1. $\varphi = \mathrm{id}$;
2. $\varphi^2 = \mathrm{id}$;
3. As $n \to \infty$ the sequence φ^n tends to a constant function $\equiv u \in]a, b[$ in $\mathscr{A}(]a, b[)$.

Theorem continued

(d) If u is the fixed point of φ then the above cases in (c) correspond to:

1. $\varphi'(u) = 1$;
2. $\varphi'(u) = -1$;
3. $|\varphi'(u)| < 1$.

Moreover, C_φ is uniformly mean ergodic and the projection $P := \lim_{N\in\mathbb{N}} \frac{1}{N}\sum_{n=1}^{N} C_{\varphi^n}$ is of the following form:

1. $P = \mathrm{id}$;
2. $P(f) = \frac{f+f\circ\varphi}{2}$, im $P = \{f : f = f\circ\varphi\}$, ker $P = \{f : f = -f\circ\varphi\}$;
3. $P(f) = f(u)$, im $P =$ the set of constant functions, ker $P = \{f : f(u) = 0\}$.

Remarks. The case $d > 1$.

- If Ω is an annulus in $\mathbb{R}^2$, then rotations φ (which have no fixed points) produce power bounded composition operators with arbitrary long cycles (for any $k \in \mathbb{N}$ we can choose a rotation φ such that $C_{\varphi^k} = \text{id}$ but $C_{\varphi^n} \neq \text{id}$ for every $0 < n < k$).

- Taking rotations with respect to $\eta \cdot \pi$ where η is irrational, we get a non-cyclic C_φ not satisfying condition (c) 3. of the above Theorem.

- If we consider such an "irrational" rotation on the disc in $\mathbb{R}^2$ we get a non-cyclic C_φ which does not satisfy (d) even though φ has a fixed point.

References

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