Operator-weighted composition operators between weighted spaces of vector-valued analytic functions

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Abstract

We investigate several properties of operator-weighted composition maps $W_{\psi,\varphi} : f \mapsto \psi(f \circ \varphi)$ on unweighted $H(\mathbb{D}, X)$ and weighted $H_v^\infty(\mathbb{D}, X)$ spaces of vector valued holomorphic functions on the unit disc $\mathbb{D}$. Here φ is an analytic self-map of $\mathbb{D}$ and ψ is an analytic operator-valued function on $\mathbb{D}$. We characterize when the operator is continuous, maps a neighbourhood into a bounded set or maps bounded sets into relatively compact sets. In this way we extend results due to Laitila and Tylli for the case of Banach valued functions. This more general setting permits us to compare the results in the unweighted and weighted case. New examples are provided, especially when the spaces X and Y are Köthe echelon spaces. They show the differences between the present setting and the case of functions taking values in Banach spaces.

1 Introduction and preliminaries

The purpose of this paper is to investigate qualitative properties of operator-weighted composition operators on spaces of vector valued analytic functions. To describe the content of the paper we need some notation. If X is a Hausdorff locally convex space, $H(\mathbb{D}, X)$ denotes the space of all analytic functions $f : \mathbb{D} \to X$. If $X = \mathbb{C}$ we simply write $H(\mathbb{D})$. This space is endowed with the topology of uniform convergence on the compact subsets of $\mathbb{D}$; which makes $H(\mathbb{D})$ a (nuclear) Fréchet space. Our notation for locally convex spaces is standard and we refer the reader to [14, 15, 21]. For a locally convex space X, $cs(X)$ is the set of all continuous seminorms on X and $\mathcal{B}(X)$ the set of all bounded absolutely convex subsets of X. Given locally convex spaces X and Y, we write $L(X,Y)$ (or $L_b(X,Y)$ if we want to be more specific) the space of all continuous linear maps from X into Y endowed with the topology of uniform convergence on the bounded subsets of X. If X and Y are Banach spaces, the topology on $L_b(X,Y)$ is the operator norm. A linear operator $T : X \to Y$ is called *bounded* if it maps a neighbourhood of X into a bounded set in Y. Clearly every bounded operator T is continuous. An operator $T : X \to Y$ is called *Montel* if it maps bounded sets in X into relatively compact subsets of Y, and it is called *compact* (cf. [15]) if it maps a neighbourhood of X into a relatively compact subset of Y. If X and Y are Banach spaces, an operator $T : X \to Y$ is Montel if and only if it is compact.

We describe the object of our study. Let X and Y be complete, barrelled locally convex spaces. For an analytic self-map φ of $\mathbb{D}$ and an analytic map $\psi : \mathbb{D} \to L_b(X, Y)$ we consider the operator

$$W_{\psi,\varphi} : H(\mathbb{D}, X) \to H(\mathbb{D}, Y), \ f \mapsto \psi(f \circ \varphi).$$

This notation will be kept through the whole article. In Section 2 we show that it is well defined and continuous and we characterize when it is bounded, Montel or compact. Since $H(\mathbb{D})$ is not a Banach space, it is natural to investigate the unweighted operator $W_{\psi,\varphi} : H(\mathbb{D}, X) \to H(\mathbb{D}, Y)$ for analytic functions with values in spaces X and Y more general than Banach spaces. Our main result in Section 2 is Theorem 9. It requires a lemma on compact sets, Lemma 8, that uses tensor product techniques and a result due to Ruess [23]. Concrete examples in case ψ is constant and in case X and Y are Köthe echelon spaces are given in Sections 2 and 4.

In Section 3 we turn our attention to the setting of weighted spaces of vector valued holomorphic functions, extending the work of Laitila and Tylli [16]. A *weight* $v : \mathbb{D} \to]0, \infty[$ is a strictly positive, continuous, radial ($v(z) = v(|z|)$ for every $z \in \mathbb{D}$) function, that is non-increasing with respect to $|z|$. Here are the most important examples of weights:

- The *standard weights* are $v(z) = (1 - |z|)^\alpha, \ \alpha > 0$.

- $v(r) = \exp(-\frac{1}{(1-r)^\alpha}), \ \alpha > 0$.

- $v(r) = (1 - \log(1 - r))^{-\alpha}, \ \alpha > 0$.

For a weight v and a locally convex space X, we define

$$H_v^\infty(\mathbb{D}, X) := \{f \in H(\mathbb{D}, X); \ \sup_{z \in \mathbb{D}} v(z)q(f(z)) < \infty \ \text{for all} \ q \in cs(X)\}.$$

It is a Banach space in case X is Banach, and it is denoted by $H_v^\infty(\mathbb{D})$ if X is the set of complex numbers. The isomorphic classification of the spaces $H_v^\infty(\mathbb{D})$ was recently obtained by Lusky [19]. Obviously, in case that $v \equiv 1$ the space $H_v^\infty(\mathbb{D}, X)$ coincides with the space $H^\infty(\mathbb{D}, X)$ of bounded analytic functions $f : \mathbb{D} \to X$. When X and Y are Banach spaces, Laitila and Tylli characterized properties such as boundedness, compactness and weak compactness of $W_{\psi,\varphi} : H_v^\infty(\mathbb{D}, X) \to H_w^\infty(\mathbb{D}, Y)$ in terms of the weights v and w as well as the inducing maps. Related results for similar operators between weighted spaces of analytic functions with values in a Banach space defined by systems of weights were obtained by Manhas [20]. In Section 3 we present characterizations of continuous, bounded or Montel operators $W_{\psi,\varphi}$ in case X and Y are complete, barrelled spaces. Our main result is Theorem 15. Examples in Section 4 and Theorems 17 and 23 emphasize the different behaviour between the weighted and the unweighted cases and between the Banach valued and Fréchet valued functions. Our results are also a far reaching extension of some theorems in [6] and [8]. New techniques, and a systematic use of the Cesàro means of the Taylor polynomials, are needed in the present more general setting.

The operators investigated in this paper contain several classes of concrete linear operators. For example, in case that $X = Y = \mathbb{C}$ and ψ is an analytic map $\mathbb{D} \to \mathbb{C}$ we obtain the weighted composition operator $f \mapsto \psi \cdot (f \circ \varphi)$ which combines the classical composition operator $C_\varphi : f \mapsto f \circ \varphi$ with the pointwise multiplication operator $M_\psi : f \mapsto \psi \cdot f$. For more information about composition operators, we refer the reader to the excellent monographs of Cowen and MacCluer [12] and Shapiro [24]. Many authors have studied composition

operators, weighted composition operators and multiplication operators on various spaces of analytic functions. We refer, for instance, to the following articles and to the references therein [4], [5], [6], [7], [8], [9], [10], [11], [17], [18] and [25].

2 Operator-weighted composition operators acting between $H(\mathbb{D}, X)$ and $H(\mathbb{D}, Y)$

The following auxiliary result is necessary in the case of holomorphic functions with values in barrelled locally convex spaces. It is trivial for Banach valued functions, because the duality bilinear function $(x, x') \to \langle x, x' \rangle$ is continuous on $X \times X'_b$ in this case.

Lemma 1 *Let X be a complete barrelled locally convex space. Let $h_1 : \mathbb{D} \to X$ and $h_2 : \mathbb{D} \to X'_b$ be holomorphic mappings. Then*

$$
\begin{aligned}
h : \mathbb{D} &\longrightarrow \mathbb{C}, \\
z &\longmapsto h(z) = \langle h_1(z), h_2(z) \rangle
\end{aligned}
$$

belongs to $H(\mathbb{D})$.

Proof. First, for $z \neq z_0 \in \mathbb{D}$,

$$
\begin{aligned}
\frac{h(z) - h(z_0)}{z - z_0} &= \frac{\langle h_1(z), h_2(z) \rangle - \langle h_1(z_0), h_2(z_0) \rangle}{z - z_0} \\
&= \langle \frac{h_1(z) - h_1(z_0)}{z - z_0}, h_2(z) \rangle + \langle h_1(z_0), \frac{h_2(z) - h_2(z_0)}{z - z_0} \rangle.
\end{aligned}
$$

Since $\lim_{z \to z_0} \frac{h_2(z) - h_2(z_0)}{z - z_0} = h'_2(z_0)$ in X'_b, we get

$$
\lim_{z \to z_0} \langle h_1(z_0), \frac{h_2(z) - h_2(z_0)}{z - z_0} \rangle = \langle h_1(z_0), h'_2(z_0) \rangle.
$$

Now, to complete the proof, we have to show that the first summand tends to $\langle h'_1(z_0), h_2(z_0) \rangle$. To see this, fix a sequence $(z_k)_{k \in \mathbb{N}} \subset \mathbb{D}$ such that $z_k \to z_0$, $z_k \neq z_0$. Put $x_k := \frac{h_1(z_k) - h_1(z_0)}{z_k - z_0}$ for every $k \in \mathbb{N}$ and $x_0 = h'_1(z_0)$. Since $h_1 : \mathbb{D} \to X$ is holomorphic, we know that $x_k \to x_0$ in X. We may assume that there is $0 < r_0 < 1$ with $|z_k - z_0| \leq r_0$. Then $\{h_2(z_k)\} \cup \{h_2(z_0)\} \subset h_2(\overline{B(z_0, r_0)}) =: K \subset X'_b$ is compact and $x'_k := h_2(z_k) \to h_2(z_0) =: x'_0$ in X'_b. Given $\varepsilon > 0$, we apply the equicontinuity of K to find $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$,

$$
|\langle x_0, x'_k - x'_0 \rangle| < \frac{\varepsilon}{2} \quad \text{and} \quad \sup_{u \in K} |u(x_k - x_0)| < \frac{\varepsilon}{2}.
$$

Hence

$$
\begin{aligned}
|\langle x_k, x'_k \rangle - \langle x_0, x'_0 \rangle| &= |\langle x_k - x_0 + x_0, x'_k \rangle - \langle x_0, x'_0 \rangle| \\
&\leq |\langle x_k - x_0, x'_k \rangle| + |\langle x_0, x'_k - x'_0 \rangle| \\
&\leq \sup_{u \in K} |u(x_k - x_0)| + |\langle x_0, x'_k - x'_0 \rangle| < \varepsilon
\end{aligned}
$$

for every $k \geq k_0$. The proof is complete. $\qquad\square$

Proposition 2 *Let X and Y be complete barrelled locally convex spaces. Let $\varphi : \mathbb{D} \to \mathbb{D}$ and $\psi : \mathbb{D} \to L_b(X, Y)$ be analytic mappings. Then the operator*

$$W_{\psi,\varphi} : \begin{array}{ccc} H(\mathbb{D}, X) & \longrightarrow & H(\mathbb{D}, Y) \\ f & \longmapsto & (W_{\psi,\varphi}f)(z) = \psi(z)[f(\varphi(z))] \end{array}$$

is well-defined.

Proof. We have to show that $W_{\psi,\varphi}f \in H(\mathbb{D}, Y)$ for every $f \in H(\mathbb{D}, X)$. Since Y is complete, by a classical result of Grothendieck [14, Section 16.7], it is enough to prove that $y' \circ W_{\psi,\varphi}f \in H(\mathbb{D})$ for every $y' \in Y'$. Fix $y' \in Y'$ and consider $T_{y'} : L_b(X, Y) \to X_b'$, $A \to y' \circ A$. Then $T_{y'}$ is linear and continuous. Since $\psi \in H(\mathbb{D}, L_b(X, Y))$ we have $T_{y'} \circ \psi \in H(\mathbb{D}, X_b')$. Moreover $f \circ \varphi \in H(\mathbb{D}, X)$ if $f \in H(\mathbb{D}, X)$. We apply Lemma 1 to obtain that the map from $\mathbb{D}$ to $\mathbb{C}$ given by

$$z \longmapsto y'(\psi(z)[(f \circ \varphi)(z)]) = \langle (y' \circ \psi)(z), (f \circ \varphi)(z) \rangle$$

is holomorphic. $\qquad\square$

The following auxiliary linear operator plays an important role in our characterizations. It was introduced in the case of Banach valued functions by Laitila and Tylli in Section 3 of [16].

Definition 3 *Let $\psi : \mathbb{D} \to L_b(X, Y)$ be analytic. We consider the operator*

$$T_\psi : \begin{array}{ccc} X & \longrightarrow & H(\mathbb{D}, Y), \\ x & \longmapsto & T_\psi(x) : \mathbb{D} \to Y, \quad T_\psi(x)(z) = \psi(z)[x] \end{array}$$

For $x \in X$ the operator $R_x : L_b(X, Y) \longrightarrow Y$, $A \mapsto Ax$ is linear and continuous and satisfies $T_\psi(x) = R_x \circ \psi$. Thus, T_ψ is holomorphic as a composition of a linear continuous operator with a holomorphic map. Moreover, T_ψ is linear by the definition of R_x.

Remark 4 (i) The auxiliary operator T_ψ inherits the ideal properties of $W_{\psi,\varphi}$. Indeed, the operator $A : X \to H(\mathbb{D}, X)$, $(Ax)(z) = f_x(z) = x$ is linear and continuous and $T_\psi = W_{\psi,\varphi} \circ A$, from where the conclusion follows.

(ii) All the operators $\psi(z) : X \to Y$, $z \in \mathbb{D}$ inherit the properties of T_ψ. To see this, consider the evaluation operator $\delta_z : H(\mathbb{D}, Y) \to Y$, $f \to f(z)$. Since δ_z is continuous and $\delta_z(T_\psi(x)) = T_\psi(x)(z) = \psi(z)[x]$ for every $x \in X$ and every $z \in \mathbb{D}$, the conclusion follows.

Proposition 5 *Let X and Y be complete barrelled locally convex spaces. Let $\varphi : \mathbb{D} \to \mathbb{D}$ and $\psi : \mathbb{D} \to L_b(X, Y)$ be analytic mappings. Then the operator $W_{\psi,\varphi} : H(\mathbb{D}, X) \to H(\mathbb{D}, Y)$ is continuous.*

Proof. Fix a compact subset K of $\mathbb{D}$ as well as a seminorm $q \in cs(Y)$. Since $\psi : \mathbb{D} \to L_b(X, Y)$ is holomorphic and X is barrelled, the set $\psi(K)$ is relatively compact in $L_b(X, Y)$, hence equicontinuous. Find a seminorm $p \in cs(X)$ such that

$$q(\psi(z)[x]) \leq p(x) \text{ for every } z \in K \text{ and every } x \in X.$$

Finally,
$$\sup_{z \in K} q(\psi(z)[f(\varphi(z))]) \le \sup_{z \in K} p(f(\varphi(z))) \le \sup_{\xi \in \varphi(K)} p(f(\xi)).$$

This shows that the operator is continuous. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Theorem 6 *Let X and Y be complete barrelled locally convex spaces. Let $\varphi : \mathbb{D} \to \mathbb{D}$ and $\psi : \mathbb{D} \to L_b(X, Y)$, $\psi \ne 0$, be analytic mappings. Then the operator $W_{\psi,\varphi} : H(\mathbb{D}, X) \to H(\mathbb{D}, Y)$ is bounded if and only if the following conditions hold:*

(a) $T_\psi : X \to H(\mathbb{D}, Y)$ is bounded,

(b) There is $0 < r_0 < 1$ such that $|\varphi(z)| \le r_0$ for every $z \in \mathbb{D}$.

Proof. First we assume that conditions (a) and (b) hold. From (a) it follows that there is a seminorm $p \in cs(X)$ such that for every compact set $K \subset \mathbb{D}$ and for every $q \in cs(Y)$ we can find $\lambda > 0$ such that

$$\sup_{z \in K} q(T_\psi(x)(z)) = \sup_{z \in K} q(\psi(z)[x]) \le \lambda p(x) \text{ for every } x \in X.$$

Select $K_0 := \{z \in \mathbb{C}; \ |z| \le r_0\}$ with r_0 as in condition (b) and define

$$P(f) := \sup_{\xi \in K_0} p(f(\xi)) \in cs(H(\mathbb{D}, X)).$$

For every compact set $K \subset \mathbb{D}$ and $q \in cs(Y)$ we put

$$Q_{K,q}(W_{\psi,\varphi} f) := \sup_{z \in K} q(\psi(z)[f(\varphi(z))])$$

and obtain, from (b),

$$Q_{K,q}(W_{\psi,\varphi} f) \le \lambda \sup_{z \in K} p(f(\varphi(z))) \le \lambda \sup_{\xi \in K_0} p(f(\xi)).$$

This shows that $W_{\psi,\varphi} : H(\mathbb{D}, X) \to H(\mathbb{D}, Y)$ is bounded.

It remains to show the converse. Obviously T_ψ is bounded by Remark 4. Since $W_{\psi,\varphi} : H(\mathbb{D}, X) \to H(\mathbb{D}, Y)$ is bounded, we find $0 < r_0 < 1$ and $p \in cs(X)$ such that for every $0 < r < 1$ and every $q \in cs(Y)$ there is $\lambda > 0$ such that for every $f \in H(\mathbb{D}, X)$ we have

$$\sup_{|z| \le r} q(\psi(z)[f(\varphi(z))]) \le \lambda \sup_{|\xi| \le r_0} p(f(\xi)).$$

Applying this inequality for the maps $f_k : \mathbb{D} \to X$ given by $f_k(z) = z^k x$, $k \in \mathbb{N}_0$, which clearly belong to the space $H(\mathbb{D}, X)$ for each $x \in X$, we obtain

$$\sup_{|z| \le r} |\varphi(z)|^k q(\psi(z)[x]) = \sup_{|z| \le r} q(\psi(z)[\varphi(z)^k x]) \le \lambda \sup_{|z| \le r_0} p(z^k x) = \lambda r_0^k p(x) \qquad (2.1)$$

for every $q \in cs(Y)$, $k \in \mathbb{N}_0$ and $x \in X$. In particular, for $k = 0$ and $0 < r < 1$ arbitrary, we get

$$\sup_{|z| \le r} q(\psi(z)[x]) \le \lambda p(x)$$

for every $q \in cs(Y)$, which implies $\psi(z)[x] = 0$ for every $z \in \mathbb{D}$ if $p(x) = 0$. Now, since $\psi \neq 0$ by assumption, there are $z_0 \in \mathbb{D}, x_0 \in X$ such that $\psi(z_0)[x_0] \neq 0$. In particular, $p(x_0) > 0$. Find $q_0 \in cs(Y)$ with $q_0(\psi(z_0)[x_0]) > 0$.

Next, we apply (2.1) to get, for each $0 < r < 1$,

$$0 < q_0(\psi(z_0)[x_0]) \sup_{|z| \leq r} |\varphi(z)|^k \leq \lambda r_0^k p(x_0),$$

for every $k \in \mathbb{N}_0$. Hence

$$\sup_{|z| \leq r} |\varphi(z)|^k \leq \frac{\lambda r_0^k p(x_0)}{q_0(\psi(z_0)[x_0])},$$

for every $k \in \mathbb{N}_0$. Taking the k-th roots and letting r go to 1 we obtain $|\varphi(z)| \leq r_0$ for every $z \in \mathbb{D}$, and (b) follows. $\qquad \square$

In our next results, we use the following notation (cf. [15]): $H(\mathbb{D})'_{co}$ is the dual of $H(\mathbb{D})$ endowed with the topology of uniform convergence on the convex compact sets, and $L_e(H(\mathbb{D})'_{co}, Y)$ is the space of all linear and continuous operators endowed with the topology of uniform convergence on the equicontinuous subsets of $H(\mathbb{D})'$. We have $L_e(H(\mathbb{D})'_{co}, Y) = H(\mathbb{D})\varepsilon Y$ if Y is complete.

Lemma 7 *Let Y be a complete locally convex space. A bounded subset $H \subset L_e(H(\mathbb{D})'_{co}, Y)$ is precompact if and only if $H(U^\circ)$ is precompact in Y for every $U \in \mathcal{U}_0(H(\mathbb{D}))$.*

Proof. By [23, Theorem 1.5], a subset $H \subset L_e(H(\mathbb{D})'_{co}, Y)$ is precompact if and only if (a) $H(U^\circ)$ is precompact in Y for every $U \in \mathcal{U}_0(H(\mathbb{D}))$ and (b) $H'(y') := \{y' \circ h; \ h \in H\}$ is precompact in $H(\mathbb{D})$ for every $y' \in Y'$. Since $H(\mathbb{D}, Y) \cong L_e(H(\mathbb{D})'_{co}, Y)$ by [14, Section 16.7], every bounded subset H of $L_e(H(\mathbb{D})'_{co}, Y)$ satisfies (b) by Montel's theorem. $\qquad \square$

Lemma 8 *Let Y be a complete locally convex space and let C be a bounded subset of $H(\mathbb{D}, Y)$. The set C is precompact if and only if for every compact subset K of $\mathbb{D}$ the set $C(K) := \{f(z); \ z \in K, f \in C\}$ is precompact in Y.*

Proof. To present the proof we must briefly recall the isomorphism $H(\mathbb{D}, Y) \cong L_e(H(\mathbb{D})'_{co}, Y)$ ([14, Section 16.7]). If $T \in L_e(H(\mathbb{D})'_{co}, Y)$, the corresponding element in $H(\mathbb{D}, Y)$ is defined by $f_T(z) := T(\delta_z)$, where $\delta_z \in H(\mathbb{D})'$ is the evaluation at the point $z \in \mathbb{D}$. Conversely if $g \in H(\mathbb{D}, Y)$, the operator $T_g \in L_e(H(\mathbb{D})'_{co}, Y)$ is the unique continuous extension of the linear map defined as $T_g(\delta_z) := g(z)$. We use this identification in the rest of the proof.

First assume that C is a precompact subset of $H(\mathbb{D}, Y)$. Fix a compact subset K of $\mathbb{D}$. We prove that $C(K)$ is precompact in Y. Set

$$U := \{f \in H(\mathbb{D}); \ \sup_{z \in K} |f(z)| \leq 1\} = \{\delta_z; \ z \in K\}^\circ,$$

the polar of $\{\delta_z; \ z \in K\} \subset H(\mathbb{D})'$ taken in $H(\mathbb{D})$. Hence $U^\circ = \{\delta_z; \ z \in K\}^{\circ\circ} = \overline{\Gamma\{\delta_z; \ z \in K\}}$. Since C is precompact in $H(\mathbb{D}, Y)$, we apply Lemma 7 to conclude that $C(U^\circ)$ is precompact in Y. But

$$C(\{\delta_z; \ z \in K\}) = \{f(\delta_z) = f(z); \ f \in C, \ z \in K\} \subset C(U^\circ).$$

and $C(K)$ is precompact, too.

To prove the converse we apply Lemma 7. Fix $U \in \mathcal{U}_0(H(\mathbb{D}))$ and select $\varepsilon > 0$ and a

compact subset K of $\mathbb{D}$ such that $\varepsilon U^\circ \subset \overline{\Gamma\{\delta_z;\ z \in K\}}$, the closure taken in the topology $\sigma(H(\mathbb{D})', H(\mathbb{D}))$. We have to show that $C(U^\circ)$ is precompact in Y. By assumption, the set $D := \overline{\Gamma\{f(z);\ z \in K,\ f \in C\}}$ is precompact in Y. Next, $\varepsilon C(U^\circ) = C(\varepsilon U^\circ) = \cup_{h \in C} h(\varepsilon U^\circ)$. Moreover $h(\varepsilon U^\circ) \subset h(\overline{\Gamma\{\delta_z;\ z \in K\}})$ and $h : H(\mathbb{D})'_{co} \to Y$ continuous. Thus

$$h(\overline{\Gamma\{\delta_z;\ z \in K\}}) \subset \overline{h(\Gamma(\{\delta_z;\ z \in K\}))} \subset \overline{\Gamma h(\{\delta_z;\ z \in K\})} \subset D$$

since $h \in C$. Therefore,

$$\bigcup_{h \in C} h(\varepsilon U^\circ) \subset D$$

and $C(U^\circ)$ is precompact in Y. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 9 *Let X and Y be complete barrelled locally convex spaces. Let $\varphi : \mathbb{D} \to \mathbb{D}$ and $\psi : \mathbb{D} \to L_b(X, Y)$ be analytic maps. The following assertions are equivalent:*

(a) $W_{\psi,\varphi} : H(\mathbb{D}, X) \to H(\mathbb{D}, Y)$ is Montel,

(b) $T_\psi : X \to H(\mathbb{D}, Y)$ is Montel,

(c) $\psi(z) : X \longrightarrow Y$ is Montel for all $z \in \mathbb{D}$.

Proof. Remark 4 implies that (c) follows from (a).

To prove that (c) implies (b), assume that $\psi(z) : X \to Y$ is a Montel operator for every $z \in \mathbb{D}$. In order to prove that T_ψ is a Montel operator, we approximate it by a sequence of Montel operators in $L_b(X, H(\mathbb{D}, Y))$. The function ψ is holomorphic with values in $L_b(X, Y)$, and then

$$\psi(z) = \sum_{m=0}^{\infty} A_m z^m,$$

where $A_m \in L_b(X, Y)$ and the series converges in $L_b(X, Y)$ uniformly on the compact sets of $\mathbb{D}$. First, we check that the coefficients A_m are Montel operators from X to Y. Indeed, $A_0 = \psi(0)$ which is a Montel operator. The rest of the coefficients are Montel operators by the Cauchy integral formula for derivatives.

Now, we consider the Taylor polynomials of ψ, $\psi_n(z) = \sum_{m=0}^{n} A_m z^m$, $n = 0, 1, 2, \ldots$, and the corresponding auxiliary operators T_{ψ_n}. We will see that T_{ψ_n} is a Montel operator for each n, and that T_{ψ_n} tends to T_ψ in $L_b(X, H(\mathbb{D}, Y))$ when n tends to infinity. Let B be a bounded set in X, then $A_m(B)$ is relatively compact in Y for each $m = 0, 1, \ldots, n$. The closure of the absolutely convex hull of the union of these sets is compact in Y, that is, the set

$$K := \overline{\Gamma\Big(\bigcup_{m=0}^{n} A_m(B)\Big)}$$

is compact in Y. Consequently, if $0 < R_0 < 1$, we obtain

$$\Big\{ \sum_{m=0}^{n} A_m(x) z^m\ :\ |z| \le R_0,\ x \in B \Big\} \subset n \cdot K,$$

which proves that T_{ψ_n} is a Montel operator. To conclude the proof, we see that T_{ψ_n} tends to T_ψ in $L_b(X, H(\mathbb{D}, Y))$. If B is a bounded set in X, $0 < R < 1$ and q is a continuous seminorm in Y, we have

$$\sup_{x \in B} \sup_{|z| \leq R} q\Big((T_\psi - T_{\psi_n})(x)[z]\Big) = \sup_{x \in B} \sup_{|z| \leq R} q\Big(\sum_{m=n+1}^{\infty} A_m(x)z^m\Big),$$

that tends to 0 when n tends to infinity, since $\psi(z) = \sum_{m=0}^{\infty} A_m z^m$ in $L_b(X, Y)$ uniformly on compact sets.

It remains to show that (b) implies (a). Take a bounded set B in $H(\mathbb{D}, X)$. Since B is bounded for every compact set K in $\mathbb{D}$ and every continuous seminorm p in X, we have

$$\sup_{z \in K} \sup_{f \in B} p(f(z)) < \infty.$$

In particular, as $\varphi(K)$ is a compact set in $\mathbb{D}$, we also have

$$\sup_{z \in K} \sup_{f \in B} p(f(\varphi(z))) < \infty$$

for every continuous seminorm p in X. Therefore, the set

$$B_K := \{f(\varphi(z)) : z \in K, f \in B\}$$

is bounded in X, for every compact set K in $\mathbb{D}$. By hypothesis, $T_\psi(B_K)$ is precompact in $H(\mathbb{D}, Y)$. By Lemma 8, for every compact set L in $\mathbb{D}$, the set

$$\{\psi(\xi)[f(\varphi(z))] : \xi \in L, z \in K, f \in B\}$$

is a precompact set in Y. Therefore, the set $\{\psi(z)[f(\varphi(z))] : z \in K, f \in B\}$ is precompact in Y, for each compact set K in $\mathbb{D}$. Since $W_{\psi,\varphi}(f)[z] = \psi(z)[f(\varphi(z))]$ for $f \in B$ and $z \in K$, by Lemma 8 again, we conclude that $W_{\psi,\varphi}(B)$ is precompact in $H(\mathbb{D}, Y)$. Then, (b) implies (a).

$\square$

Proposition 10 *Let X, Y be complete barrelled locally convex spaces. Let $\varphi : \mathbb{D} \to \mathbb{D}$ and $\psi : \mathbb{D} \to L_b(X, Y)$ be analytic maps. Then the following assertions are equivalent:*

1. *The operator $W_{\psi,\varphi} : H(\mathbb{D}, X) \to H(\mathbb{D}, Y)$ is compact,*

2. *The following conditions hold:*

   (a) *$T_\psi : X \to H(\mathbb{D}, Y)$ is compact,*
   (b) *There is $0 < r_0 < 1$ such that $|\varphi(z)| \leq r_0$ for every $z \in \mathbb{D}$.*

Proof. First, we suppose that $W_{\psi,\varphi}$ is compact. Since T_ψ inherits the properties of $W_{\psi,\varphi}$ by Remark 4 (i), we have that T_ψ is compact. Moreover, the condition (b) follows from Theorem 6, since the operator $W_{\psi,\varphi}$ is bounded.

To prove that (2) implies (1), we select r_0 as in condition 2.(b) and set $K_0 = \{\rho : |\rho| \leq r_0\}$. From 2.(a) there exists a continuous seminorm p on X such that $T_\psi(U_p)$ is precompact in $H(\mathbb{D}, Y)$ where $U_p := \{x \in X : p(x) \leq 1\}$. Then

$$\{\psi(z)x : z \in K, \ p(x) \leq 1\} \tag{2.2}$$

is precompact in Y, for each compact set K in $\mathbb{D}$, by Lemma 8. We consider

$$V := \{g \in H(\mathbb{D}, X) : \sup_{\rho \in K_0} p(g(\rho)) \leq 1\} \in \mathcal{U}_0(H(\mathbb{D}, X)).$$

We show that $W_{\psi,\varphi}(V)$ is precompact in $H(\mathbb{D}, Y)$. From Lemma 8 it suffices to check that for all compact set K in $\mathbb{D}$, the set

$$\{W_{\psi,\varphi}(g)(z) : z \in K, g \in V\}$$

is precompact in Y.

Let K be a compact subset of $\mathbb{D}$. If $z \in K$ and $g \in V$, since $\varphi(z) \in K_0$, we also have $p(g(\varphi(z))) \leq 1$. Then from (2.2) we have that

$$\{W_{\psi,\varphi}(g)(z) : z \in K, g \in V\} = \{\psi(z)(g(\varphi(z))) : z \in K, g \in V\}$$

is precompact in Y. $\qquad\square$

Now, we treat the case when the operator-valued function ψ is constant and equal to a fixed non-zero operator $L : X \to Y$. The following result is an easy consequence of our results above.

Corollary 11 *Let X, Y barrelled locally convex spaces. If $L : X \to Y$ is a non-zero linear and continuous operator and $\psi : \mathbb{D} \to L_b(X, Y)$ is the constant operator-valued holomorphic function equal to L for all $z \in \mathbb{D}$, we have:*

(i) $W_{\psi,\varphi} : H(\mathbb{D}, X) \to H(\mathbb{D}, Y)$ is continuous,

(ii) $W_{\psi,\varphi} : H(\mathbb{D}, X) \to H(\mathbb{D}, Y)$ is bounded if and only if

    (a) L is bounded, and
    (b) There is $r_0 \in (0, 1)$ such that $\varphi(\mathbb{D}) \subset r_0 \mathbb{D}$.

(iii) $W_{\psi,\varphi} : H(\mathbb{D}, X) \to H(\mathbb{D}, Y)$ is Montel if and only if L is Montel,

(iv) $W_{\psi,\varphi} : H(\mathbb{D}, X) \to H(\mathbb{D}, Y)$ is compact if and only if

    (a) L is compact, and
    (b) There is $r_0 \in (0, 1)$ such that $\varphi(\mathbb{D}) \subset r_0 \mathbb{D}$.

3 Operator-weighted composition operators acting weighted spaces $H_v^\infty(\mathbb{D}, X)$ and $H_w^\infty(\mathbb{D}, Y)$

In this section we study the operator-weighted composition operators acting between weighted spaces. In the formulation of our results we need the associated weights of [2]. For a weight v on $\mathbb{D}$ the corresponding *associated weight* $\tilde{v}$ is defined by

$$\tilde{v}(z) := \frac{1}{\sup\{|f(z)|;\ f \in H_v^\infty(\mathbb{D}), \|f\|_v \leq 1\}} = \frac{1}{\|\delta_z\|_{H_v^\infty(\mathbb{D})'}}, \ z \in \mathbb{D},$$

where $\|f\|_v := \sup_{z \in \mathbb{D}} v(z)|f(z)|$ and δ_z denotes the point evaluation of $z \in \mathbb{D}$. By [2] we know that associated weights are continuous, $\tilde{v} \geq v > 0$ and that for each $z \in \mathbb{D}$ we can find $f_z \in H_v^\infty(\mathbb{D})$, $\|f\|_v \leq 1$, such that $|f_z(z)| = \frac{1}{\tilde{v}(z)}$. It is well known that $H_{\tilde{v}}^\infty(\mathbb{D})$ is isometrically isomorphic to $H_v^\infty(\mathbb{D})$. Applying the Hahn-Banach theorem we conclude

Lemma 12 *Let X be a complete barrelled locally convex space. For every $p \in cs(X)$ and every $f \in H_v^\infty(\mathbb{D}, X)$ we have*

$$\sup_{\xi \in \mathbb{D}} \tilde{v}(\xi)p(f(\xi)) = \sup_{z \in \mathbb{D}} v(z)p(f(z)).$$

Proposition 13 *Let X and Y be complete barrelled locally convex spaces. Let $\varphi : \mathbb{D} \to \mathbb{D}$ and $\psi : \mathbb{D} \to L_b(X, Y)$ be analytic maps. The operator $W_{\psi,\varphi} : H_v^\infty(\mathbb{D}, X) \to H_w^\infty(\mathbb{D}, Y)$, $f \to (W_{\psi,\varphi}f)(z) = \psi(z)[f(\varphi(z))]$ is continuous if and only if the set*

$$\left\{ \frac{w(z)}{\tilde{v}(\varphi(z))} \psi(z); \ z \in \mathbb{D} \right\}$$

is equicontinuous on $L_b(X, Y)$.

Proof. First, assume that the set $\{ \frac{w(z)}{\tilde{v}(\varphi(z))}\psi(z); \ z \in \mathbb{D} \}$ is equicontinuous on $L_b(X, Y)$. For each $q \in cs(Y)$ we can find $p \in cs(X)$ such that

$$\frac{w(z)}{\tilde{v}(\varphi(z))}q(\psi(z)[x]) \le p(x) \text{ for every } x \in X \text{ and every } z \in \mathbb{D}. \tag{3.1}$$

Thus, for $z \in \mathbb{D}$ and $f \in H_v^\infty(\mathbb{D}, X)$, we get

$$
\begin{aligned}
w(z)q(\psi(z)[f(\varphi(z))]) &= \tilde{v}(\varphi(z))\frac{w(z)}{\tilde{v}(\varphi(z))}q(\psi(z)[f(\varphi(z))]) \\
&\underset{(3.1)}{\le} \tilde{v}(\varphi(z))p(f(\varphi(z))) \le \sup_{\xi \in \mathbb{D}} \tilde{v}(\xi)p(f(\xi)).
\end{aligned}
$$

and the operator $W_{\psi,\varphi} : H_v^\infty(\mathbb{D}, X) \to H_w^\infty(\mathbb{D}, Y)$ is continuous.

We prove the converse by contradiction: Assume that the set $\{ \frac{w(z)}{\tilde{v}(\varphi(z))}\psi(z); \ z \in \mathbb{D} \}$ is not equicontinuous on $L_b(X, Y)$. Since X is barrelled, this means that we can find an element $x \in X$ such that the set $\{ \frac{w(z)}{\tilde{v}(\varphi(z))}\psi(z)[x]; \ z \in \mathbb{D} \}$ is not bounded in Y. Thus there exists $y' \in Y'$ such that $\{ \frac{w(z)}{\tilde{v}(\varphi(z))}y'(\psi(z)[x]); \ z \in \mathbb{D} \}$ is not bounded in $\mathbb{C}$, and we can find a sequence $(z_n)_n \subset \mathbb{D}$ such that

$$\frac{w(z_n)}{\tilde{v}(\varphi(z_n))}|y'(\psi(z_n)[x])| \ge n$$

for every $n \in \mathbb{N}$. We select $f_n \in H(\mathbb{D})$ such that $|f_n| \le \frac{1}{v}$, $f_n(\varphi(z_n)) = 1/\tilde{v}(\varphi(z_n))$, and define $g_n : \mathbb{D} \to X$ by

$$g_n(z) := f_n(z)x$$

for every $n \in \mathbb{N}$, with x chosen above. The sequence $(g_n)_n$ is bounded in $H_v^\infty(\mathbb{D}, X)$ since for every $p \in cs(X)$ and every $z \in \mathbb{D}$ we have

$$v(z)p(g_n(z)) = v(z)|f_n(z)|p(x) \le p(x).$$

By assumption $(W_{\psi,\varphi}g_n)_n$ is a bounded sequence in $H_w^\infty(\mathbb{D}, Y)$ and

$$W_{\psi,\varphi}(g_n)(z) = \psi(z)[g_n(\varphi(z))] = f_n(\varphi(z))\psi(z)[x].$$

10

With $y' \in Y'$ selected as above, we put $q(s) := |y'(s)| \in cs(Y)$. Then there exists a constant $M > 0$ such that for every $z \in \mathbb{D}$ and every $n \in \mathbb{N}$

$$w(z)q(W_{\psi,\varphi}g_n(z)) \le M$$

which implies $n \le \frac{w(z_n)}{\tilde{v}(\varphi(z_n))}|y'(\psi(z_n)[x])| \le M$ for every $n \in \mathbb{N}$. This is a contradiction. $\quad\square$

Recall that a subset C of $L(X,Y)$ is called *equibounded* if there is a neighbourhood U in X such that $\{T(x) \mid x \in U, \ T \in C\}$ is bounded in Y.

Proposition 14 *Let X and Y be complete barrelled locally convex spaces. Moreover, let $\varphi : \mathbb{D} \to \mathbb{D}$ and $\psi : \mathbb{D} \to L_b(X,Y)$ be analytic maps. Then the operator $W_{\psi,\varphi} : H_v^\infty(\mathbb{D}, X) \to H_w^\infty(\mathbb{D}, Y)$ is bounded if and only if the set*

$$\left\{ \frac{w(z)}{\tilde{v}(\varphi(z))}\psi(z); \ z \in \mathbb{D} \right\}$$

is equibounded in $L_b(X,Y)$.

Proof. It is very similar to the proof of Proposition 13, once you have in mind that $\{\frac{w(z)}{\tilde{v}(\varphi(z))}\psi(z); \ z \in \mathbb{D}\}$ is equibounded in $L_b(X,Y)$ if and only if there is $p \in cs(X)$ such that for every $q \in cs(Y)$ there is a constant $\lambda_q > 0$ with

$$q\left(\frac{w(z)}{\tilde{v}(\varphi(z))}\psi(z)[x] \right) \le \lambda_q \ p(x) \tag{3.2}$$

for every $x \in X$ and every $z \in \mathbb{D}$. $\quad\square$

Let X be a complete barrelled locally convex space. In the proof of Theorem 15 we denote, for every $k \in \mathbb{N}_0$, q_k denote the map

$$\begin{array}{rccc} q_k : & H(\mathbb{D}, X) & \longrightarrow & X \\ & f & \longmapsto & \frac{f^{(k)}(0)}{k!} \end{array} \tag{3.3}$$

which is continuous, by the Cauchy Integral Formula for derivatives.

Theorem 15 *Let X and Y be complete barrelled locally convex spaces. Moreover, let $\varphi : \mathbb{D} \to \mathbb{D}$ and $\psi : \mathbb{D} \to L_b(X,Y)$ be analytic maps such that the operator $W_{\psi,\varphi} : H_v^\infty(\mathbb{D}, X) \to H_w^\infty(\mathbb{D}, Y)$ is continuous. The weighted composition operator $W_{\psi,\varphi}$ is Montel if, and only if, the following conditions are satisfied:*

(a) *$T_\psi : X \to H_w^\infty(\mathbb{D}, Y)$ is Montel.*

(b) *For every $B \in \mathcal{B}(X)$, $q \in cs(Y)$ and $\varepsilon > 0$ there is $r_0 \in]0,1[$ such that if $|\varphi(z)| > r_0$ and $x \in B$, then we have the following inequality*

$$\frac{w(z)}{\tilde{v}(\varphi(z))}q(\psi(z)[x]) \le \varepsilon.$$

Proof. Let us first assume that conditions (a) and (b) are satisfied. We will approximate the operator $W_{\psi,\varphi}$ by a sequence of Montel operators in $L_b(H_v^\infty(\mathbb{D}, X), H_w^\infty(\mathbb{D}, Y))$. To do this we consider the Cesàro sums given by

$$
\begin{array}{rccc}
C_m: & H_v^\infty(\mathbb{D}, X) & \longrightarrow & H_v^\infty(\mathbb{D}, X) \\
& f & \longmapsto & \frac{1}{m+1} \sum_{k=0}^m P_k f,
\end{array}
$$

where $P_k f(z) = \sum_{\ell=0}^k \frac{f^{(\ell)}(0)}{\ell!} z^\ell$ is the k-th Taylor polynomial of f. First we want to show that $W_{\psi,\varphi} C_m$ is a Montel operator acting from $H_v^\infty(\mathbb{D}, X)$ to $H_w^\infty(\mathbb{D}, Y)$. The definition gives

$$
\begin{aligned}
(W_{\psi,\varphi} C_m f)(z) & = \psi(z)\big[C_m f(\varphi(z))\big] = \psi(z)\Big[\frac{1}{m+1} \sum_{k=0}^m \Big(\sum_{\ell=0}^k \frac{f^{(\ell)}(0)}{\ell!} \varphi(z)^\ell\Big)\Big] \\
& = \frac{1}{m+1} \sum_{k=0}^m \Big(\sum_{\ell=0}^k \varphi(z)^\ell \psi(z) \frac{f^{(\ell)}(0)}{\ell!}\Big) \\
& = \frac{1}{m+1} \sum_{k=0}^m \sum_{\ell=0}^k \varphi(z)^\ell T_\psi q_\ell(f)(z),
\end{aligned}
$$

Since T_ψ is a Montel operator and the q_ℓ are continuous we obtain that $W_{\psi,\varphi} C_m$ is a Montel operator for every $m \in \mathbb{N}_0$, as desired. Now, we show that

$$
W_{\psi,\varphi} = \lim_{m \to \infty} W_{\psi,\varphi} C_m \tag{3.4}
$$

in $L_b(H_v^\infty(\mathbb{D}, X), H_w^\infty(\mathbb{D}, Y))$. To do this, we fix a bounded set C in $H_v^\infty(\mathbb{D}, X)$ as well as $q \in cs(Y)$ and have to show

$$
\sup_{f \in C} \sup_{z \in \mathbb{D}} w(z) q\left[W_{\psi,\varphi} f(z) - W_{\psi,\varphi} C_m f(z)\right] \to 0 \quad \text{as} \quad m \to \infty. \tag{3.5}
$$

We set

$$
\begin{aligned}
\tilde{B} & := \{\tilde{v}(\varphi(z)) f(\varphi(z)) \,; \, f \in C, z \in \mathbb{D}\}, \\
B_m & := \{\tilde{v}(\varphi(z)) \cdot C_m f(\varphi(z)), \, f \in C, z \in \mathbb{D}, \}, \quad m \in \mathbb{N},
\end{aligned}
$$

and we take

$$
B := \tilde{B} \cup \bigcup_{m=0}^\infty B_m.
$$

We claim that B is bounded in X. Indeed, if $p \in cs(X)$, there is a constant M_p such that

$$
\sup_{z \in \mathbb{D}} \tilde{v}(z) p(f(z)) \le M_p
$$

for each $f \in C$. Moreover, for each $m \in \mathbb{N}$, $z \in \mathbb{D}$, and $f \in C$, we can apply [1, Lemma 1.1] to get

$$
\sup_{z \in \mathbb{D}} \tilde{v}(\varphi(z)) p\big(C_m f(\varphi(z))\big) \le \sup_{z \in \mathbb{D}} \tilde{v}(\varphi(z)) p\big(f(\varphi(z))\big) \le M_p
$$

Hence $p(\xi) \le M_p$ for each $\xi \in B$.

Fix $\varepsilon > 0$. Select $r_0 \in]0, 1[$ for $B, q, \frac{\varepsilon}{2} > 0$ as in (b). To prove (3.5) we split the estimate as follows

$$
\sup_{z \in \mathbb{D}} w(z) q\left(W_{\psi,\varphi} f(z) - W_{\psi,\varphi} C_m f(z)\right) \le (s_1) + (s_2),
$$

12

where

$$(s_1) = \sup_{|\varphi(z)| > r_0} w(z)\, q\big(W_{\psi,\varphi} f(z) - W_{\psi,\varphi} C_m f(z)\big)$$

and

$$(s_2) = \sup_{|\varphi(z)| \leq r_0} w(z)\, q\big(W_{\psi,\varphi} f(z) - W_{\psi,\varphi} C_m f(z)\big).$$

We analyze first (s_1).

$$w(z)\, q\big(W_{\psi,\varphi} f(z) - W_{\psi,\varphi} C_m f(z)\big) \leq w(z)\, q\big(W_{\psi,\varphi} f(z)\big) + w(z) q\big(W_{\psi,\varphi} C_m f(z)\big)$$

For $|\varphi(z)| > r_0$, we have

$$\begin{aligned} w(z) q\big(W_{\psi,\varphi} f(z)\big) &= w(z) q\big(\psi(z)\big[f(\varphi(z))\big]\big) \\ &= \frac{w(z)}{\tilde{v}(\varphi(z))} q\big(\psi(z)\big[\tilde{v}(\varphi(z)) f(\varphi(z))\big]\big) \leq \frac{\varepsilon}{2} \end{aligned}$$

since $\tilde{v}(\varphi(z)) f(\varphi(z)) \in B$. Now, if $|\varphi(z)| > r_0$,

$$w(z) q\big(W_{\psi,\varphi} C_m f(z)\big) \leq \frac{w(z)}{\tilde{v}(\varphi(z))} q\big(\psi(z)\big[\tilde{v}(\varphi(z)) \cdot C_m f(\varphi(z))\big]\big) \leq \frac{\varepsilon}{2}.$$

Therefore, $(s_1) \leq \varepsilon$ for each m. Next, we study (s_2). Observe first, that for $f \in C$ we have at least formally,

$$W_{\psi,\varphi} f(z) - W_{\psi,\varphi} C_m f(z) = \frac{1}{m+1}\Big(\sum_{k=0}^{m}\Big(\sum_{\ell=k+1}^{\infty} \varphi(z)^\ell \psi(z)\Big[\frac{f^{(\ell)}(0)}{\ell!}\Big]\Big)\Big).$$

Then

$$(s_2) \leq \sup_{|\varphi(z)| \leq r_0} w(z) \frac{1}{m+1} \sum_{k=0}^{m}\Big(\sum_{\ell=k+1}^{\infty} |\varphi(z)|^\ell q\big(T_\psi\big(\frac{f^{(\ell)}(0)}{\ell!}\big)z\big)\Big)$$

By assumption T_ψ is Montel, hence it also is continuous. There is $p \in cs(X)$ such that the Cauchy Integral Formula for derivatives yields

$$w(z) q\big(T_\psi\big(\frac{f^{(\ell)}(0)}{\ell!}\big)z\big) \leq p\big(\frac{f^{(\ell)}(0)}{\ell!}\big) \leq \frac{1}{r^\ell} \max_{|\zeta|=r} p(f(\zeta)) \leq \frac{1}{r^\ell v(r)} \sup_{\zeta \in \mathbb{D}} v(\zeta) p(f(\zeta)).$$

Then, if $r > r_0$, as $|\varphi(z)| \leq r_0$ in (s_2) we obtain

$$\begin{aligned} (s_2) &\leq \sup_{z \in \mathbb{D}} v(z) p(f(z)) \Big(\frac{1}{m+1}\sum_{k=0}^{m}\Big(\sum_{\ell=k+1}^{\infty}\big(\frac{r_0}{r}\big)^\ell \frac{1}{v(r)}\Big)\Big) \\ &= \sup_{z \in \mathbb{D}} v(z) p(f(z)) \frac{1}{v(r)}\Big[\sum_{\ell=0}^{\infty}\big(\frac{r_0}{r}\big)^\ell - \frac{1}{m+1}\sum_{k=0}^{m} S_k\Big], \end{aligned}$$

where $S_k = \sum_{\ell=0}^{k}\big(\frac{r_0}{r}\big)^\ell$ is the k-th partial sum of the convergent geometric series $\sum_{\ell=0}^{\infty}\big(\frac{r_0}{r}\big)^\ell$, which shows that (s_2) tends to 0 as m tends to infinity, and (3.4) is proved.

Now, we suppose that $W_{\psi,\varphi}$ is Montel. Since T_ψ inherits the properties of $W_{\psi,\varphi}$, we obtain that T_ψ is Montel. To prove the condition (b) we argue by contradiction. If (b) does not hold then there exist an absolutely convex bounded set B in X and $q \in cs(Y)$ such that for all $n \in \mathbb{N}$ there exists $z_n \in \mathbb{D}$ with $|\varphi(z_n)| > 1 - \frac{1}{n}$ and there exists $x_n \in B$ such that the following inequality holds

$$\frac{w(z_n)}{\tilde{v}(\varphi(z_n))} q(\psi(z_n)[x_n]) > 1. \tag{3.6}$$

The set

$$M := \{f \in H_v^\infty(\mathbb{D}, X) : \ v(z)f(z) \in B \ \text{ for all } z \in \mathbb{D}\}$$

is bounded in $H_v^\infty(\mathbb{D}, X)$. Since $W_{\psi,\varphi}$ is Montel, $W_{\psi,\varphi}M$ is relatively compact in $H_w^\infty(\mathbb{D}, Y)$. For each $n \in \mathbb{N}$, select functions $f_n \in H(\mathbb{D})$ verifying

$$||f_n||_\infty \leq \frac{1}{v} \quad \text{and} \quad f_n(\varphi(z_n)) = \frac{1}{\tilde{v}(\varphi(z_n))}$$

and we define

$$h_n(z) = z^n f_n(z) x_n, \ z \in \mathbb{D}.$$

Clearly $h_n \in H_v^\infty(\mathbb{D}, X)$ and $v(z)h_n(z) \in B$ for all $z \in \mathbb{D}$. Hence $(h_n)_{n=1}^\infty \subseteq M$. Furthermore, we claim that $(h_n)_{n=1}^\infty$ converges to 0 in $H(\mathbb{D}, X)$ for the open-compact topology as n tends to infinity. In order to prove this, fix $p \in cs(X)$ and $r_0 \in]0, 1[$. We have

$$\begin{aligned}
\sup_{|z| \leq r_0} p(h_n(z)) &= \sup_{|z| \leq r_0} |z|^n p(f_n(z)x_n) \\
&\leq r_0^n \frac{1}{\tilde{v}(r_0)} \sup_{z \in \mathbb{D}} \tilde{v}(z) p(f_n(z)x_n) \\
&\leq C \, r_0^n \frac{1}{\tilde{v}(r_0)}
\end{aligned}$$

for some constant $C > 0$. Therefore $\lim_{n \to \infty} \sup_{|z| \leq r_0} p(h_n(z)) = 0$.

Since $W_{\psi,\varphi} : H(\mathbb{D}, X) \longrightarrow H(\mathbb{D}, Y)$ is continuous by Proposition 5, the sequence $(W_{\psi,\varphi}h_n)_{n=1}^\infty$ converges to 0 in $H(\mathbb{D}, Y)$ for the open-compact topology. Moreover $W_{\psi,\varphi}M$ is relatively compact in $H_w^\infty(\mathbb{D}, Y)$ and $(W_{\psi,\varphi}h_n)_{n=1}^\infty \subseteq W_{\psi,\varphi}M$. Hence $(W_{\psi,\varphi}h_n)_{n=1}^\infty$ converges to 0 in $H_w^\infty(\mathbb{D}, Y)$. But, if we take $q \in cs(Y)$ and $(z_n)_{n=1}^\infty$ satisfying (3.6), there exists n_0 such that, for $n \geq n_0$,

$$\begin{aligned}
w(z_n)q((W_{\psi,\varphi}h_n)(z_n)) &= w(z_n)q\left(\psi(z_n)[h_n(\varphi(z_n))]\right) \\
&= w(z_n)q\left(\psi(z_n)[\varphi(z_n)^n f_n(\varphi(z_n))x_n]\right) \\
&= w(z_n)q\left(\varphi(z_n)^n f_n(\varphi(z_n)) \ \psi(z_n)[x_n]\right) \\
&= w(z_n)|\varphi(z_n)|^n \frac{1}{\tilde{v}(\varphi(z_n))} \ q\left(\psi(z_n)[x_n]\right) \\
&> \left(1 - \frac{1}{n}\right)^n > \frac{1}{e},
\end{aligned}$$

which contradicts the fact that $(W_{\psi,\varphi}h_n)_{n=1}^\infty$ converges to 0 in $H_w^\infty(\mathbb{D}, Y)$. The proof is complete. $\qquad\square$

The following consequences should be compared with some results presented in [8].

Corollary 16 *Let $L : X \to Y$ be a non-zero linear and continuous operator and $\psi : \mathbb{D} \to L_b(X, Y)$ is the constant operator-valued holomorphic function equal to L for all $z \in \mathbb{D}$. The following holds:*

(i) *$W_{\psi,\varphi} : H_v^\infty(\mathbb{D}, X) \to H_w^\infty(\mathbb{D}, Y)$ is continuous if and only if the composition operator $C_\varphi : H_v^\infty(\mathbb{D}) \to H_w^\infty(\mathbb{D})$ is continuous.*

(ii) *$W_{\psi,\varphi} : H_v^\infty(\mathbb{D}, X) \to H_w^\infty(\mathbb{D}, Y)$ is bounded if and only if*

    (a) *L is bounded, and*

    (b) *$C_\varphi : H_v^\infty(\mathbb{D}) \to H_w^\infty(\mathbb{D})$ is continuous.*

(iii) *$W_{\psi,\varphi} : H_v^\infty(\mathbb{D}, X) \to H_w^\infty(\mathbb{D}, Y)$ is Montel if and only if*

    (a) *L is Montel, and*

    (b) *$C_\varphi : H_v^\infty(\mathbb{D}) \to H_w^\infty(\mathbb{D})$ is compact.*

As a concrete example, if $v(z) = (1 - |z|)^\alpha$, $w(z) = (1 - |z|)^\beta$, $\varphi(z) = z$ and $\phi(z) = L \in L(X, Y)$, L Montel, for each $z \in \mathbb{D}$, then the operator $W_{\psi,\varphi}^\infty : H_v^\infty(\mathbb{D}, X) \to H_w^\infty(\mathbb{D}, Y)$ is Montel if $0 < \alpha < \beta$, but it is continuous not Montel if $0 < \alpha = \beta$.

Complete characterizations of the continuity and compactness of composition operators $C_\varphi : H_v^\infty(\mathbb{D}) \to H_w^\infty(\mathbb{D})$ in terms of the weights v and w and the symbol φ were obtained [7], [4] and [11].

The following result gives a sufficient condition to obtain a version of the equivalence between (b) and (c) in Theorem 9 for the case of weighted spaces. Such a result should be compared with [16, Theorem 4.4], that treats the case of Banach valued holomorphic functions. In order to get the equivalence, it is necessary to put some restrictions on ψ; see Theorem 23.

For a weight function v, we define the following space:

$$H_v^0(\mathbb{D}, X) = \{ f \in H_v^\infty(\mathbb{D}, X) \; : \; \lim_{|z| \to 1-} v(z) p(f(z)) = 0 \; \text{ for all } \; p \in cs(X) \}.$$

In particular, if $v \equiv 1$ then $H_v^0(\mathbb{D}, X) = \{0\}$.

Theorem 17 *Let X, Y be complete barrelled locally convex spaces. If $\psi \in H_w^0(\mathbb{D}, L_b(X, Y))$, then the following assertions are equivalent:*

(a) *The operator $T_\psi : X \to H_w^\infty(\mathbb{D}, Y)$ is Montel,*

(b) *$\psi(z) : X \to Y$ is Montel for all $z \in \mathbb{D}$.*

Proof. Condition (a) implies condition (b) as a consequence of Remark 4 (ii).
We prove then that (b) implies (a). Since ψ is holomorphic, we can write

$$\psi(z) = \sum_{\ell=0}^\infty A_\ell z^\ell, \; A_\ell \in L(X, Y).$$

By the Cauchy Integral Formula for the coefficients A_ℓ, it follows that A_ℓ is a Montel operator from X to Y. See e.g. [22] for the integration of continuous functions with values in a complete locally convex space. We denote by $C_m \psi$ the Cesàro sums of the Taylor polynomials associated

to ψ. We check that the corresponding auxiliary operators $T_{C_m\psi}$ are Montel operators. To do this, it suffices to show that if $g : \mathbb{D} \to L(X, Y)$ is an operator-valued monomial $g(z) = Az^\ell$ where $A : X \to Y$ is a Montel operator, then the corresponding auxiliary operator $T_g : X \to H_w^\infty(\mathbb{D}, Y)$ is Montel. Fix a bounded set B in X. By assumption, $A(B)$ is relatively compact in Y. The set

$$T_g(B) = \{A(b)z^\ell \,:\, b \in B\}$$

is relatively compact in $H_w^\infty(\mathbb{D}, Y)$. Indeed, the linear operator

$$\begin{array}{rccl} S : & Y & \longrightarrow & H_w^\infty(\mathbb{D}, Y) \\ & y & \longmapsto & [z \mapsto yz^\ell] \end{array}$$

is also continuous and satisfies $T_g(B) = S(A(B))$.

Since the space of Montel operators is closed in the space of all linear and continuous operators ([15, 42.1 (3)]), the proof is complete if we show that $T_{C_m\psi}$ converges to T_ψ in $L(X, H_w^\infty(\mathbb{D}, Y))$. Let B be a bounded set in X and q a continuous seminorm on Y. We want to estimate

$$\sup_{b \in B} \sup_{z \in \mathbb{D}} w(z)q\big((C_m\psi(z) - \psi(z))(b)\big).$$

Since $\psi \in H_w^0(\mathbb{D}, L_b(X, Y))$, for each $\varepsilon > 0$ there exists $R_0 > 0$ such that for all $|z| \geq R_0$ we have

$$\sup_{b \in B} w(z)q\big(\psi(z)(b)\big) < \varepsilon/3.$$

For each $b \in B$, the function $z \mapsto \psi(z)(b)$ is in $H(\mathbb{D}, Y)$. Therefore, for each $b \in B$ and $m \in \mathbb{N}$, we can apply [1, Lemma 1.1] and Hahn-Banach theorem to conclude

$$w(z)q\big(C_m\psi(z)(b)\big) \leq \max_{|\zeta|=|z|} w(\zeta)q\big(\psi(\zeta)(b)\big).$$

Thus, for $|z| > R_0$, $b \in B$ and $m \in \mathbb{N}$,

$$w(z)q\big(\psi(z)(b)\big) < \varepsilon/3 \quad \text{and} \quad w(z)q\big(C_m\psi(z)(b)\big) < \varepsilon/3.$$

Applying again that $\psi \in H_w^\infty(\mathbb{D}, L(X, Y))$, the Cesàro sums of the Taylor polynomials $C_m\psi$ of ψ converge to ψ in the topology of $H(\mathbb{D}, L(X, Y))$. In particular, there exist $m_0 \in \mathbb{N}$ such that, for all $m \geq m_0$, we have

$$\sup_{|z| \leq R_0} \sup_{b \in B} w(z)q\big(\psi(z)(b) - C_m\psi(z)(b)\big) < \varepsilon/3.$$

Therefore, for every $\varepsilon > 0$, B bounded in X and q continuous seminorm on Y, there is m_0 (and $R_0 > 0$), such that for $m \geq m_0$, we obtain

$$\begin{aligned} \sup_{b \in B} \sup_{z \in \mathbb{D}} & w(z)q\big(T_\psi(b)(z) - T_{C_m\psi}(b)(z)\big) \\ &= \sup_{b \in B} \sup_{z \in \mathbb{D}} w(z)q\big(\psi(z)(b) - C_m\psi(z)(b)\big) \\ &\leq \sup_{b \in B}\Big\{ \sup_{|z| \geq R_0} w(z)q\big(\psi(z)(b)\big) + \sup_{|z| \geq R_0} w(z)q\big(C_m\psi(z)(b)\big) \\ &\qquad + \sup_{|z| \leq R_0} w(z)q\big(\psi(z)(b) - C_m\psi(z)(b)\big)\Big\} < \varepsilon. \end{aligned}$$

$\square$

4 Analytic dependence of diagonal operators in values of Köethe spaces

In this section we restrict our attention to operator weighted composition operators between spaces of holomorphic functions on the unit disc with values in a Köthe echelon space. The image $\psi(z)$ is a diagonal operator for each $z \in \mathbb{D}$.

We refer the reader to [21] and [3] for a detailed study of Köthe echelon spaces. We recall here a few definitions. A matrix $A = (a_n(i))_{i,n\in\mathbb{N}}$ of non-negative numbers if called a *Köthe matrix* if it satisfies the following conditions:

(1) For each $i \in \mathbb{N}$ there is $n \in \mathbb{N}$ with $a_n(i) > 0$.

(2) $a_n(i) \leq a_{n+1}(i)$ for all $i, n \in \mathbb{N}$.

For $1 \leq p < \infty$ we define

$$\lambda_p(A) := \Big\{ x \in \mathbb{C}^{\mathbb{N}} : q_n(x) := \big(\sum_{i=1}^{\infty} |x_i a_n(i)|^p \big)^{1/p} < \infty \text{ for all } n \in \mathbb{N} \Big\}$$

and for $p = \infty$ and $p = 0$:

$$\lambda_{\infty}(A) := \Big\{ x \in \mathbb{C}^{\mathbb{N}} : q_n(x) := \sup_{i\in\mathbb{N}} |x_i| a_n(i) < \infty \text{ for all } n \in \mathbb{N} \Big\},$$

$$\lambda_0(A) := \Big\{ x \in \lambda_{\infty}(A) : \lim_{i\to\infty} a_n(i) x_i = 0 \text{ for all } n \in \mathbb{N} \Big\}.$$

The spaces $\lambda_p(A)$, $1 \leq p \leq \infty$ and $p = 0$ are Fréchet spaces with a fundamental sequence of seminorms $(q_n)_{n\in\mathbb{N}}$. From now on, we write by $\lambda_p = \lambda_p(A)$ and we fix a weight function w on $\mathbb{D}$.

Proposition 18 *Let $(f_i)_{i\in\mathbb{N}}$ be a sequence in $H(\mathbb{D})$, bounded for the compact-open topology. For $1 \leq p \leq \infty$ or $p = 0$, the diagonal operator-valued function defined by*

$$\begin{aligned} \psi : \quad \mathbb{D} &\longrightarrow \quad L_b(\lambda_p, \lambda_p) \\ z &\longmapsto \quad \psi(z)(x) = (f_i(z) x_i)_{i\in\mathbb{N}} \end{aligned}$$

is well-defined, continuous and holomorphic.

Proof. It is easy to see that $\psi(z)$ is continuous for each $z \in \mathbb{D}$, since $(f_i)_{i\in\mathbb{N}}$ is bounded in the compact sets of $\mathbb{D}$. Moreover, $\psi : \mathbb{D} \to L_b(\lambda_p, \lambda_p)$ is locally bounded on $\mathbb{D}$. Indeed, the set $\psi(\{z \in \mathbb{D} : |z| \leq R\})$ is bounded in $L_b(\lambda_p, \lambda_p)$ for each $R \in]0, 1[$. To check this, fix a bounded set $B \subset \lambda_p$ and $n \in \mathbb{N}$, and estimate as follows, for $1 \leq p < \infty$,

$$\sup_{|z|\leq R} \sup_{x\in B} q_n\big(\psi(z)(x)\big) = \sup_{|z|\leq R} \sup_{x\in B} \big(\sum_{i=1}^{\infty} |a_n(i) f_i(z) x_i|^p \big)^{1/p} \leq C_R \sup_{x\in B} q_n(x) < \infty,$$

for some constant $C_R > 0$ that depends on R and the bounded sequence $(f_i)_{i\in\mathbb{N}}$. The cases $p = 0$ or $p = \infty$ are easier.

To prove that ψ is holomorphic, by a result of Grosse-Erdmann [13, Theorem 1], we have only to show that $\gamma \circ \psi : \mathbb{D} \to \mathbb{C}$ is holomorphic for each γ in a weak*-dense set $G \subset L_b(\lambda_p, \lambda_p)'$. First we consider the case $p \neq \infty$. Given $u \in (\lambda_p)'$, $y \in \lambda_p$ and $T \in L_b(\lambda_p, \lambda_p)$ we write

$$(u \otimes y)(T) := \langle T(y), u \rangle = u(T(y)),$$

which is a continuous linear functional $u \otimes y$ on $L_b(\lambda_p, \lambda_p)$. Define

$$G := \text{span} \{u \otimes y \,:\, u \in (\lambda_p)', \, y \in \lambda_p\}.$$

It is easy to see that G is a weak*-dense subset of $L_b(\lambda_p, \lambda_p)'$. It remains to show that $(u \otimes y) \circ \psi$ is holomorphic on $\mathbb{D}$ for each $u \in (\lambda_p)'$ and $y \in \lambda_p$. If $u \in (\lambda_p)'$ we can identify it with a sequence $(u_i)_{i \in \mathbb{N}}$ such that for each $y = (y_i)_{i \in \mathbb{N}} \in \lambda_p$, we have $\sum_{i=1}^{\infty} |u_i y_i| < \infty$ by the Köthe duality (see [21, Chapter 27]). Therefore

$$\big((u \otimes y) \circ \psi\big)(z) = (u \otimes y)(\psi(z)) = u\big((f_i(z)y_i\big) = \sum_{i=1}^{\infty} u_i y_i f_i(z)$$

defines an analytic function on $\mathbb{D}$.

If $p = \infty$, we repeat the proof with $G := \text{span} \{u \otimes y \,:\, u \in (\lambda_0)', \, y \in \lambda_\infty\}$. $\qquad\square$

The following essentially well-known lemma follows from the characterization of bounded sets in a Köthe echelon space in [3].

Lemma 19 *If $(\gamma_i)_{i \in \mathbb{N}}$ is a sequence in c_0, the linear and continuous diagonal operator*

$$
\begin{array}{rccc}
L: & \lambda_p & \longrightarrow & \lambda_p \\
& (x_i)_{i \in \mathbb{N}} & \longmapsto & (\gamma_i x_i)_{i \in \mathbb{N}}
\end{array}
$$

is a Montel operator.

As a consequence, if ψ is the map of Proposition 18, then $\psi(z) : \lambda_p \to \lambda_p$ is a Montel operator for each $z \in \mathbb{D}$ in case the sequence of holomorphic functions $(f_i)_{i \in \mathbb{N}}$ tends to 0 for the compact-open topology.

An application of Theorem 5, Proposition 18, and Theorem 9 to the present situation yields the following result.

Corollary 20   *(i) If $(f_i)_{i \in \mathbb{N}}$ is a bounded sequence in $H(\mathbb{D})$ for the compact-open topology, then $W_{\psi,\varphi} : H(\mathbb{D}, \lambda_p) \to H(\mathbb{D}, \lambda_p)$ is continuous.*

   (ii) If $(f_i)_{i \in \mathbb{N}}$ tends to 0 for the compact-open topology, then

     a) $\psi(z) : \lambda_p \to \lambda_p$ is a Montel operator for every $z \in \mathbb{D}$,

     b) $W_{\psi,\varphi} : H(\mathbb{D}, \lambda_p) \to H(\mathbb{D}, \lambda_p)$ is Montel,

     c) $T_\psi : \lambda_p \to H(\mathbb{D}, \lambda_p)$ is Montel.

We now treat the case of weighted spaces of holomorphic functions with values in a Köthe echelon space. To do this, we fix a sequence $(f_i)_{i \in \mathbb{N}}$ in $H_w^\infty(\mathbb{D})$ such that $\|f_i\|_w := \sup_{z \in \mathbb{D}} w(z)|f_i(z)| \leq 1$. Clearly, $(f_i)_{i \in \mathbb{N}}$ is bounded in $H(\mathbb{D})$ for the compact open topology, hence the operator $\psi : \mathbb{D} \longrightarrow L_b(\lambda_p, \lambda_p)$, $z \mapsto \psi(z)(x) = (f_i(z)x_i)_{i \in \mathbb{N}}$ is holomorphic by Proposition 18.

Our purpose now is to study the associated operator

$$T_\psi : \lambda_p \longrightarrow H_w^\infty(\mathbb{D}, \lambda_p), T_\psi(x)[z] = (f_i(z)x_i)_{i \in \mathbb{N}}.$$

We will show in Theorem 23 that T_ψ need not to be Montel even if $\psi(z)$ is Montel for each $z \in \mathbb{D}$, thus showing that the conclusion of Theorem 17 does not hold without the assumption

$\psi \in H_w^0(\mathbb{D}, L_b(X, Y))$, and that the equivalence in Theorem 9 fails in the weighted case. An example in the Banach valued case is given in [16, Example 4.1] for $f_i(z) = z^i, i \in \mathbb{N}$, $w(z) = 1$ and $\lambda_1 = \ell_1$.

Lemma 21 *If $(f_i)_{i \in \mathbb{N}}$ in $H_w^\infty(\mathbb{D})$ satisfies $\|f_i\|_w \leq 1$, then the operator*

$$T_\psi : \lambda_p \longrightarrow H_w^\infty(\mathbb{D}, \lambda_p), \ T_\psi(x)[z] = (f_i(z)x_i)_{i \in \mathbb{N}}$$

is well defined and continuous.

Proof. For every $n \in \mathbb{N}$, we have

$$\sup_{z \in \mathbb{D}} w(z) q_n\big(T_\psi(x)[z]\big) = \sup_{z \in \mathbb{D}} w(z) \big(\sum_{i=1}^\infty |a_n(i) f_i(z) x_i|^p\big)^{1/p} \leq q_n(x),$$

which proves the assertion. □

Lemma 22 *There is a sequence $(f_i)_{i \in \mathbb{N}}$ in $H_w^\infty(\mathbb{D})$ such that*

a) there is $\varepsilon > 0$ satisfying $\varepsilon < \|f_i\|_w \leq 1$ for every $i \in \mathbb{N}$,

b) $(f_i)_{i \in \mathbb{N}}$ tends to 0 for the compact-open topology.

Proof. Since $H_w^\infty(\mathbb{D})$ is infinite dimensional in the present setting, we can find a sequence $(g_i)_{i \in \mathbb{N}}$ in $H_w^\infty(\mathbb{D})$ with $\|g_i\|_w \leq 1$ for every $i \in \mathbb{N}$ which cannot have a Cauchy subsequence. Passing to a subsequence if necessary, we find $\varepsilon_0 > 0$ with $\|g_i - g_j\|_w \geq \varepsilon_0$ for each $i, j \in \mathbb{N}$. The unit ball of $H_w(\mathbb{D})$ is compact for compact-open topology, therefore there exists a subsequence $(h_j)_{j \in \mathbb{N}}$ of $(g_i)_{i \in \mathbb{N}}$, convergent to h_0 in $H(\mathbb{D})$. We put $f_j := \frac{1}{2}(h_j - h_0), j \in \mathbb{N}$. Clearly each f_j belongs to the unit ball of $H_w^\infty(\mathbb{D})$, $\|f_j - f_k\|_w = \frac{1}{2}\|h_j - h_k\|_w \geq \varepsilon_0/2$ and $(f_j)_{j \in \mathbb{N}}$ tends to 0 for the compact-open topology. □

Theorem 23 *Let $(f_i)_{i \in \mathbb{N}} \subset H_w^\infty(\mathbb{D})$ satisfy $\|f_i\|_w \leq 1$ for all $i \in \mathbb{N}$. Then*

a) If λ_p is Montel, then T_ψ is a Montel operator.

(b) If $(f_i)_{i \in \mathbb{N}}$ tends to 0 for the compact-open topology, then $\psi(z) : \lambda_p \to \lambda_p$ is a Montel operator for every $z \in \mathbb{D}$.

c) Assume that there is $\varepsilon > 0$ satisfying $\varepsilon < \|f_i\|_w$ for every $i \in \mathbb{N}$. If λ_p is not Montel, then T_ψ is not a Montel operator.

Proof. (a) If λ_p is Montel, T_ψ is Montel, because it is continuous by Lemma 21 and the bounded subsets of λ_p are relatively compact.

(b) follows from Corollary 20 (ii).

(c) If λ_p is not Montel, there is an infinite subset J of natural numbers and $m \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ there is $c_n > 0$ satisfying $a_n(j) \leq c_n a_m(j)$ for each $j \in J$ (see [21, Chapter 27]). Define

$$B := \big\{\frac{e_j}{a_m(j)} : j \in J\big\},$$

where $(e_j)_{j \in \mathbb{N}}$ is the canonical basis in λ_p. The set B is bounded in λ_p since, for every $n \in \mathbb{N}$,

$$\sup_{j \in \mathbb{N}} q_n\Big(\frac{e_j}{a_m(j)}\Big) = \sup_{j \in \mathbb{N}} \frac{a_n(j)}{a_m(j)} \leq c_n.$$

If the operator T_ψ were Montel, the set $\big(T_\psi(\frac{e_j}{a_m(j)})\big)_{j \in \mathbb{N}}$ would have a convergent subsequence in $H_w^\infty(\mathbb{D}, \lambda_p)$. But, for $i \neq j$, we have

$$\sup_{z \in \mathbb{D}} w(z) q_m\Big(T_\psi\big(\frac{e_j}{a_m(j)}\big)(z) - T_\psi\big(\frac{e_i}{a_m(i)}\big)(z)\Big)$$

$$= \sup_{z \in \mathbb{D}} w(z) q_m\Big(\psi(z)\big(\frac{e_i}{a_m(j)}\big) - \psi(z)\big(\frac{e_i}{a_m(i)}\big)\Big)$$

$$= \sup_{z \in \mathbb{D}} w(z) q_m\Big(\frac{f_j(z)}{a_m(j)} \cdot e_j - \frac{f_i(z)}{a_m(i)} \cdot e_i\Big)$$

$$= \sup_{z \in \mathbb{D}} w(z)\big(|f_i(z)|^p + |f_j(z)|^p\big)^{1/p}$$

$$\geq \max\Big(\sup_{z \in \mathbb{D}} w(z)|f_i(z)|, \ \sup_{z \in \mathbb{D}} w(z)|f_j(z)|\Big) \geq \varepsilon.$$

$\square$

Propositions 13 and 14 yield the following consequences in our present setting.

Corollary 24 *Let w be a weight, $\varphi(z) = z$ and $\psi : \mathbb{D} \to L_b(\lambda_p, \lambda_p)$, $z \mapsto \psi(z)(x) = (f_i(z)x_i)_{i \in \mathbb{N}}$, with $\|f_i\|_w \leq 1$. Then*

a) *$W_{\psi,id} : H_v^\infty(\mathbb{D}, \lambda_p) \to H_v^\infty(\mathbb{D}, \lambda_p)$ is continuous if and only if for each $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ with $m \geq n$ such that*

$$\Big\{\frac{a_n(i)}{a_m(i)} f_i \ : \ i \in \mathbb{N}\Big\}$$

is in $H^\infty(\mathbb{D})$.

b) *$W_{\psi,id} : H_v^\infty(\mathbb{D}, \lambda_p) \to H_v^\infty(\mathbb{D}, \lambda_p)$ is bounded if and only if there is $m \in \mathbb{N}$ such that, for every $n \in \mathbb{N}$, the set*

$$\Big\{\frac{a_n(i)}{a_m(i)} f_i \ : \ i \in \mathbb{N}\Big\}$$

is in $H^\infty(\mathbb{D})$.

We finish this section with an example in the setting of Köthe echelon spaces which shows that a result like Theorem 9 is not true in general if we replace being a Montel operator by being bounded.

Proposition 25 *There exists a Köthe matrix A such that*

$$\psi(z) : \quad \lambda_1(A) \quad \longrightarrow \quad \lambda_1(A)$$
$$x \quad \longmapsto \quad \psi(z)(x) = (z^i x_i)_{i \in \mathbb{N}}$$

is bounded for each $z \in \mathbb{D}$, but the corresponding auxiliary operator $T_\psi : \lambda_1(A) \to H(\mathbb{D}, \lambda_1(A))$ is not bounded.

Proof. Set $R_1 := 1$ and $R_n := \frac{2n+1}{n+1}$ for $n \geq 2$ and define a Köthe matrix A by $a_n(i) = (R_n)^i$, for each $n, i \in \mathbb{N}$. We show that the operator $\psi(z)(x) = (z^i x_i)_{i \in \mathbb{N}}$ is bounded for each $z \in \mathbb{D}$. Given z, select the smallest $m \in \mathbb{N}$ such that $\frac{m-1}{m} \leq |z| < \frac{m}{m+1}$. For every $n \in \mathbb{N}$,

$$R_n |z| < 2|z| < \frac{2m}{m+1} < R_m,$$

hence $a_n(i)|z|^i \leq a_m(i)$ for this m and every $n \in \mathbb{N}$. This implies that $\psi(z)$ is bounded.

Now suppose that the operator $T_\psi : \lambda_1 \to H(\mathbb{D}, \lambda_1)$ is bounded. There is $m \in \mathbb{N}$ such that the image $T_\psi(U_m)$ of the neighborhood of 0

$$U_m := \Big\{ x \in \lambda_1 \ : \ \sum_{i=1}^{\infty} a_m(i)|x_i| \leq 1 \Big\}$$

is bounded. Therefore, there is $m \in \mathbb{N}$ such that for every $0 < R < 1$ and $n \in \mathbb{N}$, there is $C > 0$ such that, for each $x \in U_m$,

$$\sup_{|z| \leq R} \sum_{i=1}^{\infty} R_n^i |z|^i |x_i| \leq C.$$

In particular, there is m such that for every $0 < R < 1$ and $n \in \mathbb{N}$, there is $C > 0$ satisfying, for each $i \in \mathbb{N}$,

$$\sup_{|z| \leq R} R_n^i |z|^i \leq C R_m^i.$$

Then, there is $m \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ and $0 < R < 1$, we have $R_n R \leq R_m$. Letting $n \to \infty$, there is $m \in \mathbb{N}$ such that $2R \leq 1 + \frac{m}{m+1}$ for all $0 < R < 1$, a contradiction. $\quad\square$

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