CONVERGENCE OF ARITHMETIC MEANS OF OPERATORS IN FRÉCHET SPACES

ANGELA A. ALBANESE, JOSÉ BONET* AND WERNER J. RICKER

Authors' addresses:

Angela A. Albanese: Dipartimento di Matematica e Fisica "E. De Giorgi", Università del Salento- C.P.193, I-73100 Lecce, Italy
email: angela.albanese@unisalento.it

José Bonet (corresponding author): Instituto Universitario de Matemática Pura y Aplicada IUMPA, Universitat Politècnica de València, E-46071 Valencia, Spain
email:jbonet@mat.upv.es
telephone number: +34963879497.
fax number: +34963879494.

Werner J. Ricker: Math.-Geogr. Fakultät, Katholische Universität Eichstätt-Ingolstadt, D-85072 Eichstätt, Germany
email:werner.ricker@ku-eichstaett.de

Abstract. Every Köthe echelon Fréchet space X that is Montel and not isomorphic to a countable product of copies of the scalar field admits a power bounded continuous linear operator T such that $I - T$ does not have closed range, but the sequence of arithmetic means of the iterates of T converge to 0 uniformly on the bounded sets in X. On the other hand, if X is a Fréchet space which does not have a quotient isomorphic to a nuclear Köthe echelon space with a continuous norm, then the sequence of arithmetic means of the iterates of any continuous linear operator T (for which $(1/n)T^n$ converges to 0 on the bounded sets) converges uniformly on the bounded subsets of X, i.e., T is uniformly mean ergodic, if and only if the range of $I - T$ is closed. This result extends a theorem due to Lin for such operators on Banach spaces. The connection of Browder'equality for power bounded operators on Fréchet spaces to their uniform mean ergodicity is exposed. An analysis of the mean ergodic properties of the classical Cesàro operator on Banach sequence spaces is also given.

Key words and phrases. Fréchet space, Köthe echelon space, quojection, prequojection, power bounded operator, uniformly mean ergodic operator.
Mathematics Subject Classification 2010: Primary 46A04, 47A35, 47B37; Secondary 46A11.
* Corresponding author.

1. Introduction

The purpose of this paper is to investigate the behaviour of the sequence of arithmetic means $T_{[n]} := \frac{1}{n}\sum_{m=1}^{n} T^m, n \in \mathbb{N}$, of the iterates $T^m := T \circ ... \circ T$ of a continuous linear operator $T \in \mathcal{L}(X)$ on a Fréchet space X. For undefined terminology about Fréchet spaces we refer to [9], [30], for example. A useful result of Lin [26] (see also [24]) asserts that the following conditions are equivalent for an operator T (on a Banach space X) which satisfies $\lim_{n\to\infty} ||T^n/n|| = 0$.

(1) *T is uniformly mean ergodic, i.e., there is $P \in \mathcal{L}(X)$ with* $\lim_{n\to\infty} \|T_{[n]} - P\| = 0$.
(2) *The range $(I-T)(X)$ is closed and $X = \mathrm{Ker}(I-T) \oplus (I-T)(X)$.*
(3) *$(I-T)^2(X)$ is closed.*
(4) *$(I-T)(X)$ is closed.*

It was observed in Example 2.17 of [3] that there exist power bounded, uniformly mean ergodic operators T on the Fréchet space s of rapidly decreasing sequences for which $(I-T)(X)$ is not closed. On the other hand, Theorem 4.1 of [4] provides an extension of Lin's result to those Fréchet spaces X which are quotients of countable products of Banach spaces (the so called quojections), under the additional assumption that $\mathrm{Ker}(I-T) = \{0\}$. In the present paper we undertake a careful analysis of the possible extension of Lin's result to the setting of Fréchet spaces. First, we show in Proposition 3.1 that every Montel Köthe echelon space $\lambda_p(A)$ of order $p \in [1,\infty) \cup \{0\}$ not isomorphic to a countable product of copies of the scalar field admits an operator $T \in \mathcal{L}(\lambda_p(A))$ which is power bounded and uniformly mean ergodic, but such that $I - T$ is not surjective and has dense range. The same result also holds if $\lambda_p(A)$ is non-normable, admits a continuous norm and satisfies the density condition; see Proposition 3.3. In contrast to these results, we prove in Theorem 3.5 that the conditions (1)–(4) above are equivalent for operators T defined on a Fréchet space X which does not have a separated quotient which is isomorphic to a nuclear Köthe echelon space with a continuous norm. These spaces, called prequojections, are precisely those Fréchet spaces whose strong bidual is a quojection. The proof of Theorem 3.5 is first established for quojections and then extended to prequojections. As a concrete example we investigate in Section 4 the mean ergodic properties of the classical Cesàro operator

$$C(x) = \left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)_n, \quad x = (x_n)_n \in \mathbb{C}^{\mathbb{N}},$$

in the quojection Fréchet space $\mathbb{C}^{\mathbb{N}}$ of all sequences, as well as in the Banach sequence spaces c_0, c, ℓ^p $(1 < p \leq \infty)$, bv_0 and bv_p $(1 \leq p < \infty)$. Finally, in Section 5, inspired by results in [18], [19], we investigate when the identity

$$\left\{x \in X : \left\{\sum_{k=1}^{n} T^k x\right\}_{n=1}^{\infty} \text{ is a bounded sequence in } X\right\} = (I-T)(X),$$

called Browder's equality, holds for a power bounded operator $T \in \mathcal{L}(X)$ in a locally convex space X (briefly, lcHs). The main results of Section 5 are Proposition 5.6 and Theorem 5.11 which establish the connection of Browder's equality to uniform mean ergodicity.

2. Preliminaries

Our notation for lcHs' is standard; we refer to [22, 23, 30]. The traditional reference for mean ergodic operators is [24]. More detailed information on Fréchet and Köthe echelon spaces can be found in [9], [30]. We include a few definitions and some notation below to facilitate the reading of the paper. Let X be a lcHs and Γ_X a system of continuous seminorms determining the topology of X. The strong operator topology τ_s in the space $\mathcal{L}(X)$ of all continuous linear operators from X into itself (from X into another lcHs Y we write $\mathcal{L}(X,Y)$) is determined by the family of seminorms

$$q_x(S) := q(Sx), \quad S \in \mathcal{L}(X),$$

for each $x \in X$ and $q \in \Gamma_X$, in which case we write $\mathcal{L}_s(X)$. Denote by $\mathcal{B}(X)$ the collection of all bounded subsets of X. The topology τ_b of uniform convergence on bounded sets is defined in $\mathcal{L}(X)$ via the seminorms

$$q_B(S) := \sup_{x\in B} q(Sx), \quad S \in \mathcal{L}(X),$$

for each $B \in \mathcal{B}(X)$ and $q \in \Gamma_X$; in this case we write $\mathcal{L}_b(X)$. For X a Banach space, τ_b is the operator norm topology in $\mathcal{L}(X)$. If X is metrizable and complete, then X is called a Fréchet space. In this case Γ_X can be taken countable. The identity operator on a lcHs X is denoted by I.

By X_σ we denote X equipped with its weak topology $\sigma(X,X')$, where X' is the topological dual space of X. The strong topology in X (resp. X') is denoted by $\beta(X,X')$ (resp. $\beta(X',X)$) and we write X_β (resp. X'_β); see [22, §21.2] for the definition. The strong dual space $(X'_\beta)'_\beta$ of X'_β is denoted simply by X''. By X'_σ we denote X' equipped with its weak–star topology $\sigma(X',X)$. Given $T \in \mathcal{L}(X)$, its *dual operator* $T' \colon X' \to X'$ is defined by $\langle x, T'x'\rangle = \langle Tx, x'\rangle$ for all $x \in X$, $x' \in X'$. It is known that $T' \in \mathcal{L}(X'_\sigma)$ and $T' \in \mathcal{L}(X'_\beta)$, [23, p.134].

Proposition 2.1. *Let X be a Fréchet space, X'' be its strong bidual and $T \in \mathcal{L}(X)$, in which case $T'' \in \mathcal{L}(X'')$. Then $T''(X'')$ is a closed subspace of the Fréchet space X'' if and only if $T(X)$ is a closed subspace of X.*

Proof. Suppose that $T''(X'')$ is closed in X''. Then $T''(X'')$ is also $\sigma(X'',X''')$–closed with T' continuous from $(X',\sigma(X',X'')$ into itself. Hence, T' is a weak homomorphism, [23, §32.3(2)], and so $T \in \mathcal{L}(X)$ is also a homomorphism, [23, §33.4(2), (d)⇒(a)], which implies that $T(X)$ is closed in X, [23, §33.4(2), (a)⇒(c)].

Conversely, suppose that $T(X)$ is closed in X, in which case T' is a weak homomorphism of $(X',\sigma(X',X'')$ into itself, [23, §33.4(2), (c)⇒(d)]. Accordingly, $T''(X'')$ is $\sigma(X'',X''')$–closed in X'', [23, §32.3(2)], and hence, also closed in X''. □

Let X be a lcHs and $T \in \mathcal{L}(X)$, in which case we define $T_{[0]} := I$ and

$$T_{[n]} := \frac{1}{n}\sum_{m=1}^{n} T^m, \quad n \in \mathbb{N}. \tag{2.1}$$

Then we have the identities

$$(I-T)T_{[n]} = T_{[n]}(I-T) = \frac{1}{n}(T - T^{n+1}), \quad n \in \mathbb{N}, \tag{2.2}$$

and

$$\frac{1}{n} \cdot T^n = T_{[n]} - \frac{(n-1)}{n} T_{[n-1]}, \quad n \in \mathbb{N}. \tag{2.3}$$

If $T_{[n]} \to P$ in $\mathcal{L}_s(X)$ as $n \to \infty$, so that $\tau_s\text{-}\lim_{n\to\infty} \frac{T^n}{n} = 0$, then P is a projection satisfying $TP = PT = T$ with $\operatorname{Ker} P = \overline{(I-T)(X)}$ and $P(X) = \operatorname{Ker}(I-T)$. Moreover,

$$X = \operatorname{Ker}(I-T) \oplus \overline{(I-T)(X)}, \tag{2.4}$$

[3], [40, Chap.VIII, §3, p.213]. Of course, for $S \in \mathcal{L}(X)$ we define $\operatorname{Ker} S := S^{-1}(\{0\})$. An operator $T \in \mathcal{L}(X)$ is called *power bounded* if the sequence $\{T^n\}_{n=1}^\infty$ of the iterates of T is equicontinuous in $\mathcal{L}(X)$. The operator $T \in \mathcal{L}(X)$ is said to be *mean ergodic* (resp. *uniformly mean ergodic*) if the sequence $\{T_{[n]}\}_{n=1}^\infty$ is convergent in $\mathcal{L}_s(X)$ (resp. in $\mathcal{L}_b(X)$); see [3], [24] for more details.

Proposition 2.2. *Let X be a lcHs and $T \in \mathcal{L}(X)$ be mean ergodic (resp. uniformly mean ergodic). Suppose that Y is a closed subspace of X which is T–invariant. Then the restriction $S := T|_Y \in \mathcal{L}(Y)$ is also mean ergodic (resp. uniformly mean ergodic).*

Proof. Suppose T is mean ergodic. Then there is $P \in \mathcal{L}(X)$ such that $T_{[n]} \to P$ in $\mathcal{L}_s(X)$ as $n \to \infty$. Clearly, Y is T^n–invariant, for $n \in \mathbb{N}$, the operator $S_{[n]} = T_{[n]}|_Y$, and Y is also P–invariant. Since Y is closed, it follows that $S_{[n]} \to P|_Y$ in $\mathcal{L}_s(Y)$ as $n \to \infty$ with $P|_Y \in \mathcal{L}_s(Y)$. Hence, S is mean ergodic.

The proof for T uniformly mean ergodic is similar. □

3. Arithmetic means of operators in certain classes of Fréchet spaces

We first investigate operators on Köthe echelon spaces. A sequence $A = (a_n)_n$ of functions $a_n : \mathbb{N} \to [0, \infty)$ is called a *Köthe matrix* on $\mathbb{N}$, if $0 \le a_n(i) \le a_{n+1}(i)$ for all $i,\ n \in \mathbb{N}$, and for each $i \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that $a_n(i) > 0$. To each $p \in [1, \infty)$ we associate the linear space

$$\lambda_p(A) := \Big\{ x \in \mathbb{C}^{\mathbb{N}} : q_n^{(p)}(x) := \Big(\sum_{i\in\mathbb{N}} |a_n(i) x_i|^p \Big)^{1/p} < \infty, \quad \forall n \in \mathbb{N} \Big\}. \tag{3.1}$$

We also require the linear space

$$\lambda_\infty(A) := \{ x \in \mathbb{C}^{\mathbb{N}} : q_n^{(\infty)}(x) := \sup_{i\in\mathbb{N}} a_n(i)|x_i| < \infty,\ \forall n \in \mathbb{N} \} \tag{3.2}$$

and its closed subspace (equipped with the relative topology)

$$\lambda_0(A) := \{ x \in \mathbb{C}^{\mathbb{N}} : \lim_{i\to\infty} a_n(i) x_i = 0, \qquad \forall n \in \mathbb{N} \}.$$

Elements $x \in \mathbb{C}^{\mathbb{N}}$ are denoted by $x = (x_i)_i$. The spaces $\lambda_p(A)$, for $p \in [1, \infty]$, are called *Köthe echelon spaces* (of order p); they are all Fréchet spaces (separable if $p \neq \infty$ and reflexive if $p \neq 0, 1, \infty$) relative to the increasing sequence of seminorms $q_1^{(p)} \le q_2^{(p)} \le \ldots$. For the theory of such spaces we refer to [9], [10], [22], [30], for example.

Proposition 3.1. *Let $p \in [1,\infty] \cup \{0\}$ and let A be a Köthe matrix on $\mathbb{N}$ such that $\lambda_p(A)$ is a Montel space with $\lambda_p(A) \neq \mathbb{C}^{\mathbb{N}}$. Then there is a power bounded, uniformly mean ergodic operator $T \in \mathcal{L}(\lambda_p(A))$ such that $I - T$ is not surjective and has dense range. In particular, $(I-T)(\lambda_p(A))$ is not closed.*

Proof. Since $\lambda_p(A) \subseteq \mathbb{C}^{\mathbb{N}}$ but $\lambda_p(A) \neq \mathbb{C}^{\mathbb{N}}$, it is routine to exhibit a sequence $t = (t_i)_i \in \mathbb{R}^{\mathbb{N}}$ with $t_i > 1$ for all $i \in \mathbb{N}$ and $y \in \lambda_p(A)$ such that $(t_i y_i)_i \notin \lambda_p(A)$ (see also [10, Proposition 1.8]). Set $d_i := 1 - \frac{1}{t_i}$ for all $i \in \mathbb{N}$, in which case $0 < d_i < 1$ for all $i \in \mathbb{N}$. With $d := (d_i)_i \in \mathbb{R}^{\mathbb{N}}$ define the operator $T_d \colon \lambda_p(A) \to \lambda_p(A)$ via $T_d((x_i)_i) = (d_i x_i)_i$ for $x = (x_i)_i \in \lambda_p(A)$. Then T_d is continuous and power bounded for all $p \in [1,\infty] \cup \{0\}$ as

$$q_n^{(p)}(T_d^m x) \leq q_n^{(p)}(x), \quad x \in \lambda_p(A),\, m,\, n \in \mathbb{N}.$$

Since $\lambda_p(A)$ is a Montel space, it follows that T_d is uniformly mean ergodic, [3, Proposition 2.8].

Clearly, $(I-T_d)(\lambda_p(A))$ is dense in $\lambda_p(A)$ (recall that if $\lambda_\infty(A)$ is Montel, then it coincides with $\lambda_0(A)$). If $(I-T_d)$ is surjective, there exists $x \in \lambda_p(A)$ satisfying $(I-T_d)x = y$, i.e. $x_i = t_i y_i$ for all $i \in \mathbb{N}$. This implies $x = (t_i y_i)_i \in \lambda_p(A)$, a contradiction. This shows that $(I-T_d)$ is not surjective and has dense range. Since $(I-T_d)$ is injective and $\lambda_p(A)$ is a Fréchet space, it follows that $(I-T_d)$ cannot have closed range.

□

We point out that Example 2.17 of [3] is a special case of Proposition 3.1, as every $\lambda_p(A)$ space which is Schwartz and admits a continuous norm is necessarily Montel and distinct from $\mathbb{C}^{\mathbb{N}}$.

Lemma 3.2. *Let X be a Fréchet space such that $X = X_0 \oplus X_1$ with X_0 topologically isomorphic to a Köthe Montel space $\lambda_p(A), p \in [1,\infty) \cup \{0\}$, such that $\lambda_p(A) \neq \mathbb{C}^{\mathbb{N}}$. Then X admits a power bounded uniformly mean ergodic operator T such that $I-T$ is not surjective and has dense range.*

Proof. By Proposition 3.1 there exists $S \in \mathcal{L}(X_0)$ which is power bounded and uniformly mean ergodic, $I-S$ is not surjective, and $I-S$ has dense range in X_0. Define $T \colon X \to X$ by $T(x_0 + x_1) := S(x_0)$ for all $x_0 \in X_0$ and $x_1 \in X_1$. Then $T \in \mathcal{L}(X)$ with T power bounded and uniformly mean ergodic. Since $I - S \colon X_0 \to X_0$ is not surjective, it follows that $I-T$ is not surjective. Finally, as $(I-T)(X) \supseteq (I-S)(X_0) \oplus X_1$ and $(I-S)(X_0)$ is dense in X_0, we conclude that $I-T$ has dense range in X. □

Recall that a Fréchet space X satisfies the *density condition* if the bounded subsets of X'_β are metrizable; see [8], [9] for this condition in Fréchet and Köthe echelon spaces.

Proposition 3.3. *Let $p \in [1,\infty) \cup \{0\}$ and $\lambda_p(A)$ be a Köthe echelon space which is non–normable, admits a continuous norm, and satisfies the density condition. Then there is $T \in \mathcal{L}(\lambda_p(A))$ which is power bounded and uniformly mean ergodic, but $I-T$ is not surjective and has dense range. In particular, $(I-T)(\lambda_p(A))$ is not closed.*

Proof. We first observe that $\lambda_p(A)$ has a continuous norm if and only if there is $m \in \mathbb{N}$ such that $a_m(i) > 0$ for all $i \in \mathbb{N}$. So, we may assume that $a_n(i) > 0$ for each $i, n \in \mathbb{N}$. Clearly, $\lambda_p(A) \neq \mathbb{C}^{\mathbb{N}}$. On the other hand, as $\lambda_p(A)$ is non–normable and satisfies the density condition, it follows from [12, Corollary 2.4] that there exists an infinite set $J \subseteq \mathbb{N}$ such that the sectional (hence, complemented) subspace $\lambda_p(J, A)$ is Schwartz, hence Montel. The result now follows from Lemma 3.2 and Proposition 3.1. □

We now consider a class of Fréchet spaces in which the extension of Lin's result, [26], does hold. Every Fréchet space X is a projective limit of continuous linear operators $S_k : X_{k+1} \to X_k$, for $k \in \mathbb{N}$, with each X_k a Banach space. If it is possible to choose X_k and S_k such that each S_k is surjective and X is isomorphic to $\operatorname{proj}_j(X_j, S_j)$, then X is called a *quojection*, [6, Section 5]. Quojections are characterized by the fact that every quotient with a continuous norm is a Banach space, [6, Proposition 3]. This implies that a quotient of a quojection is again a quojection. Banach spaces and countable products of Banach spaces are clearly quojections. Actually, every quojection is the quotient of a countable product of Banach spaces, [11]. In [33] Moscatelli gave the first examples of quojections which are not isomorphic to countable products of Banach spaces. Concrete examples of quojections are the sequence space $\omega = \mathbb{C}^{\mathbb{N}}$, the function spaces $L^p_{loc}(\Omega)$, with $1 \leq p \leq \infty$ and Ω an open subset of $\mathbb{R}^n$, and $C^{(m)}(\Omega)$, with $m \in \mathbb{N}_0$ and Ω an open subset of $\mathbb{R}^n$, when equipped with their canonical lc–topology. In fact, the above function spaces are isomorphic to countable products of Banach spaces. The spaces of continuous functions $C(X)$, with X a σ–compact completely regular topological space, endowed with the compact open topology, are also examples of quojections. Domański constructed a completely regular topological space X such that the Fréchet space $C(X)$ is a quojection which is not isomorphic to a complemented subspace of a product of Banach spaces [17, Theorem]. For further information on quojections we refer to the survey paper [31] and the references therein; see also [6], [16]. A *prequojection* is a Fréchet space X such that X'' is a quojection. Every quojection is a prequojection. A Fréchet space X is a prequojection if and only if X has no Köthe nuclear quotient which admits a continuous norm, [6, 15, 36, 38]. This implies that a quotient of a prequojection is again a prequojection. So, every complemented subspace of a prequojection is also a prequojection. The problem of the existence of prequojections which are not quojections arose in a natural way in [6] and was solved in [7], [15], [32], [34]; see the survey paper [31] for further information. An example of a space of continuous functions on a suitable topological space which is a non–quojection prequojection is given in [2].

The following lemma was suggested by the referee.

Lemma 3.4. *Let E and F be two quojection Fréchet spaces, $R \in \mathcal{L}(F, E)$ be surjective and $A \in \mathcal{L}(E)$. For given operators $\{R_n\}_{n=1}^{\infty} \subset \mathcal{L}(E)$ and, with $H_n := AR_n$, for $n \in \mathbb{N}$, suppose that the following conditions are satisfied:*

(i) $H_n = R_n A$ *for all* $n \in \mathbb{N}$.
(ii) $R = \tau_b - \lim_{n\to\infty} H_n R$ *in* $\mathcal{L}_b(F, E)$.

(iii) *There is a fundamental sequence of seminorms* $(||.||_j)_j$ *on* E *such that all the operators* $A, \{R_n\}_{n=1}^\infty, \{H_n\}_{n=1}^\infty$ *are continuous from the semi-normed space* $(E, ||.||_j)$ *into itself, for each* $j \in \mathbb{N}$.

Then A *is invertible, i.e.,* A *is bijective with* $A^{-1} \in \mathcal{L}(E)$.

Proof. We set $E_j := E/\operatorname{Ker}||.||_j$ endowed with the quotient topology. Since E is a quojection, E_j is a Banach space. We fix $j \in \mathbb{N}$ and let $\hat{A}_j, \hat{H}_{n,j}, \hat{R}_{n,j}$ and $\hat{I}_j$ denote the canonically induced operators in $\mathcal{L}(E_j)$. Since R is continuous and open, there is a seminorm $||.||$ on F such that $R(U) = U_j$, where $U = ||.||^{-1}([0,1])$ and $U_j = ||.||_j^{-1}([0,1])$. Since F is a quojection Fréchet space, we can apply [16, Proposition 1] to find a bounded subset B of F such that $\hat{B} = B + \operatorname{Ker}||.||$ is the unit ball of $F/\operatorname{Ker}||.||$. Therefore $\hat{B}_j = R(B) + \operatorname{Ker}||.||_j$ is the unit ball of E_j. Since $R = \tau_b - \lim_{n\to\infty} H_n R$ in $\mathcal{L}_b(F, E)$ (by condition (ii)), it follows that $\hat{H}_{n,j} \to \hat{I}_j$ as $n \to \infty$ in the norm of the Banach space E_j. Hence, there is $n \in \mathbb{N}$ such that $\hat{H}_{n,j}$ is invertible in $\mathcal{L}(E_j)$. By condition (i), $\hat{H}_{n,j} = \hat{A}_j \hat{R}_{n,j} = \hat{R}_{n,j} \hat{A}_j$, so that $\hat{A}_j$ is also invertible in $\mathcal{L}(E_j)$. Since j is arbitrary, we can conclude that A is invertible in $\mathcal{L}(E)$. Indeed, the surjectivity of A follows as in the last part of the proof of Theorem 4.1 in [4], and the injectivity is proved by a direct argument. □

Condition (iii) in Lemma 3.4 need not imply that $\{R_n\}_{n=1}^\infty$ and $\{H_n\}_{n=1}^\infty$ are equicontinuous sets in $\mathcal{L}(E)$.

The proof of the following result illustrates the versatility of Lemma 3.4 when applied in different situations and shows clearly why the "quojection property" is crucial.

Theorem 3.5. *Let* X *be a prequojection Fréchet space and* $T \in \mathcal{L}(X)$ *such that* $\tau_b\text{-}\lim_{n\to\infty} \frac{T^n}{n} = 0$. *The following conditions are equivalent.*

(1) T *is uniformly mean ergodic.*
(2) $(I-T)(X)$ *is closed and* $X = \operatorname{Ker}(I-T) \oplus (I-T)(X)$.
(3) $(I-T)^2(X)$ *is closed.*
(4) $(I-T)(X)$ *is closed.*

Proof. Case (I). X is a quojection.

(1)⇒(2). By assumption, there is a projection $P \in \mathcal{L}(X)$ such that τ_b-$\lim_{n\to\infty} T_{[n]} = P$. This implies that the decomposition (2.4) holds. It remains to show that $(I-T)(X)$ is closed. Let $Y := \overline{(I-T)(X)}$. Then Y is also a quojection Fréchet space as a complemented subspace of the quojection Fréchet space X. Moreover, Y is easily seen to be T–invariant. So, the restriction map $S := T|_Y \in \mathcal{L}(Y)$ and $S_{[n]} \to 0$ in $\mathcal{L}_b(Y)$ as $n \to \infty$. Clearly, $\operatorname{Ker}(I-S) = \{0\}$ and $\frac{S^n}{n} = \frac{T^n}{n} \to 0$ in $\mathcal{L}_b(Y)$ as $n \to \infty$.

Let $\{r_j\}_{j=1}^\infty$ be a fundamental increasing sequence of seminorms generating the lc–topology of X. Since $\frac{T^n}{n} \to 0$ in $\mathcal{L}_b(X)$ as $n \to \infty$ and X is a Fréchet space, the sequence $\{\frac{T^n}{n}\}_{n=1}^\infty$ is equicontinuous. So, for each $j \in \mathbb{N}$ there exists $c_j > 0$ such that

$$r_j\left(\frac{T^n x}{n}\right) \le c_j r_{j+1}(x), \quad x \in X,\, n \in \mathbb{N}; \tag{3.3}$$

there is no loss in generality by assuming that r_{j+1} can be chosen.

Define q_j on X by $q_j(x) := \max\left\{r_j(x), \sup_{n\in\mathbb{N}} r_j\left(\frac{T^n x}{n}\right)\right\}$, for $x \in X$. Then (3.3) ensures that $\{q_j\}_{j=1}^\infty$ is also a fundamental increasing sequence of seminorms generating the locally convex topology of X for which

$$q_j(Tx) \leq 2q_j(x), \quad x \in X,\, j \in \mathbb{N}. \tag{3.4}$$

For each $j \in \mathbb{N}$, set $||.||_j := q_j|_Y$ and observe that $\{||.||_j\}_{j=1}^\infty$ also generates the locally convex topology of Y. We use Lemma 3.4 with $F = E = Y$ and $R = I \in \mathcal{L}(Y,Y)$. Set $A := I - S$ and consider the sequences $\{R_n\}_{n=1}^\infty$ and $\{H_n\}_{n=1}^\infty$ in $\mathcal{L}(Y)$ given by $R_n := \frac{1}{n}\sum_{m=0}^{n-1}\sum_{h=0}^{m} S^h$ and $H_n := I - S_{[n]}$, for $n \in \mathbb{N}$. Observe that $H_n = AR_n = R_nA$ for all $n \in \mathbb{N}$. Then condition (1) implies that $H_n \to I = R$ in $\mathcal{L}_b(Y)$ and so condition (ii) of Lemma 3.4 holds. Moreover, (3.4) yields the validity of condition (iii) in Lemma 3.4. Therefore $I-S$ is invertible and so, surely surjective. We have

$$Y = (I-S)(Y) = (I-T)(Y) \subseteq (I-T)(X) \subseteq Y.$$

Thus, $Y = (I-T)(X)$, i.e., $(I-T)(X)$ is closed in X.

(2)⇒(3). The proof follows as in [26, Theorem,(2)⇒(3)].

(3)⇒(4). The proof follows as in [26, Theorem,(3)⇒(4)].

(4)⇒(1). Define $Y := (I-T)(X)$. By assumption Y is closed in X, hence it is a quojection Fréchet space, being a quotient space of the quojection Fréchet space X. Set $S := T|_Y$. As Y is T–invariant, $S \in \mathcal{L}(Y)$.

Let $\{q_j\}_{j=1}^\infty$ be the fundamental increasing sequence of seminorms generating the locally convex topology of X as defined in (1)⇒(2) and set $||.||_j := q_j|Y$ for all $j \in \mathbb{N}$. We can again consider the sequences $\{R_n\}_{n=1}^\infty$ and $\{H_n\}_{n=1}^\infty$ in $\mathcal{L}(Y)$ given by $R_n := \frac{1}{n}\sum_{m=0}^{n-1}\sum_{h=0}^{m} S^h$ and $H_n := I - S_{[n]}$, for $n \in \mathbb{N}$. Now we use Lemma 3.4 with $F = X, E = Y$ and $R = I - T$. Again, for $A := I - S$, we have $H_n = AR_n = R_nA$ for all $n \in \mathbb{N}$ and with all the operators involved being $||.||_j$-continuous for each $j \in \mathbb{N}$. Moreover,

$$H_n(I-T) = I - T + \frac{1}{n}(T^{n+1} - T), \quad n \in \mathbb{N},$$

which implies $\lim_{n\to\infty} H_n(I-T) = I - T$ in $\mathcal{L}_b(X,Y)$. Since $R := (I-T) \in \mathcal{L}(X,Y)$ is surjective and the operators A, $\{R_n\}_{n=1}^\infty$ and $\{H_n\}_{n=1}^\infty$ (in $\mathcal{L}(Y)$) satisfy all the assumptions of Lemma 3.4, we can apply it to conclude that $A = (I-S)$ is invertible.

Since $I - S$ is surjective, we have that

$$(I-T)(X) = Y = (I-S)(Y) = (I-T)(Y) = (I-T)^2(X).$$

Therefore, given $x \in X$ there is $y \in Y$ such that $(I-T)x = (I-T)y$, i.e., $(I-T)(x-y) = 0$. It follows that $x = (x-y) + y$ with $(x-y) \in \mathrm{Ker}(I-T)$ and $y \in Y = (I-T)(X)$. This shows that $X = \mathrm{Ker}(I-T) \oplus (I-T)(X)$. In particular, $\mathrm{Ker}(I-T)$ is also a quojection Fréchet space. We show that $T_{[n]} \to P$ in $\mathcal{L}_b(X)$ as $n \to \infty$, where P is the projection on X with $P(X) = \mathrm{Ker}(I-T)$ and $\mathrm{Ker}\, P = (I-T)(X)$. Indeed, $T_{[n]} = I$ on $\mathrm{Ker}(I-T)$ and so $T_{[n]} \to I$ in $\mathcal{L}_b(\mathrm{Ker}(I-T))$. As every bounded subset B of X can be written as $B = B_1 + B_2$ with $B_1 \in \mathcal{B}(\mathrm{Ker}(I-T))$ and $B_2 \in \mathcal{B}((I-T)(X))$, it remains to prove that $\lim_{n\to\infty} T_{[n]} = 0$ in $\mathcal{L}_b(Y)$. The surjective operator $I - T\colon X \to (I-T)(X) = Y$ lifts bounded sets by [30, Lemma 26.13], since the quojections X and $\mathrm{Ker}(I-T)$ are quasinormable Fréchet spaces. Therefore, for each bounded set C in $(I -$

$T)(X)$ there exists a bounded set B in X such that $(I-T)(B) \supseteq C$. This yields $p_C(T_{[n]}) = \sup_{c\in C} p(T_{[n]}c) \to 0$ for $n \to \infty$, because of $T_{[n]}(I-T) = \frac{1}{n}(T-T^{n+1})$, the inclusion $T_{[n]}(C) \subset T_{[n]}(I-T)(B)$ and τ_b-$\lim_{n\to\infty} \frac{T^n}{n} = 0$.

Case (II). X is a prequojection.

By definition X'' is a quojection Fréchet space and also $T'' \in \mathcal{L}(X'')$. Moreover, $\frac{T^n}{n} \to 0$ in $\mathcal{L}_b(X)$ as $n \to \infty$ implies that also $\frac{(T'')^n}{n} \to 0$ in $\mathcal{L}_b(X'')$ as $n \to \infty$.

(1)⇒(2). Since T is uniformly mean ergodic, $T_{[n]} \to P$ in $\mathcal{L}_b(X)$ as $n \to \infty$ with $P \in \mathcal{L}(X)$ a projection and $X = \mathrm{Ker}(I-T) \oplus \overline{(I-T)(X)}$. On the other hand, it is routine to check that $(T'')_{[n]} \to P''$ in $\mathcal{L}_b(X'')$ as $n \to \infty$ and so, by the result (1)⇒(2) already proved for quojections in Case (I), we have that $(I-T'')(X'') = (I-T)''(X'')$ is closed in X''. Then Proposition 2.1 implies that $(I-T)(X)$ is closed in X. Therefore, $X = \mathrm{Ker}(I-T) \oplus (I-T)(X)$.

(2)⇒(3) and (3)⇒(4) follow as in the proof of [26, Theorem].

(4)⇒(1). If $(I-T)(X)$ is closed in X, then Proposition 2.1 implies that $(I-T'')(X'') = (I-T)''(X'')$ is closed in X''. So, by (4)⇒(1) of Case (I), we have that T'' is uniformly mean ergodic. Since $T''|_X = T$, it follows that T is uniformly mean ergodic. □

Remark 3.6. Let X be a prequojection Fréchet space and $T \in \mathcal{L}(X)$. The four equivalent conditions of Theorem 3.5 are also equivalent to the condition:

(5) $(I-T)^k(X)$ is closed for (some) every $k \geq 2$.

This answers a question raised by Richard Aron. The proof is as follows.

(2)⇒(5). Let $k \in \mathbb{N}$ with $k \geq 2$. Clearly, $(I-T)^k(X) \subseteq (I-T)^{k-1}(X)$. Conversely, let $z \in (I-T)^{k-1}(X) = (I-T)^{k-2}((I-T)(X))$ and so there is $y \in (I-T)(X)$ such that $z = (I-T)^{k-2}y$. Now, condition (2) ensures that $y = (I-T)x$ with $x = x_0 + x_1$ for some $x_0 \in \mathrm{Ker}(I-T)$ and $x_1 \in (I-T)(X)$. It follows that $y = (I-T)x = (I-T)x_1$ and hence,

$$z = (I-T)^{k-2}y = (I-T)^{k-1}x_1 \in (I-T)^k(X).$$

So, $(I-T)^k(X) = (I-T)^{k-1}(X)$.

Iterating this argument it follows that

$$(I-T)^k(X) = (I-T)^{k-1}(X) = (I-T)^{k-2}(X) = \ldots = (I-T)(X)$$

and so $(I-T)^k(X)$ is closed.

(5)⇒(4). The proof as for Banach spaces (see [29]) and proceeds by induction on k. We first show that $(I-T)^{k-1}(X)$ is closed. To do this we claim that $(I-T)^{k-1}(X)+\mathrm{Ker}(I-T)$ is closed (note that $(I-T)^{k-1}(X)\cap\mathrm{Ker}(I-T) = \{0\}$). Indeed, if $y_n = (I-T)^{k-1}(x_n) + z_n \to y$ for $n \to \infty$, with $\{x_n\}_{n=1}^\infty \subset X$ and $\{z_n\}_{n=1}^\infty \subset \mathrm{Ker}(I-T)$, then $(I-T)y_n = (I-T)^k x_n \to (I-T)y$ for $n \to \infty$. Since $\{(I-T)^k x_n\}_{n=1}^\infty \subset (I-T)^k(X)$ and $(I-T)^k(X)$ is closed, it follows that $(I-T)y \in (I-T)^k(X)$ and so, $(I-T)y = (I-T)^k w$ for some $w \in X$. Thus, $y - (I-T)^{k-1}w \in \mathrm{Ker}(I-T)$, i.e., $y = (I-T)^{k-1}w + (y-(I-T)^{k-1}w) \in (I-T)^{k-1}(X) + \mathrm{Ker}(I-T)$. This establishes the claim.

The facts that $(I-T)^{k-1}(X) + \mathrm{Ker}(I-T)$ is closed, $\mathrm{Ker}(I-T)$ is closed and $(I-T)^{k-1}(X) \cap \mathrm{Ker}(I-T) = \{0\}$ imply that $(I-T)^{k-1}(X)$ is also closed.

Iterating this procedure, one shows that $(I-T)(X)$ is closed.

Remark 3.7. For X a Banach space the condition $\frac{T^n}{n} \to 0$ in $\mathcal{L}_b(X)$ as $n \to \infty$ is known to be necessary to conclude that T is uniformly mean ergodic if $(I-T)(X)$ is closed. Indeed, let $X \neq \{0\}$ be any lcHs, select $x_0 \in X$ and $x_0' \in X'$ such that $x_0'(x_0) = 1$ and define $T(x) := rx_0'(x)x_0$, where $|r| > 1$. Denote by X_0 the one-dimensional linear span of x_0. Then $(I-T)(X) = X_0$ is surely closed in X. Moreover, $T_{[n]}x_0 = \frac{r(1-r^n)}{n(1-r)}x_0$, for $n \in \mathbb{N}$, and so T is not mean ergodic. Of course, $\frac{1}{n}T^n x_0 = \frac{r^n}{n}x_0$ fails to converge to 0 as $n \to \infty$.

Remark 3.8. Let X be a quojection Fréchet space and $T \in \mathcal{L}(X)$ satisfy $\frac{T^n}{n} \to 0$ in $\mathcal{L}_b(X)$ as $n \to \infty$. Define $Y := \overline{(I-T)(X)}$ and $S := T|_Y \in \mathcal{L}(X)$. In the proof of Theorem 3.5 the following facts were established.

(1) If $\{T_{[n]}\}_{n=1}^\infty$ converges in $\mathcal{L}_b(X)$, then $I - S \colon Y \to Y$ is bijective and $(I-T)(X)$ is closed in X.
(2) If $(I-T)(X)$ is closed in X, then $I-S \colon Y \to Y$ is bijective and $\{T_{[n]}\}_{n=1}^\infty$ converges in $\mathcal{L}_b(X)$.

Corollary 3.9. *Let X be a prequojection Fréchet space and $T \in \mathcal{L}(X)$.*

(1) *If T is power bounded and uniformly mean ergodic, then the sequence $\{nT_{[n]}y\}_{n=1}^\infty$ is bounded for every $y \in (I-T)(X) = \overline{(I-T)(X)}$.*
(2) *Suppose that τ_b-$\lim_{n\to\infty} \frac{T^n}{n} = 0$. If $Y := \overline{(I-T)(X)}$ is a prequojection Fréchet space and the sequence $\{nT_{[n]}y\}_{n=1}^\infty$ is bounded for every $y \in Y$, then T is uniformly mean ergodic.*

Proof. (1). Uniform mean ergodicity of T and (2.3) imply that $\frac{T^n}{n} \to 0$ in $\mathcal{L}_b(X)$ as $n \to \infty$. So, by Theorem 3.5, $(I-T)(X)$ is closed in X and hence, $(I-T)(X) = \overline{(I-T)(X)}$.

Fix $y \in (I-T)(X)$. Then $y = (I-T)x$ for some $x \in X$. Then (2.2) yields $nT_{[n]}y = (T - T^{n+1})x$ for every $n \in \mathbb{N}$. Since T is power bounded, i.e., the sequence $\{T^n\}_{n=1}^\infty$ is equicontinuous, it follows that $\{nT_{[n]}y\}_{n=1}^\infty$ is a bounded set in X.

(2). Since Y is T–invariant, the restriction $S := T|_Y$ belongs to $\mathcal{L}(Y)$ and $\frac{S^n}{n} = \frac{T^n}{n} \to 0$ in $\mathcal{L}_b(Y)$ as $n \to \infty$. Moreover, the sequence $\{nS_{[n]}\}_{n=1}^\infty$ is equicontinuous as Y is barrelled and $\{nS_{[n]}y\}_{n=1}^\infty$ is a bounded set in Y for every $y \in Y$. This implies that $S_{[n]} \to 0$ in $\mathcal{L}_b(Y)$ as $n \to \infty$, i.e., S is uniformly mean ergodic. Now, Theorem 3.5 applied to the prequojection Fréchet space Y ensures that $(I-S)(Y)$ is closed in Y. On the other hand, we have also that $\frac{(S'')^n}{n} = \frac{(S^n)''}{n} \to 0$ and $(S'')_{[n]} = (S_{[n]})'' \to 0$ in $\mathcal{L}_b(Y'')$ as $n \to \infty$ too. So, as Y'' is a quojection Fréchet space, we can apply Theorem 4.1,(iii)⇒(ii), of [4] to conclude that the operator $I - S'' = (I-S)'' \colon Y'' \to Y''$ is surjective, i.e., $(I-S'')(Y'') = (I-S)''(Y'') = Y''$. This ensures that $(I-S)(Y)$ is dense in Y. Since $(I-S)(Y)$ is both closed and dense in Y, we obtain that $(I-S)(Y) = Y$. It follows that $Y = (I-S)(Y) \subseteq (I-T)(X) \subseteq Y$ which implies that $Y = (I-T)(X)$ and so, $(I-T)(X)$ is closed in X. Applying again Theorem 3.5 to the prequojection Fréchet space X, we can conclude that T is uniformly mean ergodic. □

An examination of the proof of Theorem 2.7 in [24, p.90] shows that the power boundedness of $T \in \mathcal{L}(X)$ assumed there can be replaced with the weaker condition $\lim_{n\to\infty} \|T^n/n\| = 0$ to yield the following criterion.

Fact. *Let X be a Banach space and $T \in \mathcal{L}(X)$ satisfy $\lim_{n\to\infty} \|T^n/n\| = 0$. Then T is uniformly mean ergodic if and only if*

(6) *Either 1 is in the resolvent set of T or 1 is a simple pole of the resolvent map of T.*

In particular, condition (6) is equivalent with each of the conditions (1)–(4) in Lin's Theorem as stated in Section 1. We now clarify the role of condition (6) for Fréchet spaces.

Let X be a Fréchet space and $T \in \mathcal{L}(X)$. The Banach resolvent set $\rho_B(T)$ of T consists of all $\lambda \in \mathbb{C}$ such that $R(\lambda, T) := (\lambda I - T)^{-1}$ exists in $\mathcal{L}(X)$, whereas the largest open subset $\rho(T)$ of the Riemann sphere $\mathbb{C} \cup \{\infty\}$ in which $\lambda \mapsto R(\lambda, T)$ is holomorphic (i.e., locally has a τ_b–convergent power series expansion in the lc–algebra $\mathcal{L}_b(X)$) is called the resolvent set of T; its complement $\sigma(T)$ is called the spectrum of T, [39, Definition 1.1]. Note that $\rho(T) \setminus \{\infty\} \subseteq \rho_B(T)$. A point $\lambda_0 \in \sigma(T)$ is called a simple pole of $R(\cdot, T)$ if there is a punctured disc $D(\lambda_0, r) := \{z \in \mathbb{C} : 0 < |z - \lambda_0| < r\} \subseteq \rho(T)$, for some $r > 0$, and $0 \neq P \in \mathcal{L}(X)$ such that $\lambda \mapsto R(\lambda, T) - (\lambda - \lambda_0)^{-1} P$ has a holomorphic, $\mathcal{L}_b(X)$–valued extension from $D(\lambda_0, r)$ to the open disc $D(\lambda_0, r) \cup \{\lambda_0\}$. The point ∞ is never a pole of $R(\cdot, T)$, [39, Satz 2.8].

Proposition 3.10. *Let X be a prequojection Fréchet space and $T \in \mathcal{L}(X)$ satisfy τ_b-$\lim_{n\to\infty} \frac{T^n}{n} = 0$. If T satisfies condition (6), then T is uniformly mean ergodic.*

Proof. If 1 is a simple pole of $R(\cdot, T)$, then the uniform mean ergodicity of T follows from [39, Korollar 4.1.2]. On the other hand, if $1 \in \rho(T)$, then $(I - T)$ is an isomorphism on X and so $(I - T)(X) = X$ is closed. Then Theorem 3.5 ensures that T is uniformly mean ergodic. □

The converse of Proposition 3.10 fails in general, even in (non–normable) quojection Fréchet spaces.

Example 3.11. Let $X = \mathbb{C}^{\mathbb{N}}$ be the Fréchet space of all sequences with the seminorms $q_k \colon X \to [0, \infty)$, for $k \in \mathbb{N}$, where $q_k(x) = \max_{1 \le j \le k} |x_j|$, for $x = (x_n)_n \in X$, in which case X is Montel and a quojection. Define $T \in \mathcal{L}(X)$ by $Tx := (\gamma_n x_n)_n$, for $x \in X$, where $\gamma_1 := 1$ and $0 < \gamma_n \uparrow 1$ in $[0, 1)$ for $n \ge 2$. Then T is power bounded and so satisfies τ_b-$\lim_{n\to\infty} \frac{T^n}{n} = 0$. As X is Montel, T is uniformly mean ergodic, [3, Proposition 2.8]. Direct calculation shows that $\sigma(T) = \{\gamma_n\}_{n=1}^{\infty}$ and, for $\lambda \in \rho(T) \setminus \{\infty\}$, that

$$R(\lambda, T)x = \left(\frac{x_n}{(\lambda - \gamma_n)} \right)_n, \quad x \in X.$$

Accordingly, T fails to satisfy condition (6) because $1 \notin \rho(T)$, actually $\mathrm{Ker}(I - T) = \{x \in X : \ x_n = 0, \ \forall n \ge 2\}$, and 1 also fails to be a simple pole of $R(\cdot, T)$ as there is no punctured disc, centred at 1, which is contained in $\rho(T)$.

4. The Cesàro operator in sequence spaces

In this section we investigate the mean ergodic properties of the *Cesàro operator* $C\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$, which is defined in the Fréchet space $\mathbb{C}^{\mathbb{N}}$ (see Example 3.11) by

$$C(x) = \left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)_n, \quad x = (x_n)_n \in \mathbb{C}^{\mathbb{N}}.$$

Then C is a bicontinuous isomorphism of $\mathbb{C}^{\mathbb{N}}$ onto itself with $C^{-1}\colon X \to X$ given by

$$C^{-1}(y) = (ny_n - (n-1)y_{n-1})_n, \quad y = (y_n)_n \in X, \tag{4.1}$$

where we set $y_{-1} := 0$. Denote by $\mathbf{1}$ the constant sequence $(1, 1, \ldots) \in \mathbb{C}^{\mathbb{N}}$. The element of $\mathbb{C}^{\mathbb{N}}$ with 1 in its n-th coordinate and 0 elsewhere is denoted by e_n, for $n \in \mathbb{N}$.

Proposition 4.1. *The Cesàro operator $C\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ is power bounded, uniformly mean ergodic and satisfies $\frac{C^n}{n} \to 0$ in $\mathcal{L}_b(\mathbb{C}^{\mathbb{N}})$ as $n \to \infty$. Moreover,* $\mathrm{Ker}(I-C) = \mathrm{span}\{\mathbf{1}\}$ *and* $(I - C)(\mathbb{C}^{\mathbb{N}}) = \{x \in \mathbb{C}^{\mathbb{N}} :\ x_1 = 0\} = \overline{\mathrm{span}}\{e_r\}_{r\geq 2}$ *is closed.*

Proof. Set $X = \mathbb{C}^{\mathbb{N}}$. That C is continuous and power bounded follows from

$$q_k(C^n x) \leq q_k(x), \quad x \in X, k,\, n \in \mathbb{N}. \tag{4.2}$$

Moreover, (4.2) implies that τ_b-$\lim_{n\to\infty} \frac{C^n}{n} = 0$. Since X is Montel, C is uniformly mean ergodic, [3, Proposition 2.8]. So, by Theorem 3.5 we have that $(I - C)(X)$ is closed in X and $X = \mathrm{Ker}(I - C) \oplus (I - C)(X)$.

We show that $\mathrm{Ker}(I - C) = \mathrm{span}\{\mathbf{1}\}$ is the one–dimensional subspace of X consisting of all constant sequences and so $I - C$ is necessarily not surjective. To establish the claim, suppose that $Cx = x$ for some $x \neq 0$. Equating coordinates yields

$$\frac{x_1 + x_2 + \ldots x_n}{n} = x_n, \quad n \in \mathbb{N},$$

which implies that $x_n = x_1$ for all $n \in \mathbb{N}$. Since $x \neq 0$, we have $x_1 \neq 0$ and so $x = x_1(1, 1, 1, \ldots) = x_1\mathbf{1} \in \mathrm{span}\{\mathbf{1}\}$. As $C\mathbf{1} = \mathbf{1}$, the conclusion follows.

We finally prove that $(I - C)(X) = \{x \in X :\ x_1 = 0\} = \overline{\mathrm{span}}\{e_r\}_{r\geq 2}$. Clearly $(I - C)(X) \subseteq \{x \in X :\ x_1 = 0\}$. Since $\{e_r\}_{r=1}^{\infty}$ is a basis of X (absolute even) it is routine to check that $\{x \in X :\ x_1 = 0\} = \overline{\mathrm{span}}\{e_r\}_{r\geq 2}$. In view of this, it remains to show that $e_r \in (I - C)(X)$, for $r \geq 2$. This follows from the identities $e_{r+1} = (I - C)y_r$, for $r \in \mathbb{N}$, with

$$y_r := e_{r+1} - \frac{1}{r}\sum_{k=1}^{r} e_k, \quad r \in \mathbb{N}, \tag{4.3}$$

which can be established via direct calculation. Accordingly, $\{e_r\}_{r\geq 2} \subseteq (I - C)(X)$. This completes the proof. $\square$

Nothing seems to be available in the literature concerning the mean ergodic properties of C acting in Banach sequence spaces. So, we proceed to examine such properties of C in the classical sequence spaces c_0, c, ℓ^p $(1 < p \leq \infty)$, bv_0 and bv_p $(1 \leq p < \infty)$.

Let $1 < p < \infty$. It is known [21, Theorem 326, p.239] that the Cesàro operator maps the Banach space ℓ^p continuously into itself, which we denote by $C^{(p)}\colon \ell^p \to \ell^p$, and that $\|C^{(p)}\| = p'$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, its spectrum is given by

$$\sigma(C^{(p)}) = \left\{ \lambda \in \mathbb{C} :\ |\lambda - \frac{p'}{2}| \leq \frac{p'}{2} \right\}$$

and $C^{(p)}$ has *no eigenvalues*, [25, Theorems 1&2]. In particular, the spectral radius of $C^{(p)}$ is then $r(C^{(p)}) := \sup\{|\lambda| :\ \lambda \in \sigma(C^{(p)})\} = p'$ and $p' \in \sigma(C^{(p)})$. So, $r(C^{(p)}) = \|C^{(p)}\| = p'$. Since ℓ^p is a Banach lattice and $C^{(p)}$ is a *positive operator* on ℓ^p, it follows from an old result of Karlin, [24, p.93], that $C^{(p)}$ is *not* uniformly mean ergodic. Actually, more is true.

Proposition 4.2. *The Cesàro operator $C^{(p)}\colon \ell^p \to \ell^p$ fails to be power bounded and is not mean ergodic. Moreover,* $\mathrm{Ker}(I - C^{(p)}) = \{0\}$ *and* $(I - C^{(p)})(\ell^p) = \overline{\mathrm{span}}\{e_r\}_{r\geq 2} = \{x \in \ell^p :\ x_1 = 0\}$ *is closed.*

Proof. By the spectral mapping theorem $\sigma((C^{(p)})^n) = \{\lambda^n :\ \lambda \in \sigma(C^{(p)}\}$, for $n \in \mathbb{N}$. Since $p' \in \sigma(C^{(p)})$ it follows that $(p')^n \in \sigma((C^{(p)})^n)$, for $n \in \mathbb{N}$, and so

$$(p')^n \leq r((C^{(p)})^n) \leq \|(C^{(p)})^n\| \leq \|C^{(p)}\|^n = (p')^n, \quad n \in \mathbb{N}.$$

Accordingly,

$$\sup_{n\in\mathbb{N}} \frac{\|(C^{(p)})^n\|}{n} = \sup_{n\in\mathbb{N}} \frac{(p')^n}{n} = \infty \quad (\text{as } p' > 1).$$

So, $C^{(p)}$ is not power bounded. Since $C^{(p)}$ has no eigenvalues, we have $\mathrm{Ker}(I - C^{(p)}) = \{0\}$. Moreover, according to the discussion after the Auxiliary Theorem 3 on p.356 of [20], the subspace $(I - C^{(p)})(\ell^p)$ is closed in ℓ^p. In particular, since the vectors $\{y_r\}_{r=1}^{\infty}$ given by (4.3) also belong to ℓ^p, we can argue as in the proof of Proposition 4.1 to conclude that

$$(I - C^{(p)})(\ell^p) = \overline{\mathrm{span}}\{e_r\}_{r\geq 2} = \{x \in \ell^p :\ x_1 = 0\}.$$

So, $(I - C^{(p)})(\ell^p)$ is a proper, closed subspace of ℓ^p. This together with $\mathrm{Ker}(I - C^{(p)}) = \{0\}$ ensure that $C^{(p)}$ is not mean ergodic. ☐

The operator $C^{(p)}\colon \ell^p \to \ell^p$ provides another example of an operator T in a reflexive Banach space X such that $(I - T)(X)$ is a *closed* (proper) subspace of X but, T is even *not mean ergodic.* Of course, via (2.3), $C^{(p)}$ fails the necessary condition $\sup_n \|T^n/n\| < \infty$ for mean ergodicity; see also Remark 3.7.

The situation is more interesting for the sequence spaces c_0, c and ℓ^∞. It is routine to check that the Cesàro operator maps each of these into itself. Denote these operators by $C^{(\infty)}\colon \ell^\infty \to \ell^\infty$, $C^{(c)}\colon c \to c$ and $C^{(0)}\colon c_0 \to c_0$.

Proposition 4.3. *The Cesàro operators $C^{(\infty)}\colon \ell^\infty \to \ell^\infty$, $C^{(c)}\colon c \to c$ and $C^{(0)}\colon c_0 \to c_0$ are power bounded but not mean ergodic. Moreover,*

$$\mathrm{Ker}(I - C^{(0)}) = \{0\},\ \mathrm{Ker}(I - C^{(\infty)}) = \mathrm{Ker}(I - C^{(c)}) = \mathrm{span}\{\mathbf{1}\},$$

and $(I - C^{(0)})(c_0)$ is not closed with

$$\overline{(I - C^{(0)})(c_0)} = \overline{\mathrm{span}}\{e_r\}_{r\geq 2} = \{x \in c_0 :\ x_1 = 0\}. \tag{4.4}$$

Proof. Direct calculation shows that $\|C^{(\infty)}\| = \|C^{(c)}\| = \|C^{(0)}\| = 1$. Accordingly, each of these 3 operators is power bounded. It is known that $\sigma(C^{(0)}) = \{\lambda \in \mathbb{C} : |\lambda - \frac{1}{2}| \leq \frac{1}{2}\}$ and that $C^{(0)}$ has no eigenvalues, [25, Theorem 3]. Hence, $\mathrm{Ker}(I - C^{(0)}) = \{0\}$. Since $\{e_n\}_{n=1}^{\infty}$ is a basis for c_0, the same argument as for $\mathbb{C}^{\mathbb{N}}$ establishes (4.4). If $C^{(0)}$ were mean ergodic, then we would have

$$c_0 = \mathrm{Ker}(I - C^{(0)}) \oplus \overline{(I - C^{(0)})(c_0)} = \{0\} \oplus \overline{\mathrm{span}}\{e_r\}_{r\geq 2}$$

which is *not* the case. Accordingly, $C^{(0)}$ is power bounded but not mean ergodic.

Moreover, $(I - C^{(0)})(c_0)$ is not closed. If it were, then [26, Theorem] together with the fact that $\lim_{n\to\infty} \|(C^{(0)})^n/n\| = 0$ would imply that $C^{(0)}$ is uniformly mean ergodic and hence, also mean ergodic. But, this is not the case. So, $(I - C^{(0)})(c_0)$ is not closed.

We point out that $\sigma(C^{(\infty)}) = \sigma(C^{(c)}) = \{\lambda \in \mathbb{C} : |\lambda - \frac{1}{2}| \leq \frac{1}{2}\}$ and also that $\mathrm{Ker}(I - C^{(\infty)}) = \mathrm{Ker}(I - C^{(c)}) = \mathrm{span}\{\mathbf{1}\}$, [25, Theorems 4 & 5]. It follows from Proposition 2.2 that $C^{(\infty)}$ and $C^{(c)}$ fail to be mean ergodic, because $C^{(0)}$ is not mean ergodic and c_0 is a closed $C^{(c)}$-invariant subspace of c and a closed $C^{(\infty)}$-invariant subspace of ℓ^∞. □

Consider now the space bv_0 consisting of all $x \in c_0$ such that

$$\|x\|_{bv_0} := \sum_{k=1}^{\infty} |x_{k+1} - x_k| < \infty, \quad x = (x_n)_n,$$

which is known to be a Banach space relative to $\|\cdot\|_{bv_0}$. Observe that bv_0 is isometrically isomorphic to ℓ^1 via the isometry $\Phi\colon bv_0 \to \ell^1$ defined by $\Phi(x) := (x_{n+1} - x_n)_n$, for $x = (x_n)_n \in bv_0$. So, bv_0 is weakly sequentially complete with the Schur property.

According to [35, Lemma 1.2 & Theorem 2.2] the Cesàro operator C maps bv_0 into itself, has norm $\|C_{bv_0}\| = 1$, and $\sigma(C_{bv_0}) = \{\lambda \in \mathbb{C} : |\lambda - 1/2| \leq 1/2\}$, where $C_{bv_0}\colon bv_0 \to bv_0$ denotes the corresponding Cesàro operator.

Proposition 4.4. *The Cesàro operator $C_{bv_0}\colon bv_0 \to bv_0$ is power bounded but not mean ergodic. Moreover,* $\mathrm{Ker}(I - C_{bv_0}) = \{0\}$ *and* $(I - C_{bv_0})(bv_0)$ *is not closed.*

Proof. Since $\|C_{bv_0}\| = 1$, C_{bv_0} is power bounded. As $bv_0 \subseteq c_0$ and $C^{(0)}\colon c_0 \to c_0$ has no eigenvalues, also C_{bv_0} has no eigenvalues, [35, Corollary 1.5]. So, $\mathrm{Ker}(I - C_{bv_0}) = \{0\}$. It is routine to verify that $Y := \{x \in bv_0 : x_1 = 0\}$ is a proper closed subspace of bv_0. Since the first row of the matrix for $I - C_{bv_0}$ is identically zero, it follows that $(I - C_{bv_0})(bv_0) \subseteq Y$ and so $\overline{(I - C_{bv_0})(bv_0)} \subseteq Y$. If C_{bv_0} were mean ergodic, then

$$bv_0 = \mathrm{Ker}(I - C_{bv_0}) \oplus \overline{(I - C_{bv_0})(bv_0)} = \{0\} \oplus \overline{(I - C_{bv_0})(bv_0)} \subseteq Y,$$

which is not the case. Hence, C_{bv_0} is not mean ergodic. So, C_{bv_0} fails to be uniformly mean ergodic and is power bounded. It follows from [26, Theorem] that $(I - C_{bv_0})(bv_0)$ is not closed. □

Remark 4.5. Using the fact that $\{\sum_{k=1}^{n} e_k\}_{n=1}^{\infty}$ is a basis for bv_0 and that $(I - C_{bv_0})y_r = e_{r+1}$, for $r \in \mathbb{N}$, with $\{y_r\}_{r=1}^{\infty} \subseteq bv_0$ given by (4.3), it can be shown that $\overline{(I - C_{bv_0})(bv_0)} = \{x \in bv_0 : x_1 = 0\}$.

For $1 \le p < \infty$ the space bv_p consists of all $x \in \mathbb{C}^{\mathbb{N}}$ such that

$$\|x\|_{bv_p} := \left(|x_1|^p + \sum_{k=1}^{\infty} |x_{k+1} - x_k|^p \right)^{1/p}, \quad x = (x_n)_n.$$

These Banach spaces are studied in detail in [5]. According to [5, Theorem 2.2] the Banach space bv_p is isometrically isomorphic to ℓ^p, $1 \le p < \infty$ via the map $\Phi_p \colon bv_p \to \ell^p$ defined by $\Phi_p(x) := (x_1, x_2 - x_1, x_3 - x_2, \ldots)$, $x \in bv_p$. Hence, bv_p is *reflexive* for all $1 < p < \infty$. Moreover, bv_1 (usually denoted by bv) is weakly sequentially complete with the Schur property.

Proposition 4.6. *bv_0 is a proper closed subspace of bv.*

Proof. For each $n \in \mathbb{N}$, let $\xi_n \colon \ell^1 \to \mathbb{C}$ be the n-th coordinate functional, i.e., $\xi_n(u) = u_n$, for $u = (u_n)_n \in \ell^1$. It follows that

$$\varphi_n := \sum_{k=1}^{n} (\xi_k \circ \Phi_1) \colon bv \to \mathbb{C}, \quad n \in \mathbb{N},$$

are bounded linear functionals on bv. But, direct calculation shows that $\varphi_n(x) = x_n$, for $x \in bv$, i.e., the coordinate functionals satisfy

$$\{\varphi_n\}_{n=1}^{\infty} \subseteq (bv)'. \tag{4.5}$$

On the other hand, it is routine to verify that

$$\|x\|_{\infty} \le \|x\|_{bv}, \quad x \in bv. \tag{4.6}$$

In particular, $bv \subseteq \ell^{\infty}$ with a continuous inclusion.

Suppose that $\{z^{(k)}\}_{k=1}^{\infty} \subseteq bv_0$ converges to z in bv. Then $\{z^{(k)}\}_{k=1}^{\infty}$ is Cauchy relative to $\|\cdot\|_{bv}$. It follows from (4.6) that $\{z^{(k)}\}_{k=1}^{\infty} \subseteq bv_0 \subseteq c_0$ is Cauchy relative to $\|\cdot\|_{\infty}$. By completeness of c_0 there exists $u \in c_0$ with $\|z^{(k)} - u\|_{\infty} \to 0$ as $k \to \infty$. Also, $\|z^{(k)} - z\|_{bv} \to 0$ as $k \to \infty$. Coordinate functionals on c_0 are continuous and, by (4.5), the coordinate functionals on bv are also continuous. It follows that $z = u$ and so $z \in bv \cap c_0$, i.e., $z \in bv_0$. Hence, bv_0 is closed in bv. Finally, $bv \setminus c_0 \neq \emptyset$ as it contains $\mathrm{span}(\{\mathbf{1}\})$. $\square$

According to [1, Theorem 3.1 & Theorem 3.3] the Cesàro operator maps bv_p into itself (denote it by C_{bv_p}), has norm $\|C_{bv_p}\| = 1$ and $\sigma(C_{bv_p}) = \{\lambda \in \mathbb{C} : |\lambda - 1/2| \le 1/2\}$ for all $1 \le p < \infty$.

Proposition 4.7. *Each Cesàro operator $C_{bv_p} \colon bv_p \to bv_p$, $1 < p < \infty$, is power bounded and mean ergodic, whereas $C_{bv} \colon bv \to bv$ is power bounded but not mean ergodic.*

Proof. As $\|C_{bv_p}\| = 1$ for all $1 \le p < \infty$, each operator C_{bv_p} is power bounded for $1 \le p < \infty$. Since bv_p is reflexive for $1 < p < \infty$, the operator C_{bv_p} is mean ergodic for each $1 < p < \infty$. On the other hand, since bv_0 is a closed C_{bv}-invariant subspace of bv, we can apply Proposition 2.2 and Proposition 4.4 to conclude that C_{bv} is not mean ergodic. $\square$

5. The Browder equality and Uniform Mean Ergodicity

Let X be a lcHs and $T \in \mathcal{L}(X)$. We recall from Section 1, with the notation

$$S(T) := \left\{ x \in X : \left\{ \sum_{k=1}^{n} T^k x \right\}_{n=1}^{\infty} \in \mathcal{B}(X) \right\},$$

that the identity (when it holds)

$$S(T) = (I - T)(X) \tag{5.1}$$

is called *Browder's equality*. Inspired by [18], [19], we investigate the validity of (5.1) in relation to uniform mean ergodicity of T.

Lemma 5.1. *Let X be a lcHs and $T \in \mathcal{L}(X)$ be a power bounded operator. Then, for each $B \in \mathcal{B}(X)$, we have $\overline{(I-T)(B)} \subseteq S(T)$.*

Proof. As T is power bounded, given $p \in \Gamma_X$ there exist $c > 0$ and $q \in \Gamma_X$ such that

$$p(T^n x) \leq c q(x), \quad x \in X,\ n \geq 0. \tag{5.2}$$

Fix $B \in \mathcal{B}(X)$. From (5.2) it follows that $C(B) := \cup_{n=0}^{\infty} T^n(B)$ belongs to $\mathcal{B}(X)$. By (2.2) we have, for each $b \in B$ and $n \in \mathbb{N}$, that

$$\begin{aligned} p\left(\left(\sum_{m=1}^{n} T^m \right) (I-T)(b) \right) &= p(Tb - T^{n+1}b) \leq p(Tb) + p(T^{n+1}b) \\ &\leq 2cq(b) \leq 2c \sup_{x \in C(B)} q(x). \end{aligned}$$

This implies, for each $p \in \Gamma_X$, that

$$\sup_{n \in \mathbb{N}} \sup_{b \in B} p\left(\left(\sum_{m=1}^{n} T^m \right) (I-T)(b) \right) \leq 2c \sup_{x \in C(B)} q(x) < \infty. \tag{5.3}$$

Accordingly, $(I-T)(B) \subseteq S(T)$.

Next, let $y \in \overline{(I-T)(B)}$. Then there exists a net $(b_\alpha)_{\alpha \in A} \subseteq B$ with $\lim_\alpha (I-T)b_\alpha = y$ in X. Fix $p \in \Gamma_X$ and select $q \in \Gamma_X$ according to (5.2). For each $n \in \mathbb{N}$, there exists $\alpha_n \in A$ so that

$$q(y - (I-T)b_{\alpha_n}) < \frac{1}{n}. \tag{5.4}$$

Combining (5.2), (5.3) and (5.4), we obtain, for each $n \in \mathbb{N}$, that

$$\begin{aligned} p\left(\sum_{m=1}^{n} T^m y \right) &\leq p\left(\left(\sum_{m=1}^{n} T^m \right) (y - (I-T)b_{\alpha_n}) \right) + p\left(\left(\sum_{m=1}^{n} T^m \right) (I-T)b_{\alpha_n} \right) \\ &\leq ncq(y - (I-T)b_{\alpha_n}) + p\left(\left(\sum_{m=1}^{n} T^m \right) (I-T)b_{\alpha_n} \right) \\ &< c + 2c \sup_{x \in C(B)} q(x). \end{aligned}$$

Accordingly, for each $y \in \overline{(I-T)(B)}$ we have $\sup_{n\in\mathbb{N}} p\left(\sum_{m=1}^n T^m y\right) \leq c + 2c\sup_{x\in C(B)} q(x)$ and hence, that

$$\sup_{y\in\overline{(I-T)(B)}} \sup_{n\in\mathbb{N}} p\left(\sum_{m=1}^n T^m y\right) \leq c + 2c \sup_{x\in C(B)} q(x) < \infty.$$

As $p \in \Gamma_X$ is arbitrary, we have $\overline{(I-T)(B)} \subseteq S(T)$. This completes the proof. □

Let X be a lcHs and $T \in \mathcal{L}(X)$ be power bounded. By [40, Ch. VIII, §3, Theorem 1], we have

$$\overline{(I-T)(X)} = \{x \in X : \lim_{n\to\infty} T_{[n]}x = 0\}. \tag{5.5}$$

Corollary 5.2. *Let X be a lcHs and $T \in \mathcal{L}(X)$ be a power bounded operator. Then*

$$(I-T)(X) \subseteq S(T) \subseteq \overline{(I-T)(X)}. \tag{5.6}$$

In particular, if $S(T)$ is closed, then

$$S(T) = \overline{(I-T)(X)} = \{x \in X : \lim_{n\to\infty} T_{[n]}x = 0\}.$$

Proof. As $(I-T)(X) = \cup_{x\in X}(I-T)(\{x\})$, Lemma 5.1 implies that $(I-T)(X) \subseteq S(T)$. On the other hand, if $x \in S(T)$, then $\{nT_{[n]}x\}_{n=1}^\infty \in \mathcal{B}(X)$ and so $\lim_{n\to\infty} T_{[n]}x = 0$, i.e., $x \in \overline{(I-T)(X)}$ by (5.5). □

In [28] Lin and Sine showed the existence of a mean ergodic operator T acting on a Banach space X for which neither of the inclusions in (5.6) is an equality.

We are now interested in the *Browder's equality* (5.1) for the lcHs setting. In [14, Lemma 5] Browder proved that (5.1) holds for every power bounded operator T acting in a reflexive Banach space. To extend this to lcHs' (see Corollary 5.4 below) we require the following fact.

Proposition 5.3. *Let X be a lcHs for which there exists a coarser lcH–topology τ on X such that every τ-closed, absolutely convex, bounded subset of X is τ–compact. Let $T \in \mathcal{L}(X)$ be any power bounded operator such that $T\colon (X,\tau) \to (X,\tau)$ is continuous. Then $S(T) = (I-T)(X)$.*

Proof. By Corollary 5.2 we have $(I-T)(X) \subseteq S(T)$. Now fix $x \in S(T)$ and define $B := \{\sum_{m=1}^n T^m x : n \in \mathbb{N}\}$ so that $B \in \mathcal{B}(X)$. Let C be the τ-closed, absolutely convex hull of $B \cup \{x\}$. Then C is τ–compact by assumption. For each $n \in \mathbb{N}$ set $x_n := x - T_{[n]}x$. Since $x - x_n = \frac{1}{n}(nT_{[n]}x)$ and $x \in S(T)$, we have $x_n \to x$ in X as $n \to \infty$. Moreover, $x_n = (I-T)[\frac{1}{n}\sum_{m=1}^n (I + T + \ldots + T^{m-1})x] \in (I-T)(X)$, for $n \in \mathbb{N}$, with each $y_n := \frac{1}{n}\sum_{m=1}^n(\sum_{k=0}^{m-1} T^k x) \in 2C$ satisfying $(I-T)y_n = x_n$. By τ-compactness there is a τ-cluster point $y \in 2C$ of the sequence $\{y_n\}_{n=1}^\infty$. As T is τ–continuous, $(I-T)y$ is also a τ-cluster point of the sequence $\{x_n\}_{n=1}^\infty = \{(I-T)y_n\}_{n=1}^\infty$. On the other hand, $x = \lim_{n\to\infty} x_n$ in X and so also $x = \lim_{n\to\infty} x_n$ in (X,τ) as τ is a coarser topology on X. Thus $x = (I-T)y$ is the only τ-cluster point of $\{x_n\}_{n=1}^\infty = \{(I-T)y_n\}_{n=1}^\infty$, i.e., $x \in (I-T)(X)$. By the arbitrariness of x, the inclusion $S(T) \subseteq (I-T)(X)$ also holds. The proof is thereby complete. □

A lcHs X is called *semireflexive* if the algebraic identity $X = X''$ holds. Equivalently, every $B \in \mathcal{B}(X)$ is relatively compact in X_σ. Given a lcHs X, let $\tau(X', X)$ denote the Mackey topology on X' induced by the dual pair $\langle X, X' \rangle$, in which case we write X'_τ.

Corollary 5.4. *Let $X := (X, \eta)$ be a lcHs and $T \in \mathcal{L}(X)$ be a power bounded operator.*

(1) *If X is semireflexive, then $S(T) = (I - T)(X)$.*
(2) *Suppose that there exist a barrelled lcHs Y and $R \in \mathcal{L}(Y)$ such that $X = Y'$ and $T = R'$. If either η is compatible for the duality $\langle Y, Y' \rangle$, i.e., $\sigma(Y', Y) \subseteq \eta \subseteq \tau(Y', Y)$, or $\eta = \beta(Y', Y)$, then $S(T) = (I - T)(X)$.*

Proof. (1) Apply Proposition 5.3 with $\tau = \sigma(X, X')$.

(2) Since Y is barrelled, the properties of being equicontinuous, relatively compact in Y'_σ, bounded in Y'_σ and bounded in Y'_β are equivalent for any subset of Y'. Set $\tau := \sigma(X, Y)$.

Suppose that $\eta = \beta(Y', Y)$. Then $T = R'$ with $R \in \mathcal{L}(Y)$ ensures that $T\colon (X, \tau) \to (X, \tau)$ is continuous, i.e., $T \in \mathcal{L}(Y'_\sigma)$; see Section 2. Moreover, since the bounded subsets of X are equicontinuous in Y', the Alaoglu-Bourbaki Theorem implies that τ satisfies the assumptions of Proposition 5.3 needed to conclude that $S(T) = (I - T)(X)$.

On the other hand if $Y'_\sigma \subseteq X \subseteq Y'_\tau$, then the closure of any absolutely convex set in Y' is the same for $\sigma(Y', Y)$ as for the compatible lc–topology η. Moreover, the η-bounded subsets of Y' coincide with the $\sigma(Y', Y)$-bounded subsets and hence, are equicontinuous. Again, via the Alaoglu-Bourbaki Theorem, we can apply Proposition 5.3 to deduce that $S(T) = (I - T)(X)$. ☐

Remark 5.5. (i) Corollary 5.4(1) is Browder's result for Banach spaces, [14, Lemma 5]. Corollary 5.4(2), for $\eta = \beta(Y', Y)$, extends a result of Lin for dual Banach spaces, [27, Theorem 3.1]; see also [28, Theorem 5].

(ii) In Corollary 5.4(2), for the case when $\eta = \beta(Y', Y)$, it is possible to choose Y quasibarrelled. For, in this case the equicontinuous subsets of Y' are precisely the bounded sets in Y'_σ, [22, p.368]. Hence, for $\tau = \sigma(Y', Y)$, every τ-closed, absolutely conves set which is bounded in $(X, \eta) = Y'_\beta$ is equicontinuous and so the same argument as in the proof of Corollary 5.4(2) applies.

The following result should be compared with Theorem 1.1 in [18].

Proposition 5.6. *Let X be a prequojection Fréchet space and $T \in \mathcal{L}(X)$ be a power bounded operator. Then the following conditions are equivalent.*

(1) *T is uniformly mean ergodic.*
(2) *$(I - T)(X)$ is closed.*
(3) *$S(T)$ is closed and $\overline{(I - T)(X)}$ is a prequojection Fréchet space.*
(4) *$S(T)$ is a complemented subspace of X.*

Proof. (1)⇔(2). This follows from Theorem 3.5.

(2)⇒(3). By (5.6) we can conclude that $S(T)$ is closed. Moreover, $\overline{(I - T)(X)} = (I - T)(X)$ is a prequojection Fréchet space being a quotient of the prequojection Fréchet space X via the map $(I - T)$; see [6], [15], [36], [38].

(3)⇒(1). This follows from Corollary 3.9(2).

(1)$\Rightarrow$(4). By Theorem 3.5 the uniform mean ergodicity of T ensures that $(I-T)(X)$ is closed with $X = \mathrm{Ker}(I-T) \oplus (I-T)(X)$. By Corollary 3.9(1) it follows that $S(T) = (I-T)(X)$ and hence, $S(T)$ is a complemented subspace of X.

(4)$\Rightarrow$(3). Being a complemented subspace of X, $S(T)$ is a prequojection Fréchet space and is clearly closed. Then (5.6) implies that $S(T) = \overline{(I-T)(X)}$, i.e., $\overline{(I-T)(X)}$ is a prequojection. □

As a consequence of Corollary 5.4(1) and Proposition 5.6 and recalling that reflexive prequojection Fréchet spaces are quojections, we obtain

Corollary 5.7. *Let X be a reflexive quojection Fréchet space and $T \in \mathcal{L}(X)$ be a power bounded operator. Then the following conditions are equivalent.*

(1) *T is uniformly mean ergodic.*
(2) *$S(T)$ is closed.*

Proof. (1)$\Rightarrow$(2). This is immediate from Proposition 5.6.

(2)$\Rightarrow$(1). Since X is reflexive, by Corollary 5.4(1) we have $S(T) = (I-T)(X)$. Then the assumption on $S(T)$ yields that $(I-T)(X)$ is closed in X. In particular, $(I-T)(X)$ is a Fréchet space. So, $(I-T)(X)$ is a quojection Fréchet space being a quotient of the quojection Fréchet space X via the map $(I-T)$. The result now follows from (3)$\Rightarrow$(1) of Proposition 5.6. □

Remark 5.8. The validity of Proposition 5.6 and Corollary 5.7 remains confined to the setting of prequojection Fréchet spaces. Indeed, consider the operator T constructed in Proposition 3.1 and acting in the Köthe Montel space $\lambda_p(A) \neq \mathbb{C}^{\mathbb{N}}$, $p \in [1,\infty] \cup \{0\}$. Then T is power bounded and uniformly mean ergodic, but $I-T$ is not surjective and has dense range. So, $(I-T)(\lambda_p(A))$ is a proper subspace of $\overline{(I-T)(\lambda_p(A))} = \lambda_p(A)$. On the other hand, since $\lambda_p(A)$ is reflexive, Corollary 5.4(1) implies that $S(T) = (I-T)(\lambda_p(A))$ and so $S(T)$ is not closed.

For Banach spaces our last two results are Proposition 2.1 and Theorem 2.3 of [19].

Lemma 5.9. *Let X be a lcHs and $T \in \mathcal{L}(X)$ be a power bounded operator. Then T is mean ergodic if and only if*

$$(I-T)[\overline{(I-T)(X)}] = (I-T)(X). \tag{5.7}$$

Proof. If T is mean ergodic, then (2.4) clearly implies (5.7).

Suppose now that (5.7) holds. By (5.5) we have

$$Y := \overline{(I-T)(X)} = \{z \in X : \lim_{n\to\infty} T_{[n]}z = 0\}.$$

Fix $x \in X$. Since $(I-T)(X) = (I-T)(Y)$, there is $y \in Y$ such that $(I-T)x = (I-T)y$, i.e., $T(x-y) = x-y$. Thus,

$$x - y = \frac{1}{n}\sum_{m=1}^{n} T^m(x-y) = \frac{1}{n}\sum_{m=1}^{n} T^m x - \frac{1}{n}\sum_{m=1}^{n} T^m y, \quad n \in \mathbb{N}.$$

But, $y \in Y$ and so $\frac{1}{n}\sum_{m=1}^{n} T^m y = T_{[n]}y \to 0$ as $n \to \infty$. Hence, $\lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} T^m x = x - y$ exists, i.e., $Px := \lim_{n\to\infty} T_{[n]}x$ exists, for all $x \in X$. Since $\{T_{[n]}\}_{n=1}^{\infty}$ is equicontinuous (as T is power bounded), it follows that $P \in \mathcal{L}(X)$, i.e., T is mean ergodic. □

Remark 5.10. Let X be a semireflexive lcHs and $T \in \mathcal{L}(X)$ be power bounded. Then T is mean ergodic, [13, Proposition 3.3], and so (5.7) necessarily holds; see Lemma 5.9. Then Corollary 5.4(1) ensures that

$$S(T) = (I - T)[\overline{(I - T)(X)}]. \tag{5.8}$$

Let X be a lcHs. A sequence $\{x_n\}_{n=1}^{\infty}$ is called a *basis* for X if for every $x \in X$ there is a unique sequence $(\alpha_n)_{n=1}^{\infty}$ of scalars such that the series $\sum_{n=1}^{\infty} \alpha_n x_n$ converges to x.

Theorem 5.11. *For a Fréchet space X with a basis the following assertions are equivalent.*

(1) *X is reflexive.*
(2) *Every power bounded operator in X is mean ergodic.*
(3) *Every power bounded operator $T \in \mathcal{L}(X)$ satisfies* (5.8).

Proof. (1)⇔(2). See [3, Theorem 1.4].

(1)⇒(3). See Remark 5.10.

(3)⇒(1). By (5.6) we have, for each power bounded $T \in \mathcal{L}(X)$, that $(I - T)[\overline{(I - T)(X)}] \subseteq \overline{(I - T)(X)} \subseteq S(T)$. So, by the assumption of (5.8) holding we have that $(I - T)[\overline{(I - T)(X)}] = (I - T)(X) = S(T)$. According to Lemma 5.9 every power bounded operator in X is then mean ergodic. By [3, Theorem 1.4] it follows that X is reflexive. □

For non-reflexive Fréchet spaces with a basis, another result in the spirit of (1) ⇔ (2) in the previous theorem occurs in [37, Theorem 12].

Acknowledgement. The research of José Bonet was partially supported by MEC and FEDER Project MTM 2007-62643, GV Project Prometeo/2008/101 (Spain) and ACOMP/2012/090.

The authors are indebted to the referee for helpful remarks.

References

[1] A.M. Akhmedev, F. Başar, *The fine spectrum of the Cesàro operator C_1 over the sequence space bv_p $(1 \le p < \infty)$.* Math. J. Okayama Univ. **50** (2008), 135–147.
[2] A.A. Albanese, *A Fréchet space of continuous functions which is a prequojection.* Bull. Soc. Roy. Sci. Liége **60** (1991), 409–417.
[3] A.A. Albanese, J. Bonet, W.J. Ricker, *Mean ergodic operators in Fréchet spaces.* Ann. Acad. Sci. Fenn. Math. **34** (2009), 401–436.
[4] A.A. Albanese, J. Bonet, W.J. Ricker, *C_0–semigroups and mean ergodic operators in a class of Fréchet spaces.* J. Math. Anal. Appl. **365** (2010), 142–157.
[5] F. Başar, B. Altay, *On the space of sequences of p–bounded variation and related matrix mappings.* Ukrainian Math. J. **55** (2003), 136–147.
[6] S.F. Bellenot, E. Dubinsky, *Fréchet spaces with nuclear Köthe quotients.* Trans. Amer. Math. Soc. **273** (1982), 579–594.
[7] E. Berhens, S. Dierolf, P. Harmand, *On a problem of Bellenot and Dubinsky.* Math. Ann. **275** (1986), 337–339.
[8] K.D. Bierstedt, J. Bonet, *Stefan Heinrich's density condition for Fréchet spaces and the characterization of distinguished Köthe echelon spaces.* Math. Nachr. **135** (1988), 140–180.
[9] K.D. Bierstedt, J. Bonet, *Some aspects of the modern theory of Fréchet spaces.* RACSAM **97** (2003), 159–188.

[10] K.D. Bierstedt, R.G. Meise, W.H. Summers, Köthe sets and Köthe sequence spaces, In: *"Functional Analysis, Holomorphy and Approximation Theory"* (Rio de Janeiro, 1980), North Holland Math. Studies **71** (1982), pp. 27–91.

[11] J. Bonet, M. Maestre, G. Metafune, V.B. Moscatelli, D. Vogt, *Every quojection is the quotient of a countable product of Banach spaces.* In: "Advances in the Theory of Fréchet spaces", T. Terzioğlu (Ed.), NATO ASI Series, **287**, Kluwer Acad. Publ., Dordrecht, 1989, pp. 355-356.

[12] J. Bonet, W.J. Ricker, *The canonical spectral measure in Köthe echelon spaces.* Integral Equations Operator Theory **53** (2005), 477-496.

[13] J. Bonet, B. de Pagter, W.J. Ricker, *Mean ergodic operators and reflexive Fréchet lattices.* Proc. Roy. Soc. Edinburgh (Sect. A), **141** (2011), 897–920.

[14] F.E. Browder, *On the iteration of transformations in noncompact minimal dynamical systems.* Proc. Amer. Math. Soc. **9** (1958), 773–780.

[15] S. Dierolf, V.B. Moscatelli, *A note on quojections.* Funct. Approx. Comment. Math. **17** (1987), 131–138.

[16] S. Dierolf, D.N. Zarnadze, *A note on strictly regular Fréchet spaces.* Arch. Math. **42** (1984), 549–556.

[17] P. Domański, *Twisted Fréchet spaces of continuous functions.* Results Math. **23** (1993), 45–48.

[18] V. Fonf, M. Lin, A. Rubinov, *On the uniform ergodic theorem in Banach spaces that do not contain duals.* Studia Math. **121** (1996), 67–85.

[19] V. Fonf, M. Lin, P. Wojtaszczyk, *Poisson's equation and characterizations of reflexivity of Banach spaces.* Colloquium Math. **124** (2011), 225–235.

[20] M. Gonzalez, *The fine spectrum of the Cesàro operator in* ℓ^p $(1 < p < \infty)$. Arch. Math. **44** (1985), 355–358.

[21] G.H. Hardy, J.E. Littlewood, G. Polya, *Inequalities.* Cambridge Univ. Press, Cambridge, 2nd Ed. reprinted, 1964.

[22] G. Köthe, *Topological Vector Spaces I.* 2nd Rev. Ed., Springer Verlag, Berlin-Heidelberg-New York, 1983.

[23] G. Köthe, *Topological Vector Spaces II.* Springer Verlag, Berlin-Heidelberg-New York, 1979.

[24] U. Krengel, *Ergodic Theorems.* Walter de Gruyter, Berlin, 1985.

[25] G. Leibowitz, *Spectra of discrete Cesàro operators.* Tamkang J. Math. **3** (1972), 123–132.

[26] M. Lin, *On the uniform ergodic theorem.* Proc. Amer. Math. Soc. **43** (1974), 337–340.

[27] M. Lin, *On quasi–compact Markov operators.* Ann. of Probab. **2** (1974), 464–475.

[28] M. Lin, R. Sine, *Ergodic theory and the functional equation* $(I - T)x = y$. J. Operator Theory **10** (1983), 153–166.

[29] M. Mbekhta, J. Zemanek, *Sur le théoreme ergodique uniforme et le spectre.* C.R. Acad. Sci. Paris **317** (1993), 1155–1158.

[30] R. Meise, D. Vogt, *Introduction to Functional Analysis.* Clarendon Press, Oxford, 1997.

[31] G. Metafune, V.B. Moscatelli, *Quojections and Prequojections.* In "Advances in the Theory of Fréchet spaces", T. Terzioğlu (Ed.), NATO ASI Series, **287**, Kluwer Academic Publishers, Dordrecht, 1989, pp. 235–254.

[32] G. Metafune, V.B. Moscatelli, *Prequojections and their duals.* In: "Progress in Functional Analysis", K.D. Bierstedt, J. Bonet, J. Horváth and M. Maestre (Eds.), North Holland, Amsterdam, 1992, pp. 215-232.

[33] V.B. Moscatelli, *Fréchet spaces without norms and without bases.* Bull. London Math. Soc. **12** (1980), 63–66.

[34] V.B. Moscatelli, *Strongly non–norming subspaces and prequojections.* Studia Math. **95** (1990), 249–254.

[35] J.I. Okutoi, *On the spectrum of* C_1 *as an operator on* bv_0. J. Austral. Math. Soc. (Series A) **48** (1990), 79–86.

[36] S. Önal, T. Terzioğlu, *Unbounded linear operators and nuclear Köthe quotients.* Arch. Math. **54** (1990), 576–581.

[37] K. Piszczek, *Quasi-reflexive Frechet spaces and mean ergodicity.* J. Math. Anal. Appl. **361** (2010), 224-233.

[38] D. Vogt, *On two problems of Mityagin.* Math. Nachr. **141** (1989), 13–25.

[39] V. Wrobel, *Spektraltheorie stetiger Endomorphismen eines lokalkonvexen Raumes.* Math. Ann. **234** (1978), 193–208.
[40] K. Yosida, *Functional Analysis.* Springer–Verlag, Berlin, 1980.