Classical operators on weighted Banach spaces of entire functions

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On joint work with María José Beltrán and Carmen Fernández
Investigate the dynamics of the operators of

**Differentiation:** \( Df := f' \)

**Integration:** \( Jf(z) := \int_0^z f(\xi) d\xi, \ z \in \mathbb{C} \)

on weighted Banach spaces of entire functions.

- \( D \) and \( J \) are continuous on \((H(\mathbb{C}), \text{co})\), where \( \text{co} \) denotes the compact-open topology.
- \( DJf = f \) and \( JDf(z) = f(z) - f(0) \ \forall f \in H(\mathbb{C}), \ z \in \mathbb{C} \).
**Weights**

A weight $v$ on $\mathbb{C}$ is a strictly positive continuous function on $\mathbb{C}$ which is radial, i.e. $v(z) = v(|z|)$, $z \in \mathbb{C}$, $v(r)$ is non-increasing on $[0, \infty[$ and rapidly decreasing, that is, it satisfies $\lim_{r \to \infty} r^n v(r) = 0$ for each $n \in \mathbb{N}$.

For $r \geq 0$ and $f \in \mathcal{H}(\mathbb{C})$, consider

$$M_p(f, r) := \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \right)^{1/p}$$

for $1 \leq p < \infty$ and

$$M_\infty(f, r) := \sup_{|z|=r} |f(z)|, \ r \geq 0.$$  

Note that for each $1 \leq p < \infty$ and each $n \in \mathbb{N}$, we have

$$M_p(z^n, r) = M_\infty(z^n, r) \text{ for each } r > 0.$$
For a weight $v$ and $1 \leq p \leq \infty$, set

$$B_{p,\infty}(v) := \{ f \in H(\mathbb{C}) : \sup_{r > 0} v(r) M_p(f, r) < \infty \}$$

and

$$B_{p,0}(v) := \{ f \in H(\mathbb{C}) : \lim_{r \to \infty} v(r) M_p(f, r) = 0 \}.$$

Both are Banach spaces with the norm

$$|||f|||_{p,v} := \sup_{r > 0} v(r) M_p(f, r).$$

In case $p = \infty$, these spaces are usually denoted by $H^\infty_v(\mathbb{C})$ and $H^0_v(\mathbb{C})$, respectively.

We have

$$B_{p,0}(v) \subseteq B_{p,\infty}(v) \subseteq B_{1,\infty}(v) \subseteq H(\mathbb{C})$$

with continuous inclusions for every $1 \leq p \leq \infty$. 

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Structure of the spaces

- The polynomials are included in $B_{p,0}(v)$ for all $1 \leq p \leq \infty$ and they are even dense. In particular, $B_{p,0}(v)$ is separable.

- For $1 < p < \infty$, the monomials are a Schauder basis of $B_{p,0}(v)$, but this is not satisfied in general for $p \in \{1, \infty\}$.

- For every $1 \leq p \leq \infty$ the bidual of $B_{p,0}(v)$ is isometrically isomorphic to $B_{p,\infty}(v)$.
The space $H_v^\infty(\mathbb{C}) = B_{\infty,\infty}(v)$ is isomorphic either to $H^\infty$ or to $\ell_\infty$. The characterization is in terms of a technical condition on the weight $v$.

The space $H_v^0(\mathbb{C}) = B_{\infty,0}(v)$ has a basis.
Weighted spaces for exponential weights

Let $1 \leq p \leq \infty$. The space $B_{p,q}(a,\alpha)$, $q = 0$ or $q = \infty$, denotes the Bergman space associated to the following weight:

- $v_{a,\alpha}(r) = e^{-\alpha}$, $r \in [0,1]$, $v_{a,\alpha}(r) = r^a e^{-\alpha r}$, $r \geq 1$, if $a < 0$ and $v_{a,\alpha}(r) = (a/\alpha)^a e^{-a}$, $r \in [0,a/\alpha]$, $v_{a,\alpha}(r) = r^a e^{-\alpha r}$, $r \geq a/\alpha$, if $a > 0$.

- In case $a = 0$, $v_{0,\alpha}(r) = e^{-\alpha r}$, and we write $B_{p,q}(\alpha)$.

The norms will be denoted by $|||_{p,a,\alpha}$ and $|||_{p,\alpha}$. If, in addition, $p = \infty$, we simply write $|||_{a,\alpha}$ and $|||_{\alpha}$.

Especially important for us are $H^\infty_\alpha(\mathbb{C}) := B_{\infty,\infty}(\alpha)$ and $H^0_\alpha(\mathbb{C}) := B_{\infty,0}(\alpha)$.
Exponential functions in the space

The following result is useful in connection with the existence of periodic points for the operators of integration or differentiation.

**Proposition (Bonet, Bonilla)**

The following conditions are equivalent for a weight \( v \) and \( 1 \leq p \leq \infty \):

(i) \( \{e^{\theta z} : |\theta| = 1\} \subset B_{p,0}(v) \).

(ii) There is \( \theta \in \mathbb{C}, |\theta| = 1 \), such that \( e^{\theta z} \in B_{p,0}(v) \).

(iii) \( \lim_{r \to \infty} v(r) \frac{e^r}{r^{2p}} = 0 \).
For a Banach space $X$, we write

$$\mathcal{L}(X) := \{ T : X \to X \text{ linear and continuous } \}.$$ 

Given $T \in \mathcal{L}(X)$, the pair $(X, T)$ is a linear dynamical system.

**Definitions**

- Let $x \in X$. The *orbit of $x$ under $T$* is the set
  
  $$\text{Orb}(x, T) := \{ x, Tx, T^2x, \ldots \} = \{ T^n x : n \geq 0 \}.$$ 

- $x \in X$ is a *periodic point* if $\exists n \in \mathbb{N}$ such that $T^n x = x$. 
For a Banach space $X$ and $T \in \mathcal{L}(X)$, we say

**Definitions**

- $T$ **topologically mixing** $\iff \forall U, V \neq \emptyset$ open, $\exists n_0 : T^n U \cap V \neq \emptyset$ \forall n \geq n_0$.
- $T$ **hypercyclic** $\iff \exists x \in X, \text{Orb}(T, x) := \{x, Tx, T^2x, \ldots \}$ is dense in $X \Rightarrow X$ separable.

**Definition (Godefroy, Shapiro)**

$T$ is **chaotic** if
- $T$ has a dense set of periodic points,
- $T$ is hypercyclic.
Dynamics of linear operators

For a Banach space $X$ and $T \in \mathcal{L}(X)$, we define

**Definitions**

- $T$ *power bounded* $\iff \sup_n \|T^n\| < \infty$
- $T$ *Cesàro power bounded* $\iff \sup_n \left\| \frac{1}{n} \sum_{k=1}^{n} T^k \right\| < \infty$
- $T$ *mean ergodic* $\iff$
  \[
  \forall x \in X, \ \exists P_x := \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} T^k x \in X
  \]
- $T$ *uniformly mean ergodic* $\iff$
  \[
  \left\{ \frac{1}{n} \sum_{k=1}^{n} T^k \right\}_n
  \]
  converges in the operator norm.
Classical results

Theorem. Mac Lane (1952).

\( D : H(\mathbb{C}) \to H(\mathbb{C}) \) is hypercyclic, i.e.,

\[ \exists f_0 \in H(\mathbb{C}) : \forall f \in H(\mathbb{C}), \exists (n_k)_k \subseteq \mathbb{N} \text{ such that} \]

\[ f_0^{(n_k)} \to f \text{ uniformly on compact sets.} \]

Proposition.

The integration operator \( J : H(\mathbb{C}) \to H(\mathbb{C}) \) is not hypercyclic. In fact, for each \( f \in H(\mathbb{C}) \), the sequence \((J^nf)_n\) converges to 0 in \( H(\mathbb{C}) \).
$\mathcal{P}$ is the space of polynomials.

### Proposition.

Let $T : (\mathcal{H}(\mathbb{C}), \tau_{co}) \to (\mathcal{H}(\mathbb{C}), \tau_{co})$ be a continuous linear operator such that $T(\mathcal{P}) \subset \mathcal{P}$, let $v$ be a weight and $1 \leq p \leq \infty$. The following conditions are equivalent:

1. $T(B_{p,\infty}(v)) \subset B_{p,\infty}(v)$.
2. $T : B_{p,\infty}(v) \to B_{p,\infty}(v)$ is continuous.
3. $T(B_{p,0}(v)) \subset B_{p,0}(v)$.
4. $T : B_{p,0}(v) \to B_{p,0}(v)$ is continuous.

Moreover, in this case the norm and the spectrum of the operators coincide.

**Harutyunyan, Lusky, 2008**: The continuity of $D$ and $J$ on $H^{\infty}_v(\mathbb{C})$ is determined by the growth or decline of $v(r)e^{\alpha r}$ for some $\alpha > 0$ in an interval $[r_0, \infty[$.
Proposition.

Let \( v \) be a weight function such that \( \sup_{r>0} \frac{v(r)}{v(r+1)} < \infty \) and let \( 1 \leq p \leq \infty \). Then the differentiation operators \( D : B_{p,\infty}(v) \rightarrow B_{p,\infty}(v) \) and \( D : B_{p,0}(v) \rightarrow B_{p,0}(v) \) are continuous.

Proposition.

Let \( v \) be a weight such that \( v(r) = e^{-\alpha r} \) for some \( \alpha > 0 \) and let \( 1 \leq p \leq \infty \). The operator \( J \) is continuous on \( B_{p,\infty}(v) \) and \( B_{p,0}(v) \) with \( \|\|J^n\|\|_{p,v} = 1/\alpha^n \) for each \( n \).
Proposition.
Assume that the integration operator $J : B_{p,0}(v) \to B_{p,0}(v)$ is continuous for some $1 \leq p \leq \infty$. The operator $J$ is not hypercyclic and it has no periodic points different from 0.

Theorem (Bonet, Bonilla)
Assume that the differentiation operator $D : B_{p,0}(v) \to B_{p,0}(v)$ is continuous for some $1 \leq p \leq \infty$. The following conditions are equivalent:

(i) $D : B_{p,0}(v) \to B_{p,0}(v)$ satisfies the hypercyclicity criterion.
(ii) $D : B_{p,0}(v) \to B_{p,0}(v)$ is hypercyclic.
(iii) $\lim \inf_{n \to \infty} \frac{\|z^n\|_{\infty,v}}{n!} = 0$
Theorem (Bonet, Bonilla)

Assume that the differentiation operator $D : B_{p,0} \to B_{p,0}$ is continuous for some $1 \leq p \leq \infty$. The following conditions are equivalent:

(i) $D : B_{p,0}(v) \to B_{p,0}(v)$ is mixing.

(ii) $\lim_{n \to \infty} \frac{\|z^n\|_{\infty,v}}{n!} = 0$.

Theorem (Bonet, Bonilla)

Let $v$ be a weight function such that the differentiation operator $D : B_{p,0} \to B_{p,0}$ is continuous for some $1 \leq p \leq \infty$. The following conditions are equivalent:

(i) $D : B_{p,0}(v) \to B_{p,0}(v)$ is chaotic.

(ii) $D : B_{p,0}(v) \to B_{p,0}(v)$ has a periodic point different from 0.

(iii) $\lim_{r \to \infty} v(r) \frac{e^r}{r^{2p}} = 0$. 

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Corollary.

The operator $D : B_{\infty,0}(a, \alpha) \to B_{\infty,0}(a, \alpha)$ satisfies

- $0 < \alpha < 1 \implies D$ is not hypercyclic and has no periodic point different from 0.

- $\alpha = 1 \implies$ if $a < 1/2$, then $D$ is topologically mixing, and if $a \geq 1/2$, $D$ is not hypercyclic. It has no periodic point different from 0 iff $a \geq 0$.

- $\alpha > 1 \implies D$ is chaotic and topologically mixing.
From now on, to simplify the notation and the exposition, we will concentrate on the operators $D$ and $J$ defined on the spaces $H^\infty_v(\mathbb{C}) = B_{\infty,\infty}(v)$ and $H^0_v(\mathbb{C}) = B_{\infty,0}(v)$.

More general results are available, but will not be mentioned in the lecture.
If \( v(r) = r^a e^{-\alpha r} \) (\( \alpha > 0, \ a \in \mathbb{R} \)) for \( r \geq r_0 \):

\[
\|z^n\|_{a,\alpha} \approx \left( \frac{n+a}{e\alpha} \right)^{n+a}, \text{ with equality for } a = 0.
\]

**Proposition.**

For \( a > 0 \):

\[
\|D^n\|_{a,\alpha} = O \left( n! \left( \frac{e\alpha}{n-a} \right)^{n-a} \right) \quad \text{and} \quad n! \left( \frac{e\alpha}{n+a} \right)^{n+a} = O(\|D^n\|_{a,\alpha}).
\]

For \( a \leq 0 \):

\[
\|D^n\|_{a,\alpha} \approx n! \left( \frac{e\alpha}{n+a} \right)^{n+a}
\]

and the equality holds for \( a = 0 \).
Proposition.

For every $\alpha > 0$ and $a \in \mathbb{R}$, the spectrum $\sigma_{a,\alpha}(D) = \alpha \overline{D}$.

Proposition.

Let $\nu$ be a weight such that $D$ is continuous on $H^\infty_v(\mathbb{C})$ and that $\nu(r)e^{\alpha r}$ is non increasing for some $\alpha > 0$. If $|\lambda| < \alpha$, the operator $D - \lambda I$ is surjective on $H^\infty_v(\mathbb{C})$ and on $H^0_v(\mathbb{C})$ and it even has a continuous linear right inverse

$$K_\lambda f(z) := e^{\lambda z} \int_0^z e^{-\lambda \xi} f(\xi) d\xi, \quad z \in \mathbb{C}.$$ 

This was proved by Atzmon, Brive (2006) for $a = 0$. 
Proposition.

For the weight $v(r) = r^a e^{-\alpha r}$ ($\alpha > 0$, $a \in \mathbb{R}$) for $r$ big enough, we have

- $\|J^n\|_{a,\alpha} \cong 1/\alpha^n$, with the equality for $a = 0$.

- $\sigma_{a,\alpha}(J) = (1/\alpha)\mathbb{D}$.

- $J - \lambda I$ is not surjective on $B_{\infty,\infty}(a, \alpha)$ or $B_{\infty,0}(a, \alpha)$ if $|\lambda| \leq 1/\alpha$. 
Proposition.

Let $T = D$ or $T = J$ and assume $T : H^\infty_v(\mathbb{C}) \to H^\infty_v(\mathbb{C})$ is continuous. The following conditions are equivalent:

(i) $T : H^\infty_v(\mathbb{C}) \to H^\infty_v(\mathbb{C})$ is uniformly mean ergodic.

(ii) $T : H^0_v(\mathbb{C}) \to H^0_v(\mathbb{C})$ is uniformly mean ergodic.

(iii) $\lim_{N \to \infty} \|T + \cdots + T^N\|_v = 0$.

Moreover, if $1 \in \sigma_v(T)$, then $T$ is not uniformly mean ergodic.
Mean ergodicity. Two useful results.

**Theorem (Lin)**

Let $T \in \mathcal{L}(X)$ such that $\|T^n/n\| \to 0$. Then,

$$T \text{ uniformly mean ergodic } \iff (I - T)X \text{ is closed}.$$ 

**Theorem (Lotz)**

Let $T \in \mathcal{L}(H_\alpha^\infty)$ such that $\|T^n/n\| \to 0$. Then,

$$T \text{ mean ergodic } \iff T \text{ uniformly mean ergodic}.$$ 

$H_\alpha^\infty$ is a Grothendieck Banach space with the Dunford-Pettis property, since it is isomorphic to $\ell_\infty$ by a result due to Galbis.
Recall

\[ f \in H^\infty_\alpha(\mathbb{C}) \iff \sup_{z \in \mathbb{C}} |f(z)| \exp(-\alpha |z|) < \infty \]

and

\[ f \in H^0_\alpha(\mathbb{C}) \iff \lim_{|z| \to \infty} |f(z)| \exp(-\alpha |z|) = 0. \]
Mean ergodicity of the differentiation operator.

Theorem.

Let \( v(r) = e^{-\alpha r}, \ r \geq 0. \)

- \( D \) is power bounded on \( H^\infty_\alpha(\mathbb{C}) \) or \( H^0_\alpha(\mathbb{C}) \) if and only if \( \alpha < 1. \)
- \( D \) is uniformly mean ergodic on \( H^\infty_\alpha(\mathbb{C}) \) and \( H^0_\alpha(\mathbb{C}) \) if \( \alpha < 1. \)
- \( D \) not mean ergodic if \( \alpha > 1, \) and
- \( D \) is not mean ergodic on \( H^\infty_1(\mathbb{C}) \) and not uniformly mean ergodic on \( H^0_1(\mathbb{C}). \)
Theorem.

Let $\nu(r) = e^{-\alpha r}$, $r \geq 0$.

- $J$ is never hypercyclic.
- $J$ is power bounded on $H^\infty_\alpha(\mathbb{C})$ or $H^0_\alpha(\mathbb{C})$ if and only if $\alpha \geq 1$.
- If $\alpha > 1$, $J$ is uniformly mean ergodic on $H^\infty_\alpha(\mathbb{C})$ and $H^0_\alpha(\mathbb{C})$.
- $J$ is not mean ergodic on these spaces if $\alpha < 1$.
- If $\alpha = 1$, then $J$ is not mean ergodic on $H^\infty_1(\mathbb{C})$, and mean ergodic but not uniformly mean ergodic on $H^0_1(\mathbb{C})$. 
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(1) Is the operator of differentiation $D$ mean ergodic on $H^0_1(\mathbb{C})$?

In other words:

Assume that $f \in H(\mathbb{C})$ satisfies $\lim_{|z| \to \infty} |f(z)| \exp(-|z|) = 0$. Does it follow that

$$\lim_{n \to \infty} \frac{1}{n} \sup_{z \in \mathbb{C}} |f'(z) + \cdots + f^{(n)}(z)| \exp(-|z|) = 0?$$
(2) Are there mean ergodic operators on a separable Banach space that are hypercyclic?

It is clear that no power bounded operator can be hypercyclic. However, there are examples of mean ergodic operators $T$ on a Banach space such that the sequence $(\|T^n\|)_n$ tends to infinity. Classical examples are due to Hille in 1945. A general construction was presented by Tomilov and Zemanek in 2004.


