División de distribuciones atemperadas de una variable

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Aim of the lecture

Study the problem of division for tempered distributions of one variable

We report on joint work with Leonhard Frerick (Univ. Trier, Germany) and Enrique Jordá (UPV)

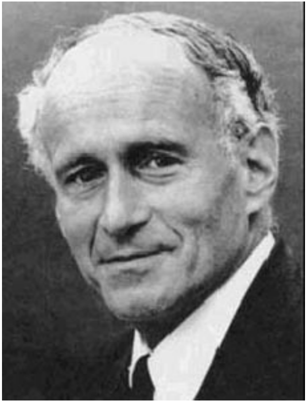
A problem of L. Schwartz

Problem of division of distributions. Schwartz (1951).

Given a distribution $S \in \mathcal{D}'(\mathbb{R}^d)$ and a polynomial P, is there $T \in \mathcal{D}'(\mathbb{R}^d)$ such that $S = PT$?

- This problem is equivalent to show that $M_P : \mathcal{E}(\mathbb{R}^d) \to \mathcal{E}(\mathbb{R}^d)$, $f \mapsto Pf$ has closed range for every polynomial P
- Schwartz observed that the problem was equivalent to the analogous one for tempered distributions: Given $S \in S'(\mathbb{R}^d)$ and P a polynomial, is there $T \in S'(\mathbb{R}^d)$ such that $S = PT$?

Laurent Schwartz



Positive solution for the problem of division

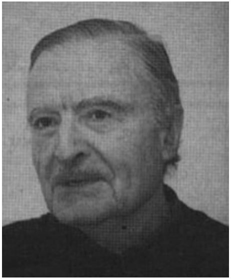
Solution

1. S. Lojasiewicz solved Schwartz's problem in 1958. He proved that $M_F : \mathcal{E}(\mathbb{R}^d) \to \mathcal{E}(\mathbb{R}^d)$ has closed range for each real analytic function F.
2. L. Hörmander (also in 1958) proved independently that $M_P : S'(\mathbb{R}^d) \to S'(\mathbb{R}^d)$, $T \mapsto PT$ is surjective.
   In particular, he showed that every linear partial differential operator with constant coefficients has a tempered fundamental solution.

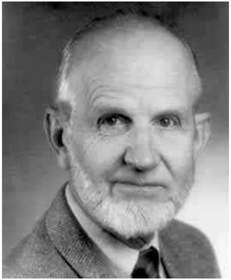
Our aim

Divide tempered distributions by functions which are not polynomials or even not real analytic.

Stalislaw Lojasiewicz



Lars Hörmander



Definitions and notation. A crash course in distribution theory.

A crash course is a rapid and intense course of training or research (usually undertaken in an **emergency**).

- $\mathcal{E}(\mathbb{R}^d)$ is the space of $\mathbb{C}$-valued C^∞ functions on $\mathbb{R}^d$.
- It is endowed with the topology of uniform convergence on compact sets of all the derivatives. It is a Fréchet space, i.e. a complete metrizable locally convex space.
- $\mathcal{D}(K)$ subspace of functions with support contained in K.
- $\mathcal{D}(\mathbb{R}^d)$ is the subspace of $\mathcal{E}(\mathbb{R}^d)$ of functions with compact support
- $\mathcal{D}'(\mathbb{R}^d)$ is the space of distributions.

Definition

A distribution $u \in \mathcal{D}'(\mathbb{R}^d)$ is a linear form $u : \mathcal{D}(\mathbb{R}^d) \to \mathbb{C}$ such that

$$\forall K \subset\subset \mathbb{R}^d \ \exists n \in \mathbb{N} \cup \{0\} \ \exists C > 0 \ \forall \varphi \in \mathcal{D}(K) :$$

$$|u(\varphi)| \le C \sup_{0 \le |\alpha| \le n} \sup_{x \in K} |\varphi^{(\alpha)}(x)|$$

Examples.

- Every $F \in L^1_{loc}(\mathbb{R}^d)$, defines a distribution via integration.
- The linear map $\delta : \mathcal{D}(\mathbb{R}^d) \to \mathbb{C}, \delta(\varphi) := \varphi(0)$, satisfies $\delta \in \mathcal{D}'(\mathbb{R}^d)$.
- (Differentiation) $u \in \mathcal{D}'(\mathbb{R})$, $u^{(k)}(\varphi) := (-1)^k u(\varphi^{(k)}), k \in \mathbb{N}$.
- $H'(\varphi) = -H(\varphi') = -\int_{\mathbb{R}} H(x)\varphi'(x)dx = -\int_0^\infty \varphi'(x)dx = \varphi(0) = \delta(\varphi)$

Multiplication of functions by distributions

Multiplication

If $f \in \mathcal{E}(\mathbb{R}^d)$ and $T \in \mathcal{D}'(\mathbb{R}^d)$, we define $fT \in \mathcal{D}'(\mathbb{R}^d)$ by

$$\langle fT, \varphi \rangle = \langle T, f\varphi \rangle,$$

for $\varphi \in \mathcal{D}(\mathbb{R}^d)$.

Examples.

- $(g\delta) = g(0)\delta$, $x\delta = 0$.
- $f(x) := \log|x| \in L^1_{loc}(\mathbb{R})$. We have $(u_f)' = (\log|x|)' = \mathrm{vp}\frac{1}{x}$. Here $\mathrm{vp}\frac{1}{x}(\varphi) := \lim_{\varepsilon \to 0} \int_{|x|>\varepsilon} \frac{\varphi(x)}{x} dx$.
- $x\mathrm{vp}\frac{1}{x} = 1$.
- $0 = (\delta x)\mathrm{vp}\frac{1}{x} \neq \delta(x\mathrm{vp}\frac{1}{x}) = \delta$.
- $x\delta' = -\delta$.

Schwartz's results on multiplication by $F \in \mathcal{E}(\mathbb{R})$

For each $F \in \mathcal{E}(\mathbb{R}^d)$, the multiplication operator $M_F : \mathcal{E}(\mathbb{R}^d) \to \mathcal{E}(\mathbb{R}^d)$ is continuous. In general it is neither injective nor surjective.

The elements T of the dual of $\mathcal{E}(\mathbb{R}^d)$ are the distributions $T \in \mathcal{E}'(\mathbb{R}^d)$ with compact support. The transpose of $M_F : \mathcal{E}(\mathbb{R}^d) \to \mathcal{E}(\mathbb{R}^d)$ is the multiplication of distributions with compact support by F.

Theorem. Schwartz.

The linear map $M_x : \mathcal{D}'(\mathbb{R}) \to \mathcal{D}'(\mathbb{R})$, $T \to xT$, is surjective and its kernel coincides with the linear span of the Dirac's measure δ.

Behind the scenes: A function $f \in \mathcal{E}(\mathbb{R})$ can be written as $f = xg, g \in \mathcal{E}(\mathbb{R})$, if and only if $f(0) = 0$.

Theorem. Schwartz.

The following conditions are equivalent for $F \in \mathcal{E}(\mathbb{R})$:

(a) For each $S \in \mathcal{D}'(\mathbb{R})$ there is $T \in \mathcal{D}'(\mathbb{R})$ such that $FT = S$.

(b) For each $S \in \mathcal{E}'(\mathbb{R})$ there is $T \in \mathcal{E}'(\mathbb{R})$ such that $FT = S$.

(c) $M_F : \mathcal{E}(\mathbb{R}) \to \mathcal{E}(\mathbb{R})$ is injective and has closed range.

(d) If $(f_k)_k \subset \mathcal{E}(\mathbb{R})$ satisfies that $(Ff_k)_k$ tends to 0 in $\mathcal{E}(\mathbb{R})$, then $(f_k)_k$ tends to 0 in $\mathcal{E}(\mathbb{R})$.

(e) F has only isolated zeros of finite order.

Schwartz's results on multiplication by $F \in \mathcal{E}(\mathbb{R})$

Comments on the Proof of Schwartz's Theorem

1. (a) $\Leftrightarrow$ (b) follows from a partition of the unity argument.
2. (b) $\Leftrightarrow$ (c) is a consequence of the closed range theorem for Fréchet spaces, and (c) $\Leftrightarrow$ (d) is a consequence of the open mapping theorem.
3. The following result due to Whitney and conjectured by Schwartz was important in this connection.

Theorem. Whitney.

Let I be an ideal in $\mathcal{E}(\mathbb{R}^d)$. The closure of I coincides with the set of all functions $f \in \mathcal{E}(\mathbb{R}^d)$ such that for each $a \in \mathbb{R}^d$ there is $g \in I$ such that $f^{(\alpha)}(a) = g^{(\alpha)}(a)$ for each multi-index α.

The space S of Schwartz.

Definition.

- For $f \in \mathcal{E}(\mathbb{R}^d)$ we define

$$\|f\|_s := \max_{|\alpha| \leq s} \sup_{x \in \mathbb{R}^d} (1 + |x|^2)^s |f^{(\alpha)}(x)|, \quad s \in \mathbb{N} \cup \{0\}.$$

- $S(\mathbb{R}^d) := \{f \in \mathcal{E}(\mathbb{R}^d) : \|f\|_s < \infty \, \forall s \in \mathbb{N}\}$.
  It is a Fréchet space.

- **Example:** $f(x) = e^{-x^2/2} \in S(\mathbb{R})$.
- Its topological dual $S'(\mathbb{R}^d)$ is the **space of tempered distributions.**

- **Relevance:** The Fourier transform $\mathcal{F}$ is a continuous linear bijection on $S(\mathbb{R}^d)$ with continuous inverse. If $f \in S(\mathbb{R}^d)$, then $\hat{f} \in S(\mathbb{R}^d)$ and f can be recovered from $\hat{f}$ by the inversion formula. Moreover the Fourier transform can be defined for tempered distributions.

The space $O_M(\mathbb{R}^d)$ of multipliers of $S(\mathbb{R}^d)$.

Question: What are the functions $F \in \mathcal{E}(\mathbb{R}^d)$ such that the multiplication operator M_F maps $S(\mathbb{R}^d)$ into $S(\mathbb{R}^d)$?

$$O_M(\mathbb{R}^d) := \{f \in \mathcal{E}(\mathbb{R}^d) : \forall \alpha \in \ \exists C_\alpha > 0, j \in \mathbb{N} : \ |f^{(\alpha)}(x)| \leq C_\alpha (1+|x|^2)^j\}.$$

Example: $f(x) = \exp(i\pi x^2), x \in \mathbb{R}$, belongs to $O_M(\mathbb{R})$.

Theorem. Schwartz.

The following conditions are equivalent for $F \in \mathcal{E}(\mathbb{R}^d)$:

1. $F \in O_M(\mathbb{R}^d)$.
2. $Ff \in S(\mathbb{R}^d)$ for each $f \in S(\mathbb{R}^d)$.
3. $FT \in S'(\mathbb{R}^d)$ for each $T \in S'(\mathbb{R}^d)$.

In this case, the multiplication operator $M_F : S(\mathbb{R}^d) \to S(\mathbb{R}^d)$ is continuous.

Statement of the problem of division for tempered distributions.

Problem.

Characterize those $F \in O_M(\mathbb{R}^d), F \neq 0$, such that for each $v \in S'(\mathbb{R}^d)$ there is $u \in S'(\mathbb{R}^d)$ such that $Fu = v$.

Two preliminary observations.

- $F \in O_M(\mathbb{R}^d), F \neq 0$, satisfies that the multiplication operator $u \to Fu$ is surjective on $S'(\mathbb{R}^d)$ if and only if the operator $M_F : S(\mathbb{R}^d) \to S(\mathbb{R}^d)$ has closed range. An argument that requires Whitney's theorem on the closure of principal ideals is needed to show that M_F is injective if it has closed range.
- If $F \in O_M(\mathbb{R}^d), F \neq 0$, satisfies that the multiplication operator $M_F : S(\mathbb{R}^d) \to S(\mathbb{R}^d)$ has closed range, then there is $m \in \mathbb{N}$ such that every zero of F has order less or equal than m. In particular, if $d = 1$, Rolle's theorem implies that F has only isolated zeros.

Our main result. Division of tempered distributions of one variable.

Set $I_{x,T} := [x - 1/(1+|x|^2)^T, x + 1/(1+|x|^2)^T]$, $T \in \mathbb{N}$ and $x \in \mathbb{R}$.
Denote by $Z(F)$ the set of zeros of F.

Theorem. Bonet, Frerick, Jordá (JFA, 2012)

For a multiplier $F \in O_M(\mathbb{R}), F \neq 0$, the following are equivalent:

(1) $M_F : S(\mathbb{R}) \to S(\mathbb{R})$ has closed range.

(2) The range of $M_F : S(\mathbb{R}) \to S(\mathbb{R})$ is complemented in $S(\mathbb{R})$.

(3) There exist $N \in \mathbb{N}$ and $T, c > 0$ such that F satisfies the following two conditions for each $x \in \mathbb{R}$:

  (a) The cardinality of the set $Z_F \cap I_{x,T}$, the zeros counted with their multiplicities, is smaller than N.

  (b) $(1+|x|^2)^T |F(x)| \geq c \prod_{i=1}^k |x - x_i|$, $(x_i)_{i=1}^k$ being the zeros of F in $I_{x,T}$ counting multiplicities.

A consequence.

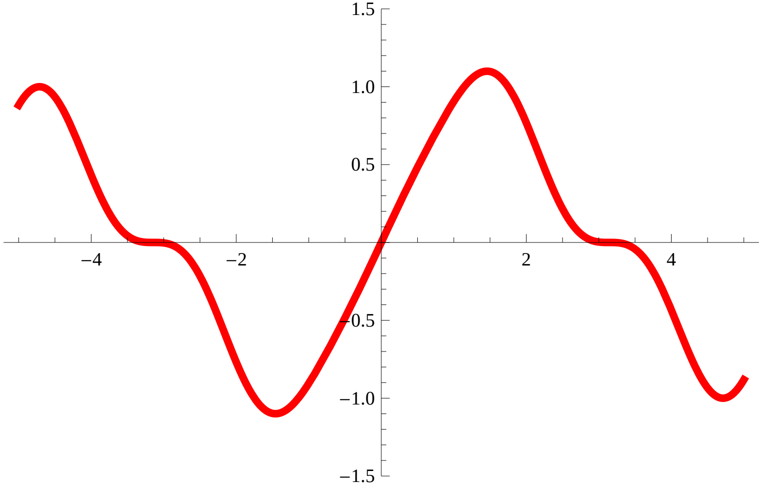
The following consequence of the main Theorem is useful to decide whether a multiplier F satisfies that M_F has closed range in $S(\mathbb{R})$.

Corollary

If $F \in O_M(\mathbb{R})$ satisfies that $M_F : S(\mathbb{R}) \to S(\mathbb{R})$ has closed range and there is $t > 0$ such that $|x - y| > t$ for each $x, y \in Z_F$, $x \neq y$, then there exist $C, T > 0$ such that $(1 + x^2)^T F^{(o(x))}(x) \geq C$ for each $x \in Z_F$. Here $o(x)$ is the order of a zero x of F, and we understand $o(x) = 0$ if x is not a zero of F.

Example. The function $F(x) := \sin^3 x + e^{-x^2} \sin x$ belongs to $O_M(\mathbb{R})$ and has all the zeros of order one. The corollary implies that $M_F : S(\mathbb{R}) \to S(\mathbb{R})$ does not have closed range, since $|F'(n\pi)| = e^{-(n\pi)^2}$ for each $n \in \mathbb{N}$.

Gráfica de $F(x) := \sin^3 x + e^{-x^2} \sin x$



Hörmander sufficient conditions (1958)

Given $f \in \mathcal{E}(\mathbb{R}^d)$, denote $N_0 = \mathbb{R}^d$,
$N_k = \{x \in \mathbb{R}^d : f^{(\alpha)}(x) = 0 \text{ if } |\alpha| < k\}$. (zeros of order at least k).
Clearly $N_{k+1} \subset N_k$.

Hörmander sufficient conditions

$M_F : S(\mathbb{R}^d) \to S(\mathbb{R}^d)$ has closed range if F satisfies the following conditions:

(i) There exits k such that $N_k = \emptyset$.

(ii) For each $j \in \mathbb{N}$, $0 \leq l \leq k-1$ there exist constants c, μ_1 and μ_2 such that

$$\sum_{|\beta|=jl} |F^j(x)^{(\beta)}| \geq c\frac{d(x, N_{l+1})^{\mu_1}}{(1+|x|^2)^{\mu_2}},$$

with $d(x, \emptyset) = 1$.

Hörmander proved that condition (ii) always holds for polynomials.

A relevant example.

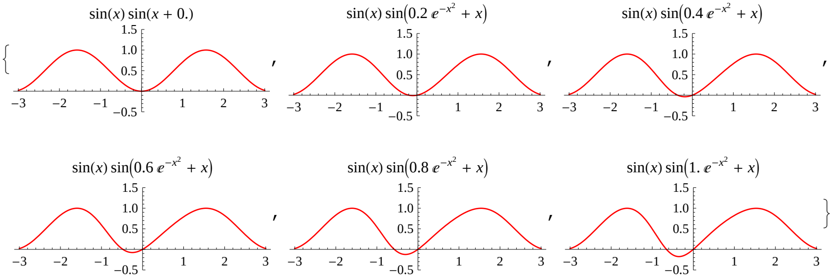
Example

If $\gamma > 0$ is small enough to ensure that $x \to x + \gamma e^{-x^2}$ a diffeomorphism, then $F(x) = \sin(x)\sin(x + \gamma e^{-x^2})$ satisfies (i) but not (ii) of Hörmander, but $M_F : S(\mathbb{R}) \to S(\mathbb{R})$ has closed range, as follows from our Theorem.

Consequence

Condition (ii) of Hörmander for F is not necessary for $M_F : S(\mathbb{R}) \to S(\mathbb{R})$ to have closed range.

Gráfica de $F(x) = \sin(x)\sin(x + 20e^{-x^2})$



Complemented range. Langenbruch's Result.

Remark

Langenbruch had proved in 1990 that condition (1) in our Theorem does not imply (2) for multiplication operators $M_F : S(\mathbb{R}^d) \to S(\mathbb{R}^d)$ in the case of several variables. He characterized those polynomials P of d variables such that $M_P : S(\mathbb{R}^d) \to S(\mathbb{R}^d)$ has complemented image, or equivalently when the division $S \to S/P$ can be obtained by means of a continuous linear operator.

Examples

1. $P(x_1, x_2, x_3) = x_3(x_1^2 + x_2^2) - x_1^3$ $(d = 3)$ and $P(x_1, x_2) = x_1^2 - x_2^2 + ix_1$ $(d = 2)$. **NO**.
2. $P(x_1, x_2) = x_1^2 - x_2^2$ $(d = 2)$ and $P(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 - x_3^2 - x_4^2$ $(d = 4)$. **YES**.

Let $F \in \mathcal{E}(\mathbb{R})$. The multiplication operator $M_F : \mathcal{E}(\mathbb{R}) \to \mathcal{E}(\mathbb{R})$ has closed range if and only if the range is complemented in $\mathcal{E}(\mathbb{R})$.

This follows from Whitney's and Schwartz theorems and a partition of the unity argument.

The result for $S(\mathbb{R})$ does not follow from this result. The range of the projection given by Schwartz result need not lie in $S(\mathbb{R})$.

The case of several variables $M_F : \mathcal{E}(\mathbb{R}^d) \to \mathcal{E}(\mathbb{R}^d)$ is much more difficult. It is related to the study of real analytic manifolds. Very deep work was done by E. Bierstone, P. Milman and G. Schwarz.

A useful tool in our proof. Hermite's interpolation.

Divided differences

Let f be a C^∞ function defined on an interval I in $\mathbb{R}$. If $x \neq y$, the divided difference $f[x,y]$ of f is defined as $\frac{f(x)-f(y)}{x-y}$ and $f[x,x] = f'(x)$.

Inductively we define

$$f[x,\ldots,x] = \frac{f^{(n)}(x)}{n!}$$

and

$$f[x_1,\ldots,x_n] = \frac{f[x_2,\ldots x_n] - f[x_1,\ldots x_{n-1}]}{x_n - x_1}.$$

A useful tool in our proof. Hermite's interpolation.

Hermite problem for interpolation

Given a C^∞ function f on an interval I in $\mathbb{R}$ and $\{x_1, \ldots, x_n\} \subset I$, the polinomial interpolating f and its derivatives in case some points are repeated, is the following

$$P(x) = f(x_1) + f[x_1, x_2](x - x_1) + \cdots f[x_1, \ldots, x_n](x - x_1) \cdots (x - x_{n-1}).$$

This interpolation combines Lagrange's polynomial in case the points are different and Taylor's polynomial if we consider only one point several times.

The Hermite polynomial satisfies

Remainder Formula

For each $x \in I$ there exists $y \in I$ such that
$f(x) - P(x) = \frac{f^{(n)}(y)}{n!}(x - x_1) \cdots (x - x_n)$

Consequence of the Remainder Formula

Let f be a C^∞ function defined on an interval I in $\mathbb{R}$. If $Z_f \cap I = \{x_1, \ldots, x_n\}$ are the zeros of f counted with the multiplicities, then for each $x \in I$ there exists $y \in I$ such that
$f(x) = (x - x_1) \cdots (x - x_n)\frac{f^{(n)}(y)}{n!}$

Statement of the main Theorem repeated

Set $I_{x,T} := [x - 1/(1+|x|^2)^T, x + 1/(1+|x|^2)^T]$, $T \in \mathbb{N}$ and $x \in \mathbb{R}$. Denote by $Z(F)$ the set of zeros of F.

Theorem. Bonet, Frerick, Jordá (JFA, 2012)

For a multiplier $F \in O_M(\mathbb{R}), F \neq 0$, the following are equivalent:

(1) $M_F : S(\mathbb{R}) \to S(\mathbb{R})$ has closed range.

(2) The range of $M_F : S(\mathbb{R}) \to S(\mathbb{R})$ is complemented in $S(\mathbb{R})$.

(3) There exist $N \in \mathbb{N}$ and $T, c > 0$ such that F satisfies the following two conditions for each $x \in \mathbb{R}$:

  (a) The cardinality of the set $Z_F \cap I_{x,T}$, the zeros counted with their multiplicities, is smaller than N.

  (b) $(1+|x|^2)^T |F(x)| \geq c \prod_{i=1}^k |x - x_i|$, $(x_i)_{i=1}^k$ being the zeros of F in $I_{x,T}$ counting multiplicities.

Comments on the Proof of the Theorem

- If M_F has closed range, by the open mapping theorem there are $s \in \mathbb{N}$ and $C > 0$ be such that

$$\left\| \frac{f}{F} \right\|_0 \leq C \|f\|_s$$

for every $f \in rg(M_F)$ (with the corresponding extensions at Z_F).

- Condition (3) (a) is proved by contradiction using Hermite's interpolation and Markov's inequality for polynomials. This is the most difficult part of the proof.
The proof of (b) uses (a) and a direct estimate with certain cut-off functions.
We mention some details of the proof of (a).

A few ideas about the proof of the main Theorem

Assume that (3) (a) does not hold.
We can find s and

- a sequence (y_n) with $|y_n| \to \infty$,
- a sequence $(\varepsilon_n)_{n=1}^{\infty}$ such that $(1 + |y_n|)^T \varepsilon_n$ tends to zero for each $T > 0$.
- If we denote $I_n := [y_n - \varepsilon_n, y_n + \varepsilon_n]$, then

$$k_n := \sum_{x \in I_n} o_F(x) > s.$$

Moreover, shrinking the intervals I_n if necessary, we can assume the sequence $(k_n)_n$ bounded.

Consider $Z_F \cap I_n = \{x_1, \ldots, x_{k_n}\}$, the zeros counted with their multiplicities, and set

$$g_n(x) := \prod_{1 \leq i \leq k_n} (x - x_i).$$

By the remainder formula for the Hermite interpolation, for some $C > 0$ and $l > 0$,

$$\sup_{y \in I_n} \left| \frac{F(y)}{g_n(y)} \right| \leq C(1 + |y_n|^2)^l.$$

A few ideas about the proof of the main Theorem

Multiplying by an appropriate cut-off function ϕ_n, the function $h_n := g_n \phi_n$ satisfies

$h_n/F \in \mathcal{D}(\mathbb{R}) \subset S(\mathbb{R})$, and h_n is in the range of M_F.

On $J_n :=]y_n - \varepsilon_n/2, y_n + \varepsilon_n/2[$, in which $h_n = g_n$, we have

$$\inf_{y \in J_n} \left| \frac{h_n(y)}{F(y)} \right| \geq \frac{1}{C(1 + |y_n|^2)^l}.$$

A few ideas about the proof of the main Theorem

Applying Markov's inequality and using that h_n is in the range of M_F, we get

$$\left\| \frac{h_n}{F} \right\|_0 \leq C(1 + |y_n|^2)^s \varepsilon_n.$$

Evaluating at $y \in J_n$, we conclude

$$\frac{1}{C^2} \leq (1 + |y_n|^2)^{s+l} \varepsilon_n,$$

a contradiction.

A few ideas about the proof of the main Theorem

Comments on the Proof of the Theorem

- Conversely, if F satisfies condition (3), we first prove, using Hermite's interpolation, the remainder formula and conditions (a) and (b) that the estimate

$$\left\| \frac{f}{F} \right\|_0 \leq C \|f\|_s$$

holds. Applying this fact inductively to F^{2^n} we prove (1).

- To prove that (1) implies (2) a partition of unity and a diffeomorphism on $\mathbb{R}$ are constructed to exhibit a continuous projection form $S(\mathbb{R})$ onto the range of M_F.

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