UNIFORM CONVERGENCE AND SPECTRA OF OPERATORS IN A CLASS OF FRÉCHET SPACES

ANGELA A. ALBANESE, JOSÉ BONET AND WERNER J. RICKER

Abstract. Well known Banach space results (eg. due to J. Koliha and to Y. Katznelson/L. Tzafriri), which relate conditions on the spectrum of a bounded operator T to the operator norm convergence of certain sequences of operators generated by T, are extended to the class of quojection Fréchet spaces. These results are then applied to establish various mean ergodic theorems for continuous operators acting in such Fréchet spaces and which belong to certain operator ideals, eg. compact, weakly compact, Montel.

1. Introduction

Given a Banach space X and a continuous linear operator T on X, there are various classical results which relate conditions on the spectrum $\sigma(T)$ of T with the operator norm convergence of certain sequences of operators generated by T. For instance, if $\lim_{n\to\infty} \frac{\|T^n\|_{op}}{n} = 0$, with $\| \ \|_{op}$ denoting the operator norm, (even $\frac{T^n}{n} \to 0$ in the weak operator topology suffices), then necessarily $\sigma(T) \subseteq \overline{\mathbb{D}}$, where $\mathbb{D} := \{z \in \mathbb{C} \colon |z| < 1\}$, [22, p.709, Lemma 1]. The stronger condition $\lim_{n\to\infty} \|T^n\|_{op} = 0$ is equivalent to the requirement that both $\sigma(T) \subseteq \mathbb{D}$ and $\lim_{n\to\infty} \frac{\|T^n\|_{op}}{n} = 0$ hold, [29]. An alternate condition, namely that $\{T^n\}_{n=1}^\infty$ is a convergent sequence relative to the operator norm, is equivalent to the requirement that the three conditions $\lim_{n\to\infty} \frac{\|T^n\|_{op}}{n} = 0$, the range $(I-T)^m(X)$ is closed in X for some $m \in \mathbb{N}$ and $\Gamma(T) \subseteq \{1\}$ are satisfied, [33]. Here $\Gamma(T) := \sigma(T) \cap \mathbb{T}$ with $\mathbb{T} := \{z \in \mathbb{C} \colon |z| = 1\}$ being the boundary of $\mathbb{D}$. Such results as above are often related to the uniform mean ergodicity of T, meaning that the sequence of averages $\left\{\frac{1}{n}\sum_{m=1}^n T^m\right\}$ of T is operator norm convergent. For instance, if $\lim_{n\to\infty} \frac{\|T^n\|_{op}}{n} = 0$ and $1 \in \rho(T) := \mathbb{C} \setminus \sigma(T)$, then T is uniformly mean ergodic, [31, p.90, Theorem 2.7]. Or, if $\lim_{n\to\infty} \frac{\|T^n\|_{op}}{n} = 0$, then T is uniformly mean ergodic if and only if $(I-T)(X)$ is closed, [32].

Our first aim is to extend results of the above kind to the class of all Fréchet spaces referred to as prequojections; this is achieved in Section 3. The extension to the class of all Fréchet spaces is not possible; see Proposition 3.10 below and [7, Example 3.11], for instance. We point out that a classical result of Katznelson and Tzafriri stating, for any Banach-space-operator T satisfying $\sup_{n\in\mathbb{N}} \|T^n\|_{op} < \infty$, that $\lim_{n\to\infty} \|T^{n+1} - T^n\|_{op} = 0$ if and only if $\Gamma(T) \subseteq \{1\}$, [28], is also extended to prequojection Fréchet spaces; see Theorem 3.13.

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Our second aim is inspired by well known applications of the above mentioned Banach space results to determine the uniform mean ergodicity of operators T which satisfy $\lim_{n\to\infty}\frac{\|T^n\|_{op}}{n}=0$ and belong to certain *operator ideals*, such as the compact or weakly compact operators; see, for example, [22, Ch. VIII, §8], [31, Ch. 2, §2.2], [23, Theorem 6.1],where T can even be quasi–compact. An extension of such a mean ergodic result to the class of quasi–precompact operators acting in various locally convex Hausdorff spaces is presented in [40]. For prequojection Fréchet spaces, this result is further extended to the (genuinely) larger class of quasi–Montel operators; see Proposition 4.10, Remark 4.11 and Theorem 4.13. A mean ergodic theorem for Cesàro bounded, weakly compact operators (and also reflexive operators) in a certain class of locally convex spaces (which includes all Fréchet spaces) is also presented; see Proposition 4.1 and Remark 4.2(ii).

2. Preliminaries and spectra of operators

Let X be a lcHs and Γ_X a system of continuous seminorms determining the topology of X. The strong operator topology τ_s in the space $\mathcal{L}(X)$ of all continuous linear operators from X into itself (from X into another lcHs Y we write $\mathcal{L}(X,Y)$) is determined by the family of seminorms $q_x(S):=q(Sx)$, for $S\in\mathcal{L}(X)$, for each $x\in X$ and $q\in\Gamma_X$, in which case we write $\mathcal{L}_s(X)$. Denote by $\mathcal{B}(X)$ the collection of all bounded subsets of X. The topology τ_b of uniform convergence on bounded sets is defined in $\mathcal{L}(X)$ via the seminorms $q_B(S):=\sup_{x\in B}q(Sx)$, for $S\in\mathcal{L}(X)$, for each $B\in\mathcal{B}(X)$ and $q\in\Gamma_X$; in this case we write $\mathcal{L}_b(X)$. For X a Banach space, τ_b is the operator norm topology in $\mathcal{L}(X)$. If Γ_X is countable and X is complete, then X is called a Fréchet space. The identity operator on a lcHs X is denoted by I.

By X_σ we denote X equipped with its weak topology $\sigma(X,X')$, where X' is the topological dual space of X. The strong topology in X (resp. X') is denoted by $\beta(X,X')$ (resp. $\beta(X',X)$) and we write X_β (resp. X'_β); see [34, IV, Ch. 23] for the definition. The strong dual space $(X'_\beta)'_\beta$ of X'_β is denoted simply by X''. By X'_σ we denote X' equipped with its weak–star topology $\sigma(X',X)$. Given $T\in\mathcal{L}(X)$, its *dual operator* $T'\colon X'\to X'$ is defined by $\langle x,T'x'\rangle=\langle Tx,x'\rangle$ for all $x\in X$, $x'\in X'$. It is known that $T'\in\mathcal{L}(X'_\sigma)$ and $T'\in\mathcal{L}(X'_\beta)$, [30, p.134].

For a Fréchet space X and $T\in\mathcal{L}(X)$, the *resolvent set* $\rho(T)$ of T consists of all $\lambda\in\mathbb{C}$ such that $R(\lambda,T):=(\lambda I-T)^{-1}$ exists in $\mathcal{L}(X)$. Then $\sigma(T):=\mathbb{C}\setminus\rho(T)$ is called the *spectrum* of T. The *point spectrum* $\sigma_p(T)$ consists of all $\lambda\in\mathbb{C}$ such that $(\lambda I-T)$ is not injective. Unlike for Banach spaces, it may happen that $\rho(T)=\emptyset$. For example, let $\omega=\mathbb{C}^{\mathbb{N}}$ be the Fréchet space equipped with the lc–topology determined via the seminorms $\{q_n\}_{n=1}^\infty$, where $q_n(x):=\max_{1\le j\le n}|x_j|$, for $x=(x_j)_{j=1}^\infty\in\omega$. Then the unit left shift operator $T\colon x\mapsto(x_2,x_3,x_4,\ldots)$, for $x\in\omega$, belongs to $\mathcal{L}(\omega)$ and, for every $\lambda\in\mathbb{C}$, the element $(1,\lambda,\lambda^2,\lambda^3,\ldots)\in\omega$ is an eigenvector corresponding to λ.

For a Fréchet space X, the natural imbedding $\Phi\colon X\to X''$ is an isomorphism of X onto the closed subspace $\Phi(X)$ of X''. Moreover, we always have

$$S''\circ\Phi=\Phi\circ S,\quad S\in\mathcal{L}(X),\tag{2.1}$$

that is, S'' is an extension of S.

The following result will be required in the sequel. Since the proof is standard we omit it. The polar of a set $\mathcal{U} \subseteq X$ is denoted by $\mathcal{U}^\circ \subseteq X'$.

Lemma 2.1. *Let X be a Fréchet space.*

(i) *Let $\{p_j\}_{j=1}^\infty \subseteq \Gamma_{X''}$ be a fundamental, increasing sequence which determines the lc–topology of X''. For each $j \in \mathbb{N}$ define q_j on X via $q_j := p_j \circ \Phi$. Then $\{q_j\}_{j=1}^\infty \subseteq \Gamma_X$ is a fundamental, increasing sequence which determines the lc–topology of X.*

(ii) *Let $\{r_j\}_{j=1}^\infty \subseteq \Gamma_X$ be a fundamental, increasing sequence which determines the lc–topology of X. For each $j \in \mathbb{N}$, let r_j'' denote the Minkowski functional (in X'') of the bipolar of $\mathcal{U}_j := r_j^{-1}([0,1]) \subseteq X$. Then $\{r_j''\}_{j=1}^\infty \subseteq \Gamma_{X''}$ is a fundamental, increasing sequence which determines the lc–topology of X''. Moreover, for each $j \in \mathbb{N}$, we have*

$$r_j(x) = \sup_{x' \in \mathcal{U}_j^\circ} |\langle x, x' \rangle| \quad \text{and} \quad r_j''(x'') = \sup_{x' \in \mathcal{U}_j^\circ} |\langle x'', x' \rangle| \tag{2.2}$$

for each $x \in X$ and $x'' \in X''$. In particular, $r_j'' \circ \Phi = r_j$, i.e., the restriction of r_j'' to $X \simeq \Phi(X)$ coincides with r_j, for each $j \in \mathbb{N}$.

For Banach spaces the following fact is well known.

Lemma 2.2. *Let X be a lcHs and $\{T_n\}_{n=1}^\infty \subseteq \mathcal{L}(X)$ be an equicontinuous sequence. Then also $\{T_n''\}_{n=1}^\infty \subseteq \mathcal{L}(X'')$ is equicontinuous.*

Proof. Let $B \in \mathcal{B}(X)$. Then $C := \cup_{n=1}^\infty T_n(B) \in \mathcal{B}(X)$ as $\{T_n\}_{n=1}^\infty$ is equicontinuous. So, for all $x' \in X'$ and $n \in \mathbb{N}$, we have $T_n' x' \in X_\beta'$ with

$$p_B(T_n' x') := \sup_{x \in B} |\langle x, T_n' x' \rangle| = \sup_{x \in B} |\langle T_n x, x' \rangle| \leq \sup_{y \in C} |\langle y, x' \rangle| = p_C(x').$$

As the seminorms $\{p_B : B \in \mathcal{B}(X)\}$ generate the lc–topology of X_β', the previous inequality shows that $\{T_n'\}_{n=1}^\infty \subseteq \mathcal{L}(X_\beta')$ is equicontinuous.

Since $\{T_n'\}_{n=1}^\infty \subseteq \mathcal{L}(X_\beta')$ is equicontinuous and the lc–topology of X'' is generated by the polars of bounded subsets of X_β', the same argument as above yields that $\{T_n''\}_{n=1}^\infty \subseteq \mathcal{L}(X'')$ is equicontinuous. □

Lemma 2.3. *Let X be a Fréchet space and $T \in \mathcal{L}(X)$. Then T is an isomorphism of X onto itself if and only if T'' is an isomorphism of X'' onto itself.*

Proof. If T is an isomorphism of X onto itself, then there exists $T^{-1} \in \mathcal{L}(X)$ with $TT^{-1} = T^{-1}T = I$. It follows that $T', (T^{-1})' \in \mathcal{L}(X_\beta')$ and so $T'', (T^{-1})'' \in \mathcal{L}(X'')$. Accordingly, $I = (TT^{-1})'' = T''(T^{-1})''$ and $I = (T^{-1}T)'' = (T^{-1})''T''$. Thus, $(T'')^{-1}$ exists in $\mathcal{L}(X'')$ and $(T'')^{-1} = (T^{-1})''$, i.e., T'' is an isomorphism of X'' onto itself.

Conversely, suppose that T'' is an isomorphism of X'' onto itself. Since T'' is an extension of T (i.e., $T = T''|_X$), we see that T is one–to–one. Moreover, since X is a closed subspace of X'' (as X is a complete, barrelled lcHs), it follows that $T(X) = T''(X)$ is closed. It remains to show that $T(X) = X$. But, if $T(X) \neq X$, then there is $f \in X' \setminus \{0\}$ such that $\langle Tx, f \rangle = \langle x, T'f \rangle = 0$ for all $x \in X$. Hence, $T'f = 0$; this is a contradiction because the surjectivity of T'' implies that T' is necessarily one–to–one. □

We remark that Lemma 2.3 remains valid for X a complete barrelled lcHs.

The next result is an immediate consequence of (2.1) and Lemma 2.3.

Corollary 2.4. *Let X be a Fréchet space and $T \in \mathcal{L}(X)$. Then $\rho(T) = \rho(T'')$ and $\sigma(T) = \sigma(T'')$. Moreover,*

$$\Phi \circ R(\lambda, T) = R(\lambda, T'') \circ \Phi, \quad \lambda \in \rho(T) = \rho(T''),$$

that is, the restriction of $R(\lambda, T'')$ to the closed subspace $X \simeq \Phi(X)$ of X'' coincides with $R(\lambda, T)$. Briefly, $R(\lambda, T'')|_X = R(\lambda, T)$.

A Fréchet space X is always a projective limit of continuous linear operators $R_k : X_{k+1} \to X_k$, for $k \in \mathbb{N}$, with each X_k a Banach space. If X_k and R_k can be chosen such that each R_k is surjective and X is isomorphic to the projective limit $\operatorname{proj}_j(X_j, R_j)$, then X is called a *quojection*, [11, Section 5]. Banach spaces and countable products of Banach spaces are quojections. Actually, every quojection is the quotient of a countable product of Banach spaces, [13]. In [37] Moscatelli gave the first examples of quojections which are not isomorphic to countable products of Banach spaces. Concrete examples of quojection Fréchet spaces are $\omega = \mathbb{C}^{\mathbb{N}}$, the spaces $L^p_{loc}(\Omega)$, with $1 \le p \le \infty$, and $C^{(m)}(\Omega)$ for $m \in \mathbb{N}_0$, with $\Omega \subseteq \mathbb{R}^N$ any open set, all of which are isomorphic to countable products of Banach spaces. The spaces of continuous functions $C(\Lambda)$, with Λ a σ–compact, completely regular topological space, endowed with the compact open topology are also quojections. Domański exhibited a completely regular topological space Λ such that the Fréchet space $C(\Lambda)$ is a quojection which is not isomorphic to a complemented subspace of a product of Banach spaces, [20, Theorem]. A Fréchet space X admits a continuous norm if and only if X contains no isomorphic copy of ω, [27, Theorem 7.2.7]. On the other hand, a quojection X admits a continuous norm if and only if it is a Banach space, [11, Proposition 3]. So, a quojection is either a Banach space or contains an isomorphic copy of ω, necessarily complemented, [27, Theorem 7.2.7]. Also [19] is relevant.

Lemma 2.5. *Let X be a quojection Fréchet space and $S \in \mathcal{L}(X)$. Suppose that $X = \operatorname{proj}_j(X_j, Q_{j,j+1})$, with X_j a Banach space (having norm $\| \, \|_j$) and linking maps $Q_{j,j+1} \in \mathcal{L}(X_{j+1}, X_j)$ which are surjective for all $j \in \mathbb{N}$, and suppose, for each $j \in \mathbb{N}$, that there exists $S_j \in \mathcal{L}(X_j)$ satisfying*

$$S_j Q_j = Q_j S, \tag{2.3}$$

where $Q_j \in \mathcal{L}(X, X_j)$, $j \in \mathbb{N}$, denotes the canonical projection of X onto X_j (i.e., $Q_{j,j+1} \circ Q_{j+1} = Q_j$). Then

$$\sigma(S) \subseteq \cup_{j=1}^{\infty} \sigma(S_j) \subseteq \sigma(S) \cup \cup_{j=1}^{\infty} \sigma_p(S_j). \tag{2.4}$$

Moreover,

$$\sigma_p(S) \subseteq \cup_{j=1}^{\infty} \sigma_p(S_j). \tag{2.5}$$

If, in addition, for every $\lambda \in \rho(S)$, the resolvent operator $R(\lambda, S)$ satisfies

$$R(\lambda, S)(\operatorname{Ker} Q_j) \subseteq \operatorname{Ker} Q_j, \quad j \in \mathbb{N}, \tag{2.6}$$

then $\sigma(S) = \cup_{j=1}^{\infty} \sigma(S_j)$.

Proof. For the containments (2.4) and (2.5) we refer to [9, Lemma 6.1].

Suppose now that (2.6) holds for each $\lambda \in \rho(S)$. To establish the desired equality, let $\lambda \in \rho(S)$. Then $\lambda I - S$ is surjective. Fix $j \in \mathbb{N}$. Since $Q_j \colon X \to X_j$ is surjective, it is routine to check from the identity $(\lambda I_j - S_j)Q_j = Q_j(\lambda I - S)$ that also $\lambda I_j - S_j$ is surjective (with $I_j \in \mathcal{L}(X_j)$ the identity operator). To verify $\lambda I_j - S_j$ is injective suppose that $(\lambda I_j - S_j)y = 0$ for some $y \in X_j$, in which case $y = Q_j x$ for some $x \in X$. Accordingly,

$$Q_j(\lambda I - S)x = (\lambda I_j - S_j)Q_j x = (\lambda I_j - S_j)y = 0$$

shows that $(\lambda I - S)x \in \operatorname{Ker} Q_j$. It then follows from (2.6) that $x = R(\lambda, S)(\lambda I - S)x \in \operatorname{Ker} Q_j$, i.e., $Q_j x = 0$. Since $y = Q_j x$, we have $y = 0$. Hence, $\lambda I_j - S_j$ is injective. This establishes that $\lambda \in \rho(S_j)$. Accordingly, $\rho(S) = \cap_{j=1}^{\infty} \rho(S_j)$ as desired. $\square$

The following result occurs in [9, Lemma 6.2].

Lemma 2.6. *Let X be a quojection Fréchet space and $\{S_n\}_{n=1}^{\infty} \in \mathcal{L}(X)$. Suppose that $X = \operatorname{proj}_j(X_j, Q_{j,j+1})$, with X_j a Banach space (having norm $\| \ \|_j$) and linking maps $Q_{j,j+1} \in \mathcal{L}(X_{j+1}, X_j)$ which are surjective for all $j \in \mathbb{N}$, and suppose, for each $j,\ n \in \mathbb{N}$, that there exists $S_n^{(j)} \in \mathcal{L}(X_j)$ satisfying*

$$S_n^{(j)} Q_j = Q_j S_n, \tag{2.7}$$

where $Q_j \in \mathcal{L}(X, X_j)$, $j \in \mathbb{N}$, denotes the canonical projection of X onto X_j (i.e., $Q_{j,j+1} \circ Q_{j+1} = Q_j$). Then the following statements are equivalent.

(i) *The limit τ_b-$\lim_{n\to\infty} S_n =: S$ exists in $\mathcal{L}_b(X)$.*

(ii) *For each $j \in \mathbb{N}$, the limit τ_b-$\lim_{n\to\infty} S_n^{(j)} =: S^{(j)}$ exists in $\mathcal{L}_b(X_j)$.*

In this case, the operators $S \in \mathcal{L}(X)$ and $S^{(j)} \in \mathcal{L}(X_j)$, for $j \in \mathbb{N}$, satisfy

$$Sx = (S^{(j)} x_j)_j, \quad x = (x_j)_j \in X. \tag{2.8}$$

Moreover, (i) *and* (ii) *remain equivalent if τ_b is replaced by τ_s.*

Given any lcHs X and $T \in \mathcal{L}(X)$, let us introduce the notation

$$T_{[n]} := \frac{1}{n} \sum_{m=1}^{n} T^m, \qquad n \in \mathbb{N}, \tag{2.9}$$

for the Cesàro means of T. Then T is called *mean ergodic* precisely when $\{T_{[n]}\}_{n=1}^{\infty}$ is a convergent sequence in $\mathcal{L}_s(X)$. If $\{T_{[n]}\}_{n=0}^{\infty}$ happens to be convergent in $\mathcal{L}_b(X)$, then T will be called *uniformly mean ergodic.*

We always have the identities

$$(I - T)T_{[n]} = T_{[n]}(I - T) = \frac{1}{n}(T - T^{n+1}), \qquad n \in \mathbb{N}, \tag{2.10}$$

and also (setting $T_{[0]} := I$) that

$$\frac{1}{n} T^n = T_{[n]} - \frac{(n-1)}{n} T_{[n-1]}, \qquad n \in \mathbb{N}. \tag{2.11}$$

Some authors prefer to use $\frac{1}{n}\sum_{m=0}^{n-1} T^m$ in place of $T_{[n]}$; since

$$T_{[n]} = T\Big(\frac{1}{n} \sum_{m=0}^{n-1} T^m\Big) = \frac{1}{n}(T^n - I) + \frac{1}{n} \sum_{m=0}^{n-1} T^m, \qquad n \in \mathbb{N},$$

this leads to identical results.

Recall that $T \in \mathcal{L}(X)$ is called *power bounded* if $\{T^n\}_{n=1}^\infty$ is an equicontinuous subset of $\mathcal{L}(X)$.

The final result that we require (i.e., [9, Lemma 6.4]) is as follows.

Lemma 2.7. *Let $X = \operatorname{proj}_j(X_j, Q_{j.j+1})$ be a quojection Fréchet space and operators $S \in \mathcal{L}(X)$ and $S_j \in \mathcal{L}(X_j)$, for $j \in \mathbb{N}$, be given which satisfy the assumptions of Lemma 2.5 (with $Q_j \in \mathcal{L}(X, X_j)$, $j \in \mathbb{N}$, denoting the canonical projection of X onto X_j and $\| \; \|_j$ being the norm in the Banach space X_j).*

(i) *$S \in \mathcal{L}(X)$ is power bounded if and only if each $S_j \in \mathcal{L}(X_j)$, $j \in \mathbb{N}$, is power bounded.*

(ii) *$S \in \mathcal{L}(X)$ is mean ergodic (resp., uniformly mean ergodic) if and only if each $S_j \in \mathcal{L}(X_j)$, $j \in \mathbb{N}$, is mean ergodic (resp., uniformly mean ergodic).*

3. Spectrum, uniform convergence and mean ergodicity

A *prequojection* is a Fréchet space X such that X'' is a quojection. Every quojection is a prequojection. A prequojection is called *non–trivial* if it is not itself a quojection. It is known that X is a prequojection if and only if X'_β is a strict (LB)–space. An alternative characterization is that X is a prequojecton if and only if X has no Köthe nuclear quotient which admits a continuous norm; see [11, 18, 39, 41]. This implies that a quotient of a prequojection is again a prequojection. In particular, every complemented subspace of a prequojection is again a prequojection. The problem of the existence of non–trivial prequojections arose in a natural way in [11]; it has been solved, in the positive sense, in various papers, [12], [18], [38]. All of these papers employ the same method, which consists in the construction of the dual of a prequojection, rather than the prequojection itself, which is often difficult to describe (see the survey paper [35] for further information). However, in [36] an alternative method for constructing prequojections is presented which has the advantage of being direct. For an example of a concrete space (i.e., a space of continuous functions on a suitable topological space), which is a non–trivial prequojection, see [1].

In this section we extend to prequojection Fréchet spaces some well known results from the Banach setting which connect various conditions on the spectrum $\sigma(T)$, of a continuous linear operator T, to the operator norm convergence of certain sequences of operators generated by T. Such results have well known consequences for the uniform mean ergodicity of T.

We begin with a construction for quojection Fréchet spaces which is needed in the sequel.

Let X be a quojection Fréchet space and $\{q_j\}_{j=1}^\infty$ be *any* fundamental, increasing sequence of seminorms generating the lc–topology of X. For each $j \in \mathbb{N}$, set $X_j := X/q_j^{-1}(\{0\})$ and endow X_j with the quotient lc–topology. Denote by $Q_j \colon X \to X_j$ the corresponding canonical (surjective) quotient map and define the quotient topology on X_j via the increasing sequence of seminorms $\{(\hat{q}_j)_k\}_{k=1}^\infty$ on X_j by

$$(\hat{q}_j)_k(Q_j x) := \inf\{q_k(y) : \, y \in X \text{ and } Q_j y = Q_j x\}, \quad x \in X, \tag{3.1}$$

for each $k \in \mathbb{N}$. Then

$$(\hat{q}_j)_k(Q_j x) \leq q_k(x), \quad x \in X, \quad k, j \in \mathbb{N}; \tag{3.2}$$

see (2.4) in [5]. Moreover,

$$(\hat{q}_j)_j(Q_j x) = q_j(x), \quad x \in X,\ j \in \mathbb{N}, \tag{3.3}$$

which implies that $(\hat{q}_j)_j$ is a norm on X_j. As noted above, since X is a quojection Fréchet space and every quotient space (of such a Fréchet space) with a continuous *norm* is necessarily Banach, [11, Proposition 3], it follows that for each $j \in \mathbb{N}$ there exists $k(j) \geq j$ such that the norm $(\hat{q}_j)_{k(j)}$ generates the lc–topology of X_j. Moreover, it is possible to choose $k(j+1) \geq k(j)$ for all $j \in \mathbb{N}$. Thus, X is isomorphic to the projective limit of the sequence $\{(X_j, (\hat{q}_j)_{k(j)})\}_{j=1}^{\infty}$ of Banach spaces with respect to the continuous, surjective linking maps $Q_{j,j+1} \colon X_{j+1} \to X_j$ defined by

$$Q_{j,j+1} \circ Q_{j+1} = Q_j, \quad j \in \mathbb{N}. \tag{3.4}$$

This particular construction will be used on various occasions in the sequel, where $\hat{B}_j$ will always denote the closed unit ball of X_j, for $j \in \mathbb{N}$. The so constructed Banach space norm $(\hat{q}_j)_{k(j)}$ of X_j will always be denoted by $\tilde{q}_j$, for $j \in \mathbb{N}$.

The following result is classical in Banach spaces, [22, p.709 Lemma 1].

Proposition 3.1. *Let X be a quojection Fréchet space and $T \in \mathcal{L}(X)$ satisfy τ_s-$\lim_{n\to\infty} \frac{T^n}{n} = 0$. Then $\sigma(T) \subseteq \overline{\mathbb{D}}$.*

In case X is a prequojection Fréchet space and τ_b-$\lim_{n\to\infty} \frac{T^n}{n} = 0$, the inclusion $\sigma(T) \subseteq \overline{\mathbb{D}}$ is again valid.

Proof. Case (I). X is a quojection.

Let $\{r_j\}_{j=1}^{\infty}$ be a fundamental, increasing sequence of seminorms generating the lc–topology of X. Since $\frac{T^n}{n} \to 0$ in $\mathcal{L}_s(X)$ as $n \to \infty$ and X is a Fréchet space, the sequence $\left\{\frac{T^n}{n}\right\}_{n=1}^{\infty}$ is equicontinuous. So, for each $j \in \mathbb{N}$ there exists $c_j > 0$ such that

$$r_j\left(\frac{T^n x}{n}\right) \leq c_j r_{j+1}(x), \quad x \in X,\, n \in \mathbb{N}; \tag{3.5}$$

there is no loss in generality by assuming that r_{j+1} can be chosen.

Define q_j on X by $q_j(x) := \max\left\{r_j(x), \sup_{n\in\mathbb{N}} r_j\left(\frac{T^n x}{n}\right)\right\}$, for $x \in X$. Then (3.5) ensures that $\{q_j\}_{j=1}^{\infty}$ is also a fundamental, increasing sequence of seminorms generating the lc-topology of X. Moreover,

$$q_j(Tx) \leq 2q_j(x), \quad x \in X,\, j \in \mathbb{N}. \tag{3.6}$$

We now apply the construction (3.1)–(3.4) to the sequence of seminorms $\{q_j\}_{j=1}^{\infty}$ to yield the corresponding sequence $\{(X_j, \tilde{q}_j)\}_{j=1}^{\infty}$ of Banach spaces and the quotient maps $Q_j \in \mathcal{L}(X, X_j)$, for $j \in \mathbb{N}$; recall that $\tilde{q}_j := (\hat{q}_j)_{k(j)}$, for $j \in \mathbb{N}$.

Fix $j \in \mathbb{N}$. Define the operator $T_j \colon X_j \to X_j$ via

$$T_j Q_j x := Q_j T x, \quad x \in X. \tag{3.7}$$

Then T_j is a well defined, continuous linear operator from X_j into X_j. Indeed, suppose $Q_j x = Q_j y$ for some $x, y \in X$, i.e., $(x-y) \in \operatorname{Ker} Q_j$, so that $q_j(x-y) = 0$. This, together with (3.6), yields $0 \leq q_j((T(x-y)) \leq 2q_j(x-y) = 0$. Since $\operatorname{Ker} Q_j = q_j^{-1}(\{0\})$, it follows that $Q_j T(x-y) = 0$ and hence, by (3.7) that $T_j Q_j(x-y) = Q_j T(x-y) = 0$. Therefore, $T_j Q_j x = T_j Q_j y$. This means that T_j

is well defined. Clearly, T_j is also linear. Moreover, (3.2), (3.6) and (3.7) imply that

$$\tilde{q}_j(T_j\hat{x}) = \tilde{q}_j(T_jQ_jx) = \tilde{q}_j(Q_jTx) \leq q_{k(j)}(Tx) \leq 2q_{k(j)}(x),$$

for all $\hat{x} \in X_j$ and $x \in X$ with $Q_jx = \hat{x}$. Taking the infimum with respect to $x \in Q_j^{-1}(\{\hat{x}\})$, it follows that

$$\tilde{q}_j(T_j\hat{x}) \leq 2\tilde{q}_j(\hat{x}), \quad \hat{x} \in X_j. \tag{3.8}$$

Since $\tilde{q}_j$ generates the quotient topology of X_j, (3.8) ensures the continuity of T_j. Moreover, it follows from (3.7) that

$$(T_j)^nQ_jx := Q_jT^nx, \quad x \in X,\, n \in \mathbb{N}. \tag{3.9}$$

The surjectivity and the continuity of Q_j together with (3.9) imply that τ_s-$\lim_{n\to\infty} \frac{(T_j)^n}{n} = 0$. Indeed, fix any $\hat{x} \in X_j$. By the surjectivity of Q_j there exists $x \in X$ such that $Q_jx = \hat{x}$. By (3.9) it follows that $\frac{(T_j)^n\hat{x}}{n} = Q_j\left(\frac{T^nx}{n}\right)$, for $n \in \mathbb{N}$. Moreover, $\frac{T^nx}{n} \to 0$ as $n \to \infty$ by assumption. So, the continuity of Q_j yields that $\lim_{n\to\infty} \frac{(T_j)^n\hat{x}}{n} = 0$ in the Banach space X_j. We can then apply Lemma 1 in [22, p.709] to obtain that $\sigma(T_j) \subseteq \overline{\mathbb{D}}$.

We have just shown that that $(\mathbb{C} \setminus \overline{\mathbb{D}}) \subseteq \cap_{j=1}^{\infty}\rho(T_j)$. Moreover, the operators T and T_j satisfy (3.7). So, we can apply Lemma 2.5 which yields $(\mathbb{C} \setminus \overline{\mathbb{D}}) \subseteq \rho(T)$, i.e., $\sigma(T) \subseteq \overline{\mathbb{D}}$.

Case (II). X is a prequojection and τ_b-$\lim_{n\to\infty} \frac{T^n}{n} = 0$.

Observe that X and X'_β are barrelled and hence, quasi-barrelled as X is a Fréchet space and X'_β is the strong dual of a prequojection Fréchet space. Since $T' \in \mathcal{L}(X'_\beta)$ and $T'' \in \mathcal{L}(X'')$, the condition τ_b-$\lim_{n\to\infty} \frac{T^n}{n} = 0$ implies that τ_b-$\lim_{n\to\infty} \frac{(T'')^n}{n} = 0$ (see [3, Lemma 2.6] or [4, Lemma 2.1]). On the other hand, X'' is a quojection Fréchet space. So, it follows from Case (I) that $\sigma(T'') \subseteq \overline{\mathbb{D}}$. Finally, Corollary 2.4 ensures that $\sigma(T) = \sigma(T'')$ and so $\sigma(T) \subseteq \overline{\mathbb{D}}$. □

Remark 3.2. For a power bounded operator $T \in \mathcal{L}(X)$ it is always the case that τ_b-$\lim_{n\to\infty} \frac{T^n}{n} = 0$ and so, whenever X is a prequojection Fréchet space, it follows from Proposition 3.1 that $\sigma(T) \subseteq \overline{\mathbb{D}}$.

For operators in Banach spaces, the following result is due to J.J. Koliha, [29].

Theorem 3.3. *Let X be a prequojection Fréchet space and $T \in \mathcal{L}(X)$. The following assertions are equivalent.*

(i) τ_b-$\lim_{n\to\infty} T^n = 0$.
(ii) *The series* $\sum_{n=0}^{\infty} T^n$ *converges in* $\mathcal{L}_b(X)$.
(iii) τ_b-$\lim_{n\to\infty} \frac{T^n}{n} = 0$ *and* $\sigma(T) \subseteq \mathbb{D}$.

Moreover, if one (hence, all) of the above conditions holds, then $I - T$ is an isomorphism of X onto X with inverse $(I - T)^{-1} = \sum_{n=0}^{\infty} T^n$ and the series converging in $\mathcal{L}_b(X)$.

Proof. Case (I). X is a quojection.

(i)⇒(ii). The assumption τ_b-$\lim_{n\to\infty} T^n = 0$ implies that τ_b-$\lim_{n\to\infty} \frac{T^n}{n} = 0$. So, we can proceed as in the proof of Proposition 3.1 to obtain that $X = \mathrm{proj}_j(X_j, Q_{j,,j+1})$ in such a way that, for every $j \in \mathbb{N}$, there exists T_j in $\mathcal{L}(X_j)$

satisfying $T_jQ_j = Q_jT$. Then also $T_j^nQ_j = Q_jT^n$, for every j, $n \in \mathbb{N}$. So, Lemma 2.6 implies that τ_b-$\lim_{n\to\infty} T_j^n = 0$ for all $j \in \mathbb{N}$. Thus, by [29, Theorem 2.1] the series $\sum_{n=0}^{\infty} T_j^n$ converges in $\mathcal{L}_b(X_j)$, for each $j \in \mathbb{N}$. With $S_n := \sum_{k=0}^{n} T^k$, for $n \in \mathbb{N}$, it follows again from Lemma 2.6 that the series $\sum_{n=0}^{\infty} T^n$ converges in $\mathcal{L}_b(X)$.

(ii)⇒(iii). The assumption clearly implies τ_b-$\lim_{n\to\infty} \frac{T^n}{n} = 0$. So, as in the proof of (i)⇒(ii), we may assume that $X = \mathrm{proj}_j(X_j, Q_{j,j+1})$ in such a way that, for every $j \in \mathbb{N}$, there exists T_j in $\mathcal{L}(X_j)$ satisfying $T_jQ_j = Q_jT$. Then also $T_j^nQ_j = Q_jT^n$, for every j, $n \in \mathbb{N}$. Since $\sum_{n=0}^{\infty} T^n$ converges in $\mathcal{L}_b(X)$ and X is a quojection, the series $\sum_{n=0}^{\infty} T_j^n$ also converges in $\mathcal{L}_b(X_j)$ for all $j \in \mathbb{N}$; see Lemma 2.6. By [29, Theorem 2.1] we have that $\sigma(T_j) \subset \mathbb{D}$ and so $\Lambda := (\mathbb{C} \setminus \overline{\mathbb{D}}) \subseteq \rho(T_j)$, for all $j \in \mathbb{N}$. Accordingly, since $T_jQ_j = Q_jT$ for all $j \in \mathbb{N}$, Lemma 2.5 yields $\Lambda \subseteq \cap_{j=1}^{\infty}\rho(T_j) \subseteq \rho(T)$, i.e., $\sigma(T) \subset \mathbb{D}$.

(iii)⇒(i). Since $\Lambda \subseteq \rho(T)$, for every $\lambda \in \Lambda$, the operator $I - \lambda^{-1}T = \lambda^{-1}(\lambda I - T) \in \mathcal{L}(X)$ is invertible, i.e., bijective with $(I - \lambda^{-1}T)^{-1} \in \mathcal{L}(X)$. On the other hand, τ_b-$\lim_{n\to\infty} \frac{(\lambda^{-1}T)^n}{n} = 0$ for every $\lambda \in \Lambda$ as τ_b-$\lim_{n\to\infty} \frac{T^n}{n} = 0$ and $|\lambda^{-1}| \leq 1$. So, by Theorem 4.1 in [5] (see also Theorem 3.5 of [7]) we can conclude that

$$\tau_b - \lim_{n\to\infty} (\lambda^{-1}T)_{[n]} = 0, \quad \lambda \in \Lambda. \tag{3.10}$$

Let $\{r_j\}_{j=1}^{\infty}$ be a fundamental, increasing sequence of seminorms generating the lc–topology of X. Arguing as in the proof of Proposition 3.1 (and adopting the notation from there) we conclude that (3.5) is satisfied. Define q_j on X by $q_j(x) := \max\left\{r_j(x), \sup_{n\in\mathbb{N}} r_j\left(\frac{T^nx}{n}\right)\right\}$, for $x \in X$. Then again (3.6) is satisfied and, for each $j \in \mathbb{N}$, there exists a continuous linear operator $T_j\colon X_j \to X_j$ satisfying both (3.7) and (3.8). Moreover, it follows from (3.7) that

$$(\lambda^{-1}T_j)^nQ_jx := Q_j(\lambda^{-1}T)^nx, \quad x \in X,\, n \in \mathbb{N},\, \lambda \in \Lambda. \tag{3.11}$$

Fix $\lambda \in \Lambda$ and consider the sequences $\{R_n\}_{n=1}^{\infty}$ and $\{H_n\}_{n=1}^{\infty}$ in $\mathcal{L}(X)$ given by $R_n := \frac{1}{n}\sum_{m=0}^{n-1}\sum_{h=0}^{m}(\lambda^{-1}T)^h$ and $H_n := I - (\lambda^{-1}T)_{[n]}$, for $n \in \mathbb{N}$. Then the operator $A := I - \lambda^{-1}T$ satisfies $H_n = AR_n = R_nA$ for all $n \in \mathbb{N}$. Moreover, (3.10) implies that $H_n \to R := I$ in $\mathcal{L}_b(X)$. Since all the assumptions of Lemma 3.4 in [7] are satisfied with $F = E = X$, $R = I \in \mathcal{L}(X, X)$ and $A = I - \lambda^{-1}T$, we can proceed as in the proof of that result to conclude, for every $j \in \mathbb{N}$, that the operator $I - \lambda^{-1}T_j$ is invertible in $\mathcal{L}(X_j)$ (hence, also $\lambda I - T_j$ is invertible), i.e., $\lambda \in \rho(T_j)$.

By the arbitrariness of $\lambda \in \Lambda$, we have that $\Lambda \subseteq \rho(T_j)$, for all $j \in \mathbb{N}$. So, there exists $\delta_j \in (0, 1)$ such that $\rho(T_j) \supset \{\lambda \in \mathbb{C} : |\lambda| \geq 1 - \delta_j\}$. It follows that

$$r(T_j) := \max\{|\lambda| : \lambda \in \sigma(T_j)\} = \lim_{n\to\infty} \sqrt[n]{\|T_j^n\|_{op}} \leq (1 - \delta_j) < 1, \quad j \in \mathbb{N},$$

and hence, that $\lim_{n\to\infty} \|T_j^n\|_{op} = 0$. Because of (3.11), with $\lambda = 1 \in \Lambda$, it follows from Lemma 2.6 (with $S_n := T^n$) that τ_b-$\lim_{n\to\infty} T^n = 0$.

Case (II). X is a prequojection.

As noted before X and X'_β are barrelled with $T' \in \mathcal{L}(X'_\beta)$ and $T'' \in \mathcal{L}(X'')$.

(i)⇒(ii). If $T^n \to 0$ in $\mathcal{L}_b(X)$ for $n \to \infty$, then an argument as for Case (II) in the proof of Proposition 3.1 shows that $(T'')^n = (T^n)'' \to 0$ in $\mathcal{L}_b(X'')$ for $n \to \infty$. Since X'' is a quojection Fréchet space, we can apply (i)⇒(ii) of Case

(I) above to conclude that the series $\sum_{n=0}^{\infty}(T'')^n$ converges in $\mathcal{L}_b(X'')$. Then also $\sum_{n=0}^{\infty} T^n$ converges in $\mathcal{L}_b(X)$ as $T''|_X = T$ and X is a closed subspace of X''.

(ii)⇒(iii). If $\sum_{n=0}^{\infty} T^n$ converges in $\mathcal{L}_b(X)$, then $\sum_{n=0}^{\infty}(T'')^n$ converges in $\mathcal{L}_b(X'')$; see [3, Lemma 2.6] or [4, Lemma 2.1]. Since X'' is a quojection Fréchet space, we can apply (ii)⇒(iii) of Case (I) above to conclude that $\sigma(T'') \subset \mathbb{D}$ (the condition τ_b-$\lim_{n\to\infty} \frac{T^n}{n} = 0$ clearly follows from the assumption). So, $\sigma(T) \subseteq \mathbb{D}$ by Corollary 2.4.

(iii)⇒(i). As already noted (cf. proof of Case (II) in Proposition 3.1) X and X'_β are barrelled (hence, quasi-barrelled) and τ_b-$\lim_{n\to\infty} \frac{(T'')^n}{n} = 0$. By Corollary 2.4, $\rho(T'') = \rho(T)$ and so $\Lambda \subseteq \rho(T'')$ by assumption. Since X'' is a quojection Fréchet space, we can apply Case (I) to conclude that τ_b-$\lim_{n\to\infty}(T'')^n = 0$. So, also τ_b-$\lim_{n\to\infty} T^n = 0$ as $T''|_X = T$ and X is a closed subspace of X''.

Finally, suppose that one (hence, all) of the above conditions hold. Then the series $\sum_{n=0}^{\infty} T^n$ converges in $\mathcal{L}_b(X)$ and so $T^n \to 0$ in $\mathcal{L}_b(X)$ for $n \to \infty$. But, for every $n \in \mathbb{N}$ we have

$$(I-T)\sum_{m=0}^{n} T^m = \sum_{m=0}^{n}(T^m - T^{m+1}) = (I - T^{n+1})$$

and so, for $n \to \infty$, we can conclude that $(I-T)\sum_{n=0}^{\infty} T^n = I$ with convergence of the series in $\mathcal{L}_b(X)$. In a similar way one shows that $\left(\sum_{n=0}^{\infty} T^n\right)(I-T) = I$, with the series again converging in $\mathcal{L}_b(X)$. □

Remark 3.4. In the proof of (iii)⇒(i) in Case (I) above, if $\inf_{j\in\mathbb{N}} \delta_j =: \delta > 0$, then it follows that $\rho(T) \supset \{\lambda \in \mathbb{C} : |\lambda| \geq (1-\delta)\}$. But, this is not the case in general as the following example shows.

Let X be a Banach space and $\{\lambda_n\}_{n=1}^{\infty} \in (0,1)$ be an increasing sequence with $\sup_{n\in\mathbb{N}} \lambda_n = 1$. Consider the quojection Fréchet space $X^{\mathbb{N}}$ (endowed with the product topology) and the operator T on $X^{\mathbb{N}}$ defined by $T(x_n)_n := (\lambda_n x_n)_n$, for $(x_n)_n \in X^{\mathbb{N}}$. It is easy to show that $T \in \mathcal{L}(X)$ and that T is even power bounded. Moreover, $\Lambda \subseteq \rho(T)$. Indeed, for a fixed $\lambda \in \Lambda$, if $x \in \mathrm{Ker}(\lambda I - T)$, then $\lambda x - Tx = 0$, i.e., $(\lambda - \lambda_n)x_n = 0$ for all $n \in \mathbb{N}$. Since $\lambda \notin \{\lambda_n\}_{n=1}^{\infty}$, it follows that $x_n = 0$ for all $n \in \mathbb{N}$ and so $x = 0$. On the other hand, if $y \in X^{\mathbb{N}}$, then $x := (y_n/(\lambda - \lambda_n))_n$ belongs to $X^{\mathbb{N}}$ and $Tx = y$. Hence, $\lambda I - T$ is bijective and so $\lambda \in \rho(T)$. Moreover, fix any $x \in X \setminus \{0\}$ and set $e_n := (\delta_{nm}x)_m$ for every $n \in \mathbb{N}$. Then $Te_n = \lambda_n e_n$ for every $n \in \mathbb{N}$. Thus, each λ_n is an eigenvalue of T.

Now, suppose that $\rho(T) \supset \{\lambda \in \mathbb{C} : |\lambda| \geq 1-\delta\}$ for some $\delta \in (0,1)$. Then $B(1,\delta/2) := \{\mu \in \mathbb{C} : |\mu - 1| < \delta/2\} \subset \rho(T)$. But $\lambda_n \to 1$ for $n \to \infty$ and hence, there is $n_0 \in \mathbb{N}$ such that $\lambda_{n_0} \in B(1,\delta/2) \subset \rho(T)$. This a contradiction as λ_{n_0} is an eigenvalue for T.

If T is uniformly mean ergodic, then (2.11) implies that τ_b-$\lim_{n\to\infty} \frac{T^n}{n} = 0$. With an extra condition the converse is also valid.

Corollary 3.5. *Let X be a prequojection Fréchet space and $T \in \mathcal{L}(X)$. If τ_b-$\lim_{n\to\infty} \frac{T^n}{n} = 0$ and $1 \in \rho(T)$, then T is uniformly mean ergodic.*

Proof. Since $1 \in \rho(T)$, the operator $I - T$ is bijective and so $(I-T)(X) = X$ is closed in X. By [7, Theorem 3.5], T is uniformly mean ergodic. In particular, as $\mathrm{Ker}(I-T) = \{0\}$, we have that $T_{[n]} \to 0$ in $\mathcal{L}_b(X)$ for $n \to \infty$. □

Remark 3.6. Let X be a prequojection Fréchet space and $T \in \mathcal{L}(X)$ satisfy τ_b-$\lim_{n\to\infty} \frac{T^n}{n} = 0$. If $1 \in \rho(T)$, then the proof of Corollary 3.5(i) shows that T is uniformly mean ergodic with τ_b-$\lim_{n\to\infty} T_{[n]} = 0$. On the other hand, if $\sigma(T) \subseteq \mathbb{D}$ (a stronger condition than $1 \in \rho(T)$), then Theorem 3.3 implies that τ_b-$\lim_{n\to\infty} T^n = 0$ and hence, again τ_b-$\lim_{n\to\infty} T_{[n]} = 0$ follows, [8, Remark 3.1]. However, the stronger conclusion that τ_b-$\lim_{n\to\infty} T^n = 0$ does not follow from Corollary 3.5(i) in general. Indeed, let $X \neq \{0\}$ be any Banach space (even finite dimensional). Then every power of $T := iI$ belongs to the set $\{-I, I, -iI, iI\}$ and so T is power bounded. This implies that τ_b-$\lim_{n\to\infty} \frac{T^n}{n} = 0$. Since $\sigma(T) = \{i\}$, surely $1 \in \rho(T)$ and so, by Corollary 3.5(i), it follows that τ_b-$\lim_{n\to\infty} T_{[n]} = 0$. However, for every $n \in \mathbb{N}$ we have $\|T^n\|_{op} = 1$ and so $\{\|T^n\|_{op}\}_{n=1}^\infty$ does *not* converge to zero. This does not contradict Theorem 3.3 as $\sigma(T)$ is not included in $\mathbb{D}$.

Remark 3.7. Let X be a prequojection Fréchet space and $T \in \mathcal{L}(X)$. We observe that:

(i) Corollary 3.5(ii) and Proposition 3.1 yield that if T is uniformly mean ergodic, then τ_b-$\lim_{n\to\infty} \frac{T^n}{n} = 0$ and $\sigma(T) \subseteq \overline{\mathbb{D}}$.
(ii) Suppose that τ_b-$\lim_{n\to\infty} \frac{T^n}{n} = 0$. If $\sigma(T) \subseteq \mathbb{D}$, then T is uniformly mean ergodic and τ_b-$\lim_{n\to\infty} T_{[n]} = 0$ (cf. Remark 3.6).

For Banach spaces the next result is due to M. Mbekhta and J. Zemànek, [33]. Recall that $\Gamma(T) := \sigma(T) \cap \mathbb{T}$.

Theorem 3.8. *Let X be a prequojection Fréchet space and $T \in \mathcal{L}(X)$. The following statements are equivalent.*

(i) $\{T^n\}_{n=1}^\infty$ *is convergent in* $\mathcal{L}_b(X)$.
(ii) τ_b-$\lim_{n\to\infty} \frac{T^n}{n} = 0$, *the linear space* $(I-T)^m(X)$ *is closed in* X *for some* $m \in \mathbb{N}$ *and* $\Gamma(T) \subseteq \{1\}$.
(iii) τ_b-$\lim_{n\to\infty}(T^n - T^{n+1}) = 0$ *and* $(I-T)^m(X)$ *is closed for some* $m \in \mathbb{N}$.

Proof. (i)$\Rightarrow$(ii). If $\{T^n\}_{n=1}^\infty$ converges in $\mathcal{L}_b(X)$ to P say, then T is uniformly mean ergodic with ergodic projection equal to P, [8, Remark 3.1]. Moreover, as $\{T^n\}_{n=1}^\infty$ is necessarily equicontinuous, it follows that τ_b-$\lim_{n\to\infty} \frac{T^n}{n} = 0$. Hence, by Theorem 3.5 and Remark 3.6 of [7] the space $(I-T)^m(X)$ is closed for every $m \in \mathbb{N}$. Moreover, by Proposition 3.1 we have $\sigma(T) \subseteq \overline{\mathbb{D}}$. To establish the remaining condition $\Gamma(T) \subseteq \{1\}$ we distinguish two cases.

(a) X is a quojection.

Let $\{r_j\}_{j=1}^\infty$ be any fundamental, increasing sequence of seminorms generating the lc-topology of X. By equicontinuity of $\{T^n\}_{n=1}^\infty$, for each $j \in \mathbb{N}$ there exists $c_j > 0$ such that

$$r_j(T^n x) \leq c_j r_{j+1}(x), \quad x \in X,\ n \in \mathbb{N}. \tag{3.12}$$

Define q_j, for each $j \in \mathbb{N}$, by $q_j(x) := \sup_{n\geq 0} r_j(T^n x)$, for $x \in X$. Then (3.12) ensures that $\{q_j\}_{j=1}^\infty$ is also a fundamental, increasing sequence of seminorms generating the lc-topology of X. Moreover, it is routine to check (using also that $T^n x \to Px$ for each $x \in X$) that

$$q_j(Tx) \leq q_j(x) \quad \text{and} \quad q_j(Px) \leq q_j(x), \quad x \in X,\ j \in \mathbb{N}. \tag{3.13}$$

With (3.13) in place of (3.6), we can argue as in the proof of Proposition 3.1 to deduce that $X = \operatorname{proj}_j(X_j, Q_{j,j+1})$ and that, for every $j \in \mathbb{N}$ there exist operators T_j and P_j in $\mathcal{L}(X_j)$ satisfying $T_jQ_j = Q_jT$ and $P_jQ_j = Q_jP$. Hence, $T_j^nQ_j = Q_jT^n$ for every j, $n \in \mathbb{N}$. Since also τ_b-$\lim_{n\to\infty} T^n = P$, it follows from Lemma 2.6 (with $S_n := T^n$ and $S := P$) that τ_b-$\lim_{n\to\infty} T_j^n = P_j$, for each $j \in \mathbb{N}$. By [33, Corollaire 3] we have that $\Gamma(T_j) \subseteq \{1\}$ for every $j \in \mathbb{N}$. This implies that $\Gamma(T) \subseteq \{1\}$. Indeed, if $\lambda \in \mathbb{T} \setminus \{1\}$, then for every $j \in \mathbb{N}$ we have $\lambda \notin \Gamma(T_j)$ and so $\lambda \in \rho(T_j)$, i.e., $\lambda \in \cap_{j=1}^\infty \rho(T_j)$. As $T_jQ_j = Q_jT$ for every $j \in \mathbb{N}$, an appeal to Lemma 2.5 yields that $\lambda \in \rho(T)$.

(b) X is a prequojection.

As noted before, X and X'_β are barrelled (hence, quasi-barrelled) with $T', P' \in \mathcal{L}(X'_\beta)$ and $T'', P'' \in \mathcal{L}(X'')$. Hence, τ_b-$\lim_{n\to\infty} T^n = P$ implies that τ_b-$\lim_{n\to\infty} (T'')^n = P''$; see [3, Lemma 2.6] or [4, Lemma 2.1]. Since X'' is a quojection Fréchet space, we can apply the result from case (a) to conclude that $\Gamma(T'') \subseteq \{1\}$ and so $\Gamma(T) \subseteq \{1\}$; see Corollary 2.4.

(ii)$\Rightarrow$(i). The assumptions τ_b-$\lim_{n\to\infty} \frac{T^n}{n} = 0$ and $(I-T)^m(X)$ closed for some $m \in \mathbb{N}$ imply that T is uniformly mean ergodic, [7, Theorem 3.4 and Remark 3.6]. In particular, $(I-T)(X)$ is closed and

$$X = \operatorname{Ker}(I-T) \oplus (I-T)(X), \tag{3.14}$$

[7, Theorem 3.4]. Moreover, Proposition 3.1 implies that $\sigma(T) \subseteq \overline{\mathbb{D}}$. It then follows from the assumption $\Gamma(T) \subseteq \{1\}$ that either $\Gamma(T) = \emptyset$ or $\Gamma(T) = \{1\}$.

If $\Gamma(T) = \emptyset$, then necessarily $\sigma(T) \subseteq \mathbb{D}$ and so, by (iii)$\Rightarrow$(i) of Theorem 3.3 we have τ_b-$\lim_{n\to\infty} T^n = 0$.

In the event that $\Gamma(T) = \{1\}$ we have that $1 \in \sigma(T)$ and so $\operatorname{Ker}(I-T) \neq \{0\}$ (otherwise, $(I-T)$ is injective and from $X = \operatorname{Ker}(I-T) \oplus (I-T)(X) = (I-T)(X)$ also surjective, i.e., $1 \in \rho(T)$). Define $Y := (I-T)(X)$ and $T_1 := T|_Y$. Then Y is a prequojection Fréchet space (being a quotient space of the prequojection X) which is T-invariant and so $T_1 \in \mathcal{L}(Y)$. The claim is that

$$\rho(T_1) = \rho(T) \cup \{1\}. \tag{3.15}$$

It follows from (3.14) that $1 \in \rho(T_1)$. Fix $\lambda \in \rho(T)$ (so that $\lambda \neq 1$). If $(\lambda I - T_1)x = 0$ for some $x \in Y$ (i.e., $(\lambda I - T)x = 0$), then $x = 0$ as $\lambda \in \rho(T)$. Hence, $(\lambda I - T_1)$ is injective. Next, let $y \in Y$. Then there exists $x \in X$ such that $(\lambda I - T)x = y$. Since $x = x_1 + x_2$ with $x_1 \in \operatorname{Ker}(I-T)$ and $x_2 \in Y$ (cf. (3.14)), it follows that $(\lambda - 1)x_1 + (\lambda I - T_1)x_2 = y$, i.e., $(\lambda - 1)x_1 = y - (\lambda I - T_1)x_2$, with $(\lambda - 1)x_1 \in \operatorname{Ker}(I-T)$ and $(y - (\lambda I - T_1)x_2) \in Y$. As $\operatorname{Ker}(I-T) \cap Y = \{0\}$ and $\lambda \neq 1$, this implies that $x_1 = 0$ and so $(\lambda I - T_1)x_2 = y$ with $x_2 \in Y$, i.e., $(\lambda I - T_1)$ is surjective. These facts show that $\lambda \in \rho(T_1)$. This establishes $\rho(T) \cup \{1\} \subseteq \rho(T_1)$.

Fix $\lambda \in \rho(T_1) \setminus \{1\}$. Suppose that $(\lambda I - T)x = 0$ for some $x \in X$. Then $x = x_1 + x_2$ with $x_1 \in \operatorname{Ker}(I-T)$ and $x_2 \in Y$ (cf. (3.14)). It follows that $(\lambda - 1)x_1 + (\lambda I - T_1)x_2 = 0$ with $(\lambda - 1)x_1 \in \operatorname{Ker}(I-T)$ and $(\lambda I - T_1)x_2 \in Y$. Arguing as in the previous paragraph, this implies that $x_1 = 0$ and $(\lambda I - T_1)x_2 = 0$. Since $x_2 \in Y$ and $\lambda \in \rho(T_1)$, we can conclude that $x = 0$, i.e., $(\lambda I - T)$ is injective. Next, let $y \in X$. Then $y = y_1 + y_2$ with $y_1 \in \operatorname{Ker}(I-T)$ and $y_2 \in Y$ (cf. (3.14)). Since $\lambda \neq 1$, the element $x_1 := \frac{y_1}{\lambda - 1} \in \operatorname{Ker}(I-T)$ exists. Moreover, $\lambda \in \rho(T_1)$ with $y_2 \in Y$ implies the existence of $x_2 \in Y$ such that

$y_2 = (\lambda I - T_1)x_2 = (\lambda I - T)x_2$. It follows that $x := (x_1 + x_2) \in X$ satisfies $(\lambda I - T)x = y$. Hence, $(\lambda I - T)$ is also surjective and so $\lambda \in \rho(T)$. Accordingly, $\rho(T_1) \subseteq \rho(T) \cup \{1\}$ is proved. This establishes (3.15).

Since $\sigma(T) \subseteq \mathbb{D} \cup \{1\}$ and (3.15) is equivalent to $\sigma(T_1) = \sigma(T) \setminus \{1\}$, it follows that $\sigma(T_1) \subseteq \mathbb{D}$. Moreover, Y is a prequojection Fréchet space and $\frac{(T_1)^n}{n} \to 0$ in $\mathcal{L}_b(Y)$ as $n \to \infty$ (because τ_b-$\lim_{n\to\infty} \frac{T^n}{n} = 0$ and $T_1 = T$ on Y). So, we can apply Theorem 3.3 to conclude that $T_1^n \to 0$ in $\mathcal{L}_b(Y)$ as $n \to \infty$. On the other hand, $T = I$ on $\mathrm{Ker}(I - T)$. These facts ensure that $T^n = I \oplus (T_1)^n \to I \oplus 0$ in $\mathcal{L}_b(X)$ because $X = \mathrm{Ker}(I - T) \oplus Y$ and $T_1 = T$ on Y.

(i)⇒(iii). If $\{T^n\}_{n=1}^\infty$ converges to some P in $\mathcal{L}_b(X)$, then T is uniformly mean ergodic with ergodic projection equal to P, [8, Remark 3.1]. Hence, by [7, Theorem 3.5 and Remark 3.6] the space $(I - T)^m(X)$ is closed for every $m \in \mathbb{N}$. Moreover, $(T^n - T^{n+1}) \to P - P = 0$ in $\mathcal{L}_b(X)$ as $n \to \infty$.

(iii)⇒(i). We first observe that

$$\frac{1}{n}\sum_{m=1}^{n}(T^m - T^{m+1}) = \frac{1}{n}(T - T^{n+1}), \quad n \in \mathbb{N}$$

This identity (together with the fact that τ_b-$\lim_{n\to\infty}(T^n - T^{n+1}) = 0$ implies for the averages that τ_b-$\lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n}(T^m - T^{m+1}) = 0$, [8, Remark 3.1]) yields τ_b-$\lim_{n\to\infty} \frac{1}{n}(T - T^{n+1}) = 0$. But, τ_b-$\lim_{n\to\infty} \frac{T}{n} = 0$ and so we can conclude that τ_b-$\lim_{n\to\infty} \frac{T^n}{n} = 0$. As also $(I - T)^m(X)$ is closed for some $m \in \mathbb{N}$, we can apply [7, Theorem 3.4 and Remark 3.6] to conclude that T is uniformly mean ergodic and, in particular, that (3.14) is valid with $(I-T)(X)$ being closed. We claim that this fact, together with the assumption that τ_b-$\lim_{n\to\infty}(T^n - T^{n+1}) = 0$, imply that $\{T^n\}_{n=1}^\infty$ converges in $\mathcal{L}_b(X)$. To see this, note that $T = I$ on $\mathrm{Ker}(I - T)$ and so $T^n = I \to I$ in $\mathcal{L}_b(\mathrm{Ker}(I - T))$ as $n \to \infty$. On the other hand, the surjective operator $(I - T)\colon X \to (I - T)(X)$ lifts bounded sets via [34, Lemma 26.13] because X and $\mathrm{Ker}(I - T)$, both being prequojections, are quasinormable Fréchet spaces [35, Proposition 2.1], [41], i.e., for every $C \in \mathcal{B}((I - T)(X))$ there exists $B \in \mathcal{B}(X)$ such that $C \subseteq (I-T)(B)$. So, for fixed $C \in \mathcal{B}((I-T)(X))$ (with corresponding set $B \in \mathcal{B}(X)$) and $p \in \Gamma_X$ (every $q \in \Gamma_{(I-T)X}$ is the restriction of some $p \in \Gamma_X$), we have

$$\sup_{y\in C} p(T^n y) \le \sup_{x\in B} p(T^n(I - T)x) = \sup_{x\in B} p((T^n - T^{n+1})x), \quad n \in \mathbb{N},$$

where $\sup_{x\in B} p((T^n - T^{n+1})x) \to 0$ as $n \to \infty$ by assumption. Set $T_1 := T|_{(I-T)(X)}$. The arbitrariness of C and p show that $(T_1)^n \to 0$ in $\mathcal{L}_b((I - T)(X))$ (after observing that $(I - T)(X)$ is T–invariant and so $T_1 = T|_{(I-T)(X)} \in \mathcal{L}((I - T)(X)))$. These facts ensure that $T^n = I \oplus (T_1)^n \to I \oplus 0$ in $\mathcal{L}_b(X)$ as $X = \mathrm{Ker}(I - T) \oplus Y$. □

Remark 3.9. In assertion (ii) of Theorem 3.8 the condition that "$(I - T)^m(X)$ *is closed in* X *for some* $m \in \mathbb{N}$" can be replaced with the condition that "T *is uniformly mean ergodic*"; see [7, Theorem 3.5 and Remark 3.6].

Theorems 3.3 and 3.8 do not necessarily hold for operators acting in general Fréchet spaces.

Proposition 3.10. *Let $p \in [1,\infty)$ or $p = 0$ and let A be a Köthe matrix on $\mathbb{N}$ such that $\lambda_p(A)$ is a Montel space with $\lambda_p(A) \neq \mathbb{C}^{\mathbb{N}}$. Then there exists an operator $T \in \mathcal{L}(\lambda_p(A))$ such that $T^n \to 0$ in $\mathcal{L}_b(\lambda_p(A))$ as $n \to \infty$ and $\Gamma(T) = \{1\}$ but, $(I-T)^m(\lambda_p(A))$ is not closed for every $m \in \mathbb{N}$.*

Proof. By the proof of Proposition 3.1 in [7] there exists $d := (d_i)_i \in \mathbb{R}^{\mathbb{N}}$ with $0 < d_i < 1$ for all $i \in \mathbb{N}$ such that the diagonal operator $T\colon \lambda_p(A) \to \lambda_p(A)$ given by $T((x_i)_i) := (d_i x_i)_i$, for $x = (x_i)_i \in \lambda_p(A)$, is *power bounded, uniformly mean ergodic* and $(I-T)(\lambda_p(A))$ is dense but, not closed in $\lambda_p(A)$. So, for every $m \in \mathbb{N}$, also $(I-T)^m(\lambda_p(A))$ is dense but not closed in $\lambda_p(A)$. To see this, note that the arguments in the proof of [7, Remark 3.6, (5)⇒(4)] are valid for any operator T satisfying $\tau_s\text{-}\lim_{n\to\infty} \frac{T^n}{n} = 0$ and acting in any Fréchet space. So, in the case that $(I-T)^m(\lambda_p(A))$ was closed for some $m \in \mathbb{N}$, we could apply [7, Remark 3.6, (5)⇒(4)] to conclude that $(I-T)(\lambda_p(A))$ is also closed; a contradiction. So $1 \in \Gamma(T)$.

We claim that $T^n \to 0$ in $\mathcal{L}_b(\lambda_p(A))$ as $n \to \infty$. Indeed, since $\{T^n\}_{n=1}^\infty$ is equicontinuous and convergence of a sequence in $\mathcal{L}_b(\lambda_p(A))$ is equivalent to its convergence in $\mathcal{L}_s(\lambda_p(A))$ (as $\lambda_p(A)$ is Montel), it suffices to show that $\lim_{n\to\infty} T^n e_j = 0$ in $\lambda_p(A)$ for each $j \in \mathbb{N}$, where $e_j := (\delta_{ij})_i \in \lambda_p(A)$. But, this is immediate because $T^n e_j = d_j^n e_j$, for all $j,\ n \in \mathbb{N}$.

It remains to show that $\Gamma(T) \subseteq \{1\}$. Set $D := \overline{\{d_i : i \in \mathbb{N}\}}$. Then $D \subseteq [0,1]$. Let $\lambda \in \mathbb{T} \setminus \{1\}$. Then $\inf_{i\in\mathbb{N}} |\lambda - d_i| =: \delta > 0$. It is routine to check that, for a fixed $y \in \lambda_p(A)$, the element $x := \left(\frac{1}{\lambda - d_i} y_i\right)_i$ belongs to $\lambda_p(A)$ and satisfies $(\lambda I - T)x = y$. This means that the operator $(\lambda I - T)$ is surjective. On the other hand $\mathrm{Ker}(\lambda I - T) = \{0\}$ which follows from $\lambda \notin \overline{\{d_i : i \in \mathbb{N}\}}$. Therefore, as $\lambda_p(A)$ is a Fréchet space, $\lambda \in \rho(T)$, i.e., $\mathbb{T} \setminus \{1\} \subseteq \rho(T)$. Since $1 \in \Gamma(T)$, it follows that $\Gamma(T) = \{1\}$. □

Concerning the example in Proposition 3.10 we note that (i) of Theorem 3.3 holds but, (iii) of Theorem 3.3 fails (as $\Gamma(T) = \{1\}$ implies that $\sigma(T) \not\subseteq \mathbb{D}$). Moreover, (i) of Theorem 3.8 holds (as $\tau_b\text{-}\lim_{n\to\infty} T^n = 0$) but, (ii) and (iii) of Theorem 3.8 fail (because $(I-T)^m(\lambda_p(A))$ is not closed in $\lambda_p(A)$ for every $m \in \mathbb{N}$). Of course, $\lambda_p(A)$ is *not* a prequojection.

A well known result of Katznelson and Tzafriri states that a power bounded operator T on a Banach space satisfies $\lim_{n\to\infty} \|T^{n+1} - T^n\|_{op} = 0$ if and only if $\Gamma(T) \subseteq \{1\}$, [28, Theorem 1 and p. 317 Remark]. In order to extend this result to prequojection Fréchet spaces (see Theorem 3.13 below) we require the following notion.

Let X be a Fréchet space and $T \in \mathcal{L}(X)$. A fundamental, increasing sequence $\{q_j\}_{j=1}^\infty \subseteq \Gamma_X$ which generates the lc–topology of X is called *T-contractively admissible* if, for each $j \in \mathbb{N}$, we have

$$q_j(Tx) \leq q_j(x), \quad x \in X. \tag{3.16}$$

Lemma 3.11. *Let X be a Fréchet space and $T \in \mathcal{L}(X)$. Then there exists a T-contractively admissible sequence of seminorms which generates the lc–topology of X if and only if T is power bounded.*

Proof. If $\{q_j\}_{j=1}^{\infty} \subseteq \Gamma_X$ is T-contractively admissible, then it is clear from (3.16) that $q_j(T^n x) \le q_j(x)$, for $x \in X$ and every $n \in \mathbb{N}_0$, $j \in \mathbb{N}$. This means precisely that $\{T^n\}_{n=1}^{\infty}$ is equicontinuous in $\mathcal{L}(X)$, i.e., T is power bounded.

Conversely, suppose that T is power bounded. Let $\{r_j\}_{j=1}^{\infty}$ be a fundamental, increasing sequence in Γ_X which generates the lc–topology of X. Via the equicontinuity of $\{T^n\}_{n=1}^{\infty}$ for every $j \in \mathbb{N}$ there exist $k(j) \ge j$ and $\alpha_j > 0$ such that

$$r_j(T^n x) \le \alpha_j r_{k(j)}(x), \quad x \in X,\ n \in \mathbb{N}.$$

Define $q_j(x) := \sup_{n \in \mathbb{N}_0} r_j(T^n x)$, for $x \in X$ and each $j \in \mathbb{N}$. Then the previous inequality implies that

$$r_j(x) \le q_j(x) \le \alpha_j r_{k(j)}(x), \quad x \in X,\ j \in \mathbb{N},$$

and so $\{q_j\}_{j=1}^{\infty} \subseteq \Gamma_X$ is a fundamental, increasing sequence determining the lc–topology of X, which clearly satisfies (3.16). That is, $\{q_j\}_{j=1}^{\infty}$ is T-contractively admissible. □

Remark 3.12. (i) For a Banach space X, Lemma 3.11 simply states that T is power bounded if and only if it is a contraction for some equivalent norm in X.

(ii) Let X be a Fréchet space and $T \in \mathcal{L}(X)$ be an isomorphism which is *bi-power bounded*, i.e., $\{T^n : n \in \mathbb{Z}\}$ is equicontinuous in $\mathcal{L}(X)$. An examination of the proof of Lemma 3.11 shows that there exists a sequence $\{q_j\}_{j=1}^{\infty} \subseteq \Gamma_X$, again called *$T$-contractively admissible*, which generates the lc–topology of X and satisfies, for each $j \in \mathbb{N}$,

$$q_j(T^n x) \le q_j(x), \quad x \in X,\ n \in \mathbb{Z}. \tag{3.17}$$

Theorem 3.13. *Let X be a prequojection Fréchet space and $T \in \mathcal{L}(X)$ be power bounded. The following assertions are equivalent.*

(i) τ_b-$\lim_{n\to\infty}(T^{n+1} - T^n) = 0$.

(ii) $\Gamma(T) \subseteq \{1\}$ *and there exists a T-contractively admissible sequence $\{p_j\}_{j=1}^{\infty} \subseteq \Gamma_X$ such that, for each $\lambda \in \mathbb{T} \setminus \{1\}$ and $j \in \mathbb{N}$, there exists $M_{\lambda,j} > 0$ satisfying*

$$p_j(R(\lambda, T)x) \le M_{\lambda,j} p_j(x), \quad x \in X. \tag{3.18}$$

.

Remark 3.14. (i) If $\Gamma(T) \subseteq \{1\}$, then necessarily $\mathbb{T} \setminus \{1\} \subseteq \rho(T)$ and so the resolvent family $\{R(\lambda, T) : \lambda \in \mathbb{T} \setminus \{1\}\}$ is defined.

(ii) If $\Gamma(T) = \emptyset$, then (i) of Theorem 3.13 follows without any further conditions. Indeed, by Remark 3.2 we actually have $\sigma(T) \subseteq \mathbb{D}$. Then Theorem 3.3 implies that τ_b-$\lim_{n\to\infty} T^n = 0$ and hence, also τ_b-$\lim_{n\to\infty}(T^{n+1} - T^n) = 0$.

(iii) If X is a Banach space and $\|\cdot\|$ is any norm in X for which T is a contraction (i.e., $\|\cdot\|$ is T-contractively admissible), then the requirement (3.18) automatically holds with $M_\lambda := \|R(\lambda, T)\|_{op}$. That is, condition (ii) in Theorem 3.13 simply reduces to $\Gamma(T) \subseteq \{1\}$ and we recover the result of Katznelson and Tzafriri.

Proof. (of Theorem 3.13) (i)⇒(ii). As usual we distinguish two cases.

Case (I). X is a quojection.

According to Lemma 3.11 there is a T-contractively admissible sequence $\{q_j\}_{j=1}^{\infty} \subseteq \Gamma_X$ satisfying (3.16) and hence, also $q_j(T^n x) \le q_j(x)$, for $x \in X$ and all $j,\ n \in \mathbb{N}$.

We proceed as in the proof of Proposition 3.1 (now using (3.16) in place of (3.6) so that (3.8) becomes $\tilde{q}_j(T_j\hat{x}) \leq \tilde{q}_j(\hat{x})$, for $\hat{x} \in X_j$ and $j \in \mathbb{N}$) to obtain that $X = \mathrm{proj}_j(X_j, Q_{j,j+1})$ in such a way that, for every $j \in \mathbb{N}$, there exists a *contraction* $T_j \in \mathcal{L}(X_j)$ satisfying $T_j Q_j = Q_j T$. Then also $T_j^n Q_j = Q_j T^n$ for all $j,\ n \in \mathbb{N}$. For each $j \in \mathbb{N}$, define $p_j(x) := \tilde{q}_j(Q_j x)$ for $x \in X$. By the properties of projective limits $\{p_j\}_{j=1}^{\infty} \subseteq \Gamma_X$ is a fundamental sequence generating the lc-topology of X. Moreover,

$$p_j(Tx) = \tilde{q}_j(Q_j Tx) = \tilde{q}_j(T_j Q_j x) \leq \tilde{q}_j(Q_j x) = p_j(x), \quad x \in X,$$

shows that $\{q_j\}_{j=1}^{\infty}$ is also T-contractively admissible. According to Lemma 2.6 (applied to the norms $\|\ \|_j := \tilde{q}_j$ and with $S_n := (T^{n+1} - T^n)$, $n \in \mathbb{N}$, and $S_n^{(j)} = (T_j^{n+1} - T_j^n)$, for $j,\ n \in \mathbb{N}$), the assumption τ_b-$\lim_{n\to\infty}(T^{n+1} - T^n) = 0$ implies that $\lim_{n\to\infty} \|T_j^{n+1} - T_j^n\|_{op} = 0$, for each $j \in \mathbb{N}$. By [28, Theorem 1] we can conclude that $\Gamma(T_j) \subseteq \{1\}$. On the other hand, $\sigma(T_j) \subseteq \overline{\mathbb{D}}$ as T_j is a contraction and so $\sigma(T_j) \subseteq \mathbb{D} \cup \{1\}$, i.e., $\rho(T_j) \supseteq \mathbb{C} \setminus (\mathbb{D} \cup \{1\})$, for $j \in \mathbb{N}$. According to Lemma 2.5 also $\rho(T) \supseteq \mathbb{C} \setminus (\mathbb{D} \cup \{1\})$, i.e., $\Gamma(T) \subseteq \{1\}$.

Concerning (3.18), fix $\lambda \in \mathbb{T} \setminus \{1\}$ and $j \in \mathbb{N}$. By the previous paragraph $\lambda \in \rho(T) \cap \rho(T_j)$. It follows from $T_j Q_j = Q_j T$ that $Q_j R(\lambda, T) = R(\lambda, T_j) Q_j$. Hence, for $x \in X$, we have

$$\begin{aligned} p_j(R(\lambda,T)x) &= \tilde{q}_j(Q_j R(\lambda,T)x) = \tilde{q}_j(R(\lambda,T_j)Q_j x) \\ &\leq \|R(\lambda,T_j)\|_{op}\tilde{q}_j(Q_j x) = \|R(\lambda,T_j)\|_{op} p_j(x) \end{aligned}$$

which establishes (3.18).

Case (II). X is a prequojection.

As noted before, X and X'_β are barrelled (hence, quasi–barrelled) with $T' \in \mathcal{L}(X'_\beta)$ and $T'' \in \mathcal{L}(X'')$. So, the assumption τ_b-$\lim_{n\to\infty}(T^{n+1} - T^n) = 0$ implies that τ_b-$\lim_{n\to\infty}((T'')^{n+1} - (T'')^n) = 0$. Moreover, X'' is a quojection Fréchet space and T'' is power bounded; see Lemma 2.2. So, the result of Case (I) yields $\Gamma(T'') \subseteq \{1\}$. But, $\Gamma(T) = \Gamma(T'')$ (see Corollary 2.4) and so $\Gamma(T) \subseteq \{1\}$.

By (i)⇒(ii) for quojections there exists a T''-contractively admissible sequence $\{p''_j\}_{j=1}^{\infty} \subseteq \Gamma_{X''}$ such that, for every $\lambda \in \mathbb{T} \setminus \{1\}$ and $j \in \mathbb{N}$, there exists $M_{\lambda,j} > 0$ satisfying

$$p''_j(R(\lambda, T'')x'') \leq M_{\lambda,j}\, p''_j(x''), \quad x'' \in X''.$$

By Lemma 2.1 and Corollary 2.4 the seminorms $p_j := p''_j \circ \Phi$, $j \in \mathbb{N}$, satisfy (3.18).

(ii)⇒(i). Case (I): X is a quojection.

Let $\{p_j\}_{j=1}^{\infty} \subseteq \Gamma_X$ be as in the statement of (ii), in which case (3.16) holds. Proceed as in Case (I) of the proof of (i)⇒(ii) to obtain that $X = \mathrm{proj}_j(X_j, Q_{j,j+1})$ in such a way that, for every $j \in \mathbb{N}$, there exists a *contraction* $T_j \in \mathcal{L}(X_j)$, satisfying $T_j Q_j = Q_j T$.

Claim 1. $\Gamma(T_j) \subseteq \{1\}$*, for every* $j \in \mathbb{N}$.

To establish this, let $\lambda \in \mathbb{T} \setminus \{1\}$. Since $\Gamma(T) \subseteq \{1\}$, it follows that $\lambda \in \rho(T)$ and hence, $\lambda I - T$ is surjective. But, also $Q_j\colon X \to X_j$ is surjective. It is then routine to check from the identity $(\lambda I_j - T_j)Q_j = Q_j(\lambda I - T)$ that $\lambda I_j - T_j$ is surjective. To verify that $\lambda I_j - T_j$ is injective suppose that $(\lambda I_j - T_j)y = 0$ for some $y \in X_j$, in which case $y = Q_j x$ for some $x \in X$. Accordingly,

$$Q_j(\lambda I - T)x = (\lambda I_j - T_j)Q_j x = (\lambda I_j - T_j)y = 0$$

shows that $(\lambda I - T)x \in \operatorname{Ker} Q_j = \operatorname{Ker} p_j$. It then follows from (3.18) that $x = R(\lambda, T)(\lambda I - T)x \in \operatorname{Ker} p_j$, i.e., $Q_j x = 0$. Since $y = Q_j x$, we have $y = 0$. Hence, $\lambda I_j - T_j$ is injective. This establishes that $\lambda \in \rho(T_j)$ and hence, Claim 1 follows as $\lambda \in \mathbb{T} \setminus \{1\}$ was arbitrary.

Fix $j \in \mathbb{N}$. From Claim 1 and the fact that T_j is a contraction, it follows from [28, Theorem 1] that $\lim_{n\to\infty} \|T_j^n - T_j^{n+1}\|_{op} = 0$. According to Lemma 2.6 (with $S_n := (T^{n+1} - T^n)$, $n \in \mathbb{N}$) we can conclude that τ_b-$\lim_{n\to\infty}(T^{n+1} - T^n) = 0$.

Case (II): X is a prequojection.

By Corollary 2.4 we have from $\Gamma(T) \subseteq \{1\}$ that $\Gamma(T'') \subseteq \{1\}$. Moreover, Lemma 2.2 implies that $T'' \in \mathcal{L}(X'')$ is power bounded.

Let $\{p_j\}_{j=1}^\infty \subseteq \Gamma_X$ be as stated in part (ii). Apply Lemma 2.1 to construct the seminorms $\{p_j''\}_{j=1}^\infty \subseteq \Gamma_{X''}$ given there. We first verify that $\{p_j''\}_{j=1}^\infty \subseteq \Gamma_{X''}$ is T''-contractively admissible. Since $\{p_j\}_{j=1}^\infty$ is T-contractively admissible, we have $T(\mathcal{U}_j) \subseteq \mathcal{U}_j$ with $\mathcal{U}_j$ the closed unit ball of p_j, i.e., $\mathcal{U}_j = p_j^{-1}([0,1])$, for $j \in \mathbb{N}$. By the Bi-polar Theorem, [34, Theorem 22.13] applied twice we have

$$T''(\mathcal{U}_j^{\circ\circ}) = T''(\overline{\mathcal{U}}_j^{\sigma}) \subseteq \overline{T(\mathcal{U}_j)}^{\sigma} \subseteq \overline{\mathcal{U}}_j^{\sigma} = \mathcal{U}_j^{\circ\circ}, \tag{3.19}$$

where $\overline{V}^{\sigma}$ denotes the closure for the weak topology $\sigma(X'', X')$ of a subset $V \subseteq X''$ (or, of $V \subseteq X \subseteq X''$). Then (3.19) implies that $p_j''(T''x'') \leq p_j''(x'')$ for each $x'' \in X''$ and $j \in \mathbb{N}$, i.e., $\{p_j''\}_{j=1}^\infty$ is T''-contractively admissible.

It follows from (3.18) that $R(\lambda, T)(\mathcal{U}_j) \subseteq \mathcal{U}_j$, for all $\lambda \in \mathbb{T} \setminus \{1\}$ and $j \in \mathbb{N}$. Using $R(\lambda, T'')|_X = R(\lambda, T)$ (c.f. Corollary 2.4) one can repeat the argument via the Bi-polar Theorem to conclude that $R(\lambda, T'')(\mathcal{U}_j^{\circ\circ}) \subseteq M_{\lambda,j}\mathcal{U}_j^{\circ\circ}$, which then implies that

$$p_j''(R(\lambda, T'')x'') \leq M_{\lambda,j} p_j''(x''), \quad x'' \in X''.$$

So, the conditions in part (ii) are satisfied for the power bounded operator $T'' \in \mathcal{L}(X'')$ with respect to $\{p_j''\}_{j=1}^\infty$. Applying (ii)$\Rightarrow$(i) for the quojection Fréchet space X'' we conclude that τ_b-$\lim_{n\to\infty}((T'')^{n+1} - (T'')^n) = 0$. But, $T''|_X = T$ with X closed in X''. So, τ_b-$\lim_{n\to\infty}(T^{n+1} - T^n) = 0$, i.e., (i) holds. $\square$

Let X be a prequojection Fréchet space and $T \in \mathcal{L}(X)$ be power bounded. By Remark 3.2 we have $\sigma(T) \subseteq \overline{\mathbb{D}}$. Suppose that T is actually bi-power bounded. Then also $\sigma(T^{-1}) \subseteq \overline{\mathbb{D}}$. Clearly, $0 \in \rho(T)$. Moreover, if $\mu \in \mathbb{D} \setminus \{0\}$, then $\frac{1}{\mu} \in \mathbb{C} \setminus \overline{\mathbb{D}}$ and so $\frac{1}{\mu} \in \rho(T^{-1})$, i.e., $\left(\frac{1}{\mu}I - T^{-1}\right) \in \mathcal{L}(X)$. It is routine to check that $R_\mu := -\frac{1}{\mu}T^{-1}\left(\frac{1}{\mu}I - T^{-1}\right) \in \mathcal{L}(X)$ satisfies $(\mu I - T)R_\mu = I = R_\mu(\mu I - T)$ and hence, $(\mu I - T)$ is invertible in $\mathcal{L}(X)$ with $(\mu I - T)^{-1} = R_\mu$. This shows that $\mathbb{D} \subseteq \rho(T)$. Accordingly, $\sigma(T) \subseteq \mathbb{T}$; for X a Banach space, see [21, Proposition 1.31], for example. Suppose now, in addition, that $\sigma(T) = \{1\}$ in which case $\sigma(T - I) = \{0\}$, i.e., T is quasinilpotent. For X a Banach space, a classical result of Gelfand–Hille then states that necessarily $T = I$; see the survey article [42] for a complete discussion of this topic. The following fact is an extension of this result.

Corollary 3.15. *Let X be a prequojection Fréchet space and $T \in \mathcal{L}(X)$ be an isomorphism which is bi-power bounded. Suppose that $\Gamma(T) = \{1\}$ and there* *exists a T-contractively admissible sequence $\{p_j\}_{j=1}^{\infty} \subseteq \Gamma_X$ such that, for each $\lambda \in \mathbb{T} \setminus \{1\}$, the inequalities (3.18) are satisfied. Then $T = I$.*

Proof. According to Theorem 3.13 we can conclude that τ_b-$\lim_{n\to\infty}(T^{n+1} - T^n) = 0$. Fix $x \in X$. For each $j \in \mathbb{N}$, it follows that

$$p_j((T-I)x) = p_j(T^{-n}T^n(T-I)x) \le p_j(T^n(T-I)x) = p_j((T^{n+1} - T^n)x)$$

for every $n \in \mathbb{N}$. Since $\lim_{n\to\infty}(T^{n+1} - T^n)x = 0$, it follows that $p_j((T-I)x) = 0$ with $j \in \mathbb{N}$ arbitrary, i.e., $Tx = x$. So, $T = I$. □

4. Operator ideals and uniform mean ergodicity

Let X, Y be lcHs'. An operator $T \in \mathcal{L}(X,Y)$ is called *Montel* (resp. *reflexive*) if T maps bounded subsets of X into relatively compact (resp. relatively weakly compact subsets) subsets of Y, [17] (resp., [16]). According to Grothendieck, [26, Chapter 5, Part 2], T is called *compact* (resp., *weakly compact*) if there exists a 0–neighbourhood $\mathcal{U} \subseteq X$ such that $T(\mathcal{U})$ is relatively compact (resp., relatively weakly compact) in Y. Clearly, the 2–sided ideal $\mathcal{M}(X,Y)$ (resp., $\mathcal{R}(X,Y)$) of all Montel (resp., reflexive) operators coincides with the 2–sided ideal $\mathcal{K}(X,Y)$ (resp., $\mathcal{WK}(X,Y)$) of all compact (resp., weakly compact) operators whenever X, Y are Banach spaces. For general lcHs' we always have $\mathcal{K}(X,Y) \subseteq \mathcal{M}(X,Y)$ but, the containment may be proper; consider the identity operator on an infinite dimensional Montel lcHs. Clearly, $\mathcal{M}(X,Y) \subseteq \mathcal{R}(X,Y)$ and $\mathcal{WK}(X,Y) \subseteq \mathcal{K}(X,Y)$. Criteria for membership of $\mathcal{M}(X,Y)$ (resp. $\mathcal{R}(X,Y)$) occur in Theorem 9.2.1 (resp. Corollary 9.3.2) of [24], for example.

In this section we present various connections between the uniform convergence of sequences of operators generated by an operator $T \in \mathcal{H}(X)$ and the uniform mean ergodicity of T, where $\mathcal{H}$ stands for one of the operator ideals $\mathcal{K}$, $\mathcal{M}$, $\mathcal{WK}$, $\mathcal{R}$.

Every compact operator T acting in a Banach space has the property that $(I-T)$ has closed range. Hence, if $\lim_{n\to\infty} \frac{\|T^n\|_{op}}{n} = 0$, then T is uniformly mean ergodic, [22, p.711, Corollary 4], [31, p.87, Theorem 2.1]. For any lcHs X and $T \in \mathcal{K}(X)$, it is also the case that $(I-T)(X)$ is a closed subspace of X, [24, Theorem 9.10.1]. Hence, if X is a prequojection Fréchet space, then Theorem 3.5 of [7] implies that T is uniformly mean ergodic whenever τ_b-$\lim_{n\to\infty} \frac{T^n}{n} = 0$ (equivalently, τ_s-$\lim_{n\to\infty} \frac{T^n}{n} = 0$ because $K \in \mathcal{K}(X)$; see Remark 4.4(ii)). Since $\mathcal{K}(X) \subseteq \mathcal{M}(X)$, the question arises of whether the same is true for $T \in \mathcal{M}(X)$? This is indeed so; see Theorem 4.5 below.

In a lcHs X all relatively $\sigma(X,X')$–compact sets and all relatively sequentially $\sigma(X,X')$–compact sets are necessarily relatively countably $\sigma(X,X')$–compact. These are the only implications between these three notions which hold in general. All three notions coincides whenever X_σ is *angelic*, [25, p.31]. Such spaces X include all Fréchet spaces (actually, all (LF)–spaces), all (DF)–spaces and many more, [25, Section 3.10], [14, Theorem 11, Examples 1.2].

Operators $T \in \mathcal{L}(X)$ for which $\{T_{[n]}\}_{n=1}^{\infty} \subseteq \mathcal{L}(X)$ is equicontinuous will be called *Cesàro bounded*; see [31, p.72] for X a Banach space.

Proposition 4.1. *Let X be a lcHs such that X_σ is angelic and $T \in \mathcal{L}(X)$.*

(i) *If* $T \in \mathcal{R}(X)$ *is Cesàro bounded and satisfies* τ_s-$\lim_{n\to\infty} \frac{T^n}{n} = 0$, *then* T *is mean ergodic.*

(ii) *If* $T \in \mathcal{M}(X)$ *is Cesàro bounded and satisfies* τ_b-$\lim_{n\to\infty} \frac{T^n}{n} = 0$, *then* T *is uniformly mean ergodic.*

Proof. (i) Fix $x \in X$. It follows from (2.10) that

$$T_{[n]}x = T_{[n]}(I-T)x + T_{[n]}Tx = \frac{1}{n}(T - T^{n+1})x + TT_{[n]}x, \qquad n \in \mathbb{N}. \tag{4.1}$$

The equicontinuity of $\{T_{[n]}\}_{n=1}^\infty$ ensures that $\{T_{[n]}x\}_{n=1}^\infty \in \mathcal{B}(X)$. Since $T \in \mathcal{R}(X)$, the set $\{T(T_{[n]}x)\}_{n=1}^\infty$ is relatively weakly compact in X. Moreover, $\lim_{n\to\infty} \frac{1}{n}(T - T^{n+1})x = 0$ in X because of τ_s-$\lim_{n\to\infty} \frac{T^n}{n} = 0$. These facts, together with X_σ being angelic and(4.1), show that $\{T_{[n]}x\}_{n=1}^\infty$ is relatively weakly (hence, relatively weakly sequentially) compact in X. Since x is arbitrary, we can apply Theorem 2.4 of [2] (an examination of its proof shows that it is not necessary to assume the barrelledness of X stated there because of the equicontinuity of $\{T_{[n]}\}_{n=1}^\infty$ assumed here) to conclude that T is mean ergodic.

(ii) By part (i) the operator T is mean ergodic, i.e., τ_s-$\lim_{n\to\infty} T_{[n]} =: P$ exists in $\mathcal{L}_s(X)$. In particular, $P = TP = PT$ (which follows from (2.10)) and so $P = T_{[n]}P = PT_{[n]}$, for $n \in \mathbb{N}$.

To establish the uniform mean ergodicity of T, fix $p \in \Gamma_X$, $\varepsilon > 0$ and $B \in \mathcal{B}(X)$. By the equicontinuity of $\{T_{[n]}\}_{n=1}^\infty$ there exist $M > 0$ and $q \in \Gamma_X$ such that

$$p((T_{[n]} - P)x) \le Mq(x), \qquad x \in X,\ n \in \mathbb{N}. \tag{4.2}$$

On the other hand, $T(B)$ is a relatively compact subset of X and so there exist $z_1, \dots, z_h \in T(B)$ such that, for every $y \in T(B)$, we have $q(y - z_i) < \varepsilon/(2M)$ for some $i \in \{1, \dots, h\}$. Hence, via (4.2) we obtain, for every $x \in B$ and $n \in \mathbb{N}$, that

$$\begin{aligned} p(T_{[n]}Tx - Px) &= p((T_{[n]} - P)Tx) \le p((T_{[n]} - P)(Tx - z_i)) + p((T_{[n]} - P)z_i) \\ &\le Mq(Tx - z_i) + p((T_{[n]} - P)z_i) < \frac{\varepsilon}{2} + p((T_{[n]} - P)z_i). \end{aligned}$$

It follows that

$$\sup_{x\in B} p(T_{[n]}Tx - Px) \le \frac{\varepsilon}{2} + \max_{i=1,\dots,h} p((T_{[n]} - P)z_i), \quad n \in \mathbb{N},$$

with $\lim_{n\to\infty} \max_{i=1,\dots,h} p((T_{[n]} - P)z_i) = 0$. The arbitrariness of $\varepsilon > 0$ implies that $\lim_{n\to\infty} \sup_{x\in B} p(T_{[n]}Tx - Px) = 0$. So, τ_b-$\lim_{n\to\infty} T_{[n]}T = P$.

Finally, the arbitrariness of $p \in \Gamma_X$ and of $B \in \mathcal{B}(X)$ together with the assumption τ_b-$\lim_{n\to\infty} \frac{T^n}{n} = 0$ imply, via (4.1), that T is uniformly mean ergodic. □

Remark 4.2. (i) Let X be a lcHs and $T \in \mathcal{L}(X)$ be mean ergodic with $P := \tau_s$-$\lim_{n\to\infty} T_{[n]}$. Then it follows from $P = PT$ that $P \in \mathcal{H}(X)$ whenever $T \in \mathcal{H}(X)$ (here, $\mathcal{H}$ stands for the operator ideal $\mathcal{K}$, $\mathcal{M}$, $\mathcal{WK}$ or $\mathcal{R}$). In particular, if $T \in \mathcal{K}(X)$, then the space $\mathrm{Fix}(T) := \{x \in X : Tx = x\} = \mathrm{Ker}(I - T) = P(X)$ is finite–dimensional, [24, Theorem 9.10.1(1)].

(ii) Let X be a lcHs such that X_σ is angelic. Then the class of all *weakly completely continuous* operators in $\mathcal{L}(X)$ in the sense of Definition 2 in [10] is precisely $\mathcal{WK}(X)$. Moreover, if X is additionally barrelled then, for any $T \in \mathcal{L}(X)$, the boundedness of the set $\{T^n\}_{n=1}^\infty$ in $\mathcal{L}_s(X)$ is equivalent to T being power bounded. In particular, T is necessarily Cesàro bounded and satisfies

$\tau_s\text{-}\lim_{n\to\infty}\frac{T^n}{n}=0$. Accordingly, the containment $\mathcal{WK}(X)\subseteq\mathcal{R}(X)$ shows that Proposition 4.1(i) is an extension of the following result of Altman, [10, Theorem].

Fact 1. *Let X be a barrelled lcHs with X_σ being angelic. Then every power bounded operator $T\in\mathcal{WK}(X)$ is mean ergodic.*

The following technical result should be compared with [17, Proposition 3.1].

Lemma 4.3. *Let X be a quojection Fréchet space, Y be a Fréchet space and $T\in\mathcal{M}(X,Y)$ (resp. $T\in\mathcal{R}(X,Y)$). Suppose that $X=\operatorname{proj}_j(X_j,Q_{j,j+1})$, with X_j a Banach space (having norm $\|\ \|_j$) and surjective linking maps $Q_{j,j+1}\in\mathcal{L}(X_{j+1},X_j)$, for all $j\in\mathbb{N}$, and that $Y=\operatorname{proj}_j(Y_j,R_{j,j+1})$, with Y_j a Banach space (having norm $|||\ |||_j$) and linking maps $R_{j,j+1}\in\mathcal{L}(Y_{j+1},Y_j)$ for all $j\in\mathbb{N}$. Then, for every $j\in\mathbb{N}$, there exist $k(j)\geq j$ and $T_j\in\mathcal{K}(X_{k(j)},Y_j)$ (resp. $T_j\in\mathcal{WK}(X_{k(j)},Y_j)$) such that*

$$R_jT=T_jQ_{k(j)}, \tag{4.3}$$

where $R_j\in\mathcal{L}(Y,Y_j)$, $j\in\mathbb{N}$, is the canonical projection of Y into Y_j (i.e., $R_{j,j+1}\circ R_{j+1}=R_j$).

Proof. If we define $q_j(x):=\|Q_jx\|_j$ for $x\in X$ and $j\in\mathbb{N}$ and $r_j(y):=|||R_jy|||_j$ for $y\in Y$ and $j\in\mathbb{N}$, then $\{q_j\}_{j=1}^\infty$ and $\{r_j\}_{j=1}^\infty$ are fundamental sequences of seminorms generating the lc–topology of X and of Y, respectively.

Fix $j\in\mathbb{N}$. The continuity of T implies that there exist $k(j)\geq j$ and $C_j>0$ satisfying

$$r_j(Tx)\leq C_jq_{k(j)}(x),\quad x\in X,$$

or equivalently, that

$$|||R_jTx|||_j\leq C_j\|Q_{k(j)}x\|_j,\quad x\in X.$$

As noted before such an inequality ensures that there exists $T_j\in\mathcal{L}(X_{k(j)},Y_j)$ defined via $R_jT=T_jQ_{k(j)}$.

Denote by $\mathcal{U}_{k(j)}$ the closed unit ball of $X_{k(j)}$. Since X is a quojection Fréchet space, there exists $B\in\mathcal{B}(X)$ such that $\mathcal{U}_{k(j)}\subseteq Q_{k(j)}(B)$, [19, Proposition 1]. Since T is Montel (resp. reflexive) and R_j is continuous, it follows from $T_j(\mathcal{U}_{k(j)})\subseteq T_j(Q_{k(j)}(B))=R_j(T(B))$, with $R_j(T(B))$ a relatively compact subset (resp. relatively weakly compact subset) of Y_j, that $T_j(\mathcal{U}_{k(j)})$ is a relatively compact (resp. relatively weakly compact) subset of Y_j. That is, $T_j\in\mathcal{K}(X_{k(j)},Y_j)$ (resp. $T_j\in\mathcal{WK}(X_{k(j)},Y_j)$). □

Remark 4.4. (i) Let $X=\operatorname{proj}_j(X_j,Q_{j,j+1})$ be a quojection Fréchet space and $T\in\mathcal{L}(X)$. Suppose, for every $j\in\mathbb{N}$, that there exists $C_j>0$ such that $q_j(Tx)\leq C_jq_j(x)$ for $x\in X$ (here, the notation is according to Lemma 4.3 and its proof with $Y:=X$). Then, for every $j\in\mathbb{N}$, there exists $T_j\in\mathcal{L}(X_j)$ satisfying $Q_jT=T_jQ_j$. So, if $T\in\mathcal{M}(X)$ (resp., $T\in\mathcal{R}(X)$), then each $T_j\in\mathcal{K}(X_j)$ (resp., $T_j\in\mathcal{WK}(X_j)$).

(ii) Let X be a Fréchet space and $T\in\mathcal{M}(X)$. Then $\tau_s\text{-}\lim_{n\to\infty}\frac{T^n}{n}=0$ if and only if $\tau_b\text{-}\lim_{n\to\infty}\frac{T^n}{n}=0$.

As $\tau_s\subseteq\tau_b$, it suffices to show $\tau_s\text{-}\lim_{n\to\infty}\frac{T^n}{n}=0$ implies $\tau_b\text{-}\lim_{n\to\infty}\frac{T^n}{n}=0$.

Since X is a Fréchet space and $\tau_s\text{-}\lim_{n\to\infty}\frac{T^n}{n}=0$, the set $\{\frac{T^n}{n}\}_{n=1}^\infty$ is equicontinuous in $\mathcal{L}(X)$, i.e., for every $p\in\Gamma_X$ there exist $q\in\Gamma_X$ and $M>0$ such that

$$p\left(\frac{T^nx}{n}\right)\leq Mq(x),\quad x\in X,\ n\in\mathbb{N}. \tag{4.4}$$

Now, fix $p \in \Gamma_X$, $B \in \mathcal{B}(X)$ and $\varepsilon > 0$. Choose $q \in \Gamma_X$ and $M > 0$ according to (4.4). Since T is a Montel operator, $T(B)$ is a relatively compact subset of X and so there exist $x_1, \ldots, x_k \in X$ such that

$$T(B) \subseteq \cup_{i=1}^{k} \left(x_i + \frac{\varepsilon}{2M} \mathcal{U}_q \right), \tag{4.5}$$

with $\mathcal{U}_q := \{x \in X : q(x) \leq 1\}$. Let $x \in B$. By (4.5) there exist $i \in \{1, \ldots, k\}$ and $z \in \mathcal{U}_q$ such that $T(x) = x_i + \frac{\varepsilon}{2M} z$. Then, by (4.4), we have for every $n > 1$ that

$$p\left(\frac{T^n x}{n}\right) = p\left(\frac{T^{n-1}}{n} T(x)\right) \leq p\left(\frac{T^{n-1} x_i}{n}\right) + \frac{\varepsilon}{2M} p\left(\frac{T^{n-1} z}{n}\right) \leq p\left(\frac{T^{n-1} x_i}{n-1}\right) + \frac{\varepsilon}{2}.$$

But, $p\left(\frac{T^{n-1} x_i}{n-1}\right) \to 0$ as $n \to \infty$. So, there exists $n_0 \in \mathbb{N}$ (depending only on x_i) such that $p\left(\frac{T^n x}{n}\right) < \varepsilon$ for every $n \geq n_0$. Since x is arbitrary and the set $\{x_1, \ldots, x_k\}$ is finite, we can conclude that $\sup_{x \in B} p\left(\frac{T^n x}{n}\right) \to 0$ for $n \to \infty$. By the arbitrariness of B and p we have τ_b-$\lim_{n\to\infty} \frac{T^n}{n} = 0$.

The following result should be compared with Proposition 4.1(ii). We point out (even if $\dim(X) < \infty$) that a Cesàro bounded operator T need *not* satisfy $\frac{T^n}{n} \to 0$ in $\mathcal{L}_s(X)$, [31, p.85].

Theorem 4.5. *Let X be a prequojection Fréchet space and $T \in \mathcal{M}(X)$. If τ_s-$\lim_{n\to\infty} \frac{T^n}{n} = 0$, then T is uniformly mean ergodic.*

Proof. Case (I). X is a quojection.

The condition τ_s-$\lim_{n\to\infty} \frac{T^n}{n} = 0$ ensures that both τ_b-$\lim_{n\to\infty} \frac{T^n}{n} = 0$ (see Remark 4.4(ii)) and that we can represent $X = \mathrm{proj}_j (X_j, Q_{j,j+1})$ such that, for every $j \in \mathbb{N}$, there exists $T_j \in \mathcal{L}(X_j)$ satisfying $Q_j T = T_j Q_j$; see the proof of Proposition 3.1. According to Lemma 4.3 and Remark 4.4(i) we have $T_j \in \mathcal{K}(X_j)$ for all $j \in \mathbb{N}$. Moreover, $\frac{T_j^n}{n} \to 0$ in $\mathcal{L}_b(X_j)$ for $n \to \infty$; see Remark 4.4(ii) and Lemma 2.6 with $S_n := \frac{T^n}{n}$, for $n \in \mathbb{N}$.

Since $T_j \in \mathcal{K}(X_j)$ and $\frac{T_j^n}{n} \to 0$ in $\mathcal{L}_b(X_j)$ for $n \to \infty$, for every $j \in \mathbb{N}$, each T_j is uniformly mean ergodic, [22, p.711 Corollary 4], which implies that T is also uniformly mean ergodic; see Lemma 2.7.

Case (II). X is a prequojection.

As noted before X and X'_β are barrelled (hence, quasi-barrelled) with $T' \in \mathcal{L}(X'_\beta)$ and $T'' \in \mathcal{L}(X'')$. So, the condition τ_b-$\lim_{n\to\infty} \frac{T^n}{n} = 0$ (see Remark 4.4(ii)) implies that τ_b-$\lim_{n\to\infty} \frac{(T'')^n}{n} = 0$. Moreover, X'' is a quojection Fréchet space. Also, Corollaries 2.3 and 2.4 of [17] yield that $T'' \in \mathcal{M}(X'')$. We can then apply Case (I) to conclude that T'' is uniformly mean ergodic. So, T is also uniformly mean ergodic as $T''|_X = T$ and X is a closed subspace of X''. □

It was noted prior to Proposition 4.1, for X a prequojection Fréchet space and $T \in \mathcal{K}(X)$, that T is uniformly mean ergodic whenever τ_b-$\lim_{n\to\infty} \frac{T^n}{n} = 0$. Since $\mathcal{K}(X) \subset \mathcal{M}(X)$ in general, Theorem 4.5 can be viewed as an extension of this fact.

Corollary 4.6. *Let X be a prequojection Fréchet space and $T \in \mathcal{M}(X)$ be power bounded. Then $\Gamma(T) \subseteq \{1\}$ if and only if τ_b-$\lim_{n\to\infty}(T^{n+1} - T^n) = 0$.*

Proof. If τ_b-$\lim_{n\to\infty}(T^{n+1} - T^n) = 0$, then Theorem 3.13 yields $\Gamma(T) \subseteq \{1\}$.

Conversely, suppose that $\Gamma(T) \subseteq \{1\}$. Since T is power bounded, $\frac{T^n}{n} \to 0$ in $\mathcal{L}_b(X)$ for $n \to \infty$ and so T is uniformly mean ergodic by Theorem 4.5. By Theorem 3.5 of [7] this is equivalent to the fact that $(I-T)(X)$ is closed in X. So, by Theorem 3.8 (ii)$\Leftrightarrow$(iii) we can conclude that τ_b-$\lim_{n\to\infty}(T^{n+1} - T^n) = 0$. $\Box$

In a Banach space X, an operator $T \in \mathcal{L}(X)$ is called *quasi–compact* if there exist $m \in \mathbb{N}$ and $K \in \mathcal{K}(X)$ such that $\|T^m - K\|_{op} < 1$, [23, §6], [31, p.88]. For example, if some power of $T \in \mathcal{L}(X)$ is compact or if some power of T has norm less than one, then T is quasi–compact. For a quasi–compact operator T it is known that τ_s–$\lim_{n\to\infty} \frac{T^n}{n} = 0$ suffices for T to be uniformly mean ergodic, [22, Ch.VIII, Corollary 8.4]. For X non–normable, the question arises of how to extend the notion of a quasi–compact operator.

According to [40, Definition 1], for a lcHs X an operator $T \in \mathcal{L}(X)$ is called *quasi–precompact* if there exists a 0–neighbourhood W such that for every 0-neighbourhood $\mathcal{U}$ in X there exist $p \in \mathbb{N}$ and a finite set $F \subseteq X$ (both depending on $\mathcal{U}$) with the property that $T^p(W) \subseteq \cup_{y\in F}(y + \mathcal{U})$. For X a Banach space, this notion coincides precisely with T being quasi–compact, [40, Theorem 3]. In [15] an operator $K \in \mathcal{L}(X)$ is called V–compact if $K(V)$ is a relatively compact subset of X, where V is some 0–neighbourhood in X. More generally, $T \in \mathcal{L}(X)$ is called *V–quasicompact*, [15, Definition 2.1], if there exist $m \in \mathbb{N}$, a V–compact operator K and $\delta \in (0, 1)$ such that $(T^m - K)(V) \in \mathcal{B}(X)$ and $(T^m - K)(V) \subseteq \delta V$.

Lemma 4.7. *Let X be a lcHs and V be any 0–neighbourhood in X. Then every V–quasicompact operator is quasi–precompact.*

Proof. Let $T \in \mathcal{L}(X)$ be V–quasicompact. Choose $m \in \mathbb{N}$, a V–compact operator K and $\delta \in (0, 1)$ such that $B := (T^m - K)(V)$ is bounded and $B \subseteq \delta V$. Then

$$(T^m - K)^2(V) = (T^m - K)(B) \subseteq (T^m - K)(\delta V) = \delta B.$$

Proceeding inductively yields

$$(T^m - K)^p(V) \subseteq \delta^{p-1}B, \quad p \in \mathbb{N}. \tag{4.6}$$

Fix $p \in \mathbb{N}$. Note that T^m and K need not commute. By expanding $(T^m - K)^p$ it can be seen that $(T^m - K)^p = T^{mp} - H_p$, where H_p is a finite sum of operators all of the form AK or $BK(T^m)^n$ with $A,\ B \in \mathcal{L}(X)$ and $n \in \{1, \dots, p-1\}$. The claim is that H_p is a V–compact operator. Indeed, since AK is always V-compact and the finite sum of V-compact operators is clearly V–compact, it suffices to show that $K(T^m)^n$ (hence, also $BK(T^m)^n$) is V–compact for all $1 \le n < p$.

For $n = 1$, observe that $T^m(V) = K(V) + B \subseteq K(V) + \delta V$ yields

$$KT^m(V) \subseteq K^2(V) + \delta K(V),$$

which is a relatively compact subset of X. For $n = 2$, we then have

$$(T^m)^2(V) \subseteq T^m(K(V) + \delta V) = T^m K(V) + \delta T^m(V)$$

and hence, that

$$K(T^m)^2(V) \subseteq KT^m K(V) + \delta KT^m(V).$$

Since both $T^m K(V)$ and $KT^m(V)$ are relatively compact, it follows that $K(T^m)^2(V)$ is also relatively compact. This argument can be continued to yield the above stated claim for all $1 \le n < p$.

Define now $W := V$ and let $\mathcal{U}$ be any convex, balanced 0–neighbourhood of X. Since B is bounded, there is $\lambda > 0$ such that $B \subseteq \frac{1}{2}\lambda\mathcal{U}$. Choose $\tilde{p} \in \mathbb{N}$ large enough to ensure that $\delta^{\tilde{p}-1}\lambda \leq 1$. It follows from (4.6) that

$$(T^{m\tilde{p}} - H_{\tilde{p}})(W) = (T^m - K)^{\tilde{p}}(V) \subseteq \delta^{\tilde{p}-1}B$$

and so

$$T^{m\tilde{p}}(W) \subseteq H_{\tilde{p}}(V) + (T^{m\tilde{p}} - H_{\tilde{p}})(W) \subseteq H_{\tilde{p}}(V) + \frac{1}{2}\delta^{\tilde{p}-1}\lambda\mathcal{U} \subseteq H_{\tilde{p}}(V) + \frac{1}{2}\mathcal{U}.$$

But, $H_{\tilde{p}}(V)$ is relatively compact and so there is a finite set $F \subseteq X$ such that $H_{\tilde{p}}(V) \subseteq \cup_{x\in F}(x + \frac{1}{2}\mathcal{U})$. Accordingly,

$$T^{m\tilde{p}}(W) \subseteq \frac{1}{2}\mathcal{U} + \cup_{x\in F}(x + \frac{1}{2}W) \subseteq \cup_{x\in F}(x + \mathcal{U}).$$

which establishes that T is quasi–precompact. $\square$

Returning to mean ergodicity, we have the following result of Pietsch, [40, Theorem 7].

Fact 2. *Let X be a complete, barrelled lcHs and $T \in \mathcal{L}(X)$ be a quasi–precompact operator satisying τ_s-$\lim_{n\to\infty} \frac{T^n}{n} = 0$. Then T is uniformly mean ergodic and* $\mathrm{Fix}(T) = \mathrm{Ker}(I - T)$ *is finite-dimensional.*

In order to be able to extend this result to a larger class of operators we recall, for a Banach space X, that $T \in \mathcal{L}(X)$ is quasi–compact if and only if there exists a sequence $\{K_n\}_{n=1}^\infty \subseteq \mathcal{K}(X)$ such that $\lim_{n\to\infty} \|T^n - K_n\| = 0$, [31, p.88 Lemma 2.4].

Definition 4.8. Let X be a lcHs. An operator $T \in \mathcal{L}(X)$ is called *quasi–Montel* (resp., *quasi–reflexive*) if there exists a sequence $\{M_n\}_{n=1}^\infty \subseteq \mathcal{M}(X)$ (resp., $\{M_n\}_{n=1}^\infty \subseteq \mathcal{R}(X)$) such that $(T^n - M_n) \to 0$ in $\mathcal{L}_b(X)$ as $n \to \infty$.

Remark 4.9. (i) Let X be a Fréchet space and $T \in \mathcal{L}(X)$ be quasi–Montel. Then $T'' \in \mathcal{L}(X'')$ is also quasi–Montel. Indeed, in the notation of Definition 4.8, we have $\{M_n''\}_{n=1}^\infty \subseteq \mathcal{M}(X'')$, [17, Corollaries 2.3 and 2.4], with $((T'')^n - M_n'') \to 0$ in $\mathcal{L}_b(X'')$ as $n \to \infty$; see [3, Lemma 2.6] or [4, Lemma 2.1].

(ii) Let X be a Fréchet space and $T \in \mathcal{L}(X)$ be quasi–Montel. Then τ_s-$\lim_{n\to\infty} \frac{T^n}{n} = 0$ if and only if τ_b-$\lim_{n\to\infty} \frac{T^n}{n} = 0$.

Again it suffices to show that τ_s-$\lim_{n\to\infty} \frac{T^n}{n} = 0$ implies τ_b-$\lim_{n\to\infty} \frac{T^n}{n} = 0$.

Arguing as in Remark 4.4(ii), for every $p \in \Gamma_X$ there exist $q \in \Gamma_X$ and $M > 0$ such that (4.4) holds. Fix $p \in \Gamma_X$, $B \in \mathcal{B}(X)$ and $\varepsilon > 0$. Choose q and M according to (4.4). Since T is a quasi–Montel operator, there is $\{M_n\}_{n=1}^\infty \subseteq \mathcal{M}(X)$ with $(T^n - M_n) \to 0$ in $\mathcal{L}_b(X)$ as $n \to \infty$. So there exists $m \in \mathbb{N}$ such that

$$\sup_{x\in B} q((T^m - M_m)x) < \frac{\varepsilon}{4M}. \tag{4.7}$$

But, $M_m \in \mathcal{M}(X)$ and so $M_m(B)$ is a relatively compact subset of X. It follows that there exist $x_1, \ldots, x_k \in X$ such that

$$M_m(B) \subseteq \cup_{i=1}^k \left(x_i + \frac{\varepsilon}{4M}\mathcal{U}_q\right), \tag{4.8}$$

where $\mathcal{U}_q := \{x \in X : q(x) \le 1\}$. From (4.7) and (4.8) it follows that

$$T^m(B) \subseteq (T^m - M_m)(B) + M_m(B) \subseteq \frac{\varepsilon}{4M}\mathcal{U}_q + \cup_{i=1}^k \left(x_i + \frac{\varepsilon}{4M}\mathcal{U}_q\right)$$
$$\subseteq \cup_{i=1}^k \left(x_i + \frac{\varepsilon}{2M}\mathcal{U}_q\right). \tag{4.9}$$

Fix $x \in B$. By (4.9) there exist $i \in \{1, \ldots, k\}$ and $z \in \mathcal{U}_q$ such that $T^m(x) = x_i + \frac{\varepsilon}{2M}z$. Then, by (4.4), for every $n > m$ we have that

$$p\left(\frac{T^n x}{n}\right) = p\left(\frac{T^{n-m}}{n}T^m(x)\right) \le p\left(\frac{T^{n-m}x_i}{n}\right) + \frac{\varepsilon}{2M}p\left(\frac{T^{n-m}z}{n}\right) \le p\left(\frac{T^{n-m}x_i}{n-m}\right) + \frac{\varepsilon}{2}.$$

But, $p\left(\frac{T^{n-m}x_i}{n-m}\right) \to 0$ as $n \to \infty$. So, there exists $n_0 \in \mathbb{N}$ (depending only on x_i) such that $p\left(\frac{T^n x}{n}\right) < \varepsilon$, for every $n \ge n_0$. Since x is arbitrary and the set $\{x_1, \ldots, x_k\}$ is finite, we can conclude that $\sup_{x \in B} p\left(\frac{T^n x}{n}\right) \to 0$ for $n \to \infty$. By the arbitrariness of B and p we have $\tau_b\text{-}\lim_{n\to\infty} \frac{T^n}{n} = 0$.

Proposition 4.10. *Let X be a prequojection Fréchet space and $T \in \mathcal{L}(X)$ satisfy $\tau_s\text{-}\lim_{n\to\infty} \frac{T^n}{n} = 0$. If T is quasi–precompact, then there exists a sequence $\{K_n\}_{n=1}^\infty \subseteq \mathcal{K}(X)$ such that $\tau_b\text{-}\lim_{n\to\infty}(T^n - K_n) = 0$. In particular, T is quasi–Montel as $\mathcal{K}(X) \subseteq \mathcal{M}(X)$.*

Proof. The completeness of X ensures that every precompact subset of X is also relatively compact. By Fact 2 the operator T is uniformly mean ergodic and so $\tau_b\text{-}\lim_{n\to\infty} \frac{T^n}{n} = 0$. By Theorem 1, Theorem 2 and Satz 10 of [40] there exist $R \in \mathcal{L}(X)$ and a projection $P \in \mathcal{L}(X)$ commuting with T such that $\dim P(X) < \infty$ and satisfying

$$T^n = R^n + T^n P, \quad n \in \mathbb{N} \tag{4.10}$$

and

$$\mathbb{C} \setminus \mathbb{D} \subseteq \rho(R). \tag{4.11}$$

Since $P \in \mathcal{K}(X)$, also $K_n := T^n P \in \mathcal{K}(X)$ for each $n \in \mathbb{N}$. Moreover, (4.10) yields $R^n = T^n(I - P) = (I - P)T^n$, for $n \in \mathbb{N}$, and so $\tau_b\text{-}\lim_{n\to\infty} \frac{R^n}{n} = 0$. Since (4.11) is equivalent to $\sigma(R) \subseteq \mathbb{D}$, it then follows from Theorem 3.3 applied to R that $\tau_b\text{-}\lim_{n\to\infty} R^n = 0$. It is then clear (see (4.10)) that $(T^n - K_n) = R^n \to 0$ in $\mathcal{L}_b(X)$ as $n \to \infty$. ☐

Remark 4.11. There exist quasi-Montel operators, even in quojection Fréchet spaces, which fail to be quasi–precompact.

(i) For $X := \omega = \mathbb{C}^{\mathbb{N}}$, define the projection $P \in \mathcal{L}(X)$ via

$$Px := (x_1, 0, x_3, 0, x_5, \ldots), \quad x = (x_n)_n \in X.$$

Since X is a Montel space, all of its bounded subsets are relatively compact. It is then clear that $P \in \mathcal{M}(X)$ and hence, P is surely quasi–Montel. Of course, $P \notin \mathcal{K}(X)$. On the other hand, since $\mathrm{Ker}(I - P)$ is infinite-dimensional, P cannot be quasi–precompact, [40, Satz 3].

(ii) Let X be as in (i) and define the diagonal operator $T \in \mathcal{L}(X)$ by

$$Tx := \left(x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \ldots\right), \quad x = (x_n)_n \in X.$$

The same argument as in (i) shows that $T \in \mathcal{M}(X)$. In this case, in contrast to (i), the space $\mathrm{Ker}(I-T) = \overline{\mathrm{span}}\{(1,0,0,\ldots)\}$ is finite-dimensional. However, T still fails to be quasi–precompact, [40, p.24].

Remark 4.12. The converse of Proposition 4.10 is not valid. Indeed, let $X := \omega$ and $T \in \mathcal{L}(X)$ be as Remark 4.11(ii), in which case X is a quojection Fréchet space. For each $n \in \mathbb{N}$, let $K_n \in \mathcal{L}(X)$ be the finite rank operator given by

$$K_n x := \left(x_1, \frac{x_2}{2^n}, \frac{x_3}{3^n}, \ldots, \frac{x_n}{n^n}, 0, 0, \ldots\right), \quad x = (x_j)_j \in X.$$

Then $\mathcal{U}_n := \{x \in X\colon \max_{1\le j\le n} |x_j| \le 1\}$ is a 0–neighbourhood in X. Since K_n has finite-dimensional range, it follows that $K_n(\mathcal{U}_n)$ is a relatively compact subset of X, i.e., $K_n \in \mathcal{K}(X)$ for each $n \in \mathbb{N}$. Direct calculations show that the sequence of operators

$$(T^n - K_n)x = \left(0, \ldots, 0, \frac{x_{n+1}}{(n+1)^n}, \frac{x_{n+2}}{(n+2)^n}, \ldots\right), \quad x = (x_j)_j \in X,$$

converges to 0 in $\mathcal{L}_s(X)$ as $n \to \infty$. Since X is a Montel space, also τ_b-$\lim_{n\to\infty}(T^n - K_n) = 0$. However, as noted in Remark 4.11(ii), the diagonal operator T is *not* quasi–compact.

In view of Remark 4.11 the following result is an extension of Fact 2 above for prequojection Fréchet spaces (without the condition $\dim \mathrm{Ker}(I-T) < \infty$).

Theorem 4.13. *Let X be a prequojection Fréchet space and $T \in \mathcal{L}(X)$. If T is a quasi–Montel operator and τ_s-$\lim_{n\to\infty} \frac{T^n}{n} = 0$, then T is uniformly mean ergodic.*

Proof. Case (I). X is a quojection.

The assumption τ_s-$\lim_{n\to\infty} \frac{T^n}{n} = 0$ ensures that we can proceed as in the proof of Proposition 3.1 to obtain $X = \mathrm{proj}_j(X_j, Q_{j,j+1})$ in such a way that, for every $j \in \mathbb{N}$, there exists T_j in $\mathcal{L}(X_j)$ satisfying $Q_j T = T_j Q_j$. Then also $Q_j T^n = T_j^n Q_j$ and $Q_j \frac{T^n}{n} = \frac{T_j^n}{n} Q_j$, for every $j,\ n \in \mathbb{N}$. So, Lemma 2.6 (with $S_n := \frac{T^n}{n}$, for $n \in \mathbb{N}$) implies that τ_s-$\lim_{n\to\infty} \frac{T_j^n}{n} = 0$ for all $j \in \mathbb{N}$.

Since T is quasi–Montel, there exists a sequence $\{M_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M}(X)$ such that τ_b-$\lim_{n\to\infty}(T^n - M_n) = 0$. From this it follows that the operator T_j, for any fixed $j \in \mathbb{N}$, is quasi–precompact. To see this, let $\tilde{q}_j$ denote the norm of X_j and $\varepsilon > 0$. Since $p_j := \tilde{q}_j \circ Q_j \in \Gamma_X$, there exists $n \in \mathbb{N}$ such that

$$\sup_{x\in B} p_j(T^n x - M_n x) < \frac{\varepsilon}{2},$$

with $B \in \mathcal{B}(X)$ chosen such that $\hat{B}_j \subseteq Q_j(B)$. Since

$$\sup_{x\in B} p_j(T^n x - M_n x) = \sup_{x\in B} \tilde{q}_j(Q_j(T^n x - M_n x)) = \sup_{x\in B} \tilde{q}_j(T_j^n Q_j x - Q_j M_n x),$$

it follows that

$$T_j^n(\hat{B}_j) \subseteq T_j^n(Q_j(B)) \subseteq Q_j(M_n(B)) + \frac{\varepsilon}{2}\hat{B}_j.$$

Hence, by the relative compactness (hence, precompactness) of $Q_j(M_n(B))$ in X_j, due to $M_n \in \mathcal{M}(X)$ and the continuity of Q_j, there exist $\hat{x}_1, \ldots, \hat{x}_k \in X_j$ such that

$$T_j^n(\hat{B}_j) \subseteq \cup_{i=1}^k (\hat{x}_i + \varepsilon \hat{B}_j).$$

The arbitrariness of ε ensures that $T_j \in \mathcal{L}(X_j)$ is quasi–precompact. As X_j is a Banach space, T_j is quasi–compact,[40, Theorem 3], and satisfies $\frac{T_j^n}{n} \to 0$ in $\mathcal{L}_s(X_j)$ for $n \to \infty$. By Fact 2, each operator T_j, for $j \in \mathbb{N}$, is uniformly mean ergodic. Then Lemma 2.7 implies that T is also uniformly mean ergodic.

Case (II). X is a prequojection.

The condition τ_s-$\lim_{n\to\infty} \frac{T^n}{n} = 0$ actually means that τ_b-$\lim_{n\to\infty} \frac{T^n}{n} = 0$ because T is quasi–Montel (see Remark 4.9(ii)). So, arguing as for Case (II) in the proof of Theorem 4.5 it follows that also τ_b-$\lim_{n\to\infty} \frac{(T'')^n}{n} = 0$. Moreover, by Remark 4.9(i) the operator T'' is quasi–Montel. Since X'' is a quojection Fréchet space, we can apply Case (I) to conclude that T'' is uniformly mean ergodic. Then T is also uniformly mean ergodic as $T''|_X = T$ with X a closed subspace of X''. $\square$

Since the only Fréchet-Montel spaces which are normable are the finite-dimensional ones, the following result may be viewed as an analogue of the fact that $\mathrm{Ker}(\lambda I - T)$ is finite-dimensional whenever T is quasi–precompact; see Definition 3 and Theorem 1 of [40].

Proposition 4.14. *Let X be a Fréchet space and $T \in \mathcal{L}(X)$ be a quasi–Montel operator. Then* $\mathrm{Ker}(\lambda I - T)$ *is a Fréchet–Montel space, for every $\lambda \in \mathbb{T}$.*

Proof. It suffices to show that $\mathrm{Fix}(T) = \mathrm{Ker}(I - T)$ is a Fréchet-Montel space. Indeed, for every $\lambda \in \mathbb{T}$, the operator $\lambda^{-1}T$ is quasi–Montel if and only if T is quasi–Montel, with $\mathrm{Ker}(\lambda I - T) = \mathrm{Fix}(\lambda^{-1}T)$.

Let $\{r_j\}_{j=1}^\infty$ be any fundamental, increasing sequence of seminorms generating the lc–topology of X. Let $\{x_k\}_{k=1}^\infty \subseteq \mathrm{Fix}(T)$ be a *bounded* sequence. Since T is quasi–Montel, there exists $\{M_n\}_{n=1}^\infty \subseteq \mathcal{M}(X)$ such that τ_b-$\lim_{n\to\infty}(T^n - M_n) = 0$ and so, for every $j \in \mathbb{N}$, we have $\sup_{k\in\mathbb{N}} r_j(x_k - M_n x_k) \to 0$ as $n \to \infty$.

Since $\{x_k\}_{k=1}^\infty$ is bounded and each operator M_n, for $n \in \mathbb{N}$, is Montel, we may construct recursively subsequences $\{x_k^n\}_{k=1}^\infty$ of $\{x_k\}_{k=1}^\infty$ such that each $\{x_k^{n+1}\}_{k=1}^\infty$ is a subsequence of $\{x_k^n\}_{k=1}^\infty$ and $\{M_n x_k^n\}_{k=1}^\infty$ converges in X for all $n \in \mathbb{N}$. Consider the diagonal sequence $\{x_k^k\}_{k=1}^\infty$. Clearly, $\{M_n x_k^k\}_{k=1}^\infty$ converges in X for each $n \in \mathbb{N}$ (by observing that $\{M_n x_k^k\}_{k=1}^\infty \subseteq \{M_n x_k^n\}_{k\geq n}^\infty$). Fix $\varepsilon > 0$ and $j \in \mathbb{N}$. Then , for every $k,\ k' \in \mathbb{N}$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} r_j(x_k^k - x_{k'}^{k'}) \qquad &\leq r_j(x_k^k - M_n x_k^k) + r_j(M_n x_k^k - M_n x_{k'}^{k'}) + r_j(M_n x_{k'}^{k'} - x_{k'}^{k'}) \\ &\leq 2 \sup_{h\in\mathbb{N}} r_j(x_h - M_n x_h) + r_j(M_n x_k^k - M_n x_{k'}^{k'}), \end{aligned}$$

with $\sup_{h\in\mathbb{N}} r_j(x_h - M_n x_h) \to 0$ as $n \to \infty$. So, there is $n_0 \in \mathbb{N}$ such that $\sup_{h\in\mathbb{N}} r_j(x_h - M_n x_h) < \varepsilon/4$ for every $n \geq n_0$. But, $\{M_{n_0} x_k^k\}_{k=1}^\infty$ converges in X and hence, there is also $k_0 \in \mathbb{N}$ such that $r_j(M_{n_0} x_k^k - M_{n_0} x_{k'}^{k'}) < \varepsilon/2$ for all $k,\ k' \geq k_0$. It follows that $r_j(x_k^k - x_{k'}^{k'}) < \varepsilon$ whenever $k,\ k' \geq k_0$. By the arbitrariness of $j \in \mathbb{N}$ and $\varepsilon > 0$ this means that $\{x_k^k\}_{k=1}^\infty$ is a Cauchy sequence in X and so it converges in X. Since X is a Fréchet space, this shows that $\mathrm{Fix}(T)$ is a Fréchet–Montel space. $\square$

Proposition 4.15. *Let X be a prequojection Fréchet space and $T \in \mathcal{L}(X)$ be a quasi–Montel operator. If τ_s-$\lim_{n\to\infty} \frac{T^n}{n} = 0$, then $(I - T)(X)$ is closed.*

Proof. By Theorem 4.13 the operator T is uniformly mean ergodic. Also τ_b-$\lim_{n\to\infty} \frac{T^n}{n} = 0$. By [7, Theorem 3.5] this is equivalent to $(I-T)(X)$ being closed in X. □

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Angela A. Albanese, Dipartimento di Matematica e Fisica “E. De Giorgi”, Università del Salento- C.P.193, I-73100 Lecce, Italy
E-mail address: angela.albanese@unisalento.it

José Bonet, Instituto Universitario de Matemática Pura y Aplicada IUMPA, Universidad Politécnica de Valencia, E-46071 Valencia, Spain
E-mail address: jbonet@mat.upv.es

Werner J. Ricker, Math.-Geogr. Fakultät, Katholische Universität Eichstätt-Ingolstadt, D-85072 Eichstätt, Germany
E-mail address: werner.ricker@ku-eichstaett.de