The Bounded Approximation Property in Fréchet Spaces
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The purpose of this seminar is to explain several results concerning the bounded approximation property for Fréchet spaces. We give a full detailed proof of an important result due to Pełczyński [Pel71] (see also [Mat77]) that asserts that every separable Fréchet space with the bounded approximation property is isomorphic to a complemented subspace of a Fréchet space with a Schauder basis. We also explain Vogt's example (cf. [Vog83]) of a nuclear Fréchet space without the bounded approximation property. This example was much simpler than the original counterexample due to Dubinski. These examples solved a long standing problem of Grothendieck. Vogt [Vog10] obtained another simple example of a nuclear Fréchet function space without the bounded approximation property. The relation of the bounded approximation property for Fréchet spaces with a continuous norm and the countably normable spaces, including several results due to Dubinski and Vogt [DuV85], is also explained.

1 Introduction

A topological vector space E is a Fréchet space if it is metrizable, complete and locally convex. We use below the abbreviation "lcs" for "locally convex space". The topology of E is defined by a fundamental system of seminorms $p_1 \leq p_2 \leq \ldots \leq p_n \leq \ldots$ satisfying that for each $x \in E, x \neq 0$, there exists $n \in \mathbb{N}$ such that $p_n(x) > 0$. Recall that for every neighbourhood of the origin $U \in \mathcal{U}_0(E)$, there exist $n \in \mathbb{N}$ and $\varepsilon > 0$ such that $\{x \in U : p_n(x) < \varepsilon\} \subset U$. We may assume that a basis of neighborhoods of the origin is given by $U_n := \{x \in E : p_n(x) < 1\}$, $n \in \mathbb{N}$. We say that B is a bounded set in E, and we write $B \in \mathfrak{B}(E)$, if $\sup_{b \in B} p_n(b) < \infty$ for every $n \in \mathbb{N}$.

Let $(p_n)_n$ and $(q_m)_m$ be fundamental system of seminorms in E and F respectively. A linear operator $T : E \to F$ is continuous operator if and only if for every $m \in N$ there exists $n \in \mathbb{N}$ and $C > 0$ such that $q_m(T(x)) \leq p_n(x)$ for every $x \in E$. We denote $L(E, F)$ the space of linear and continuous operators from E to F. A set $H \subset L(E, F)$ is *equicontinuous* if for every $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ and $C > 0$ such that $q_m(T(x)) \leq p_n(x)$ for every $x \in E$ and for every $T \in H$. Note that this condition is equivalent to the fact that $\cap_{T \in H} T^{-1}(V) \in \mathcal{U}_0(E)$ for every $V \in \mathcal{U}_0(F)$. It is also important to recall Banach-Steinhaus' theorem for Fréchet spaces: Let E be a Fréchet space. $H \subset L(E, F)$ is equicontinuous if and only if for every $x \in E$, $H(x) := \{T(x) : T \in H\}$ is a bounded set of F.

Definition 1.1 We say that E admits a continuous norm if there exists a norm $\|\cdot\| : E \to \mathbb{R}$

that is continuous for the topology of E; that is there exists a norm $\|\cdot\| : E \to \mathbb{R}$ such that there exists $n \in \mathbb{N}$ and $C > 0$ with $\|x\| \leq Cp_n(x)$ for every $x \in E$. If E has a continuous norm, we can choose a fundamental system of seminorms $p_1 \leq p_2 \leq \ldots \leq p_n \leq \ldots$ in E such that p_k is a norm for every $k \in \mathbb{N}$.

Example 1.2 1. The space $H(\Omega)$ with $\Omega \subset \mathbb{C}$ an connected open set in the complex plane endowed with the topology of uniform convergence on the compact subsets of Ω is a Fréchet space that admits a continuous norm and is not normable.

2. The space Fréchet $C^{\infty}([0,1])$ endowed by the topology given by the seminorms

$$p_n(f) := \max_{1 \leq \alpha \leq n} \sup_{x \in [0,1]} \left| f^{(\alpha)}(x) \right|,$$

also admits a continuous norm.

3. The space $\omega := \mathbb{K}^{\mathbb{N}}$ endowed by the topology given by the seminorms

$$p_n(x) := \max_{1 \leq j \leq n} |x_j|, \text{ with } x = (x_j)_j,$$

does not admit a continuous norm.

4. The space $C^{\infty}(\Omega)$ endowed by the topology given by the seminorms

$$p_n(f) := \max_{1 \leq x \leq n} \sup_{x \in K_n} \left| f^{(\alpha)}(x) \right|,$$

where $K_1 \subset K_2 \subset \ldots \subset K_n \subset \ldots$ is a fundamental sequence of compact subsets in Ω, does not admit a continuous norm.

There are two important results concerning Fréchet spaces with does not have a continuous norm.

Theorem 1.3 ***(Bessaga, Pełczyński)*** *A Fréchet space does not have a continuous norm if and only if ω is isomorphic to a complemented subspace of E.*

Theorem 1.4 ***(Eidelheit)*** *If E is a Fréchet space that is not normable then E have a isomorphic quotient in ω.*

Example 1.5 Here is a concrete example of a not normable Fréchet space with a quotient isomorphic to ω: Consider the Fréchet space $H(\mathbb{C})$ of entire functions endowed with the compact open topology. Select a sequence (z_n) in $\mathbb{C}$ such that $|z_{n+1}| > |z_n|$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} |z_n| = \infty$. The linear map $T : H(\mathbb{C}) \to \omega$ defined by $f \mapsto f(z_n)$ is surjective by Weierstrass interpolation Theorem. The map T is clearly continuous and it is open by the open mapping theorem for Fréchet spaces.

Definition 1.6 A lcs E has the *bounded approximation property* (BAP) if there exists an equicontinuous net $(A_j)_{j\in J} \subset L(E)$ with $\dim(A_j(E)) < \infty$ for every $j \in J$ and $\lim_{j\in J} A_j(x) = x$ for every $x \in E$. In other words, the net $(A_j)_{j\in J}$ converges to the identity in the space $L_s(E)$, i.e. for the topology of pointwise or simple convergence.

Remark 1.7 Let $H \subset L(E,F)$ be equicontinuous. If $N \subset E$ is a total subset of E (i.e. $\overline{\text{span}(N)} = E$), then the topologies of simple convergence on E ($\mathfrak{T}_s(E)$) and on N ($\mathfrak{T}_s(N)$) coincide in H ([Köt79, 39.4.(1)]). In particular, if E is separable and F is metrizable, then the topology $\mathfrak{T}_s(E)$ of simple convergence on E is metrizable on every equicontinuous subset H oh $L(E,F)$ ([Köt79, 39.4.(7)]).

Consequence 1.8 *If E is a separable, metrizable lcs then E has the BAP if and only if there exists $(A_n)_n \subset L(E)$ (a sequence) which is equicontinuous, dim$(A_n(E)) < \infty$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} A_n(x) = x$ for each $x \in E$.*

In case that E is barrelled metrizable and separable, E has BAP if and only if there exists $(A_n)_n \subset L(E)$ with dim$(A_n(E)) < \infty$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} A_n(x) = x$ for each $x \in E$. This is a consequence of Banach-Steinhaus Theorem.

Remark 1.9 Let H be an equicontinuous subset of $L(E,F)$. By [Köt79, 39.4.(2)] the topology $\mathfrak{T}_s(E)$ and the topology $\mathfrak{T}_c(E)$ of uniform convergence of precompact subsets of E coincide on H.

Consequence 1.10 *If E has the BAP, $A_j \to I$ with $j \in J$ uniformly on the precompact subsets of E. Accordingly, the BAP implies approximation property.*

In what follows E is a separable Fréchet space, and $p_1 \leq p_2 \leq \ldots \leq p_k \leq p_{k+1} \leq \ldots$ is a fundamental system of seminorms in E.

We assume, without loss of generality, that $U_k := \{x \in E : p_k(x) \leq 1\}$, with $k \in \mathbb{N}$, form a basis of 0-neighborhoods in E.

In case E is a Fréchet space with a continuous norm, we assume without loss of generality that all the elements of the fundamental system of seminorms $p_1 \leq p_2 \leq \ldots \leq p_k \leq p_{k+1} \leq \ldots$ are in fact norms.

Remark 1.11 If E is a separable Fréchet space with a fundamental system of seminorms $(p_k)_k$ has the BAP, we can find $(A_n)_n \subset L(E)$, with dim$(A_n(E)) < \infty$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} A_n(x) = x$ for each $x \in E$. By the Banach-Steinhaus Theorem, $(A_n)_n$ is equicontinuous. Therefore, for every $k \in \mathbb{N}$ there exists $l \geq k$ and $C_k > 0$ with $p_k(A_n(x)) = C_k p_{l(k)}(x)$ for every $x \in E$ and for every $n \in \mathbb{N}$.

Proposition 1.12 *Let E be a separable Fréchet space, then the following conditions are equivalent:*

1. *The BAP holds in E,*
2. *There exists $(A_n)_n \subset L(E)$, with dim$(A_n(E)) < \infty$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} A_n(x) = x$ for each $x \in E$,*
3. *There exists $(B_n)_n \subset L(E)$, with dim$(B_n(E)) < \infty$ for every $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} B_n(x) = x$ for each $x \in E$.*

Proof. (1) $\Rightarrow$ (2) Since E is a separable space, there exists a countable dense subset F of E Accordingly, the following topologies coincide on the equicontinuous subsets of $L(E)$:

- Uniform convergence over the compact sets of E,

- Pointwise convergence on E,
- Pointwise convergence on F.

As the topology of pointwise convergence on F is metrizable, the results holds.

(2) $\Rightarrow$ (1) Since $A_n(x)$ converges to x when n tends to infinity, then $\{A_n(x)\}_n$ is bounded in E for every $x \in E$. By Banach-Steinhaus' theorem, $\{A_n\}_n$ is equicontinuous.

(2) $\Rightarrow$ (3) Take $B_1 := A_1$ and $B_{n+1} := A_{n+1} - A_n$, for every $n \in \mathbb{N}$, to get the result.

(3) $\Rightarrow$ (2) Now set $A_n := B_1 + \ldots + B_n$ for every $n \in \mathbb{N}$. □

Remark 1.13 Let $(p_k)_k$ be a fundamental system of seminorms in E. Define $q_k(x) := \sup_{n\in\mathbb{N}} p_k\left(\sum_{i=1}^n B_i(x)\right)$, for every $x \in E$ and for every $k \in \mathbb{N}$.

Since $x = \lim_{n\to\infty} \sum_{i=1}^n B_i(x)$, we have $p_k(x) = \lim_{n\to\infty} p_k\left(\sum_{i=1}^n B_i(x)\right)$ and this implies that $p_k(x) \leq q_k(x)$ for every $x \in E$ and for every $k \in \mathbb{N}$.

Observe that $\sum_{i=1}^n B_i = A_n$ for each $n \in \mathbb{N}$. Hence $p_k\left(\sum_{i=1}^n B_i(x)\right) = p_k(A_n(x)) \leq C_k p_{l(k)}(x)$ for each $n \in \mathbb{N}$. Thus $q_k(x) \leq C_k p_{l(k)}(x)$ for every $x \in E$ and for every $k \in \mathbb{N}$. And the sequence of seminorms $(q_k)_k$ is a fundamental system of seminorms in E.

Definition 1.14 We say that $\{x_n\} \subset E$ is a Schauder basis in E with coefficient functionals $\{x'_n\}$ if:

- For every $k, n \in \mathbb{N}$, $\langle x'_k, x_n\rangle = \delta_{n,k}$
- For every $x \in E$, $x = \sum_{n=1}^{\infty} \langle x'_n, x\rangle x_n$, the series converging in E.

Example 1.15 Some spaces with Schauder basis are Köthe echelon spaces, and the Banach sequence spaces $\ell_p, 1 \leq p < \infty$, and c_0.

Proposition 1.16 *The following results holds:*

(1) If E is a lcs with the BAP and $F \subset E$ is complemented, then F has the BAP, too.

(2) If E is a barrelled lcs with a Schauder basis, then E has the BAP.

Proof. (1) Let $(A_\tau)_{\tau\in T} \subset L(E)$ be an equicontinuous net such that $\dim A_\tau(E) < \infty$ for every $\tau \in T$ and $\lim_{\tau\in T} A_\tau(x) = x$ for each $x \in E$. Let $F \subset E$ be complemented. Denote by $J : F \to E$ the canonical inclusion and by $P : E \to F$ the projection. For each $\tau \in T$, define $B_\tau : F \to F$ by $B_\tau := PA_\tau J$. Clearly, $\dim B_\tau(F) \leq \dim A_\tau(F) < \infty$ and for every $q \in \mathrm{cs}(E)$ there exists $q' \in \mathrm{cs}(E)$ such that $q(A_\tau(x)) \leq q'(x)$ for every $x \in E$ and for every $\tau \in E$. Moreover, as $P : E \to F$ is continuous, given $p \in \mathrm{cs}(E)$ there exists $q \in \mathrm{cs}(E)$ such that $p(Px) \leq q(x)$ for every $x \in E$. Then

$$p(B_\tau(x)) = p(PA_\tau J(x)) = p(PA_\tau(x)) \leq q(A_\tau(x)) \leq q'(x)$$

for every $x \in F$ and for every $\tau \in T$. Thus $(B_\tau)_{\tau\in T}$ is equicontinuous in $L(F)$. Finally, for $x \in F$,

$$\lim_{\tau\in T} B_\tau(x) = \lim_{\tau\in T} PA_\tau(x) = P\left(\lim_{\tau\in T} A_\tau(x)\right) = P(x) = x.$$

(2) Let $(x_n)_n \subset E$ be a Schauder basis with coefficient functionals $(x'_n)_n \subset E'$. That is $\langle x'_k, x_n\rangle = \delta_{k,n}$ and $x = \sum_{n=1}^{\infty} x'_n(x)\, x_n$ converges in E for each $x \in E$. Denote by $P_n : E \to E$

the map $P_n(x) := \sum_{k=1}^{n} x'_k(x)\, x_k$, which is a continuous projection onto span$(x_1,\dots,x_n)$. Since E is barrelled, $(P_n)_n$ is equicontinuous. As $\lim_{n\to\infty} P_n(x) = x$ for every $x \in E$, we conclude that E has the BAP. □

Theorem 1.17 ***(Pełczyński. 1971)*** *Every separable Fréchet space E with the BAP is isomorphic to a complemented subspace of a Fréchet space E_0 with a Schauder basis. If E has a continuous norm, E_0 can be chosen with a continuous norm.*

Proof. Fix a fundamental sequence of seminorms, $|\cdot|_1 \le |\cdot|_2 \le \dots \le |\cdot|_k \le |\cdot|_{k+1} \le \dots$ in E. By assumption there is $(A_n)_n \subset L(E)$, $\dim(A_n(E)) < \infty$ for each $n \in \mathbb{N}$, such that, $A_n \neq 0$ for each $n \in \mathbb{N}$, and $\lim_{n\to\infty} \sum_{p=1}^{n} A_p(x) = x$ in E for every $x \in E$.

We first select another (more suitable) fundamental sequence of seminorms. Set $E_p := A_p(E)$, $p = 1, 2, \dots$ and $m_p :=\dim(E_p)$, $p = 1, 2, \dots$ with $m_0 := 0$. Since $\dim(E_p) < \infty$, for each p there is $k(p) \in \mathbb{N}$ such that $k(p-1) < k(p)$ and $|\cdot|_{k(p)}$ is a norm in E_p. We set $\|\cdot\|_n := |\cdot|_{k(n)}$. Clearly, $\|\cdot\|_n \le \|\cdot\|_{n+1}$ for each n and $(\|\cdot\|_n)_n$ is a fundamental sequence of seminorms in E. Fix $n \in \mathbb{N}$ and for $j < n$, set $F_j^n := \left(\text{ Ker } \|\cdot\|_j\right) \cap E_n$. As $\|\cdot\|_j \le \|\cdot\|_{j+1}$ for each j, we have $F_{n-1}^n \subset F_{n-2}^n \subset \dots \subset F_1^n \subset E_n$. They are all closed in E_n and, since they are finite dimensional, each one is complemented in the previous one.

We select a complement in each step $F_{n-2}^n = F_{n-1}^n \oplus H_{n-2}^n$, $F_{n-3}^n = \left(F_{n-1}^n \oplus H_{n-2}^n\right) \oplus H_{n-3}^n$ and $E_n = F_1^n \oplus H_0^n$. We can write $E_n = H_0^n \oplus H_1^n \oplus \dots \oplus H_{n-2}^n \oplus F_{n-1}^n$. Selecting a element in each component and writing the projections, each $x \in E_n$ can be uniquely written as $x = \sum_{l=0}^{n-1} \sum_{k(\text{finite})} x_k^l$.

Fix a seminorm $\|\cdot\|_j$ with $j < n$ and consider each projection x_k^l of x. If $l \ge j$, $x_k^l \in H_l^n \subset F_l^n = (\text{ Ker } \|\cdot\|_l) \cap E_n$ then $x_k^l \in \text{Ker}\left(\|\cdot\|_j\right)$ (i.e. $\|x_k^l\|_j = 0$); therefore $\left\|\sum_{l=j}^{n-1} \sum_{k(\text{finite})} x_k^l\right\|_j = 0$. On the other hand, $\|\cdot\|_j$ is a norm on $H_0^n \oplus \dots \oplus H_{j-1}^n$. This implies that the projection $\sum_{r=0}^{j-1} \sum_{k(\text{finite})} x_k^r \to x_k^l$ is continuous for $0 \le l < j$ and each k. So we can find $C_j > 0$ such that

$$\left\|x_k^l\right\|_j \le C_j \left\|\sum_{r=0}^{j-1} \sum_{k(\text{finite})} x_k^r\right\|_j = C_j \left\|\sum_{r=0}^{n-1} \sum_{k(\text{finite})} x_k^r\right\|_j, \text{ for } 0 \le l < j \text{ and each } k.$$

Accordingly, we have found for E_p and m_p a family of 1-dimensional operators $B_j^p : E_p \to E_p$ with $j = 1, \dots, m_p$ such that

$$e = \sum_{j=1}^{m_p} B_j^p e \text{ for every } e \in E_p,$$

and

$$\max_{1\le j\le m_p} \left\|B_j^p e\right\|_k \le R_p \|e\|_k \text{ for every } e \in E_p \text{ and for every } k = 1, \dots, p.$$

In fact, $R_p = \max(C_1, \dots, C_p)$, selected as above.

Now, since $\lim_{n\to\infty} \sum_{p=1}^{n} A_p(x) = x$ for every $x \in X$, the sequence $\left(\sum_{p=1}^{n} A_p\right)_{n=1}^{\infty}$ is equicontinuous in $L(E)$, this means that, for every $k \in \mathbb{N}$, there exists $M_k > 0$ and $l(k) \ge k$

such that $\left\|\sum_{p=1}^{n} A_p(x)\right\|_k \leq M_k \|x\|_{l(k)}$ for every $x \in X$ and for every $n \in \mathbb{N}$. This implies

$$\|A_n(x)\|_k \leq \left\|\sum_{p=1}^{n} A_p(x)\right\|_k + \left\|\sum_{p=1}^{n-1} A_p(x)\right\|_k \leq 2M_k \|x\|_{l(k)} \text{ for each } n \in \mathbb{N} \text{ and } x \in E.$$

For each $p \in \mathbb{N}$ select $N_p \in \mathbb{N}$ with $m_p R_p \leq N_p$ and set $N_0 = 0$. Set $C_i^p := N_p^{-1} B_j^p$ with $i = rm_p + j$, $r = 0, 1, \ldots, N_p - 1$ and $j = 1, \ldots, m_p$. Observe that there are $N_p m_p$ rank-1 operators.

$$\begin{array}{ll} r = 0, & \frac{1}{N_p} B_1^p \ldots \frac{1}{N_p} B_{m_p}^p \\ r = 1, & \frac{1}{N_p} B_1^p \ldots \frac{1}{N_p} B_{m_p}^p \\ \ldots & \qquad \ldots \\ r = N_p - 1, & \frac{1}{N_p} B_1^p \ldots \frac{1}{N_p} B_{m_p}^p \end{array}$$

If $e \in E_p$, we get

$$\sum_{i=1}^{m_p N_p} C_i^p e = \sum_{r=0}^{N_p-1} \sum_{j=1}^{m_p} \frac{1}{N_p} B_j^p e = \frac{1}{N_p} \sum_{r=0}^{N_p-1} e = e \text{ for every } p \in \mathbb{N}.$$

Moreover, for $k = 1, 2, \ldots, p$, $e \in E_p$, and $1 \leq q \leq m_p N_p$, we get r and w with $0 \leq r \leq N_p - 1$, $1 \leq w \leq m_p$ such that

$$\sum_{i=1}^{q} C_i^p = r \sum_{j=1}^{m_p} N_p^{-1} B_j^p + \sum_{j=1}^{w} N_p^{-1} B_j^p.$$

Thus, for $k = 1, 2, \ldots, p$ we have

$$\begin{aligned} \left\|\sum_{j=1}^{q} C_i^p e\right\|_k &\leq \frac{r}{N_p} \left\|\sum_{j=1}^{m_p} B_j^p e\right\|_k + \frac{1}{N_p} \left\|\sum_{j=1}^{w} B_j^p e\right\|_k \leq \frac{r}{N_p} \|e\|_k \text{‘} \frac{1}{N_p} \sum_{j=1}^{w} \left\|B_j^p e\right\|_k \leq \\ &\leq \|e\|_k + \frac{1}{N_p} \sum_{j=1}^{w} R_p \|e\|_k \leq \left(1 + \frac{wR_p}{N_p}\right) \|e\|_k \leq 2 \|e\|_k, \end{aligned}$$

where $\frac{wR_p}{N_p} \leq 1$ since $1 \leq w \leq m_p$ and $m_p R_p \leq N_p$. We then obtain

$$\max_{1 \leq q \leq m_p N_p} \left\|\sum_{i=1}^{q} C_i^p e\right\|_k \leq 2 \|e\|_k \text{ for every } e \in E_p \text{ and } k = 1, \ldots, p.$$

Define now $\widetilde{A_s} := C_i^p A_p$ for $s = m_0 N_0 + \ldots + m_{p-1} N_{p-1} + i$, $p = 1, 2, \ldots$ and $i = 1, 2, \ldots, m_p N_p$. Observe that $\widetilde{A_s} \in L(E)$ since $E \overset{A_p}{\to} E_p \overset{C_i^p}{\to} E_p \hookrightarrow E$.

Claim 1.18 $\left(\sum_{s=1}^{n} \widetilde{A_s}\right)_{n=1}^{\infty}$ *is equicontinuous in* $L(E)$.

If $n \geq m_1 N_1$ there are t, q with $1 \leq q \leq m_{t+1} N_{t+1}$ such that

$$\sum_{s=1}^{n} \widetilde{A_s} = \sum_{p=1}^{t} \sum_{i=1}^{m_p N_p} C_i^p A_p(x) + \sum_{i=1}^{q} C_i^{t+1} A_{t+1}.$$

Fix $k \in \mathbb{N}$; for $x \in E$ we have, if $k \leq t+1$,

$$\begin{aligned}\left\|\sum_{s=1}^{n} \widetilde{A_s}(x)\right\|_k &\leq \left\|\sum_{p=1}^{t} \sum_{i=1}^{m_p N_p} C_i^p A_p(x)\right\|_k + \left\|\sum_{i=1}^{q} C_i^{t+1} A_{t+1}(x)\right\|_k = \\ &= \left\|\sum_{p=1}^{t} A_p(x)\right\|_k + 2\left\|A_{t+1}(x)\right\|_k \leq \\ &\leq M_k \|x\|_{l(k)} + 4M_k \|x\|_{l(k)} = 5M_k \|x\|_{l(k)}.\end{aligned}$$

And the claim follows, since this estimates holds for all n such that $k \leq t+1$, hence for all except a finite number. Consequently, we have

$$\forall k \in \mathbb{N}, \exists \omega(k), K_k > 0 : \sup_n \left\|\sum_{s=1}^{n} \widetilde{A_s}(x)\right\|_k \leq K_k \|x\|_{\omega(k)} \text{ for every } x \in E.$$

Claim 1.19 $\lim_{n\to\infty} \sum_{s=1}^{n} \widetilde{A_s}(x) = x$ *for every* $x \in E$.

First, select $t \in \mathbb{N}$ and q with $1 \leq q \leq m_{t+1}N_{t+1}$, for $n \geq m_1 N_1$ then

$$\begin{aligned}\left\|\sum_{s=1}^{n} \widetilde{A_s}(x) - x\right\|_k &\leq \left\|\sum_{p=1}^{t} \sum_{i=1}^{m_p N_p} C_i^p A_p(x) - x\right\|_k + \left\|\sum_{i=1}^{q} C_i^{t+1} A_{t+1}(x)\right\|_k \leq \\ &\leq \left\|\sum_{p=1}^{t} A_p(x) - x\right\|_k + 2\left\|A_{t+1}(x)\right\|_k \text{ if } k \leq t+1.\end{aligned}$$

Now,

$$A_{t+1}(x) = \left(\sum_{r=1}^{t+1} A_r(x) - x\right) - \left(\sum_{r=1}^{t} A_r(x) - x\right),$$

where the two expressions tends to 0 as t tends to infinity. Then, $\lim_{t\to\infty} \|A_{t+1}(x)\|_k = 0$. As n tends to infinity, then t tends also to infinity, therefore, there exists

$$\lim_{n\to\infty} \left\|\sum_{s=1}^{n} A_s(x) - x\right\|_k = \lim_{t\to\infty} \left\|\sum_{p=1}^{t} A_p(x) - x\right\|_k + \lim_{t\to\infty} \|A_t(x)\|_k = 0$$

Denote by $E_0 := \left\{y = (y(s))_{s\in\mathbb{N}} : y(s) \in \widetilde{A_s}(E) \text{ and } \sum_{s=1}^{\infty} y(s) \text{ converges in } E\right\}$, endowed with the fundamental system of seminorms

$$\|\!| (y(s))_s \|\!|_k := \sup_n \left\|\sum_{s=1}^{n} y(s)\right\|, y = (y(s))_s \in E_0.$$

It is not difficult to prove that E_0 is a Fréchet space. We prove that E_0 has a Schauder basis.

Since $\dim\left(\widetilde{A_s}(E)\right) = 1$ for each $s \in \mathbb{N}$, we choose $y_s \in \widetilde{A_s}(E)$, $y_s \neq 0$, for each $s \in \mathbb{N}$. For each $y \in \widetilde{A_s}(E)$ there is $c \in \mathbb{K}$ such that $y = cy_s$. Given $s \in \mathbb{N}$, define $e_s := (\widehat{y}(t))_{t\in\mathbb{N}}$

by $\widehat{y}(t) = 0$ if $t \neq s$ and $\widehat{y}_s(s) = y_s$. It is easy to see that $\overline{\text{span}(e_s, s \in \mathbb{N})} = E_0$. Moreover, $\| \sum_{s=1}^{n} c_s e_s \|_k \leq \| \sum_{s=1}^{n+1} c_s e_s \|_k$, for each $n, k \in \mathbb{N}$ and each $c_1, \ldots, c_{n+1} \in \mathbb{K}$. Then Theorem 14.3.6 in [Jar81, p. 298] implies that $\{e_s\}_{s\in\mathbb{N}}$ is a Schauder basis of X_0.

Now define $I : E \to E_0$ by $I(x) := \left(\widetilde{A_s}(x)\right)_{s\in\mathbb{N}}$. Since $x = \sum_{s=1}^{\infty} \widetilde{A_s}(x)$ in E, it follows that $I(x)$ is well-defined. Moreover, I and $I^{-1} : I(E) \to E$ are continuous by the estimates $\| I(x) \|_k \leq K_k \|x\|_{w_k}$ (that were proved when we showed the equicontinuity of $\left(\sum_{s=1}^{n} \widetilde{A_s}\right)_n$) and

$$\|x\|_k = \left\| \lim_{n\to\infty} \sum_{s=1}^{n} \widetilde{A_s}(x) \right\|_k \leq \sup_n \left\| \sum_{s=1}^{n} \widetilde{A_s}(x) \right\|_k = \| I(x) \|_k.$$

Observe that $I^{-1}\left((y(s))_s\right) = \sum_s y(s)$, if $(y(s))_s \in I(E)$.

Finally, we define a projection $L : E_0 \to I(E)$ by $L\left((y(s))_s\right) := \left(\widetilde{A_s}\left(\sum_{t=1}^{\infty} y(t)\right)\right)_s$. To check that L is continuous, if $\sum_{s=1}^{\infty} y(s)$ converges in E, using

$$\left\| \sum_{s=1}^{\infty} y(s) \right\|_l = \lim_{n\to\infty} \left\| \sum_{s=1}^{n} y(s) \right\|_l \leq \sup_n \left\| \sum_{s=1}^{n} y(s) \right\|_l = \| (y(s))_s \|_l,$$

for each $l \in \mathbb{N}$ and for each $y \in E_0$, then

$$\| \left(A_s \left(\sum_{t=1}^{\infty} y(t) \right) \right)_s \|_k = \sup_n \left\| \sum_{s=1}^{n} \widetilde{A_s} \left(\sum_{t=1}^{\infty} y(t) \right) \right\|_k \leq K_k \left\| \sum_{t=1}^{\infty} y(t) \right\|_{\omega_k} \leq K_k \| (y(s))_s \|_{\omega_k}.$$

Finally, $L^2 = L$, since $\sum_{n=1}^{\infty} \widetilde{A_n}(z) = z$ for every $z \in E$. □

2 Extension of injective maps. Vogt's Example of a nuclear Fréchet space without the BAP

In this section we present Vogt's counterexample [Vog83] of a nuclear Fréchet space which does not satisfy the bounded approximation property. Some results on the extension of injective continuous linear maps between normed spaces are needed first.

Let E, F be two normed spaces and let $T \in L(E, F)$. We denote $\widehat{E}, \widehat{F}$ the completion of E, F respectively. We know that $T : E \to \widehat{F}$ is a continuous map. There exists a unique continuous linear map $\widehat{T} : \widehat{E} \to \widehat{F}$ such that the restriction $\widehat{T}\Big| E$ of $\widehat{T}$ to E coincides T. It is defined as $\widehat{T}(x) := \lim_{j\to\infty} T(x_j)$, with $(x_j)_j \subset E$ and $x_j \to x$ in $\widehat{E}$ as $j \to \infty$. In general, $\widehat{T}$ need not to be injective.

Example 2.1 Let $(X, \|\cdot\|)$ be an infinite dimensional Banach space. Take $u \in X^* \backslash X'$ (i.e. $u : X \to \mathbb{K}$ is a non-continuous linear form), and define $\|x\| := |u(x)| + \|x\|$ (observe that $\|\cdot\| \leq \| \cdot \|$ in X). Clearly the identity $T : (X, \| \cdot \|) \to (X, \|\cdot\|)$ is continuous and injective. Then, there exists a unique continuous linear map $\widehat{T} : \widehat{(X, \| \cdot \|)} \to (X, \|\cdot\|)$ such that $\widehat{T}\Big| X = T$. Clearly T is surjective since $\widehat{T}(X) = T(X) = X \subset \widehat{T}\left(\widehat{(X, \| \cdot \|)}\right)$. Assume that T is injective. By the closed graph theorem, $\left(\widehat{T}\right)^{-1}$ would be continuous. This would imply $\|x\| = |U(x)| + \|x\| \leq C|x|$, for every $x \in X$, then $u \in X'$, a contradiction.

Proposition 2.2 *Let X, Y be normed spaces and let $A : (X, \|\cdot\|) \to (Y, \||\cdot\||)$ be a continuous injective linear operator. The unique continuous linear extension $\widehat{A} : \widehat{(X, \|\cdot\|)} \to \widehat{(Y, \||\cdot\||)}$ of A is injective if and only if for every $(x_j)_j \subset X$, which is $\|\cdot\|$-Cauchy in X such that* $\lim_{j\to\infty} \||Ax_j\|| = 0$ *we have* $\lim_{j\to\infty} \|x_j\| = 0$.

Proof. Let $\widehat{x} \in \widehat{(X, \|\cdot\|)}$ such that $\widehat{A}\widehat{x} = 0$ in $\widehat{(Y, \||\cdot\||)}$. Then, there exists a $(x_j)_j \subset X$, where $(x_j)_j$ is $\|\cdot\|$-Cauchy in X, such that $x_j \to \widehat{x}$ in $\widehat{(X, \|\cdot\|)}$. Using that $\widehat{A}$ is continuous $\widehat{A}x_j = Ax_j$ converges $\widehat{A}\widehat{x} = 0$ in $\widehat{(Y, \||\cdot\||)}$, hence $Ax_j \to 0$ in $(Y, \||\cdot\||)$ as $j \to \infty$. By assumption $x_j \to 0$ in X, hence $\widehat{x} = 0$. And $\widehat{A}$ is injective.

In order to show the converse, let $(x_j)_j$ be a Cauchy sequence in X such that $\lim_{j\to\infty} \||Ax_j\|| = 0$. There is $\widehat{x} \in \widehat{X}$ such that $x_j \to \widehat{x}$ in $\widehat{(X, \|\cdot\|)}$, thus $Ax_j \to \widehat{A}\widehat{x}$ in $\widehat{(Y, \||\cdot\||)}$. This implies $\widehat{A}\widehat{x} = 0$ and, since $\widehat{A}$ is injective, $\widehat{x} = 0$. Therefore $\lim_{j\to\infty} \|x_j\| = 0$. □

Lemma 2.3 ***(Vogt's main Lemma)*** *Let E be a Fréchet space with a fundamental system of seminorms $(\|\cdot\|_k)_k$. Assume that E has a continuous norm and the BAP. Then, there exists $k(0)$ such that for every $k \geq k(0)$ exists $l \geq k$ such that for every $(x_j)_j \subset E$ that is $\|\cdot\|_l$-Cauchy such that* $\lim_{j\to\infty} \|x_j\|_{k(0)} = 0$ *we have* $\lim_{j\to\infty} \|x_j\|_k = 0$.

Proof. Let $(A_\tau)_{\tau\in T}$ be an equicontinuous net in $L(E)$ such that $A_\tau(E)$ is finite dimensional for every $\tau \in T$ and $A_\tau x$ converges to x for each $x \in E$.

Let $\|\cdot\|_{k(1)}$ be a norm. Select $k(0) \geq k(1)$ and $C > 0$ such that $\|A_\tau x\|_{k(1)} \leq C\|x\|_{k(0)}$ for each $x \in E$ and each $\tau \in T$.

Since $\|\cdot\|_{k(1)}$ is a norm and $A_\tau(E)$ is finite dimensional, for each $\tau \in T$ and $k \in \mathbb{N}$ there is $C_{\tau,k} > 0$ such that

$$\|A_\tau x\|_k \leq C_{\tau,k} \|A_\tau x\|_{k(1)} \leq C_{\tau,k} C \|x\|_{k(0)} .$$

Fix now $k \geq k(0)$ and select $l \geq k$ and $D > 0$ such that $\|A_\tau x\|_k \leq D\|x\|_l$ for every $x \in E$ and for every $\tau \in T$. Let $(x_j)_j \subset E$ be a $\|\cdot\|_l$-Cauchy sequence such that $\lim_{j\to\infty} \|x_j\|_{k(0)} = 0$. Given $\varepsilon > 0$, choose $j(0) \in \mathbb{N}$ such that $\|x_j - x_{j(0)}\|_l < \varepsilon$ if $j \geq j(0)$. Given $x_{j(0)} \in E$, select $\tau \in T$ such that $\|x_{j(0)} - A_\tau x_{j(0)}\|_k < \varepsilon$. For $j \geq j(0)$, we have

$$\begin{aligned}
\|x_j\|_k &\leq \|x_j - x_{j(0)}\|_k + \|x_{j(0)} - A_\tau x_{j(0)}\| + \|A_\tau (x_{j(0)} - x_j)\| + \|A_\tau x_j\| \leq \\
&\leq \|x_j - x_{j(0)}\|_l + \varepsilon + C_{\tau,k} C \|x_{j(0)} - x_j\|_{k(0)} + C_{\tau,k} C \|x_j\|_{k(0)} \leq \\
&\leq (2\varepsilon + C_{\tau,k} C\varepsilon) + C_{\tau,k} C \|x_j\|_{k(0)} .
\end{aligned}$$

Selecting $j(1) > j(0)$ with $\|x_j\|_{k(0)} < \varepsilon$, we get $\|x_j\|_k \leq (2 + 2C_{\tau,k})\varepsilon$ for all $j \geq j(1)$. □

Example 2.4 (Vogt's example) Take $0 < \rho_{\mu,\nu} \leq 1$, $\mu, \nu \in \mathbb{N}$, with $\lim_{\mu\to 0} \rho_{\mu,\nu} = 0$ for every $\nu \in \mathbb{N}$. Denote $x = \left(x^n_{\mu,\nu}\right)_{n,\mu,\nu} \in \mathbb{K}^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$. For $p \in \mathbb{N}$, define

$$\|x\|_p := \sum_{n,\mu,\nu\leq p} \left|x^n_{\mu,\nu}\right| p^{n+\mu+\nu} + \sum_{n,\mu,\nu>p} \left|\rho_{\mu,\nu} x^n_{\mu,\nu} - x^{n+1}_{\mu,\nu}\right| p^{n+\mu+\nu}.$$

Set $E := \left\{ x = \left(x^n_{\mu,\nu}\right)_{n,\mu,\nu\in\mathbb{N}^3} : \|x\|_p < \infty \text{ for every } p \in \mathbb{N} \right\}$. It is a Fréchet space and

$$\|x\|_p \leq 2 \left(\sum_{n,\mu,\nu\leq p+1} \left|x^n_{\mu,\nu}\right| p^{n+\mu+\nu} + \sum_{n,\mu,\nu>p+1} \left|\rho_{\mu,\nu} x^n_{\mu,\nu} - x^{n+1}_{\mu,\nu}\right| p^{n+\mu+\nu} \right) =: \|x\|'_p .$$

To see this, use the inequality

$$\sum_{n,\mu,\nu=p+1} \left|\rho_{\mu,\nu} x^n_{\mu,\nu} - x^{n+1}_{\mu,\nu}\right| p^{n+\mu+(p+1)} \leq \sum_{n,\mu} \left(\left|x^n_{\mu,p+1}\right| + \left|x^{n+1}_{\mu,p+1}\right| \right) p^{n+\mu+(p+1)}.$$

The canonical map $\left(E, \|\cdot\|_{p+1}\right) \mapsto \left(E, \|\cdot\|'_p\right)$ is nuclear, then E is a nuclear space. Indeed,

$$x = \sum_{n,\mu,\nu} (e_{n,\mu,\nu} \otimes u_{n,\mu,\nu})(x),$$

where $e_{n,\mu,\nu}$ are the canonical unit vectors in $E \subset \mathbb{K}^{\mathbb{N}^3}$ and $u_{n,\mu,\nu}$ are the canonical unit vectors in the dual. Exactly in the same positions we obtain

$$\begin{array}{ll} \|e_{n,\mu,\nu}\|'_p & = \begin{cases} p^{n+\mu+\nu}, \text{ or} \\ \rho_{\mu,\nu} p^{n+\mu+\nu} \end{cases} \\ \|u_{n,\mu,\nu}\|_{p+1} & = \begin{cases} \frac{1}{(p+1)^{n+\mu+\nu}}, \text{ or} \\ \frac{1}{\rho_{\mu,\nu}(p+1)^{n+\mu+\nu}} \end{cases} \end{array} \quad \text{then} \sum_{n,\mu,\nu} \|e_{n,\mu,\nu}\|'_p \|u_{n,\mu,\nu}\|_{p+1} < \infty.$$

Now, observe that $\|\cdot\|_p$ is a norm in E for all p. Indeed, assume $x = \left(x^n_{\mu,\nu}\right) \in E$ satisfies $\|x\|_p = 0$ then $x^n_{\mu,\nu} = 0$ for all n, μ if $\nu \leq p$. In that case, $\rho_{\mu,\nu} x^{n+1}_{\mu,\nu}$ with $\nu > p$ for every n, μ then $x^n_{\mu,\nu} = \rho^n_{\mu,\nu} x^{n+1}_{\mu,\nu}$ with $\nu > p$ for every $\mu \in \mathbb{N}$. Suppose there are $\mu \in \mathbb{N}$ and $\nu > p$ such that $x^1_{\mu,\nu} \neq 0$. Select $q \in \mathbb{N}$ with $q\rho_{\mu,\nu} > 1$ and $q > \nu$. Then, since $x \in E$,

$$\infty \overset{q\geq\nu}{\geq} \|x\|_q \leq \sum_n \left|x^n_{\mu,\nu}\right| q^{n+\mu+\nu} = \sum_n (\rho_{\mu,\nu})^{n-1} \left|x^1_{\mu,\nu}\right| q^{n+\mu,\nu} = \left|x^1_{\mu,\nu}\right| \sum_n (\rho_{\mu,\nu} q)^{n-1} q^{\mu+\nu+1} = \infty,$$

and this is a contradiction. Then $x^1_{\mu,\nu} = 0$ for each μ and each $\nu > p$ implies that $x^n_{\mu,\nu} = 0$ for every $\mu, n \in \mathbb{N}$ and for every $\nu > p$. Therefore $x = 0$.

To prove that E does not have the BAP, we use Vogt's main lemma and we will prove for every p_0 and for every $q \geq p = p_0 + 1$ there exists $(x_m)_m \subset E$, which is $\|\cdot\|_q$-Cauchy, with $\|x_m\|_{p_0}$ converging 0 as m tends to infinity, but $\|x_m\|_p$ does not converge to 0. To prove this fact, given $q \geq p_0 + 1$, select $\mu \in \mathbb{N}$ such that $\rho_{\mu,p} q < 1$. Define $x_m := \sum_{n=1}^m (\rho_{\mu,p})^n e_{n,\mu,p}$, where $e_{n,\mu,\nu}$ are the canonical unit vectors in E. For $l < m$ we get

$$\|x_m - x_l\|_q = \left\| \sum_{n=l+1}^m (\rho_{\mu,p})^n e_{n,\mu,p} \right\|_q \overset{q\geq p}{=} \sum_{l+1}^m \rho^n_{\mu,p} q^{n+\mu+p} = q^{\mu+p} \sum_{l+1}^m (\rho_{\mu,p} q)^n .$$

And $(x_m)_m$ is $\|\cdot\|_q$-Cauchy, since $\rho_{\mu,p} q < 1$. On the other hand,

$$\|x_m\|_{p_0} = \sum_{n=1}^m \left|\rho_{\mu,p} x^n_{\mu,p} - x^{n+1}_{\mu,p}\right| p_0^{n+\mu+p},$$

where, for $n = 1, \ldots, m-1$, $\rho_{\mu,p} x_{\mu,p}^{n} - x_{\mu,p}^{n+1} = \rho_{\mu,p} \rho_{\mu,p}^{n} - \rho_{\mu,p}^{n+1} = 0$, therefore

$$\|x_m\|_{p_0} = \rho_{\mu,p}^{m+1} p_0^{m+\mu+p} = p_0^{\mu+p-1} (\rho_{\mu,p} p_0)^{m+1} \to 0 \text{ as } m \to \infty,$$

since $\rho_{\mu,p} p_0 < \rho_{\mu,p} q < 1$.

Finally,

$$\|x_m\|_p = \sum_{n=1}^{m} \rho_{\mu,p}^{n} p^{n+\mu+p} = p^{\mu+p} \sum_{n=1}^{m} (\rho_{\mu,p} p)^n \geq p^{\mu+p+1} \rho_{\mu,p} \text{ for every } m \in \mathbb{N}.$$

Therefore $\|x_m\|_p$ does not converge to 0 as m tends to infinity.

3 Countably normable Fréchet Spaces

Definition 3.1 A Fréchet space E is *countably normable (or countably normed)* if there exists a fundamental sequence of norms $(\|\cdot\|_k)_k$ defining the topology of E such that the inclusions $i_k : (E, \|\cdot\|_{k+1}) \to (E, \|\cdot\|_k)$ can be extended (uniquely) to an injection $\varphi_k : \widehat{(E, \|\cdot\|_{k+1})} \to \widehat{(E, \|\cdot\|_k)}$ (i.e. E is an intersection of Banach spaces).

The following result is a consequence of proposition 2.2.

Lemma 3.2 *Let X be a vector space with two norms $\|\cdot\| \leq \|\!|\cdot|\!\|$. The inclusion $i : (X, \|\!|\cdot|\!\|) \to (X, \|\cdot\|)$ extends uniquely to an injective continuous map if and only if for every $(x_n)_n \subset X$, which is $\|\!|\cdot|\!\|$-Cauchy, such that $\|x_n\| \to 0$ as n tends to infinity then $\|\!|x_n|\!\| \to 0$.*

Remark 3.3 A Fréchet space E with a continuous norm is countably normable if and only if there exists a fundamental system $(\|\cdot\|_k)_k$ of norms on E such that for every $k \in \mathbb{N}$ there exists $j > k$ such that if $(x_n)_n \subset X$ is $\|\cdot\|_j$-Cauchy and $\lim_n \|x_n\|_k = 0$, then $\lim_n \|x_n\|_j = 0$.

Indeed, if we suppose that E is countably normable then it is enough to take $j = k+1$. If the condition is satisfied, it is enough to pass to a subsequence.

As a consequence, if E is a Fréchet space which is countably normable and $F \subset E$ is a closed subspace, then F is a countably normable Fréchet space. To prove it we just take the restriction to F of the norms given by the remark on E.

Proposition 3.4 *Every Fréchet space E with a Schauder basis and a continuous norm is countably normable. Consequently, every separable Fréchet space with a continuous norm and the bounded approximation property is countably normable.*

Proof. Let $(x_n)_n$ be a Schauder basis in E with coefficient functionals $(x_n')_n$. We write

$$\begin{array}{rccc} A_n : & E & \longrightarrow & E \\ & x & \longrightarrow & \langle x_n', x\rangle x_n. \end{array}$$

We have $\dim(A_n(E)) = 1$, $A_n A_m = \delta_{n,m} A_n$ if $n \neq m$ and $x = \sum_{n=1}^{\infty} A_n(x) = \sum_{n=1}^{\infty} \langle x_n', x\rangle x_n$ converging in E for every $x \in E$.

Given a fundamental sequence of norms $(\|\cdot\|_k)_{k\in\mathbb{N}}$ in E, define $|y|_k := \sup_n \|\sum_{i=1}^{n} A_i y\|_k$ for every $y \in E$ and $k \in \mathbb{N}$. Then $(|\cdot|_k)_k$ is a fundamental sequence of seminorms in E. Indeed,

$$\|x\|_k = \lim_{n\to\infty} \left\| \sum_{i=1}^{n} A_i(x) \right\|_k \leq \sup_n \left\| \sum_{i=1}^{n} A_i(y) \right\|_k = |x|_k,$$

for every $k \in \mathbb{N}$ and for every $x \in E$. In particular $|\cdot|_k$ is a norm for each k. On the other hand, since $\left(\sum_{i=1}^{n} A_i\right)_n$ is equicontinuous in $L(E)$ by Banach-Steinhaus' Theorem, for every $k \in \mathbb{N}$, there exists $l(k) > k$ and $C_k > 0$ such that $\left\|\sum_{i=1}^{n} A_i(x)\right\|_k \leq C_k \|x\|_{l(k)}$ for every $n \in \mathbb{N}$ and for every $x \in E$. This implies that for every $k \in \mathbb{N}$ there exists $l(k) > k$ and $C_k > 0$ such that $|x|_k \leq C_k \|x\|_{l(k)}$.

Consider that map $A_j : (E, |\cdot|_k) \to (E, |\cdot|_k)$ between these normed spaces. It is continuous

$$|A_j(y)|_k = \sup_n \left\| \sum_{i=1}^{n} A_i A_j(y) \right\|_k = \|A_j(y)\|_k \leq \left\| \sum_{i=1}^{j} A_j(y) - \sum_{i=1}^{j-1} A_j(y) \right\|_k \leq 2|y|_k,$$

and if $j = 1$ then $|A_1(y)|_k \leq |y|_k$. Then there exists a unique continuous extension $A_j^k : \widehat{(E, |\cdot|_k)} \to \widehat{(E, |\cdot|_k)}$ such that $A_j^k\Big| E = A_j$ for each $j \in \mathbb{N}$. Moreover, since $A_j^k(E)$ is finite dimensional (in fact 1-dimensional), it is closed and $A_j^k\left[\widehat{(E, |\cdot|_k)}\right] \subset A_j(E) = \mathrm{span}(x_j)$. Since $A_i A_j = \delta_{i,j} A_j$ on E, by density $A_i^k A_j^k = \delta_{i,j} A_i^k$. We show now that $\left(\sum_{i=1}^{n} A_j\right)_n$ is equicontinuous on $L(E, |\cdot|_k)$. If $y \in E$, $m \in \mathbb{N}$,

$$\left\| \sum_{i=1}^{m} A_i \left(\sum_{j=1}^{n} A_j \right)(y) \right\|_k \overset{m \geq n}{=} \left\| \sum_{j=1}^{n} A_j(y) \right\|_k \leq |y|_k,$$

since

$$\begin{aligned} \left\| \sum_{j=1}^{n} A_j(y) \right\|_k &= \sup_m \left\| \sum_{i=1}^{m} \left(\sum_{j=1}^{n} A_j \right)(y) \right\|_k = \\ &= \sup_{m \geq n} \left\| \sum_{i=1}^{m} A_i \left(\sum_{j=1}^{m} A_j \right)(y) \right\|_k + \sup_{1 \leq m < n} \left\| \sum_{i=1}^{m} A_i \left(\sum_{j=1}^{m} A_j \right)(y) \right\|_k \leq 2|y|_k. \end{aligned}$$

This implies that the extensions $\left(\sum_{j=1}^{n} A_j^k\right)_n$ form also an equicontinuous set in $L\left(\widehat{(E, |\cdot|_k)}\right)$. Since $\sum_{j=1}^{n} A_j^k \to I$ pointwise in E and $\left(\sum_{j=1}^{n} A_j^k\right)_n$ is equicontinuous in $L\left(\widehat{(E, |\cdot|_k)}\right)$ then, for every $x \in \widehat{(E, |\cdot|_k)}$, $\sum_{j=1}^{n} A_j^k \widehat{x} \overset{n}{\to} \widehat{x}$ in $\widehat{(E, |\cdot|_k)}$.

We finally prove that the unique extension $\varphi_k : \widehat{\left(E, |\cdot|_{k+1}\right)} \to \widehat{(E, |\cdot|_k)}$ of the identity $\left(E, |\cdot|_{k+1}\right) \to (E, |\cdot|_k)$ is injective. Fix $\widehat{y} \in \widehat{\left(E, |\cdot|_{k+1}\right)}$ such that $\varphi_k \widehat{y} = 0$ in $\widehat{(E, |\cdot|_k)}$. We know $\widehat{y} = \sum_{n=1}^{\infty} A_n^{k+1} \widehat{y}$ the series converging in $\widehat{\left(E, |\cdot|_{k+1}\right)}$ and the decomposition is unique, since $A_i^{k+1} A_j^{k+1} = \delta_{i,j} A_i^{k+1}$ if $i \neq j$. Moreover, $A_n^{k+1}(\widehat{y}) \in E$, since $A_n(E)$ is finite dimensional in E, hence closed in $\widehat{\left(E, |\cdot|_{k+1}\right)}$.

Now $0 = \varphi_k(\widehat{y}) = \sum_{n=1}^{\infty} A_n^k(\varphi_k(\widehat{y}))$, the series converging in $\widehat{(E, |\cdot|_k)}$. Since the decomposition is unique, we have $A_n^k(\varphi_k(\widehat{y})) = 0$ for each $n \in \mathbb{N}$. We are done if we show that $A_n^{k+1}\widehat{y} = A_n^k(\varphi_k(\widehat{y}))\,(= 0)$, because this will imply $\widehat{y} = \sum_{n=1}^{\infty} A_n^{k+1} \widehat{y} = 0$.

To prove $A_n^{k+1} = A_n^k(\varphi_k(\widehat{y}))$, select $(y_s) \subset E$ such that $y_s \to \widehat{y}$ in $\widehat{\left(E, |\cdot|_{k+1}\right)}$. Then y_s converges to $\varphi_k(\widehat{y})$ in $\widehat{(E, |\cdot|_k)}$. Now A_n^k is the extension of A_n and A_n^{k+1} of A_n. Thus $A_n^{k+1}(\widehat{y}) = \lim_{s \to \infty} A_n(y_s) = A_n^k(\varphi_k(\widehat{y}))$. $\square$

The advantage of the next characterization is that it is formulated in terms of an arbitrary fundamental sequence of seminorms.

Theorem 3.5 ***(Dubinski, Vogt, 1985)*** *Let E be a Fréchet space with a continuous norm. Let $(\|\cdot\|_k)$ be a increasing sequence of norms which define the topology of E. Denote $E_k = (\widehat{E, \|\cdot\|_k})$ and $\varphi_k : E_{k+1} \to E_k$ the unique extension of the identity $i : \left(E, \|\cdot\|_{k+1}\right) \to (E, \|\cdot\|_k)$. Then, the following are equivalent:*

1. *E is countably normable*
2. *There exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ there exists $j > k$ such that if $(x_n)_n \subset E$ is $\|\cdot\|_j$-Cauchy and $\|x_n\|_{k_0} \to 0$, then $\|x_n\|_k \to 0$.*

Proof. In order to show (1) $\Rightarrow$ (2), let $(|\cdot|_k)_{k\in\mathbb{N}}$ be a fundamental sequence of norms in E such that the extensions $\varphi_k : \left(\widehat{E, |\cdot|_{k+1}}\right) \to (\widehat{E, |\cdot|_k})$ of the identity $i : \left(E, |\cdot|_{k+1}\right) \to (E, |\cdot|_k)$ are injective for each k.

Select $k_0 \in \mathbb{N}$ such that $|\cdot|_1 \leq C \|\cdot\|_{k_0}$ for some $C > 0$ (recall that both $(|\cdot|_k)$ and $(\|\cdot\|_k)_k$ are fundamental systems of seminorms of E). Fix $k \geq k_0$ and choose k' such that $\|\cdot\|_k \leq D |\cdot|_{k'}$ for some $D > 0$. Now choose j such that $|\cdot|_{k'} \leq E \|\cdot\|_j$.

Take $(x_n)_n \subset E$, which is $\|\cdot\|_j$-Cauchy and satisfies $\|x_n\|_{k_0} \to 0$ as n tends to infinity. Since $|\cdot|_1 \leq C \|\cdot\|_{k_0}$, we get $|x_n|_1 \to 0$ as $n \to \infty$. Moreover $(x_n)_n$ is $|\cdot|_{k'}$-Cauchy in E and the unique extension $\varphi_1 \circ \ldots \circ \varphi_{k'-1} : (\widehat{E, |\cdot|_{k'}}) \to (\widehat{E, |\cdot|_1})$ of the identity $i : (E, \|\cdot\|_{k'}) \to (E, \|\cdot\|_1)$ is injective. By the lemma, $|x_n|_{k'} \to 0$ as $n \to \infty$. Since $\|\cdot\|_k \leq D |\cdot|_{k'}$, we conclude $\|x_n\|_k \stackrel{n}{\to} 0$. Then, the proof of (2) is complete.

Now, in order to show (2) $\Rightarrow$ (1) we first prove the following

Claim 3.6 *Condition (2) implies that there exists $k_0 \in \mathbb{N}$ such that, for every element $x \in \cap_{k=k_0}^{\infty} \varphi_{k_0} \ldots \varphi_k (E_{k+1})$, there exist $x_k \in E_k, k \in \mathbb{N}$, such that $x_{k_0} = x$ and $x_k = \varphi_k x_{k+1}$ for every $k \geq k_0$ (i.e. x belongs to $Proj_{k\geq k_0} ((E_k)_k , \varphi_k : E_{k+1} \to E_k)$).*

Proof. For each $k \geq k_0$, we choose $j_k > k$ satisfying (2), we may select it satisfying $j_{k+1} > j_k$ for each $k \in \mathbb{N}$. Given $x \in \cap_{k=k_0}^{\infty} \varphi_{k_0} \ldots \varphi_k (E_{k+1})$, we can find for each $k > k_0$, $y_k \in E_k$ such that $x = \varphi_{k_0} \ldots \varphi_{k-1} y_k$. By (2), $\varphi_{k_0} \ldots \varphi_{k-1}$ is injective on $\varphi_k \ldots \varphi_{j_k-1} (E_{j_k})$. Indeed, $\varphi_{k_0} \ldots \varphi_{k-1} : E_k \to E_{k_0}$ and $\varphi_k \ldots \varphi_{j_k-1} : E_{j_k} \to E_k$ and $(\varphi_{k_0} \ldots \varphi_{k-1}) (\varphi_k \ldots \varphi_{j_k-1}) : E_j \to E_{k_0}$ is the unique continuous extension of the identity $i : \left(E, \|\cdot\|_{j_k}\right) \to \left(E, \|\cdot\|_{k_0}\right)$. By (2) if $(x_n)_n$ is $\|\cdot\|_{j_k}$-Cauchy and $\|x_n\|_{k_0} \to 0$, therefore $\|x_n\|_k \to 0$.

Suppose $(\varphi_{k_0} \ldots \varphi_{k-1}) (\varphi_k \ldots \varphi_{j_k-1}) (z) = 0$ with $z \in E_{j_k}$. Select $(z_n)_n \subset E$, which is $\|\cdot\|_{j_k}$-Cauchy with $z_n \to z$ in E_{j_k}. We obtain

$$(\varphi_{k_0} \ldots \varphi_{k-1}) (\varphi_k \ldots \varphi_{j_k-1}) (z_n) = z_n \to (\varphi_{k_0} \ldots \varphi_{k-1}) (\varphi_k \ldots \varphi_{j_k-1}) (z) = 0 \text{ in } E_{k_0},$$

then $\|z_n\|_{k_0} \stackrel{n}{\to} 0$; therefore, $\|z_n\|_k \stackrel{n}{\to} 0$.

On the other hand, as $z_n \stackrel{n}{\to} z$ in E_{j_k} then $z_n = (\varphi_k \ldots \varphi_{j_k-1}) (z_n) \stackrel{n}{=\!\!\to} (\varphi_k \ldots \varphi_{j_k-1}) (z)$ in E_k. Therefore $z_n \stackrel{k}{\to} \varphi_k \ldots \varphi_{j_k-1} (z)$ in E_k but $\|z_n\|_k \stackrel{n}{\to} 0$ and this implies $(\varphi_k \ldots \varphi_{j_k-1}) (z) = 0$.

We define now $x_{k_0} := x$ and $x_k := \varphi_k \ldots \varphi_{j_k-1} (y_{j_k})$ for each $k \in \mathbb{N}$. Observe first that the injectivity of $\varphi_{k_0} \ldots \varphi_{k-1}$ on $\varphi_k \ldots \varphi_{j_k-1} (E_{j_k})$ implies $\varphi_k \ldots \varphi_{j_k-1} (y_{j_k}) = \varphi_k \ldots \varphi_{j-1} (y_j)$

for all $j \geq j_k$. Both belong to $\varphi_k \dots \varphi_{j_k-1}(E_{j_k}) \subset E_k$ and both are mapped to x by $\varphi_{k_0} \dots \varphi_{k-1}$. In particular, for $j = j_{k+1}$ we get $\varphi_k \dots \varphi_{j_k-1}(y_{j_k}) = \varphi_k \dots \varphi_{j_k}\left(y_{j_{k+1}}\right)$. Now $x_{k+1} = \varphi_{k+1} \dots \varphi_{j_k}\left(y_{j_{k+1}}\right)$, thus $\varphi_k x_{k+1} = \varphi_k \varphi_{k+1} \dots \varphi_{j_k}\left(y_{j_{k+1}}\right) = \varphi_k \dots \varphi_{j_k-1}(y_{j_k}) = x_k$. And $x_{k_0+1} = \varphi_{k_0+1} \dots \varphi_{j_{k_0+1}-1}\left(y_{j_{k_0+1}}\right)$ and $\varphi_{k_0} x_{k_0+1} = \varphi_{k_0}\varphi_{k_0+1} \dots \varphi_{j_{k_0+1}}\left(y_{j_{k_0+1}}\right) = x = x_{k_0}$. And the claim is proved. □

We may assume that $k_0 = 1$ in the claim. So, for every $x \in \cap_{k=1}^{\infty} \varphi_1 \dots \varphi_k(E_{k+1})$, there exists $(x_k)_k$ such that $x_k \in E_k$, with $x_1 = x$ and $x_k = \varphi_k x_{k+1}$ for each $k \geq 1$. Set $F_k := \varphi_1 \dots \varphi_k(E_{k+1})$ with the quotient norm induced by E_{k+1}. The space $F = \cap_k F_k$ with the projective topology is a countably normed Fréchet space. Observe that $F \subset E_1$ algebraically and the injection is continuous, since each map $\varphi_1 \dots \varphi_k : E_{k+1} \to E_1$ is continuous.

We denote by $P_k : E \to E_k$ the canonical inclusion. Recall that $\|\cdot\|_1$ is a norm, hence $P_k : E \to (\widehat{E, \|\cdot\|_k})$ is injective. We show that $P_1 E = F$ (in E_1). By definition of E, $P_1 = \varphi_1 \dots \varphi_k P_{k+1}$ for each $k \in \mathbb{N}$ then $P_1 E = \varphi_1 \dots \varphi_k P_{k+1} E \subset \varphi_1 \dots \varphi_k(E_{k+1})$ for each k; therefore, $P_1 E = F$. On the other hand, if $y \in F \subset E_1$, $y = \cap_{k=1}^{\infty} \varphi_1 \dots \varphi_k(E_{k+1})$ we apply the claim to find $(x_k)_k$, $x_k \in E_k$ for each k such that $x_1 = y$ and $\varphi_k x_{k+1} = x_k$ for each $k \in \mathbb{N}$. Since E is a Fréchet space and $E = proj_k(E_k, \varphi_k)$, there is $x \in E$ with $P_k(x) = x_k$ for each k. In particular $P_1(x) = y$ and $F \subset P_1 E$. Thus $P_1 : E \to F \subset E_1$ is bijective. We know that $P_1 : E \to E_1$ is continuous and the inclusion $F \subset E_1$ is also continuous. If we prove that P_1 has closed graph as a map from E to F, the closed graph theorem implies that P_1 is a continuous and (being bijective) by the open graph theorem an isomorphism. Suppose $x_n \to x$ in E and $P_1 x_n \to y$ in F, then $P_1 x_n \to y$ in E_1 (since $F \hookrightarrow E_1$ is continuous) and $P_1 x_n \to Px$ in E_1 (since $P_1 : E \hookrightarrow E_1$ is continuous), therefore E_1 is Banach/Hausdorff and then $Px = y$. □

Consequence 3.7 *Vogt's Example 2.4 is not countably normable.*

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