# FRÉCHET SPACES WITH NO INFINITE–DIMENSIONAL BANACH QUOTIENTS

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Abstract. We exhibit examples of Fréchet Montel spaces $E$ which have a non-reflexive Fréchet quotient but such that every Banach quotient is finite dimensional. The construction uses a method developed by Albanese and Moscatelli and requires new ingredients. Some of the main steps in the proof are presented in Section 2. They are of independent interest and show for example that the canonical inclusion between James spaces $J_p \subset J_q, 1 < p < q < \infty$, is strictly cosingular. This result requires a careful analysis of the block basic sequences of the canonical basis of the dual $J_p'$ of the James space $J_p$, and permits us to show that the Fréchet space $J_{p^+} = \cap_{q>p} J_q$ has no infinite–dimensional Banach quotients. Plichko and Maslyuchenko had proved that it has no infinite–dimensional Banach subspaces.

## 1. Introduction.

It is well known that every quotient of a reflexive Banach space is also reflexive. Grothendieck [13] discovered that there are Köthe echelon spaces of order one which are Montel, hence reflexive, but have a quotient isomorphic to $\ell_1$ (see also [19, §31, p.433]). Following Grothendieck [13, Définition 2, p.99], a lcHs $X$ is called *totally reflexive* if every quotient of $X$ is reflexive. Answering a question of Grothendieck [13, Probl. 9], Valdivia proved in [29, Theorem 3] that a Fréchet space $X$ is totally reflexive if and only if it is the reduced projective limit of a sequence of reflexive Banach spaces. The aim of this paper is to give examples of non-totally reflexive, reflexive Fréchet spaces with no infinite–dimensional Banach quotients in Theorem 3.2 and non-Schwartz, Montel Fréchet spaces with no infinite–dimensional Banach quotients in Theorem 3.1. More concrete examples are given in Example 3.3. These examples answer in the negative the natural question whether every Fréchet space with a non reflexive quotient has a non reflexive Banach quotient. The question treated here is related to recent work of Bonet and Wright [6] on factorization of weakly compact operators between Banach spaces and Fréchet or (LB)–spaces. The construction uses a method to exhibit Fréchet spaces due to Albanese and Moscatelli [1]. However, the main step is accomplished in Section 2. There we prove in Theorem 2.10 that the canonical inclusion between James spaces $J_p \subset J_q, 1 < p < q < \infty$, is strictly cosingular. This results requires a careful analysis of the block basic sequences of the canonical basis of the dual $J_p'$ of the James space $J_p$, and permits us to show in Theorem 2.11 that the Fréchet space $J_{p^+} = \cap_{q>p} J_q$ and the (LB)-space $J_{p^-} = \cup_{1<q<p} J_q$ have no infinite–dimensional Banach quotients. Plichko and Maslyuchenko had proved in [28] that they have no infinite–dimensional Banach

*Key words and phrases.* Banach quotient, reflexive Fréchet space, Fréchet Montel space, James space.

*Mathematics Subject Classification 2010:* Primary 46A04, 46A25; Secondary 46A11, 46B45.

*Acknowledgement.* This research was partially supported by MEC and FEDER Project MTM2010-15200 and by GV Project Prometeo/2008/101.

subspaces. Some positive results are included in Section 4. An Appendix collects technical results needed in the proofs.

**Notation.** Let $E$ be a locally convex Hausdorff space (lcHs, briefly) and $\Gamma_E$ a system of continuous seminorms determining the topology of $E$. Denote by $\mathcal{L}(E)$ the space of of all continuous linear operators from $E$ to itself (from $E$ to another lcHs $F$ we write $\mathcal{L}(E,F)$). The collection of all bounded subsets of $E$ is denoted by $\mathcal{B}(E)$. $E_\sigma$ stands for $E$ equipped with its weak topology $\sigma(E,E')$, where $E'$ is the topological dual space of $E$. The strong topology in $E$ (resp. $E'$) is denoted by $\beta(E,E')$ (resp. $\beta(E',E)$) and we write $E_\beta$ (resp. $E'_\beta$). The strong dual space $(E'_\beta)'_\beta$ of $E'_\beta$ is denoted by $E''_\beta$. $E'_{\sigma^*}$ stands for $E'$ equipped with its weak–star topology $\sigma(E',E)$. Given $T\in\mathcal{L}(E)$, its *dual operator* $T'\colon E'\to E'$ is defined by $\langle x, T'x'\rangle = \langle Tx, x'\rangle$ for all $x\in E$, $x'\in E'$. It is known that $T'\in\mathcal{L}(E'_\sigma)$ and $T'\in\mathcal{L}(E'_\beta)$. For a quotient space of a lcHs $E$ we always mean a separated quotient of $E$ endowed with its quotient lc–topology. For two lcHs $E$ and $F$, we write $E\simeq F$ to mean that $E$ is topologically isomorphic to $F$.

For undefined notation about functional analysis and locally convex spaces, and about Banach spaces we refer to [19, 22, 13] and [10, 21, 32], respectively. Our notation for Köthe echelon spaces is as in [3].

## 2. The James Fréchet spaces $J_{p^+}$, $p\geq 1$

It is known that the space $J_{p^+}=\cap_{q>p}J_q$, $p\geq 1$, is a non–reflexive Fréchet space with no infinite-dimensional Banach subspaces, [28]. Our aim is to show that $J_{p^+}$, $p\geq 1$, is also a Fréchet space with no infinite-dimensional Banach quotients. For this purpose, we first collect some results about block basic sequences of $(e'_n)_{n=1}^\infty$ in the dual James space $J'_p$, $1<p<\infty$ ($(e'_n)_{n=1}^\infty$ denotes the canonical Schauder basis of $J'_p$). Then, we study the topological structure of the closed infinite–dimensional subspaces of $J'_p$ and hence, of the infinite–dimensional quotients of $J_p$.

Let $1<p<\infty$. For each sequence $x=(a_i)_{i=1}^\infty$ of real numbers, set

$$\|x\|_{J_p} := \frac{1}{\sqrt[p]{2}}\sup\left(|a_{k_n}-a_{k_1}|^p+\sum_{i=1}^{n-1}|a_{k_i}-a_{k_{i+1}}|^p\right)^{1/p}, \tag{2.1}$$

where the supremum is taken over all $n\in\mathbb{N}$ with $n\geq 2$, and all choices of integers $(k_i)_{i=1}^n$ with $1\leq k_1<k_2<\ldots<k_n$.

Recall that the $p^{\text{th}}$ James space is defined by

$$J_p := \{x=(a_i)_{i=1}^\infty\subseteq\mathbb{R} : \ x\in c_0 \text{ and } \|x\|_{J_p}<\infty\}.$$

Then $(J_p, \|\ \|_{J_p})$ is a Banach space. The unit vectors $e_n:=(\delta_{in})_{i=1}^\infty$, $n\in\mathbb{N}$, form a monotone and shrinking basis for $J_p$ with respect to the norm (2.1). So, the biorthogonal functionals $(e'_n)_{n=1}^\infty\subseteq J'_p$ associated with $(e_n)_{n=1}^\infty$ form a boundedly complete basis of the dual space $J'_p$ of $J_p$. The sequence $(x_n)_{n=1}^\infty\subseteq J_p$ defined by $x_n=\sum_{i=1}^n e_i$, for $n\in\mathbb{N}$, is a boundedly complete basis for $J_p$, with biorthogonal functionals $(x'_n)_{n=1}^\infty\subseteq J'_p$ given by $x'_n=e'_n-e'_{n+1}$, for $n\in\mathbb{N}$. The bidual space $J''_p$ of $J_p$ consists of all sequences $(a_i)_{i=1}^\infty$ of real numbers for which the variation norm (2.1) is finite. Since the finiteness of the norm (2.1) implies the existence of $\lim_{i\to\infty}a_i$, one infers that $J''_p$ is the linear span of $J_p$ (more precisely, of the canonical image of $J_p$ in $J''_p$) and the functional $x''_0$ defined by $x''_0(e'_n)=1$ for all $n\in\mathbb{N}$, i.e., the functional which corresponds to the sequence $\mathbf{1}=(1,1,1,\ldots)$. It follows that the canonical image of $J_p$ in its bidual space $J''_p$ has codimension 1 and hence, $J_p$ is *quasi–reflexive* *of order 1.* In particular, the space $(J_p, \| \, \|_{J_p})$ is isometric to its bidual space $J_p''$. For $p = 2$ these results have been shown in [15] (or see [21, Example 1.d.2, pp.25–26]); for an arbitrary $1 < p < \infty$ the proofs are similar.

Now, we observe that

**Proposition 2.1.** *Let $1 < p < \infty$ and $x' = \sum_{i=1}^{\infty} a_i e_i' \in J_p'$. Then the following assertions hold.*

(i) *If $a_i \geq 0$ for all $i \in \mathbb{N}$, then $\|x'\|_{J_p}' = \sum_{i=1}^{\infty} a_i$.*

(ii) $\|x'\|_{J_p}' \geq \frac{1}{\sqrt[q]{2}} (\sum_{i=1}^{\infty} |a_i|^q)^{1/q}$, *where $1 < q < \infty$ is the conjugate exponent of $p$.*

*Proof.* (i). Since $\|\mathbf{1}\|_{J_p}'' = 1$, we have $\sum_{i=1}^{\infty} a_i = \langle \mathbf{1}, x' \rangle \leq \|x'\|_{J_p}'$ and hence, $0 \leq \sum_{i=1}^{\infty} a_i < \infty$. On the other hand, $\|x'\|_{J_p}' \leq \sum_{i=1}^{\infty} a_i$. So, the result follows.

(ii). Let $q$ be the conjugate exponent of $p$. Observe that (2.1) and $(|x| + |y|)^p \leq 2^{p-1}(|x|^p + |y|^p)$, for $x$, $y \in \mathbb{R}$, imply that

$$\| \sum_{i=1}^{n} \alpha_i e_i \|_{J_p} \leq \sqrt[q]{2} (\sum_{i=1}^{n} |\alpha_i|^p)^{1/p}$$

for every $n \in \mathbb{N}$ and $(\alpha_i)_{i=1}^n \subseteq \mathbb{R}$. Set $\alpha_i := |a_i|^{q-1} \mathrm{sign}\,(a_i)$ for all $i \in \mathbb{N}$. Then, it follows that

$$\| \sum_{i=1}^{n} |a_i|^{q-1} \mathrm{sign}\,(a_i) e_i \|_{J_p} \leq \sqrt[q]{2} (\sum_{i=1}^{n} |a_i|^{(q-1)p})^{1/p} = \sqrt[q]{2} (\sum_{i=1}^{n} |a_i|^q)^{1/p}, \quad n \in \mathbb{N},$$

and hence,

$$\sum_{i=1}^{n} |a_i|^q = \langle \sum_{i=1}^{n} |a_i|^{q-1} \mathrm{sign}\,(a_i) e_i, x' \rangle \leq \|x'\|_{J_p}' \sqrt[q]{2} (\sum_{i=1}^{n} |a_i|^q)^{1/p}, \quad n \in \mathbb{N}.$$

So, we obtain that

$$\|x'\|_{J_p}' \geq \frac{1}{\sqrt[q]{2}} (\sum_{i=1}^{n} |a_i|^q)^{1/q}, \quad n \in \mathbb{N},$$

and the thesis follows letting $n \to \infty$. □

Let $X$ be a Banach space. A sequence $(x_n)_{n=1}^{\infty}$ in $X$ is called *semi–normalized* if there exists a constant $C > 0$ such that $C^{-1} \leq ||x_n|| \leq C$ for all $n \in \mathbb{N}$. A sequence $(x_n)_{n=1}^{\infty}$ in $X$ is called *basic sequence* if it is a Schauder basis for its closed linear span $[(x_n)_{n=1}^{\infty}]$. If $(x_n)_{n=1}^{\infty}$ is a Schauder basis for $X$, a sequence $(y_n)_{n=1}^{\infty}$ in $X$ is called *block basic sequence* of $(x_n)_{n=1}^{\infty}$ if $y_k = \sum_{i=n_k+1}^{n_{k+1}} a_i x_i$, $k \in \mathbb{N}$, for some sequence $0 = n_1 < n_2 < \ldots n_k < n_{k+1} < \ldots$ of integers and some sequence $(a_n)_{n=1}^{\infty}$ of scalars.

**Proposition 2.2.** *Let $1 < p < \infty$ and $(z_k)_{k=1}^{\infty}$ be a block basic sequence in $J_p'$ with $z_k = \sum_{i=n_k+1}^{n_{k+1}} a_i e_i'$. Suppose that $\sum_{i=n_k+1}^{n_{k+1}} a_i = K > 0$ for all $k \in \mathbb{N}$. Then the following estimate holds:*

$$\| \sum_{k=1}^{\infty} b_k z_k \|_{J_p}' \geq K \| \sum_{k=1}^{\infty} b_k e_k' \|_{J_p}'. \tag{2.2}$$

*If, in addition, $a_i \geq 0$ for all $i \in \mathbb{N}$, then the converse estimates also holds, i.e.,*

$$\| \sum_{k=1}^{\infty} b_k z_k \|_{J_p}' \leq \sqrt[q]{2} K \| \sum_{k=1}^{\infty} b_k e_k' \|_{J_p}', \tag{2.3}$$

*where $1 < q < \infty$ is the conjugate exponent of $p$.*

*Proof.* Let $x = \sum_{i=1}^{\infty} c_i e_i \in J_p$. Observe that $\|x\|_{J_p} = \|\sum_{k=1}^{\infty} c_k (\sum_{i=n_k+1}^{n_{k+1}} e_i)\|_{J_p}$ and that

$$\langle \sum_{k=1}^{\infty} c_k \sum_{i=n_k+1}^{n_{k+1}} e_i, \sum_{k=1}^{\infty} b_k z_k \rangle = \sum_{k=1}^{\infty} b_k c_k \sum_{i=n_k+1}^{n_{k+1}} a_i = K \sum_{k=1}^{\infty} b_k c_k = K \langle \sum_{k=1}^{\infty} c_k e_k, \sum_{k=1}^{\infty} b_k e'_k \rangle.$$

From where it follows

$$K|\langle \sum_{k=1}^{\infty} c_k e_k, \sum_{k=1}^{\infty} b_k e'_k \rangle| \leq \|x\|_{J_p} \cdot \| \sum_{k=1}^{\infty} b_k z_k \|'_{J_p},$$

which implies

$$K \| \sum_{k=1}^{\infty} b_k e'_k \|'_{J_p} = K \sup_{\|x\|_{J_p}=1} |\langle \sum_{k=1}^{\infty} c_k e_k, \sum_{k=1}^{\infty} b_k e'_k \rangle| \leq \| \sum_{k=1}^{\infty} b_k z_k \|'_{J_p},$$

i.e., (2.2) is proved.

Suppose $a_i \geq 0$ for all $i \in \mathbb{N}$ and define $\overline{c}_k := \frac{1}{K} \sum_{i=n_k+1}^{n_{k+1}} a_i c_i$ for every $k \in \mathbb{N}$. Then (2.1) implies that

$$\| \sum_{k=1}^{\infty} \overline{c}_k e_k \|_{J_p} = \| \sum_{k=1}^{\infty} \overline{c}_k ( \sum_{i=n_k+1}^{n_{k+1}} e_i ) \|_{J_p} \leq \sqrt[q]{2} \| \sum_{k=1}^{\infty} ( \sum_{i=n_k+1}^{n_{k+1}} c_i e_i ) \|_{J_p}$$

and hence,

$$\begin{aligned} |\langle \sum_{k=1}^{\infty} c_k e_k, \sum_{k=1}^{\infty} b_k z_k \rangle| &= |\sum_{k=1}^{\infty} b_k \sum_{i=n_k+1}^{n_{k+1}} a_i c_i| = K |\sum_{k=1}^{\infty} b_k \overline{c}_k| \\ &= K |\langle \sum_{k=1}^{\infty} \overline{c}_k e_k, \sum_{k=1}^{\infty} b_k e'_k \rangle| \leq K \| \sum_{k=1}^{\infty} \overline{c}_k e_k \|_{J_p} \cdot \| \sum_{k=1}^{\infty} b_k e'_k \|'_{J_p} \\ &\leq \sqrt[q]{2} K \| \sum_{k=1}^{\infty} c_k e_k \|_{J_p} \cdot \| \sum_{k=1}^{\infty} b_k e'_k \|'_{J_p}. \end{aligned}$$

This implies

$$\| \sum_{k=1}^{\infty} b_k z_k \|'_{J_p} = \sup_{\|x\|_{J_p}=1} |\langle \sum_{k=1}^{\infty} c_k e_k, \sum_{k=1}^{\infty} b_k z_k \rangle| \leq \sqrt[q]{2} K \| \sum_{k=1}^{\infty} b_k e'_k \|'_{J_p}$$

and (2.3) is proved. □

An immediate consequence of Proposition 2.2 is the following result.

**Corollary 2.3.** *Let $1 < p < \infty$. Then the unit vector basis $(e'_n)_{n=1}^{\infty}$ of $J'_p$ is equivalento to each of its subsequences, i.e., it is spreading.*

**Proposition 2.4.** *Let $1 < p < \infty$ and $(z_k)_{k=1}^{\infty}$ a semi–normalized block basic sequence in $J'_p$ with $z_k = \sum_{i=n_k+1}^{n_{k+1}} a_i e'_i$. Suppose that $\sum_{i=n_k+1}^{n_{k+1}} a_i = 0$ for all $k \in \mathbb{N}$. Then $(z_k)_{k=1}^{\infty}$ is equivalent to the unit vector basis of $\ell_q$, with $1 < q < \infty$ the conjugate exponent of $p$, and $[(z_k)_{k=1}^{\infty}]$ is a complemented subspace of $J'_p$.*

*Proof.* Since $(e'_n)_{n=1}^{\infty}$ is spreading, we may assume that $a_{n_k} = 0$ for all $k \in \mathbb{N}$.

Consider the operator $P \colon J_p \to J_p$ defined by

$$P(\sum_{i=1}^{\infty} \alpha_i e_i) := \sum_{k=1}^{\infty} \alpha_{n_{k+1}} \left( \sum_{i=n_k+1}^{n_{k+1}} e_i \right), \quad (\alpha_i)_{i=1}^{\infty} \in J_p.$$

Then $P$ is an idempotent operator (i.e., a projection on $J_p$) such that $\mathrm{Im}P \simeq J_p$ and $\mathrm{Ker}\, P = [\{e_j : \ j \neq n_k \ \forall k \in \mathbb{N}\}]$ is reflexive, see [7, Corollary 3 and Theorem 5] for $p = 2$, [20, Lemma 4.2] for $1 < p < \infty$. Moreover, $\mathrm{Ker}\, P$ is isomorphic to the direct sum $\left(\oplus J_p^{(n_k)}\right)_p$ (in the sense of $\ell_p$) of $\{J_p^{(n_k)}\}_{k=1}^\infty$, where $J_p^{(n_k)}$ is the finite–dimensional Banach space defined by

$$J_p^{(n_k)} := [e_{n_k+1}, e_{n_k+2}, \ldots, e_{n_{k+1}-1}], \quad k \in \mathbb{N},$$

and equipped with the norm it inherits from $J_p$, see [7, Lemma 2] for $p = 2$, [20, Proposition 4.4(iv)] for $1 < p < \infty$. Observe that $J_p^{(n_k)} \subseteq \mathrm{Ker}\, P$ for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$ let $Q_k \colon J_p \to J_p$ denote the natural projection of $J_p$ (and also, of $\mathrm{Ker}\, P$) onto $J_p^{(n_k)}$ (i.e., $Q_k(\sum_{i=1}^\infty \alpha_i e_i) = \sum_{i=n_k+1}^{n_{k+1}-1} \alpha_i e_i$ for $(\alpha_i)_{i=1}^\infty \in J_p$). Then $(I - P)Q$ is again a projection of $J_p$ (and also, of $\mathrm{Ker}\, P$) onto $J_p^{(n_k)}$ for all $k \in \mathbb{N}$. So, $Q'(I - P')$ is a projection of $J'_p$ (and also, of $(\mathrm{Ker}\, P)'$) onto $(J_p^{(n_k)})'$ for all $k \in \mathbb{N}$. Since $a_{n_k} = 0$ for all $k \in \mathbb{N}$, we have that $Q'(I - P')z_k = z_k$ so that $z_k \in (J_p^{(n_k)})' \subseteq (\mathrm{Ker}\, P)'$ for all $k \in \mathbb{N}$. Since $(\mathrm{Ker}\, P)'$ is a (complemented) closed subspace of $J'_p$ and isomorphic to $\left(\oplus (J_p^{(n_k)})'\right)_q$, and the semi–normalized sequence $(z_k)_{k=1}^\infty \subseteq \left(\oplus (J_p^{(n_k)})'\right)_q$ is equivalent to the standard basis of $\ell_q$ in $\left(\oplus (J_p^{(n_k)})'\right)_q$ and $[(z_k)_{k=1}^\infty]$ is a complemented subspace of $\left(\oplus (J_p^{(n_k)})'\right)_q$, the result follows. □

**Proposition 2.5.** *Let $1 < p < \infty$ and $(z_k)_{k=1}^\infty$ a semi–normalized block basic sequence in $J'_p$ with $z_k = \sum_{i=n_k+1}^{n_{k+1}} a_i e'_i$. Suppose that $\sum_{i=n_k+1}^{n_{k+1}} a_i = K > 0$ for all $k \in \mathbb{N}$. Then $(z_k)_{k=1}^\infty$ is equivalent to the basis $(e'_k)_{k=1}^\infty$ of $J'_p$, and $[(z_k)_{k=1}^\infty]$ is a complemented subspace of $J'_p$.*

*Proof.* Proposition 2.2 implies that $\| \sum_{k=1}^\infty b_k e'_k \|'_{J_p} \leq K^{-1} \| \sum_{k=1}^\infty b_k z_k \|'_{J_p}$ for every sequence $(b_k)_{k=1}^\infty$ for which the serie $\sum_{k=1}^\infty b_k z_k$ converges in $J'_p$. To establish the other inequality, it suffices to show that the convergence of the series $\sum_{k=1}^\infty b_k e'_k$ in $J'_p$ implies the convergence of the series $\sum_{k=1}^\infty b_k z_k$ in $J'_p$. So, we first observe that

$$\begin{aligned} z_k &= \sum_{i=n_k+1}^{n_{k+1}} a_i e'_i = \frac{K}{n_{k+1} - n_k} \sum_{i=n_k+1}^{n_{k+1}} e'_i + \sum_{i=n_k+1}^{n_{k+1}} \left( a_i - \frac{K}{n_{k+1} - n_k} \right) e'_i \\ &=: u_k + v_k, \quad k \in \mathbb{N}, \end{aligned}$$

where $\|u_k\|'_{J_p} = K$ for all $k \in \mathbb{N}$ by Proposition 2.1(i) and hence, there is $M > 0$ such that $\|v_k\|'_{J_p} \leq M$ as $(z_k)_{k=1}^\infty$ is a semi–normalized sequence. If now $\sum_{k=1}^\infty b_k e'_k$ converges in $J'_p$, then from Proposition 2.2 it follows that $\sum_{k=1}^\infty b_k u_k$ converges in $J'_p$, and from Propositions 2.1(ii) and 2.5 that $\sum_{k=1}^\infty b_k v_k$ converges in $J'_p$. Hence, $\sum_{k=1}^\infty b_k z_k$ converges in $J'_p$. Thus, $(z_k)_{k=1}^\infty$ is equivalento to $(e'_k)_{k=1}^\infty$.

As in the proof of [2, Proposition 6], we can show that the operator $P \colon J_p \to J_p$ defined by

$$P(\sum_{i=1}^\infty c_i e_i) := \frac{1}{K} \sum_{k=1}^\infty \left( \sum_{i=n_k+1}^{n_{k+1}} a_i c_i \right) \left( \sum_{i=n_k+1}^{n_{k+1}} e_i \right), \quad (c_i)_{i=1}^\infty \in J_p,$$

is a projection on $J_p$ with dual map $P'$ given by

$$P'\left(\sum_{i=1}^{\infty} d_i e_i'\right) = \sum_{k=1}^{\infty}\left(\frac{1}{K}\sum_{i=n_k+1}^{n_{k+1}} d_i\right) z_k.$$

So, $\mathrm{Im}P' = [(z_k)_{k=1}^{\infty}]$ and hence, $[(z_k)_{k=1}^{\infty}]$ is a complemented subspace of $J_p'$. □

**Remark 2.6.** For $p = 2$ the results collected above are known and due to Andrew [2].

The following result is the basic step towards Theorem 2.10 and seems new even for the case $p = 2$.

**Theorem 2.7.** *Let $1 < p, q < \infty$ with $q$ the conjugate exponent of $p$. If $X$ is an infinite–dimensional closed subspace of $J_p'$, then $X$ contains a subspace isomorphic to $\ell_q$ and complemented in $J_p'$.*

*Proof.* Since $J_p'$ is a quasi–reflexive space, each infinite–dimensional closed subspace of $J_p'$ contains an infinite–dimensional reflexive subspace, see [14, Lemma 2]. So, $X$ contains a sequence of norm one vectors $(y_k)_{k=1}^{\infty}$ which $\sigma(J_p', J_p'')$–converges to 0. By [21, Proposition 1.a.12] we may assume that $(y_k)_{k=1}^{\infty}$ is equivalent to a semi–normalized block basic sequence $(z_k)_{k=1}^{\infty}$ of $(e_k')_{k=1}^{\infty}$. Consequently, $(z_k)_{k=1}^{\infty}$ also $\sigma(J_p', J_p'')$–converges to 0 and hence, $\langle \mathbf{1}, z_k\rangle \to 0$ as $k \to \infty$: i.e., suppose, for every $k \in \mathbb{N}$, that $z_k = \sum_{i=n_k+1}^{n_{k+1}} a_i e_i'$, we have $\delta_k := \sum_{i=n_k+1}^{n_{k+1}} a_i \to 0$ as $k \to \infty$.

To finish the proof we have to consider two cases: (a) $\delta_k = 0$ for infinitely many indices $k$; (b) there exists $k_0 \in \mathbb{N}$ such that $\delta_k \neq 0$ for every $k \geq k_0$.

(a) Let $(k_j)_{j=1}^{\infty}$ be a strictly increasing sequence of positive integers such that $\delta_{k_j} = 0$ for every $j \in \mathbb{N}$. Then, from Proposition 2.4 it follows that $(z_{k_j})_{j=1}^{\infty}$ is equivalent to the unit vector basis of $\ell_q$ and, $[(z_{k_j})_{j=1}^{\infty}]$ is a complemented subspace of $J_p'$. Since $(y_{k_j})_{j=1}^{\infty}$ is equivalent to $(z_{k_j})_{j=1}^{\infty}$, $(y_{k_j})_{j=1}^{\infty}$ is also equivalent to the unit vector basis of $\ell_q$ and $[(y_{k_j})_{j=1}^{\infty}]$ is a complemented subspace of $J_p'$.

(b) By passing to a subsequence if necessary, we may suppose that each $\delta_k > 0$ and $\sum_{k=1}^{\infty} \delta_k =: \delta < 1$. Then we may write

$$z_k = \frac{\delta_k}{n_{k+1} - n_k}\sum_{i=n_k+1}^{n_{k+1}} e_i' + \sum_{n_k+1}^{n_{k+1}}\left(a_i - \frac{\delta_k}{n_{k+1} - n_k}\right) e_i' =: u_k + v_k, \quad k \in \mathbb{N},$$

with $\|u_k\|_{J_p}' = \delta_k$ by Proposition 2.1(i) and $\sum_{n_k+1}^{n_{k+1}}\left(a_i - \frac{\delta_k}{n_{k+1}-n_k}\right) = 0$ for every $k \in \mathbb{N}$. So, by Propositon 2.4 $(v_k)_{k=1}^{\infty}$ is equivalent to the unit vector basis of $\ell_q$ and, $[(v_k)_{k=1}^{\infty}]$ is a complemented subspace of $J_p'$. Moreover, we have

$$\sum_{k=1}^{\infty} \|z_k - v_k\|_{J_p}' = \sum_{k=1}^{\infty} \delta_k = \delta < 1.$$

It follows that the block basic sequence $(z_k)_{k=1}^{\infty}$ is equivalent to the block basic sequence $(v_k)_{k=1}^{\infty}$, [21, Proposition 1.a.9(ii)] and hence, to the unit vector basis of $\ell_q$, and that $[(z_k)_{k=1}^{\infty}]$ is complemented in $J_p'$ (taking $\delta$ small enough). Since $(y_k)_{k=1}^{\infty}$ is equivalent to $(z_k)_{k=1}^{\infty}$, $(y_k)_{k=1}^{\infty}$ is also equivalent to the unit vector basis of $\ell_q$ and, the subspace $[(y_k)_{k=1}^{\infty}]$ is complemented in $J_p'$. □

The next result for $p = 2$ is due to Andrew [2, Theorem 2].

**Theorem 2.8.** *Let $1 < p < \infty$. If $X$ is a non–reflexive subspace of $J_p'$, then $X$ contains a subspace isomorphic to $J_p'$ and, complemented in $J_p'$.*

*Proof.* Since $X$ is a non–reflexive subspace of $J_p'$, there exists a sequence of norm one vectors $(y_k)_{k=1}^\infty \subseteq X$ with no $\sigma(J_p', J_p'')$–convergent subsequences. Since $J_p$ is separable, the unit ball of $J_p'$ is $\sigma(J_p', J_p)$–compact and so, we may suppose that $(y_k)_{k=1}^\infty \subseteq X$ has a $\sigma(J_p', J_p)$–limit $y \in J_p'$, by eventually passing to a subsequence.

We first consider the case when $y \in X$. In such a case, we may suppose $y = 0$. Since $(y_k)_{k=1}^\infty$ does not $\sigma(J_p', J_p'')$–converges to 0, there exists $x \in J_p''$ so that $\langle x, y_k\rangle \not\to 0$ as $k \to \infty$. But, $x = x_0 + \lambda\mathbf{1}$ for some $x_0 \in J_p$ and $\lambda \in \mathbb{R}$. So, the $\sigma(J_p', J_p)$–convergence of $(y_k)_{k=1}^\infty$ to 0 implies that $\lambda \neq 0$ and that $\langle \mathbf{1}, y_k\rangle \not\to 0$ as $k \to \infty$. It follows that $\langle \mathbf{1}, y_k\rangle > \bar\varepsilon$ for some $\bar\varepsilon > 0$, by eventually passing to a subsequence. We can now construct a closed subspace of $X$ which is isomorphic to $J_p'$ and complemented in $J_p'$.

Assume $y_k = \sum_{i=1}^\infty a_i^{(k)} e_i'$, for $k \in \mathbb{N}$. Then $1 \geq \sum_{i=1}^\infty a_i^{(k)} = \langle \mathbf{1}, y_k\rangle > \bar\varepsilon$ for $k \in \mathbb{N}$. Fix $\varepsilon \in ]0,1[$, $(\varepsilon_k)_{k=1}^\infty \subseteq ]0,\infty)$ so that $\sum_{k=1}^\infty \varepsilon_k < \varepsilon/2$. Pick $n_2 \in \mathbb{N}$ such that $\lambda_1 := \sum_{i=1}^{n_2} a_i^{(1)} > \bar\varepsilon$ and $\|\sum_{i>n_2} a_i^{(1)} e_i'\|_{J_p}' < \varepsilon_1\bar\varepsilon$. If we set $z_1 = \lambda_1^{-1}\sum_{i=1}^{n_2} a_i^{(1)} e_i'$, then $\|z_1 - \lambda_1^{-1} y_1\|_{J_p}' < \lambda_1^{-1}\bar\varepsilon\varepsilon_1 < \varepsilon_1$. Now, the $\sigma(J_p', J_p)$–convergence of $(y_k)_{k=1}^\infty$ to 0 ensures that there exists $k_2 > k_1 := 1$ such that

$$\sup_{i=1}^{n_2} |a_i^{(k)}| < \frac{\varepsilon_2\bar\varepsilon}{2n_2}, \quad k \geq k_2.$$

This implies that

$$\sum_{i>n_2} a_i^{(k)} > \frac{\bar\varepsilon}{2} \quad \text{and} \quad \|\sum_{i=1}^{n_2} a_i^{(k)} e_i'\|_{J_p}' < \frac{\varepsilon_2\bar\varepsilon}{2}, \quad k \geq k_2.$$

Hence, there exists $n_3 > n_2$ ($n_1 := 0$) such that $\lambda_2 := \sum_{i=n_2+1}^{n_3} a_i^{(k_2)} > \frac{\bar\varepsilon}{2}$ and $\|\sum_{i>n_3} a_i^{(k_2)} e_i'\|_{J_p}' < \frac{\varepsilon_2\bar\varepsilon}{2}$. If we set $z_2 = \lambda_2^{-1}\sum_{i=n_2+1}^{n_3} a_i^{(k_2)} e_i'$, then $\|z_2 - \lambda_2^{-1} y_{k_2}\|_{J_p}' < \lambda_2^{-1}\bar\varepsilon\varepsilon_2 < 2\varepsilon_2$. Proceeding in this way, we construct a sequence $(\lambda_j)_{j=1}^\infty$ of scalars, a sequence $(z_j)_{j=1}^\infty$ of vectors and two increasing sequences $(k_j)_{j=1}^\infty$ and $(n_j)_{j=1}^\infty$ of positive integers such that $\frac{\bar\varepsilon}{2} < \lambda_j \leq 1$, $z_j = \lambda_j^{-1}\sum_{i=n_j+1}^{n_{j+1}} a_i^{(k_j)} e_i'$ and $\|z_j - \lambda_j^{-1} y_{k_j}\|_{J_p}' < 2\varepsilon_j$ for every $j \in \mathbb{N}$. It follows that, for each $j \in \mathbb{N}$, $1 - \varepsilon < \lambda_j^{-1} - 2\varepsilon_j < \|z_j\| < 2\varepsilon_j + \lambda_j^{-1} < \varepsilon + \frac{2}{\bar\varepsilon}$, $\langle \mathbf{1}, z_j\rangle = 1$, and $\sum_{j=1}^\infty \|z_j - \lambda_j^{-1} y_{k_j}\|_{J_p}' < \varepsilon$. This implies that $(\lambda_j^{-1} y_{k_j})_{j=1}^\infty \subseteq X$ is equivalent to $(z_j)_{j=1}^\infty$. But, by Proposition 2.5 the semi–normalized block basic sequence $(z_j)_{j=1}^\infty$ is equivalent to the basis $(e_j')_{j=1}^\infty$ of $J_p'$, and $[(z_j)_{j=1}^\infty]$ is a complemented subspace of $J_p'$. So, also $(\lambda_j^{-1} y_{k_j})_{j=1}^\infty \subseteq X$ is equivalent to the basis $(e_j')_{j=1}^\infty$ of $J_p'$, and $[(y_j)_{j=1}^\infty] \subseteq X$ is a complemented subspace of $J_p'$ (as soon as $\varepsilon$ is small enough).

We now consider the case when $X$ contains no sequences $\sigma(J_p', J_p)$–converging to 0 but failing to $\sigma(J_p', J_p'')$–converge to 0. In such a case, we observe that $(y_k)_{k=1}^\infty$ is a sequence of norm one vectors of $X \oplus [y]$ with no $\sigma(J_p', J_p'')$–convergent subsequences and $\sigma(J_p', J_p)$–convergent to $y \in X \oplus [y]$. So, proceeding as above, we can conclude that there exists a semi–normalized sequence $(z_k)_{k=1}^\infty \subseteq X \oplus [y]$ such that $(z_k)_{k=1}^\infty$ is $\sigma(J_p', J_p)$–convergent to 0, $\langle \mathbf{1}, z_k\rangle \not\to 0$ as $k \to \infty$, $(z_k)_{k=1}^\infty$ is equivalent to the basis $(e_j')_{j=1}^\infty$ of $J_p'$, and $[(z_k)_{k=1}^\infty] \subseteq X \oplus [y]$ is a complemented subspace of $J_p'$. Since $(z_k)_{k=1}^\infty \subseteq X \oplus [y]$, we have, for every $k \in \mathbb{N}$, that $z_k = u_k + a_k y$ with $u_k \in X$ and $a_k \in \mathbb{R}$. The assumption on $X$ ensures that we may suppose that $(a_k)_{k=1}^\infty$ has a non–zero cluster point. So, by perturbing and eventually passing to a subsequence, we may assume that $z_k = u_k + ay$ with $a \neq 0$ for every $k \in \mathbb{N}$. It follows that $z_k - z_{k+1} = y_k - y_{k+1} \in X$ for every $k \in \mathbb{N}$. On the other hand, $(z_k - z_{k+1})_{k=1}^{\infty}$ is equivalent to $(e'_k - e'_{k+1})_{k=1}^{\infty}$. So, $[(y_k - y_{k+1})_{k=1}^{\infty}] \simeq [(e'_k - e'_{k+1})_{k=1}^{\infty}]$. Since $(e'_k - e'_{k+1})_{k=1}^{\infty}$ is the dual basis of the boundedly complete basis $(x_k)_{k=1}^{\infty}$ for $J_p$ and hence, $[(e'_k - e'_{k+1})_{k=1}^{\infty}]' \simeq J_p$, [21, Proposition 1.b.4], we obtain that $[(y_k - y_{k+1})_{k=1}^{\infty}]' \simeq J_p$. Since the predual of $J_p$ is isomorphic to $J'_p$, [11, Theoreme 10], it follows that $[(y_k - y_{k+1})_{k=1}^{\infty}] \simeq J'_p$ and hence, $X$ contains an isomorphic copy of $J'_p$. Moreover, $[(y_k - y_{k+1})_{k=1}^{\infty}]$ is a one–codimensional closed subspace of $[(z_k)_{k=1}^{\infty}]$ and hence, it is complemented in $[(z_k)_{k=1}^{\infty}]$. Since $[(z_k)_{k=1}^{\infty}]$ is complemented in $J'_p$, it follows that also $[(y_k - y_{k+1})_{k=1}^{\infty}]$ is complemented in $J'_p$. □

Let $T\colon E \to F$ be a continuous linear operator between Banach spaces $E$ and $F$. The operator $T$ is called *strictly singular* if the restriction of $T$ to any infinite–dimensional subspace of $E$ is not an isomorphism, [18]. The operator $T$ is called *strictly cosingular* if the only Banach spaces $G$ for which there exist surjective continuous linear operators (i.e., quotient maps) $q_E\colon E \to G$ and $q_F\colon F \to G$ such that $q_E = q_F \circ T$ are finite–dimensional, [25]. The class of strictly singular operators is somewhat related by duality to the class of strictly cosingular operators. Indeed, if the dual operator $T'$ of an operator $T$ is strictly singular (strictly cosingular, resp.), then $T$ is strictly cosingular (strictly singular, resp.), [25]. However, the converse statements are not true in general. But, we clearly have:

**Remark 2.9.** Let $E$, $F$ be Banach spaces and $T \in \mathcal{L}(E, F)$. If $E$ is reflexive, then the operator $T$ is strictly singular (strictly cosingular, resp.) if and only if its dual operator $T'$ is strictly cosingular (strictly singular, resp.). Indeed, the reflexivity of $E$ implies that the bidual operator $T''$ coincides with $T$ and hence, the results follows by duality.

Let $1 < p, q < \infty$ with $p < q$. Since $\| \,\|_{J_p}$ is defined as the supremum of $\ell_p$–norms of certain sequences, $\| \,\|_{\ell_q} \leq \| \,\|_{\ell_p}$ implies $\| \,\|_{J_q} \leq \| \,\|_{J_p}$ on $J_p$. So, $J_p \subseteq J_q$ and the canonical inclusion map $\iota_p^q\colon J_p \hookrightarrow J_q$ is continuous with norm 1 and dense range. Moreover, the operator $\iota_p^q$ is strictly singular (it easily follows from the fact that each subspace of $J_p$ contains a subspace isomorphic to $\ell_p$, [28, Proposition 1]). On the other hand, Theorem 2.7 ensures that each subspace of $J'_q$ contains a subspace isomorphic to $\ell_q$ and hence, implies that the dual operator $(\iota_p^q)'\colon J'_q \hookrightarrow J'_p$ is also strictly singular. Therefore, we easily obtain that

**Theorem 2.10.** *Let $1 < p < q < \infty$ and $\iota_p^q\colon J_p \hookrightarrow J_q$ denote the canonical inclusion map. Then $\iota_p^q$ and its dual map $(\iota_p^q)'$ are both strictly singular and strictly cosingular.*

*Proof.* Since $(\iota_p^q)'$ is strictly singular by Theorem 2.7, $\iota_p^q$ is strictly cosingular, [25]. Hence, it remains to show only that $(\iota_p^q)'$ is strictly cosingular. This follows from the fact the bidual map $(\iota_p^q)''\colon (J_p)'' \hookrightarrow (J_q)''$ is strictly singular as $(J_p)'' \simeq J_p$, $(J_q)'' \simeq J_q$ and, each subspace of $J_p$ contains a subspace isomorphic to $\ell_p$, [28, Proposition 1]. □

Recall that, for $1 \leq p < \infty$, $J_{p^+}$ denotes the Fréchet space $\cap_{q>p} J_q$ with its natural projective topology (i.e., if $q_n \downarrow p$, then $J_{p^+} = \cap_{n=1}^{\infty} J_{q_n}$) and that, for $1 < p \leq \infty$, $J_{p^-}$ denotes the (LB)–space $\cup_{1<q<p} J_q$ again endowed with its natural inductive topology (i.e., if $1 < q_n \uparrow p$, then $J_{p^-} = \cup_{n=1}^{\infty} J_{q_n}$), [28]. In the sequel, for each $q > p$ ($q < p$, resp.), we denote by $\iota_q\colon J_{p^+} \hookrightarrow J_q$ ($\iota^q\colon J_q \hookrightarrow J_{p^-}$, resp.) the canonical inclusion map. So, for $p < q_2 < q_1$, $\iota_{q_1} = \iota_{q_2}^{q_1} \circ \iota_{q_2}$, and, for $1 < q_1 < q_2 < p$, $\iota^{q_1} = \iota^{q_2} \circ \iota_{q_1}^{q_2}$.

In [28, Propostion 2, Theorem] it is shown that the locally convex spaces $J_{p^+}$ and $J_{p^-}$ are quasi-reflexive of order one with no infinite–dimensional Banach spaces. Now, as an immediate consequence of Theorem 2.10 we obtain the following results.

**Theorem 2.11.** *Let $1 \leq p < \infty$. Then $J_{p^+}$ and $J_{p^-}$ $(p > 1)$ are non-reflexive locally convex spaces with no infinite–dimensional Banach quotients.*

*Proof.* For the space $J_{p^+}$: Suppose that the Banach space $X$ is a quotient of $J_{p^+}$ and denote by $Q\colon J_{p^+} \to X$ the quotient map. Since $J_{p^+} = \cap_{n=1}^{\infty} J_{q_n}$ with $q_n \downarrow p$, there exists $n_0 \in \mathbb{N}$ so that $X$ is a quotient of $J_{q_{n_0}}$, i.e., there exists a quotient map $Q_{n_0}\colon J_{q_{n_0}} \to X$ such that $Q = Q_{n_0} \circ \iota_{q_{n_0}}$. Then $X$ must also to be a quotient of $J_{q_{n_0+1}}$ (here, $q_{n_0+1} < q_{n_0}$), i.e., there exists a quotient map $Q_{n_0+1}\colon J_{q_{n_0+1}} \to X$ such that $Q = Q_{n_0+1} \circ \iota_{q_{n_0+1}}$. It follows that $Q_{n_0} \circ \iota_{q_{n_0+1}}^{q_{n_0}} \circ \iota_{q_{n_0+1}} = Q_{n_0} \circ \iota_{q_{n_0}} = Q = Q_{n_0+1} \circ \iota_{q_{n_0+1}}$ and hence, $Q_{n_0} \circ \iota_{q_{n_0+1}}^{q_{n_0}} = Q_{n_0+1}$ as $\iota_{q_{n_0+1}}$ has dense range. By Theorem 2.10 this equality forces $X$ to be finite–dimensional.

For the space $J_{p^-}$ $(p > 1)$: Suppose that the Banach space $X$ is a quotient of $J_{p^-}$ and denote by $Q\colon J_{p^-} \to X$ the quotient map. Since $J_{p^-}$ is a (DF)–space, there exists $B \in \mathcal{B}(J_{p^-})$ such that $Q(B) \supseteq B_X$, where $B_X$ denotes the closed unit ball of $X$. This together with the regularity of the (LB)–space $J_{p^-}$ (see, [28]) imply that $X$ must be a quotient of $J_{q_{n_0}}$ for some $n_0 \in \mathbb{N}$, i.e., there exists $Q_{n_0}\colon J_{q_{n_0}} \to X$ such that $Q_{n_0} = Q \circ \iota^{q_{n_0}}$. Then $X$ must also to be a quotient of $J_{q_{n_0+1}}$ (here, $q_{n_0+1} > q_{n_0}$), i.e., there exists a quotient map $Q_{n_0+1}\colon J_{q_{n_0+1}} \to X$ such that $Q_{n_0+1} = Q \circ \iota^{q_{n_0+1}}$. It follows that $Q_{n_0} = Q \circ \iota^{q_{n_0}} = Q \circ \iota^{q_{n_0+1}} \circ \iota_{q_{n_0}}^{q_{n_0+1}} = Q_{n_0+1} \circ \iota_{q_{n_0}}^{q_{n_0+1}}$. By Theorem 2.10 this equality forces $X$ to be finite–dimensional. □

## 3. Reflexive (Montel) Fréchet spaces with no infinite–dimensional Banach quotients

The aim of this section is to exhibit examples of non–totally reflexive, reflexive Fréchet spaces and of non–Schwartz, Montel Fréchet spaces with no infinite–dimensional Banach quotients. The construction is based on a method given in [1].

Let $(E_n, j_{n+1}^n)_{n=1}^{\infty}$ be a reduced projective sequence of Banach spaces such that the Fréchet space $E := \operatorname{proj}_n E_n \subseteq J_{p^+}$ and the inclusion map $u\colon E \to J_{p^+}$ is continuous with dense range, for some $p \geq 1$. For each $n \in \mathbb{N}$, denote by $j_n\colon E \to E_n$ the canonical projection of $E$ into $E_n$ so that $j_{n+1}^n \circ j_{n+1} = j_n$ and, by $\| \, \|_n$ a norm defining the topology of $E_n$. If $J_{p^+} = \cap_{n=1}^{\infty} J_{q_n}$ with $q_n \downarrow p$, then we may suppose that, for every $n \in \mathbb{N}$ there exists a continuous linear map $u_n\colon E_{n+1} \to J_{q_n}$ with dense range such that $\iota_{q_n} \circ u = u_n \circ j_{n+1}$ (eventually, by passing to a subsequence).

Further, let $(L_n)_{n=1}^{\infty}$ be a reduced projective sequence of normal Banach sequence spaces such that $L_{n+1} \subseteq L_n$ and the inclusion map $\rho_{n+1}^n\colon L_{n+1} \hookrightarrow L_n$ is continuous with dense range. Let $L := \cap_{n=1}^{\infty} L_n$ be the Fréchet space endowed with its natural projective topology. For each $n \in \mathbb{N}$, denote by $\rho_n\colon L \to L_n$ the canonical inclusion of $L$ into $L_n$ so that $\rho_{n+1}^n \circ \rho_{n+1} = \rho_n$ and, by $| \, |_n$ a norm defining the topology of $L_n$. Moreover, suppose that each normal Banach sequence space $L_n$ satisfies the property $(\varepsilon)$, i.e., $|a - ((a_k)_{k\leq h}, (0)_{k>h})|_n \to 0$ as $h \to \infty$ for every $a = (a_k)_{k=1}^{\infty} \in L_n$ (cf. Appendix).

Following [1], for every $n \in \mathbb{N}$, we define the Banach space

$$M_n := L_n((X_n)_{k<n}, (J_{q_n})_{k\geq n}),$$

i.e.,

$$M_n = \{((x_k)_{k<n}, (y_k)_{k\geq n}) : \ x_k \in E_n, \ y_k \in J_{q_n}, \ ((\|x_k\|_n)_{k<n}, (\|y_k\|_{J_{q_n}})_{k\geq n}) \in L_n\}$$

with the topology generated by the norm

$$r_n((x_k)_{k<n}, (y_k)_{k\geq n}) := |((\|x_k\|_n)_{k<n}, (\|y_k\|_{J_{q_n}})_{k\geq n})|_n.$$

Then the maps $j^n_{n+1}$, $\iota^{q_n}_{q_{n+1}}$, together with the maps $u_n$ induce continuous linear maps $U^n_{n+1}\colon M_{n+1} \to M_n$ defined by

$$U^n_{n+1}((x_k)_{k<n+1}, (y_k)_{k\geq n+1}) = ((j^n_{n+1}(x_k))_{k<n}, u_n(x_n), (\iota^{q_n}_{q_{n+1}}(y_k))_{k\geq n+1}). \tag{3.1}$$

So, we may form the reduced projective limit

$$M(E, J_{p^+}; L) := \operatorname{proj}_n M_n$$

which is a Fréchet space, [1, Proposition 1]. Note that the definition above is not as general as the one given in [1]. Constructions of Fréchet spaces of this type have been used since the paper of Moscatelli [24] several times to exhibit variuos counterexamples. Bonet and Dierolf thoroughly investigated this type of constructions in [4]. We refer the reader to the survey paper [5] for more information on the topic.

Recall that a Fréchet space $E$ is called *Schwartz* (*infra–Schwartz*, resp.) if for every $n \in \mathbb{N}$ there is $m > n$ such that the canonical map $j^n_m\colon E_m \to E_n$ ($j^n_m := j^{m-1}_m \circ \ldots \circ j^n_{n+1}$) is compact (weakly compact, resp.), [17]. Infra–Schwartz Fréchet spaces are projective limits of sequences of reflexive Banach spaces, [17, 7.5.3]. Valdivia proved in [29, Theorem 3] that a Fréchet space $E$ is infra–Schwartz if and only if it is totally reflexive, i.e., every quotient of $E$ is reflexive.

Combining [1, Theorem 2, Corollaries 2 and 3] with Theorem 2.10, we easily obtain the following results.

**Theorem 3.1.** *Let $1 \leq p < \infty$. If $L$ and $E$ are Schwartz Fréchet spaces, then the space $M(E, J_{p^+}; L)$ is a non–Schwartz, non–totally reflexive, Montel Fréchet space with no infinite–dimensional Banach quotient.*

*Proof.* Theorem 2 together with Corollary 2 in [1] ensure that the Fréchet space $M(E, J_{p^+}; L)$ is Montel but not Schwartz and not totally reflexive. Now suppose that the Banach space $X$ is a quotient of $M(E, J_{p^+}; L)$. Then, using the same arguments of the proof of Theorem 2.11, one obtains that there exist $n_0 \in \mathbb{N}$ and quotient maps $Q_{n_0}\colon M_{n_0} \to X$, $Q_{n_0+1}\colon M_{n_0+1} \to X$ such that $Q_{n_0} \circ U^{n_0}_{n_0+1} = Q_{n_0+1}$. Since $L$ and $E$ are both Schwartz Fréchet spaces, there is no loss of generality in assuming that all the maps $j^n_{n+1}\colon E_{n+1} \to E_n$, $u_n\colon E_{n+1} \to J_{q_n}$ and $\rho^n_{n+1}\colon L_{n+1} \hookrightarrow L_n$ are compact and that all the dual spaces $L'_n$ satisfy the property $(\varepsilon)$. On the hand hand, by Theorem 2.10 the maps $\iota^{q_n}_{q_{n+1}}\colon J_{q_{n+1}} \hookrightarrow J_{q_n}$ are strictly cosingular. This implies by (3.1) that the maps $U^n_{n+1}$ are also strictly cosingular in virtue of Proposition 5.4 in the Appendix and Remark 2.9. Hence, the equality $Q_{n_0} \circ U^{n_0}_{n_0+1} = Q_{n_0+1}$ forces $X$ to be finite–dimensional. □

**Theorem 3.2.** *Let $1 \leq p < \infty$. If $L$ is a Schwartz Fréchet space, $E$ is a totally reflexive Fréchet space such that the maps $j^n_{n+1}$ and $u_n$ are strictly cosingular, then the space $M(E, J_{p^+}; L)$ is a non–totally reflexive, reflexive Fréchet space with no infinite–dimensional Banach quotients.*

*Proof.* Theorem 2 together with Corollary 3 in [1] ensure that the Fréchet space $M(E, J_{p^+}; L)$ is reflexive but not totally reflexive. Suppose that the Banach space $X$ is a quotient of $M(E, J_{p^+}; L)$. Then, as in the proof of Theorem 3.1, we obtain that there exist $n_0 \in \mathbb{N}$ and quotient maps $Q_{n_0}\colon M_{n_0} \to X$, $Q_{n_0+1}\colon M_{n_0+1} \to X$

such that $Q_{n_0} \circ U^{n_0}_{n_0+1} = Q_{n_0+1}$. Since $L$ ($E$, resp.) is a Schwartz Fréchet space (infra–Schwartz Fréchet space, resp.), there is no loss of generality in assuming that all the maps $\rho^n_{n+1} \colon L_{n+1} \hookrightarrow L_n$ are compact and that all the dual spaces $L'_n$ satisfy the property ($\varepsilon$). On the hand hand, by Theorem 2.10 the maps $\iota^{q_n}_{q_{n+1}} \colon J_{q_{n+1}} \hookrightarrow J_{q_n}$ are strictly cosingular. This implies via (3.1) that the maps $U^n_{n+1}$ are also strictly cosingular by Proposition 5.4 in the Appendix and Remark 2.9. So, the equality $Q_{n_0} \circ U^{n_0}_{n_0+1} = Q_{n_0+1}$ implies that $X$ is finite–dimensional. □

We observe that non–Schwartz, Montel Fréchet spaces with no infinite–dimensional Banach quotients have been already exhibited in [1, §4, Theorem 3]. But, as observed in [1, §4, Remark 9], all such spaces are totally reflexive. So, Theorem 3.1 ensures the existence of non–totally reflexive, non–Schwartz, Montel Fréchet spaces with no infinite–dimensional Banach quotients.

Applying Theorems 3.1 and 3.1 we obtain that

**Example 3.3.** (1) Let $E$ be the nuclear Fréchet space $s = \cap_{n=1}^{\infty} \ell_1(j^n)$ of all rapidly decreasing sequences. Then, for every $1 \le p < \infty$, the space $M(s, J_{p^+}; s)$ is a non–Schwartz, non–totally reflexive, Montel Fréchet space with no finite–dimensional Banach quotients.

(2) Let $E$ be the totally reflexive Fréchet space $\ell_{p^+} = \cap_{q>p} \ell_q$ for some $p > 1$ (see [23] for more information on this space). Then, for every for $q \in ]p, \infty)$, the space $M(\ell_{p^+}, J_{q^+}; s)$ is a non–totally reflexive, reflexive Fréchet space with no finite–dimensional Banach quotients.

## 4. Some positive results

Theorem 2.11 ensures that the space $J_{p^+}$, $p \ge 1$, is an example of a *non–reflexive Fréchet space with no infinite–dimensional Banach quotients.* In this section we show that in some case the weaker condition *every Banach quotient is reflexive* implies that the underlying Fréchet space *is reflexive or totally reflexive.*

**Proposition 4.1.** *Let $A$ be a Köthe matrix and $E = \lambda_p(A)$ with $p \in \{0, 1, \infty\}$. If every Banach quotient space of $E$ is reflexive, then $E$ is quasinormable and totally reflexive.*

*Proof.* The result follows from [31, Ch. 2, §2, 3(14) and 3(15), p.226] for $p = 1$ and from [31, Ch. 2, §4, 3(11) and 3(12), p.266] for $p = 0$.

Set $E = \lambda_\infty(A)$ and suppose $E$ is not quasinormable. Then [9, Theorem 2] implies that $E$ has a Banach quotient space $X$ which is isomorphic to $c_0$. So, $E$ has a Banach quotient space which is not reflexive. Next, suppose that $E = \lambda_\infty(A)$ is not reflexive. Therefore, $E$ is not Montel and hence, $E$ admits a sectional subspace $X$ which is isomorphic to $\ell_\infty$, [31, Ch. 2, §4, 2(2) and 2(7), pp.262–264]. As $X$ is a sectional subspace of $E$, $X$ is a complemented subspace of $E$ and hence, a quotient space of $E$. Finally, since $E$ is reflexive and quasinormable, by a result of Grothendieck [13, Corollary 3, p.115] we have $E$ is totally reflexive. □

A lcHs $E$ is said to have the *Grothendieck property* if every sequence in $E'$ which is convergent in $(E', \sigma(E', E))$ is also convergent in $(E', \sigma(E', E''))$. Clearly, every reflexive lcHs sayisfies the Grothendieck property.

**Proposition 4.2.** *Let $E$ be a quasinormable Fréchet space with the Grothendieck property. Then $E$ is totally reflexive.*

*Proof.* Suppose $E$ is not totally reflexive. Then a result of Valdivia [29, Theorem 2] implies that $E$ has a Fréchet quotient space with a Schauder basis (hence, a separable quotient space) which is not reflexive. This contradicts [9, Proposition 4]. $\square$

**Proposition 4.3.** *Let $E$ be a separable and $\sigma(E,E')$–sequentially complete Fréchet space. If every Banach quotient space of $E$ is reflexive, then $E$ is also reflexive.*

*Proof.* Suppose that $E$ is not reflexive. Then $E$ contains a bounded sequence $(x_n)_{n=1}^\infty$ with no $\sigma(E,E')$–convergent subsequences. So, there exists a subsequence $(x_{k_n})_{n=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that $(x_{k_n})_{n=1}^\infty$ is either $\sigma(E,E')$–Cauchy or equivalent to the unit vector basis of $\ell_1$, [8, Lemma 3] (see also [30]). Since $E$ is $\sigma(E,E')$–sequentially complete, the sequence $(x_{k_n})_{n=1}^\infty$ is necessarily equivalent to the unit vector basis of $\ell_1$. Then $X = [(x_{k_n})_{n=1}^\infty]$ is a closed subspace of $E$ isomorphic to $\ell_1$. This implies that the Banach space $C([0,1])$ is isomorphic to a quotient space of $E$ and the proof follows now similarly as in the case of Banach spaces, [26, Theorem 3.4], [10, Ch.XI, pp.213-214]. Indeed, consider the Banach space $C(\Delta)$ where $\Delta$ denotes the Cantor set (recall that $C(\Delta)$ and $C([0,1])$ are isomorphic). Since $C(\Delta)$ is a separable Banach space, $C(\Delta)$ is a quotient space of $\ell_1$ via some continuous linear operator $q$. On the other hand, $C(\Delta)$ is a closed subspace of $\ell_\infty(\Delta)$. So, $q \in \mathcal{L}(\ell_1, \ell_\infty(\Delta))$ and hence, we may extend $q$ to a continuous linear operator $Q\colon E \to \ell_\infty(\Delta)$, [16, Corollary 7.4.5, p.133]. It follows that $\overline{Q(E)}$ is a separable closed subspace of $\ell_\infty(\Delta)$ and hence, it is isometric to a closed subspace of $C(\Delta)$. On the other hand, $q(\ell_1) = Q(\ell_1)$ is a closed subspace of $\overline{Q(E)}$ isomorphic to $C(\Delta)$ (actually, $= C(\Delta)$). So, by a result of Pelczynski, [27], [10, Ch.XI, p.214], we can conclude that $q(\ell_1) = Q(\ell_1)$ contains a closed subspace $Y$ which is isometric to $C(\Delta)$ and complemented in $\overline{Q(E)}$ by a norm one projection $P$. Set $R = P \circ Q$. Then $R \in \mathcal{L}(E, Y)$ and is a quotient map with $R(E) = Y \simeq C(\Delta)$. This completes the proof.

We have so shown that the non–reflexive Banach space $C([0,1])$ is a quotient space of $E$; a contradiction. $\square$

**Corollary 4.4.** *Let $E$ be a separable, $\sigma(E,E')$–sequentially complete and quasi-normable Fréchet space. If every Banach quotient space of $E$ is reflexive, then $E$ is totally reflexive.*

*Proof.* From Proposition 4.3 it follows that $E$ is reflexive and hence, $E$ satisfies the Grothendieck property. Then $E$ is necessarily totally reflexive by Proposition 4.2. $\square$

## 5. Appendix

In this section we prove a technical result which was needed in the proof of the main Theorems 3.1 and 3.2.

Let $(L, ||\ ||)$ be a normal Banach sequence space, i.e., a Banach sequence space satisfying the following properties:

($\alpha$) $\varphi \subseteq L \subseteq \omega$ and the inclusion $(L, ||\ ||) \hookrightarrow \omega$ is continuous,

($\beta$) $\forall a = (a_k)_{k=1}^\infty \in L$, $\forall b = (b_k)_{k=1}^\infty \in \omega$ such that $|b_k| \leq |a_k|$ for $\forall k \in \mathbb{N}$, we have $b \in L$ and $||b|| \leq ||a||$.

We say that a normal Banach sequence space $(L, ||\ ||)$ satisfies the property ($\varepsilon$) if

($\varepsilon$) $||a - ((a_k)_{k\leq n}, (0)_{k>n})|| \to 0$ as $n \to \infty$ for every $a = (a_k)_{k=1}^\infty \in L$.

If the sequence space $(L, ||\ ||)$ satisfies ($\varepsilon$), then the vectors $e_n := (\delta_{kn})_{k=1}^\infty \in L$ form a Schauder basis for $L$ and its topological dual can be identified with its $\alpha$–dual [19, §30]; hence, $(L', ||\ ||')$ is also a normal Banach sequence space. Typical examples of normal Banach sequence spaces are the Banach spaces $\ell_p$, $1 \leq p \leq \infty$, $c_0$ and their diagonal transforms. In particular, the Banach spaces $\ell_p$, for $1 \leq p < \infty$, and $c_0$ satisfy $(\varepsilon)$.

Further, let $(E_k, \| \, \|_k)_{k=1}^{\infty}$ be a sequence of Banach spaces. Then the Banach space $E := L((E_k)_{k \in \mathbb{N}})$ is defined as the linear space

$$E = \{(x_k)_{k=1}^{\infty} \in \prod_{k=1}^{\infty} E_k : \ (\|x_k\|_k)_{k=1}^{\infty} \in L\}$$

endowed with the norm $r(x) := \|(\|x_k\|_k)_{k=1}^{\infty}\|$ for $x = (x_k)_{k=1}^{\infty} \in E$.

For each $n \in \mathbb{N}$, let $J_n \colon \prod_{k=1}^{n} E_k \to E$ and $P_n \colon E \to \prod_{k=1}^{n} E_k$ be the continuous linear maps respectively defined by $J_n(x_k)_{k \geq n} := ((x_k)_{k \leq n}, (0)_{k > n})$ and $P_n(x_k)_{k=1}^{\infty} := (x_k)_{k \leq n}$. Clearly, we have $P_n \circ J_n = I$ for every $n \in \mathbb{N}$.

**Lemma 5.1.** *Let $(L, || \, ||)$ be a normal Banach sequence space satisfying the property $(\varepsilon)$ and $(E_k, \| \, \|_k)_{k=1}^{\infty}$ be a sequence of Banach spaces. If $X$ is an infinite–dimensional closed subspace of $E := L((E_k)_{k \in \mathbb{N}})$. Then either $X$ contains an infinite–dimensional closed subspace which is isomorphic to a subspace of some $E_k$ or $X$ contains an infinite–dimensional closed subspace which is isomorphic to a subspace of $L$.*

*Proof.* For each $n \in \mathbb{N}$, let introduce the map $Q_n \colon E \to E_n$ defined by $Q_n(x_k)_{k=1}^{\infty} := x_n$. If there exists $n_0 \in \mathbb{N}$ so that the restriction map $Q_{n_0}|_X$ is not strictly singular, then $X$ contains an infinite–dimensional closed subspace which is isomorphic to a subspace of $E_{n_0}$.

Suppose that each $Q_n|_X$ is strictly singular and in this case, for every $n \in \mathbb{N}$ the restriction of $P_n = \sum_{k=1}^{n} Q_k$ to $X$ is also strictly singular, [12, Theorem III.2.4, p.86]. Let $(d_n)_{n=1}^{\infty}$ be any decreasing sequence of positive numbers convergent to 0 with $d_1 < 1/2$ and, for each $n \in \mathbb{N}$, let $\tau_n := ||((1)_{k \leq n}, (0)_{k > n})||$. By $(\alpha)$ we may suppose that each $\tau_n \geq 1$. Next, let $y^0 \in X$ with $r(y^0) = 1$. Since $(L, || \, ||)$ satisfies the property $(\varepsilon)$, we can choose $k_1 \in \mathbb{N}$ so that $r(y^0 - J_{k_1} P_{k_1} y^0) < d_1$. Since $P_{k_1}|_X$ is strictly singular, there is $y^1 \in X$ with $r(y^1) = 1$ such that $\sum_{k=1}^{k_1} ||y_k^1||_k < d_1/\tau_{k_1}$. Applying again the property $(\varepsilon)$, we can choose $k_2 > k_1$ such that $r(y^1 - J_{k_2} P_{k_2} y^1) < d_2$. Iterating this procedure, we can find a sequence $(y^n)_{n \geq 0} \subseteq X$, $y^n = (y_k^n)_{k=1}^{\infty}$, and an increasing sequence $(k_n)_{n=1}^{\infty}$ of positive integers satisfying the following properties:

$$r(y^n) = 1, \ r(y^n - J_{k_{n+1}} P_{k_{n+1}} y^n) < d_{n+1}, \quad n \geq 0, \tag{5.1}$$

$$\sum_{k=1}^{k_n} ||y_k^n||_k < d_n/\tau_{k_n}, \quad n \in \mathbb{N}. \tag{5.2}$$

For $n = 0$ define $z_k^0 := y_k^0$ if $1 \leq k \leq k_1$ and $z_k^0 := 0$ otherwise. For $n \geq 1$, define $z^n := y_k^n$ if $k_n < k \leq k_{n+1}$ and $z_k^n := 0$ otherwise. Then, for every $n \geq 0$, we have $z^n := (z_k^n)_{k=1}^{\infty} \in E$, $r(z^n) \leq 1$ by (5.1) and $(\beta)$, $r(J_{k_n} P_{k_n} y^n) \leq \tau_{k_n} \sum_{k=1}^{k_n} ||y_k^n||_k < d_n$ by (5.2) and $(\beta)$. So, by (5.1) it follows that

$$r(z^0) = r(J_{k_1} P_{k_1} y^0 - y^0 + y^0) \geq r(y^0) - r(y^0 - J_{k_1} P_{k_1} y^0) > 1 - d_1 > 0,$$

$$\begin{aligned} r(z^n) &= r(J_{k_{n+1}} P_{k_{n+1}} y^n - J_{k_n} P_{k_n} y^n + y^n - y^n) \\ &\geq r(y^n) - r(J_{k_n} P_{k_n} y^n) - r(y^n - J_{k_{n+1}} P_{k_{n+1}} y^n) \\ &> 1 - d_n - d_{n+1} \geq 1 - 2d_1 > 0, \quad n \geq 1. \end{aligned}$$

Moreover, for every $m < n$ and for evry choice $(a_k)_{k=0}^m$ of scalars we have

$$r\left(\sum_{k=0}^{m} a_k z^k\right) \le r\left(\sum_{k=0}^{n} a_k z^k\right),$$

as it is easily to verify. Hence, $(z^n)_{n\ge 0}$ is a semi–normalized basic sequence of $E$. Define $Z := [(z^n)_{n\ge 0}]$. Then $Z$ is isomorphic to a closed subspace of $L$. To show this, we observe that the vectors $e_n := (\delta_{nk})_{k=1}^\infty$, for $n \in \mathbb{N}$, form a Schauder basis of $L$ via $(\varepsilon)$ and that the vectors $w_0 = \sum_{k=1}^{k_1} ||y_k^0||_k e_k$ and $w_n := \sum_{k=k_n+1}^{k_{n+1}} ||y_k^n||_k e_k$, for $n \ge 1$, form a semi–normalized block basic sequence of $(e_n)_{n=1}^\infty$. Let $W := [(w_n)_{n\ge 0}]$ and $S\colon Z \to W$ be the operator defined by setting

$$S(\sum_{k=0}^{\infty} a_k z^k) = \sum_{k=0}^{\infty} a_k w_k, \quad z = \sum_{k=0}^{\infty} a_k z^k \in Z.$$

Then $W$ is a closed subspace of $L$ and $S$ is an isomorphism onto by $(\beta)$.

Finally, put $Y := [(y^n)_{n\ge 0}] \subseteq X$ and observe that, by (5.1) we have

$$\sum_{k=0}^{\infty} r(y^n - z^n) = \sum_{k=0}^{\infty} r(y^n - z^n) \le \sum_{k=1}^{\infty} d_k.$$

So, if we choose the sequence $(d_n)_{n=1}^\infty$ satisfying $\sum_{k=1}^\infty d_k < c << 1$, then by [21, Proposition 1.a.9(i)] we obtain that $(y^n)_{n\ge 0}$ is a basic sequence equivalent to $(z_n)_{n\ge 0}$. Therefore, $Y \simeq Z \simeq W$ and the proof is complete. □

**Proposition 5.2.** *Let $(L, ||\ ||_L)$, $(M, ||\ ||_M)$ be normal Banach sequence spaces satisfying the property $(\varepsilon)$, $(E_k, ||\ ||_k)_{k=1}^\infty$, $(F_k, |\ |_k)_{k=1}^\infty$ be sequences of Banach spaces and, for each $k \in \mathbb{N}$, let $T_k \in \mathcal{L}(E_k, F_k)$ with $K := \sup_{k\in\mathbb{N}} ||T_k|| < \infty$. If $L$ is compactly embedded in $M$ and each operator $T_k$ is strictly singular, then the continuous linear operator $T\colon L((E_k)_{k\in\mathbb{N}}) \to M((F_k)_{k\in\mathbb{N}})$ defined by $T((x_k)_{k=1}^\infty) = (T_k x_k)_{k=1}^\infty$ is also strictly singular.*

*Proof.* Define $E := L((E_k)_{k\in\mathbb{N}})$ and $F := M((F_k)_{k\in\mathbb{N}})$ and denote by $r$ and $t$ the norm of $E$ and $F$ respectively. Suppose that $X$ is an infinite–dimentional closed subspace of $E$ such that $T|_X$ is an isomorpshim into. Then there exist $c$, $d > 0$ such that

$$cr(x) \le t(Tx) \le dr(x), \quad x \in X, \tag{5.3}$$

and hence, $T(X)$ is an infinite–dimensional closed subspace of $F$. By Lemma 5.1 the space $T(X)$ must contain an infinite–dimensional closed subspace which is isomorphic either to a closed subspace of $F_{k_0}$, for some $k_0 \in \mathbb{N}$, or to a closed subspace of $M$.

Suppose that $T(X)$ contains an infinite–dimensional closed subspace $Z$ which is isomorphic to a closed subspace of $F_{k_0}$. Hence, there exist $c'$, $d' > 0$ such that

$$c'|z_{k_0}|_{k_0} \le t(z) \le d'|z_{k_0}|_{k_0}, \quad z = (z_k)_{k=1}^\infty \in z. \tag{5.4}$$

Set $Y := (T|_X)^{-1}(Z) \subseteq X$. Then $Y$ is also an infinite–dimensional closed subspace of $X$ such that

$$||y_{k_0}||_{k_0} \le \tau^{k_0} r(y) \le \tau^{k_0} c^{-1} t(Ty) \le \tau^{k_0} c^{-1} d' |T_{k_0} y_{k_0}|_{k_0} \le \tau^{k_0} c^{-1} d' ||T_{k_0}||\, ||y_{k_0}||_{k_0}$$

for all $y = (y_k)_{k=1}^\infty \in Y$, as it follows from (5.3) and (5.4) (here, $\tau^{k_0} := ||e_{k_0}||_L$). The inequalities above ensure that $Q_{k_0}(Y)$ is an infinite–dimensional subspace of $E_{k_0}$ such that $T_{k_0}|_{Q_{k_0}(Y)}$ is an isomorphism into; a contradiction.

Next, suppose that $T(X)$ contains an infinite–dimensional closed subspace which is isomorphic to a closed subspace of $M$. So, as it follows from the proof of Lemma 5.1, this is the case when $X$ contains a sequence $(y^n)_{n\geq 0}$ such that $t(Ty^n) = 1$ for $n \geq 0$, $(Ty^n)_{n\geq 0}$ is a semi–normalized basic sequence and $t(J_{k_{n+1}}P_{k_{n+1}}Ty^n - J_{k_n}P_{k_n}Ty^n) > 1 - 2d_1 > 0$ for $n \geq 1$ and for a suitable increasing sequence $(k_n)_{n=1}^{\infty}$ of positive integers. But, the inclusion map $L \hookrightarrow M$ is compact and hence, for any $\epsilon > 0$ there exists $\overline{k} \in \mathbb{N}$ such that $||((0)_{k\geq\overline{k}}, (a_k)_{k>\overline{k}})||_M \leq \epsilon||((0)_{k\geq\overline{k}}, (a_k)_{k>\overline{k}})||_L$ for all $a = ((0)_{k\geq\overline{k}}, (a_k)_{k>\overline{k}}) \in L$. Therefore, by $(\beta)$ and (5.3) it follows that

$$\begin{aligned} 1 - 2d_1 &< t(J_{k_{n+1}}P_{k_{n+1}}Ty^n - J_{k_n}P_{k_n}Ty^n) \\ &\leq K||((0)_{k\leq k_n}, (||y_k^n||_k)_{k_n<k\leq k_{n+1}}, (0)_{k>k_{n+1}})||_M \\ &\leq \epsilon K||((0)_{k\leq k_n}, (||y_k^n||_k)_{k_n<k\leq k_{n+1}}, (0)_{k>k_{n+1}})||_L \\ &\leq \epsilon K r(y^n) \leq \epsilon K c^{-1} \end{aligned}$$

for all $n \geq \overline{n}$ with $\overline{n} := \min\{n \in \mathbb{N} : k_n \geq \overline{k}\}$. So, we obtain again a contradiction as we can take $\epsilon$ small enough that $\epsilon K c^{-1} < 1 - 2d_1$. This completes the proof. □

**Remark 5.3.** Let $(L, || \ ||_L)$, $(M, || \ ||_M)$, $(E_k, \| \ \|_k)_{k=1}^{\infty}$, $(F_k, | \ |_k)_{k=1}^{\infty}$ and $(T_k)_{k=1}^{\infty}$ as in Proposition 5.2. If all the maps $T_k$ are weakly compact, then $T$ is also weakly compact.

**Proposition 5.4.** *Let $(L, || \ ||_L)$ and $(M, || \ ||_M)$ be normal Banach sequence spaces such that L and M together with their duals satisfy the property $(\varepsilon)$, $(E_k, \| \ \|_k)_{k=1}^{\infty}$, $(F_k, | \ |_k)_{k=1}^{\infty}$ be sequences of Banach spaces and, for each $k \in \mathbb{N}$, let $T_k \in \mathcal{L}(E_k, F_k)$ with $K := \sup_{k\in\mathbb{N}} ||T_k|| < \infty$. If $L$ is compactly embedded in $M$ with dense range and each dual operator $T_k'$ is strictly singular, then the continuous linear operator $T\colon L((E_k)_{k\in\mathbb{N}}) \to M((F_k)_{k\in\mathbb{N}})$ defined by $T((x_k)_{k=1}^{\infty}) = (T_k x_k)_{k=1}^{\infty}$ is strictly cosingular.*

*Proof.* The proof follows by duality from Proposition 5.2 after having observed that the spaces $(L', || \ ||_L')$, $(M', || \ ||_M')$ are normal Banach sequence spaces satisfying the property $(\varepsilon)$, that $(L((E_k)_{k\in\mathbb{N}}))' = L'((E_k')_{k\in\mathbb{N}})$ and $(M((F_k)_{k\in\mathbb{N}}))' = M'((F_k')_{k\in\mathbb{N}})$ [4, Lemma 2.2], that $M'$ is compactly embedded in $L'$ and that $T'(y') = (T_k' y_k')_{k=1}^{\infty}$ for $y' \in M'((F_k')_{k\in\mathbb{N}})$. □

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