A note about Volterra operators on weighted Banach spaces of entire functions

José Bonet and Jari Taskinen

Abstract

We characterize boundedness, compactness and weak compactness of Volterra operators V_g acting between different weighted Banach spaces $H_v^\infty(\mathbb{C})$ of entire functions with sup-norms in terms of the symbol g; thus we complement recent work by Bassallote, Contreras, Hernández-Mancera, Martín and Paul [3] for spaces of holomorphic functions on the disc and by Constantin and Peláez [16] for reflexive weighted Fock spaces.

1 Introduction, notation and preliminaries

The aim of this paper is to investigate boundedness and (weak) compactness of the Volterra operator when it acts between two weighted Banach spaces of entire functions $H_v^\infty(\mathbb{C})$ and $H_w^\infty(\mathbb{C})$. We reduce the problem to the study of multiplication operators between related weighted Banach spaces of entire functions, so our approach is similar to that in [3]. It enables us to give simplified proofs of some results of [16] for the Volterra operator between two weighted Fock spaces of order infinity and to obtain new results, in particular about weak compactness and about operators on the smaller spaces $H_v^0(\mathbb{C})$ and new examples. Our main results are Theorems 3.4–3.7, and concrete examples and applications are given Remark 3.11 and its corollaries. In Lemma 3.10 we also study some general properties of weight functions relevant for the theory of $H_v^\infty(\mathbb{C})$-spaces: we formulate sufficient conditions for the so called essentialness of weight functions.

In what follows $H(\mathbb{C})$ and $\mathcal{P}$ denote the space of entire functions and the space of polynomials, respectively. The space $H(\mathbb{C})$ will be endowed with the compact open topology τ_{co}. The differentiation operator $Df(z) = f'(z)$ and the integration operator $Jf(z) = \int_0^z f(\zeta)d\zeta$ are continuous on $H(\mathbb{C})$.

2010 Mathematics Subject Classification. Primary: 47G10, secondary: 30D15, 30D20, 46E15, 47B07, 47B37, 47B38.

Key words and phrases. Integral operator; Volterra operator; entire functions; growth conditions; weighted Banach spaces of entire functions; boundedness; compactness

Acknowledgement. The research of J. Bonet was partially supported by MEC and FEDER Project MTM2010-15200 and by GV Project Prometeo II/2013/013. The research of J. Taskinen was partially supported by the Väisälä Foundation of the Finnish Academy of Sciences and Letters.

Given an entire function $g \in H(\mathbb{C})$, the Volterra operator V_g with symbol g is defined on $H(\mathbb{C})$ by

$$V_g(f)(z) := \int_0^z f(\zeta) g'(\zeta) d\zeta \quad (z \in \mathbb{C}).$$

For $g(z) = z$ this reduces to the integration operator, denoted by J. Clearly V_g defines a continuous operator on $H(\mathbb{C})$. The Volterra operator for holomorphic functions on the unit disc was introduced by Pommerenke [29] and he proved that V_g is bounded on the Hardy space H^2, if and only if $g \in BMOA$. Aleman and Siskakis [1] extended this result for $H^p, 1 \leq p < \infty$, and they considered later in [2] the case of weighted Bergman spaces; see also [25]. We refer the reader to the memoir by Peláez and Rättyä [28] and the references therein. Volterra operators on weighted Banach spaces of holomorphic functions on the disc of type H^∞ have been investigated recently in [3] and this approach was influential in ours. Constantin started in [15] the study of the Volterra operator on spaces of entire functions. She characterized the continuity of V_g on the classical Fock spaces. Constantin and Peláez [16] characterize the entire functions $g \in H(\mathbb{C})$ such that V_g is bounded or compact on a large class of Fock spaces induced by smooth radial weights.

Throughout the paper, a *weight* v is a continuous function $v : [0, \infty[\to]0, \infty[$, which is non-increasing on $[0, \infty[$ and satisfies $\lim_{r \to \infty} r^m v(r) = 0$ for each $m \in \mathbb{N}$. If necessary, we extend v to $\mathbb{C}$ by $v(z) := v(|z|)$. For such a weight, the *weighted Banach spaces of entire functions* are defined by

$$H_v^\infty(\mathbb{C}) := \{f \in H(\mathbb{C}) \mid \|f\|_v := \sup_{z \in \mathbb{C}} v(|z|)|f(z)| < \infty\},$$
$$H_v^0(\mathbb{C}) := \{f \in H(\mathbb{C}) \mid \lim_{|z| \to \infty} v(|z|)|f(z)| = 0\},$$

and they are endowed with the weighted sup norm $\|\cdot\|_v$. Clearly, $H_v^0(\mathbb{C})$ is a closed subspace of $H_v^\infty(\mathbb{C})$, which contains the polynomials. Both are Banach spaces and the closed unit ball of $H_v^\infty(\mathbb{C})$ is τ_{co}-compact. The polynomials are contained and dense in $H_v^0(\mathbb{C})$ but the monomials do not in general form a Schauder basis, [21]. The Cesàro means of the Taylor polynomials satisfy $\|C_n f\|_v \leq \|f\|_v$ for each $f \in H_v^\infty(\mathbb{C})$ and the sequence $(C_n f)_n$ is $\|\cdot\|_v$-convergent to f when $f \in H_v^0(\mathbb{C})$, see [8]. Clearly, changing the value of v on a compact interval does not change the spaces and gives an equivalent norm. By [10, Ex 2.2], the bidual of $H_v^0(\mathbb{C})$ is isometrically isomorphic to $H_v^\infty(\mathbb{C})$. Spaces of this type appear in the study of growth conditions of analytic functions and have been investigated in various articles, see e.g. [8, 9, 11, 19, 21, 22] and the references therein.

The space $H_v^\infty(\mathbb{C})$ is denoted as the weighted Fock space $\mathcal{F}_\infty^\phi$ of order infinity (i.e. with sup-norms) in [16] with $v(z) = \exp(-\phi(|z|))$, and $\phi : [0, \infty[\to]0, \infty[$ is a twice continuously differentiable increasing function. The operator V_g is denoted by T_g in [16].

For an entire function $f \in H(\mathbb{C})$, we denote by $M(f, r) := \max\{|f(z)| \mid |z| = r\}$. Using the notation O and o of Landau, $f \in H_v^\infty(\mathbb{C})$ if and only if $M(f, r) = O(1/v(r)), r \to \infty$, and $f \in H_v^0(\mathbb{C})$ if and only if $M(f, r) = o(1/v(r)), r \to \infty$.

To clarify the notation, f' denotes the usual complex derivative, if f is an analytic function, and the partial derivative with respect to the variable $r \in [0, \infty[$, if f is a weight (which is still defined in the entire plane) or its inverse. By C, C', c

(respectively, C_n) etc. we denote positive constants (resp. constant depending on the index n), the value of which may vary from place to place.

2 Multiplication operators

In our study of Volterra operators we need the characterizations of boundedness and (weak) compactness of multiplication operators $M_h(f) := hf, h \in H(\mathbb{C})$, between weighted Banach spaces $H_v^\infty(\mathbb{C})$ of entire functions. These characterizations are well-known when the operators act on spaces of holomorphic functions defined on the unit disc; see e.g. [13] and [17].

We derive the results for the case of spaces of entire functions using the so called *associated weight* (see [9]) as an important tool. For a weight v, the associated weight $\tilde{v}$ is defined by

$$\tilde{v}(z) := \Big(\sup \big\{ |f(z)| \mid f \in H_v^\infty(\mathbb{C}), \|f\|_v \leq 1 \big\} \Big)^{-1} = \big(\|\delta_z\|_v \big)^{-1}, \ z \in \mathbb{C},$$

where δ_z denotes the point evaluation of z. By [9, Properties 1.2] we know that the associated weight is continuous, radial, that $\tilde{v} \geq v > 0$ holds and that for each $z \in \mathbb{D}$ we can find $f_z \in H_v^\infty$, $\|f_z\|_v = 1$ with $|f_z(z)|\tilde{v}(z) = 1$. It is also shown in [9, Observation 1.12] that $H_{\tilde{v}}^\infty(\mathbb{C})$ coincides isometrically with $H_v^\infty(\mathbb{C})$. Under the present assumptions on the weights, it is also true that $H_{\tilde{v}}^0(\mathbb{C})$ coincides isometrically with $H_v^0(\mathbb{C})$. Indeed, since $v \leq \tilde{v}$ and $H_{\tilde{v}}^\infty(\mathbb{C})$ coincides isometrically with $H_v^\infty(\mathbb{C})$, we find that $H_{\tilde{v}}^0(\mathbb{C})$ is a closed subspace of $H_v^0(\mathbb{C})$. By the assumption $\lim_{r\to\infty} r^m v(r) = 0$ for each $m \in \mathbb{N}$, and this implies $z^m \in H_{\tilde{v}}^\infty(\mathbb{C})$ for all $m \in \mathbb{N}$. Therefore $z^m \in H_{\tilde{v}}^0(\mathbb{C})$ for each $m \in \mathbb{N}$. Thus the polynomials $\mathcal{P}$ are contained in $H_{\tilde{v}}^0(\mathbb{C})$. Since $\mathcal{P}$ is dense in $H_v^0(\mathbb{C})$, the conclusion follows. Observe that we have also shown that $\lim_{r\to\infty} r^m \tilde{v}(r) = 0$ for each $m \in \mathbb{N}$.

A weight v is called *essential*, if there is $C > 0$ such that $v(z) \leq \tilde{v}(z) \leq Cv(z)$ for each $z \in \mathbb{C}$. A weight v is essential, if and only if there is $c > 0$ such that for each $z_0 \in \mathbb{C}$ there is $f_0 \in H(\mathbb{C})$ with properties $|f_0(z_0)| \geq c/v(z_0)$ and $|f_0(z)| \leq 1/v(z)$ for all $z \in \mathbb{C}$. It follows from [14, Lemma 1] or [23, Theorem 17 and Lemma 46] that the weight $v(z) = \exp(-\alpha|z|^p), z \in \mathbb{C}, \alpha > 0, p > 0$, is essential; see Lemma 3.10 and Remark 3.11 of this paper for more details.

Proposition 2.1 *Let v and w be weights. The following conditions are equivalent for an entire function $h \in H(\mathbb{C})$:*

(1) $M_h : H_v^\infty(\mathbb{C}) \to H_w^\infty(\mathbb{C})$ *is continuous.*

(2) $M_h : H_v^0(\mathbb{C}) \to H_w^0(\mathbb{C})$ *is continuous.*

(3) $\sup \frac{w(z)|h(z)|}{\tilde{v}(z)} < \infty$.

(4) $\sup \frac{\tilde{w}(z)|h(z)|}{\tilde{v}(z)} < \infty$.

Proof. (1) ⇒ (2). By (1), $z^n h \in H_w^\infty(\mathbb{C})$ for each $n \in \mathbb{N}$. This implies $z^n h \in H_w^0(\mathbb{C})$ for each $n \in \mathbb{N}$. Since the polynomials $\mathcal{P}$ are dense in $H_v^0(\mathbb{C})$, we have $M_h(H_v^0(\mathbb{C})) \subset H_w^0(\mathbb{C})$.

(2) ⇒ (1). Fix $f \in H_v^\infty(\mathbb{C})$. There is a sequence $(p_n)_n \in \mathcal{P}$ such that $\|p_n\| \leq \|f\|$ for each $n \in \mathbb{N}$ and $p_n \to f$ in $H(\mathbb{C})$ for the compact open topology (cf. [8]). Then $hp_n \to hf$ in $H(\mathbb{C})$ for the compact open topology, and by (2), there is $C > 0$ such that $\|hp_n\|_w \leq C$ for each $n \in \mathbb{N}$. This implies $\|hf\|_w \leq C$; also $M_h(H_v^\infty(\mathbb{C})) \subset H_w^\infty(\mathbb{C})$ holds.

Clearly (4) implies (3).

(3) ⇒ (1). Assume that $\frac{\tilde{w}(z)h(z)}{\tilde{v}(z)} \leq D$ for all $z \in \mathbb{C}$. Given $f \in H_v^\infty(\mathbb{C})$ with $\|f\|_v \leq 1$, we have $f \leq 1/\tilde{v}$ on $\mathbb{C}$. Hence $w|hf| \leq \frac{w|h|}{\tilde{v}}\tilde{v}|f| \leq D$, and $hf \in H_w^\infty(\mathbb{C})$. The conclusion follows from the closed graph theorem.

(1) ⇒ (4). By (1), the transpose map $M_h^t : (H_w^\infty(\mathbb{C}))' \to (H_v^\infty(\mathbb{C}))'$ is continuous. It is easy to see that the set $\{\tilde{w}\delta_z \mid z \in \mathbb{C}\}$ is bounded in $(H_w^\infty(\mathbb{C}))'$. Since $M_h^t(\delta_z) = h(z)\delta_z$ for each $z \in \mathbb{C}$, one can find $D > 0$ such that

$$\frac{\tilde{w}(z)|h(z)|}{\tilde{v}(z)} \leq \tilde{w}(z)|h(z)|\|\delta_z\|_v \leq \|M_h^t(\tilde{w}(z)\delta_z)\| \leq D.$$

□

A sequence $(z_j)_j$ in $\mathbb{C}$ is called interpolating for $H_v^\infty(\mathbb{C})$, if for every sequence $(\alpha_j)_j$ with $\sup_{j\in\mathbb{N}} v(z_j)|\alpha_j| < \infty$, there is $g \in H_v^\infty(\mathbb{C})$ such that $f(z_j) = \alpha_j$ for each $j \in \mathbb{N}$. Examples of weights v such that every discrete sequence in $\mathbb{C}$ has a subsequence, which is interpolating for $H_v^\infty(\mathbb{C})$, are given in [7, Proposition 9]. The result is based on [23]. This property holds true for example for $v(z) = e^{-\alpha|z|^p}, \alpha > 0, p > 0$.

Proposition 2.2 *Let v and w be weights. The following conditions are equivalent for an entire function $g \in H(\mathbb{C})$:*

(1) $M_h : H_v^\infty(\mathbb{C}) \to H_w^\infty(\mathbb{C})$ *is compact.*

(2) $M_h : H_v^0(\mathbb{C}) \to H_w^0(\mathbb{C})$ *is compact.*

(3) $\lim_{|z|\to\infty} \frac{w(z)|h(z)|}{\tilde{v}(z)} = 0$.

(4) $\lim_{|z|\to\infty} \frac{\tilde{w}(z)|h(z)|}{\tilde{v}(z)} = 0$.

If, moreover every discrete sequence in $\mathbb{C}$ has a subsequence that is interpolating for $H_{\tilde{v}}^\infty(\mathbb{C})$, these four conditions are also equivalent to

(5) $M_h : H_v^\infty(\mathbb{C}) \to H_w^\infty(\mathbb{C})$ *is weakly compact.*

(6) $M_h : H_v^0(\mathbb{C}) \to H_w^0(\mathbb{C})$ *is weakly compact.*

Proof. If $M_h : H_v^\infty(\mathbb{C}) \to H_w^\infty(\mathbb{C})$ is compact, then it is continuous and proposition 2.1 implies that $M_h : H_v^0(\mathbb{C}) \to H_w^0(\mathbb{C})$ is compact. Conversely, if $M_h : H_v^0(\mathbb{C}) \to H_w^0(\mathbb{C})$ is compact, there is a compact subset K of $H_w^0(\mathbb{C})$ such that the unit ball B_v^0 of $H_v^0(\mathbb{C})$ satisfies $M_h(B_v^0) \subset K$. By [8] or [10], since K is compact for the compact open topology, the unit ball B_v^∞ of $H_v^\infty(\mathbb{C})$ satisfies $M_h(B_v^\infty) \subset M_h(\overline{B_v^0}) \subset \overline{M_h(B_v^0)} \subset K$, the closure taken for the compact open topology. This implies condition (1).

Clearly (4) implies (3). To show that (3) implies (1) it is enough to show that if a sequence $(f_k)_k$ is bounded in $H_v^\infty(\mathbb{C})$ and $f_k \to 0$ for the compact open topology, then $hf_k \to 0$ in $H_w^\infty(\mathbb{C})$ (see e.g. [30, Section 2.4]). To see that this holds, set $M := \sup_{k\in\mathbb{N}} \|f_k\|_v$, and given $\varepsilon > 0$, select $R > 0$ such that, for $|z| > R$, $\frac{w(z)h(z)}{\tilde{v}(z)} < \varepsilon/(2M)$. Set $C := \sup_{|z|\le R} w(z)|h(z)|$. Since $f_k \to 0$ for the compact open topology, there is $k_0 \in \mathbb{N}$ such that, for $k \ge k_0$, we have $|f_k(z)| < \varepsilon/(2C)$ if $|z| \le R$. Thus, if $k \ge k_0$ and $z \in \mathbb{C}$, we have $w(z)|h(z)f_k(z)| < \varepsilon$.

To complete the proof of the equivalence of the first four conditions, we show that (2) implies (4). By assumption (and Schauder's theorem), $M_h^t : (H_w^0(\mathbb{C}))' \to (H_v^0(\mathbb{C}))'$ is compact. Since $H_w^0(\mathbb{C}) = H_{\tilde{w}}^0(\mathbb{C})$, for each $f \in H_w^0(\mathbb{C})$ and each $\varepsilon > 0$ there is $R > 0$ such that if $|z| > R$, then $|\tilde{w}(z)\delta_z(f)| = \tilde{w}(z)|f(z)| < \varepsilon$; i.e. $\lim_{|z|\to\infty} \tilde{w}(z)\delta_z(f) = 0$ for the weak* topology $\sigma((H_w^0(\mathbb{C}))', H_w^0(\mathbb{C}))$. Since the set $\{\tilde{w}\delta_z \mid z \in \mathbb{C}\}$ is bounded in $(H_w^\infty(\mathbb{C}))'$ and $M_h^t : (H_w^0(\mathbb{C}))' \to (H_v^0(\mathbb{C}))'$ is compact, the weak* convergence implies the norm convergence of $\lim_{|z|\to\infty} \tilde{w}(z)M_h^t(\delta_z(f)) = 0$ in $(H_v^0(\mathbb{C}))'$, which clearly implies condition (4).

Conditions (5) and (6) are both equivalent to $M_h(H_v^\infty(\mathbb{C})) \subset H_w^0(\mathbb{C})$ by [18, p. 482] and the Gantmacher theorem, since the bidual of $H_v^0(\mathbb{C})$ is isometrically isomorphic to $H_v^\infty(\mathbb{C})$ by [10]. Conditions (1)–(4) clearly imply (5) and (6). Suppose now that condition (3) is not satisfied. We can find a discrete sequence $(z_j)_j$ in $\mathbb{C}$ and $\varepsilon > 0$ such that $\frac{w(z_j|h(z_j)|}{\tilde{v}(z_j)} > \varepsilon$ for each $j \in \mathbb{N}$. By our assumption, there is a subsequence $(z_{j_k})_k$ of $(z_j)_j$ and there is $f \in H_v^\infty(\mathbb{C}) = H_{\tilde{v}}^\infty(\mathbb{C})$ such that $f(z_{j_k}) = 1/\tilde{v}(z_{j_k})$ for each $k \in \mathbb{N}$. This implies $hf \notin H_w^0(\mathbb{C})$ and (5) does not hold.

□

3 Volterra operators

In this section we present the main results, Theorems 3.4–3.7. Concrete examples and applications are given Remark 3.11 and its corollaries. The results hold under some mild technical assumptions on the weights, and it will be convenient to first formulate some results for the inverse function φ instead of the weight w itself. However, we start with results on the continuity of the integration and differentiation operators. A thorough investigation of the continuity of these operators on weighted spaces of holomorphic functions have been undertaken by Harutyunyan and Lusky in [20].

We consider the following setting. Let $\varphi : [0,\infty[\to]0,\infty[$ be a continuous non-decreasing function, which is C^1 on $[r_\varphi,\infty[$ for some $r_\varphi \ge 0$. We also suppose that the derivative φ' is non-decreasing in $[r_\varphi,\infty[$, that $\varphi'(r_\varphi) > 0$ and that $r^n = O(\varphi'(r))$ as $r \to \infty$ for each $n \in \mathbb{N}$. As a consequence of these assumptions, $r^n = O(\varphi(r))$ as $r \to \infty$ for each $n \in \mathbb{N}$. Therefore $w_\varphi(z) := 1/\varphi(|z|), z \in \mathbb{C}$, is a weight. Clearly, also the function

$$u_\varphi(z) := 1/\max\{\varphi'(r_\varphi), \varphi'(|z|)\} = \begin{cases} 1/\varphi'(r_\varphi) \ , & |z| \le r_\varphi \\ 1/\varphi'(|z|) \ , & |z| \ge r_\varphi \end{cases}$$

is a weight. We will keep this notation and these assumptions on the function φ for the rest of this section. Here is one example: If $\varphi(r) = \exp(\alpha r^p), r \geq 0, \alpha > 0, p > 0$, then $w_\varphi(z) = \exp(-\alpha|z|^p)$, $z \in \mathbb{C}$, and $u_\varphi(z) = \alpha^{-1}p^{-1}|z|^{1-p}\exp(-\alpha|z|^p)$ for $|z|$ large enough.

Recall that the integration operator J is the Volterra operator V_g with g as the identity mapping.

Proposition 3.1 *The integration operators $J : H^\infty_{u_\varphi}(\mathbb{C}) \to H^\infty_{w_\varphi}(\mathbb{C})$ and $J : H^0_{u_\varphi}(\mathbb{C}) \to H^0_{w_\varphi}(\mathbb{C})$ are continuous.*

Proof. By the proof of [4, Lemma 2.1], it is enough to show that $J : H^\infty_{u_\varphi}(\mathbb{C}) \to H^\infty_{w_\varphi}(\mathbb{C})$ is continuous. Fix $f \in H^\infty_{u_\varphi}(\mathbb{C})$ with $\|f\|_{u_\varphi} \leq 1$. We have, for $z \in \mathbb{C}, |z| \geq r_\varphi$,

$$\begin{aligned} |Jf(z)| = \Big|\int_0^1 f(tz)z dt\Big| &\leq \int_0^1 \frac{|z|}{u_\varphi(t|z|)}dt = \int_0^{|z|}\frac{1}{u_\varphi(s)}ds \\ = \int_0^{r_\varphi}\varphi'(r_\varphi)ds &+ \int_{r_\varphi}^{|z|}\varphi'(s)ds = r_\varphi\varphi'(r_\varphi) + \varphi(|z|) - \varphi(r_\varphi). \end{aligned} \tag{3.1}$$

This implies

$$w_\varphi(z)|Jf(z)| \leq 2 + \frac{r_\varphi\varphi'(r_\varphi)}{\varphi(r_\varphi)} \quad \forall\ |z| \geq r_\varphi.$$

If $|z| \leq r_\varphi$, then $w_\varphi(z)|Jf(z)| \leq \frac{r_\varphi\varphi'(r_\varphi)}{\varphi(0)}$, and the continuity of J follows. □

Proposition 3.2 *If the function φ is of smoothness C^2 on $[r_\varphi, \infty[$ for some $r_\varphi > 0$ and it satisfies $\sup_{r\geq r_\varphi}\frac{\varphi''(r)\varphi(r)}{(\varphi'(r))^2} < \infty$ in addition to the general assumptions of this section, then the differentiation operators $D : H^\infty_{w_\varphi}(\mathbb{C}) \to H^\infty_{u_\varphi}(\mathbb{C})$ and $D : H^0_{w_\varphi}(\mathbb{C}) \to H^0_{u_\varphi}(\mathbb{C})$, $Df := f'$, are continuous.*

Proof. An entire function f belongs to $H^\infty_{w_\varphi}(\mathbb{C})$ if and only if $M(f, r) = O(\varphi(r))$, when $r \to \infty$. It follows from [16, Lemma 21] that this is equivalent to $M(f', r) = O(\varphi'(r))$ for $r \to \infty$, i.e. $Df \in H^\infty_{u_\varphi}(\mathbb{C})$. The closed graph theorem then implies that $D : H^\infty_{w_\varphi}(\mathbb{C}) \to H^\infty_{u_\varphi}(\mathbb{C})$ is continuous. Finally, the argument of [4, Lemma 2.1] implies that $D : H^0_{w_\varphi}(\mathbb{C}) \to H^0_{u_\varphi}(\mathbb{C})$ is also continuous. □

The condition on the function φ in Proposition 3.2 corresponds to the condition K_p in [16]; see (3.4) in that paper. The argument behind [16, Lemma 21] can be traced back at least to [26, Theorem 2.1]. A remarkable result of Hardy is used in [26] to exhibit examples of functions φ that satisfy the assumption of Proposition 3.2. For example one can take $\varphi(r) := r^a(\log)^b\exp(cr^d + k(\log r)^m)$, for large r, where $c > 0, d > 0$ or $c = 0, k > 0, m > 1$).

As a consequence of Propositions 3.1 and 3.2 we obtain the following result, which gives a Littlewood-Paley-type formula for entire functions and growth estimates of infinite order and which should be compared with [16, Theorem 10]. The result follows directly from the proven continuity of J and D, and from $JDf = f - f(0)$.

Corollary 3.3 *Let φ be as in Proposition 3.2. An entire function f satisfies $f \in H^\infty_{w_\varphi}(\mathbb{C})$ (resp. $f \in H^0_{w_\varphi}(\mathbb{C})$) if and only if $f' \in H^\infty_{u_\varphi}(\mathbb{C})$ (resp. $f' \in H^0_{u_\varphi}(\mathbb{C})$). Moreover, there are constants $C, C', C'' > 0$ such that, for each $f \in H^\infty_{w_\varphi}(\mathbb{C})$,*

$$\|f'\|_{u_\varphi} \leq C\|f\|_{w_\varphi}$$

and

$$\|f\|_{w_\varphi} \leq C'|f(0)| + C''\|f'\|_{u_\varphi}.$$

We apply the above mentioned results to the Volterra operator.

Theorem 3.4 *Let v be a weight and let φ be as in Proposition 3.2. The following conditions are equivalent for an entire function $g \in H(\mathbb{C})$:*

(1) $V_g : H^\infty_v(\mathbb{C}) \to H^\infty_{w_\varphi}(\mathbb{C})$ is continuous.

(2) $V_g : H^0_v(\mathbb{C}) \to H^0_{w_\varphi}(\mathbb{C})$ is continuous.

(3) $\sup_{|z|\geq r_\varphi} \frac{|g'(z)|}{\varphi'(|z|)\tilde{v}(z)} < \infty$.

Proof. Assume that condition (1) holds. By Proposition 3.2, the differentiation operator $D : H^\infty_{w_\varphi}(\mathbb{C}) \to H^\infty_{u_\varphi}(\mathbb{C})$ is continuous. We can apply (1) and the identity $DV_g = M_{g'}$ to conclude that $M_{g'} : H^\infty_v(\mathbb{C}) \to H^\infty_{u_\varphi}(\mathbb{C})$ is continuous. Now condition (3) follows from Proposition 2.1, since $u_\varphi(z) = 1/\varphi'(|z|), |z| \geq r_\varphi$. Conversely, if condition (3) holds, the operator $M_{g'} : H^\infty_v(\mathbb{C}) \to H^\infty_{u_\varphi}(\mathbb{C})$ is continuous by proposition 2.1. We apply Proposition 3.1 to get that $V_g = J \circ M_{g'} : H^\infty_v(\mathbb{C}) \to H^\infty_{w_\varphi}(\mathbb{C})$ is continuous.

The equivalence of (2) and (3) is obtained in the same way, as a consequence of Propositions 3.2, 3.1 and 2.1. □

The corresponding statements on the compactness and weak compactness are as follows. The proof is very similar to the one of Theorem 3.4, and it is a consequence of Propositions 3.2, 3.1 and 2.2.

Theorem 3.5 *Let v be a weight and let φ be as in Proposition 3.2. The following conditions are equivalent for an entire function $g \in H(\mathbb{C})$:*

(1) $V_g : H^\infty_v(\mathbb{C}) \to H^\infty_{w_\varphi}(\mathbb{C})$ is compact.

(2) $V_g : H^0_v(\mathbb{C}) \to H^0_{w_\varphi}(\mathbb{C})$ is compact.

(3) $\lim_{|z|\to\infty} \frac{|g'(z)|}{\varphi'(|z|)\tilde{v}(z)} = 0$.

If, moreover every discrete sequence in $\mathbb{C}$ has a subsequence that is interpolating for $H^\infty_{\tilde{v}}(\mathbb{C})$, these three conditions are also equivalent to

(4) $V_g : H^\infty_v(\mathbb{C}) \to H^\infty_{w_\varphi}(\mathbb{C})$ is weakly compact.

(5) $V_g : H^0_v(\mathbb{C}) \to H^0_{w_\varphi}(\mathbb{C})$ is weakly compact.

Let us next reformulate the above results in terms of two quite arbitrary weights v and w, however, assuming some additional properties about the latter.

Theorem 3.6 *Let v and w be weights, and assume that for some constants $R > 0$, $C > 0$ and $0 < \delta \le 1$, the following hold for w on the interval $]R, \infty[$: (i) w is of smoothness C^2, (ii) the function $|w'(r)|r^{1+\delta}$ is non-increasing , (iii) $-\frac{w(r)w''(r)}{w'(r)^2} \le C$. The following conditions are equivalent for an entire function $g \in H(\mathbb{C})$:*

(1) $V_g : H_v^\infty(\mathbb{C}) \to H_w^\infty(\mathbb{C})$ *is continuous.*

(2) $V_g : H_v^0(\mathbb{C}) \to H_w^0(\mathbb{C})$ *is continuous.*

(3) $\sup_{|z|\ge R} \frac{w(z)^2|g'(z)|}{w'(z)\tilde{v}(z)} < \infty$.

Notice that for a weight w satisfying all assumptions of this theorem, the quantity on the left hand side of (iii) may be negative, as it is for example in the case $w(r) = e^{-r}$.

Proof. It is enough to show that the function $\varphi(z) := 1/w(z)$ satisfies the assumptions of Theorem 3.4, since $w_\varphi = w$ and since the conditions (3) in Theorems 3.4 and 3.6 are the same.

First, calculating the derivatives yields $\frac{\varphi''\varphi}{(\varphi')^2} = 2 - \frac{w''w}{(w')^2}$, hence, the corresponding assumption of Theorem 3.4 follows from (iii). Second, in view of the beginning of this section, we should show that $\varphi'(r) \ge C_n r^n$ for all n, $r \ge R$. We have by assumption (ii)

$$\begin{aligned} w(r) &= \int_r^\infty |w'(t)|dt = \int_r^\infty |w'(t)|t^{1+\delta}t^{-1-\delta}dt \\ &\le |w'(r)|r^{1+\delta}\int_r^\infty t^{-1-\delta}dt \le C|w'(r)|r \quad \forall r \ge R. \end{aligned} \tag{3.2}$$

Since w is a weight, we have $w(r) \le C_n r^{-n}$, so this and (3.2) imply

$$\varphi'(r) = -\frac{w'(r)}{w(r)^2} = \frac{1}{w(r)}\frac{|w'(r)|}{w(r)} \ge C_n' r^n r^{-1} = C_n' r^{n-1}$$

for arbitrary $n \ge 1$, $r \ge R$. This proves the claim. □

The above argument and Theorem 3.5 lead also to the following statements.

Theorem 3.7 *Let v and w be weights and assume that w satisfies the conditions (i)–(iii) of Theorem 3.6. The following conditions are equivalent for an entire function $g \in H(\mathbb{C})$:*

(1) $V_g : H_v^\infty(\mathbb{C}) \to H_w^\infty(\mathbb{C})$ *is compact.*

(2) $V_g : H_v^0(\mathbb{C}) \to H_w^0(\mathbb{C})$ *is compact.*

(3) $\lim_{|z|\to\infty} \frac{w(z)^2|g'(z)|}{w'(z)\tilde{v}(z)} = 0$.

If, moreover every discrete sequence in $\mathbb{C}$ has a subsequence that is interpolating for $H_{\tilde{v}}^\infty(\mathbb{C})$, these three conditions are also equivalent to

(4) $V_g : H_v^\infty(\mathbb{C}) \to H_w^\infty(\mathbb{C})$ *is weakly compact.*

(5) $V_g : H_v^0(\mathbb{C}) \to H_w^0(\mathbb{C})$ *is weakly compact.*

We next formulate special cases of the above theorems. For the proof of the next result it suffices to realize that the function $\varphi(r) := 1/w(r) := \exp(\alpha r^p)$ satisfies all the assumptions of Theorem 3.4 and 3.5, respectively.

Corollary 3.8 *Let v be a weight and let $w(r) := \exp(-\alpha r^p)$, where $\alpha > 0, p > 0$ are constants. The following conditions are equivalent for $g \in H(\mathbb{C})$:*

(1) $V_g : H_v^\infty(\mathbb{C}) \to H_w^\infty(\mathbb{C})$ *is continuous.*

(2) $V_g : H_v^0(\mathbb{C}) \to H_w^0(\mathbb{C})$ *is continuous.*

(3) *There exist a constant $C > 0$ such that $|g'(z)| \le C|z|^{p-1}\exp(\alpha|z|^p)\tilde{v}(|z|)$ for all $z \in \mathbb{C}$, $|z| \ge 1$.*

Moreover, the following conditions are also mutually equivalent for $g \in H(\mathbb{C})$:

(1) $V_g : H_v^\infty(\mathbb{C}) \to H_w^\infty(\mathbb{C})$ *is compact.*

(2) $V_g : H_v^0(\mathbb{C}) \to H_w^0(\mathbb{C})$ *is compact.*

(3) $|g'(z)| = o\big(|z|^{p-1}\exp(\alpha|z|^p)\tilde{v}(|z|)\big)$ *as* $|z| \to \infty$.

Let us finally consider weights, which are explicitly given in the form $v(r) = e^{-\phi(r)}$. We make the assumptions that the function $\phi : [0,\infty[\to [0,\infty[$ is of smoothness C^2, $\phi' > 0$ on $[0,\infty[$, and

$$r\phi'(r) \to \infty \ \text{ as } r \to \infty\,, \ \text{ and } \ \phi''(r) \le (1-\delta)\phi'(r)^2 \ \ \forall r \in [R_0,\infty[, \tag{3.3}$$

for some contants $R_0 \ge 0$ and $0 < \delta < 1$.

Remark 3.9 The first condition means that ϕ grows faster than logarithmically. The second condition is a monotonicity condition for ϕ''. It is not satisfied for example by $\sin r$. Another such function with positive ϕ'' can be constructed by setting $\phi''(r) = e^{-r}$ on $[0,\infty[\setminus K$, where K is a compact set containing some small neighbourhoods of the points $1, 2, 3, \ldots$ such that its measure $|K|$ satisfies $|K| \le 1/100$, and requiring $\phi''(n) = 10$ for $n \in \mathbb{N}$, $\phi''(r) \le 10$ for all r. Then, defining $\phi'(r) = \int_0^r \phi''(t)dt$ we have $\phi'(r) \le 2$ for all r and the second inequality (3.3) fails.

On the other hand, there does not exist a positive function ϕ'' with $\phi''(r) \ge c\phi'(r)^2$ ($c > 0$ constant) on an unbounded interval. This follows from the theory of the solutions of the nonlinear differential equation (think g as ϕ')

$$g' = cg^2. \tag{3.4}$$

Any solution of (3.4) with positive value $g(t)$ for some t, blows up at some finite value of r, i.e. it ceases to exist globally in r. For example, given $b > 0$, the solution of (3.4) with $g(0) = b$ is $g(r) = b/(1 - cbr)$, which is defined only on the interval $[0, 1/(cb)[$. In view of this remark, the second condition in (3.3) is indeed quite mild.

These weights seem very useful for the theory presented in this paper, as we can see from the following lemma.

Lemma 3.10 *Let ϕ satisfy the conditions around (3.3). Then, the expression $v(r) = e^{-\phi(r)}$ is a weight and it satisfies (ii) and (iii) of Theorem 3.6. In addition, if there exists constants $r_0, c > 0$ such that the inequality*

$$\phi'(r) + r\phi''(r) \geq \frac{c}{r} \tag{3.5}$$

holds for ϕ, for all $r \geq r_0$, then v is also essential.

Proof. From the first relation in (3.3) we get, for all $n \in \mathbb{N}$, for large enough $R > 0$,

$$\phi(r) \geq \int_R^r \phi'(t)dt \geq \int_R^r \frac{n}{r}dt = n \log r - n \log R,$$

hence, $v(r) \leq Cr^{-n}$ and v is a weight. Moreover,

$$\begin{aligned} &\frac{d}{dr}(-v'(r)r^{1+\delta}) = \Big((1+\delta)r^\delta \phi'(r) + r^{1+\delta}\big(\phi''(r) - \phi'(r)^2\big)\Big)e^{-\phi(r)} \\ \leq\ &\Big((1+\delta)r^\delta\phi'(r) - \delta r^{1+\delta}\phi'(r)^2\Big)e^{-\phi(r)} = \phi'(r)r^\delta\Big((1+\delta) - \delta r\phi'(r)\Big)e^{-\phi(r)} < 0 \end{aligned}$$

for large enough r, by both relations (3.3), since $r\phi'(r) \to \infty$ as $r \to \infty$. Hence, v satisfies (ii) of Theorem 3.6. Finally, again by (3.3),

$$-\frac{v(r)v''(r)}{v'(r)^2} = \frac{\phi''(r) - \phi'(r)^2}{\phi'(r)^2} \leq -\delta,$$

so that (iii) of Theorem 3.6 also holds.

To prove the essentialness of the weight we shall use a result of [14] (another approach would be contained in [23, Theorem 17 and Lemma 46]), and to this end we first observe that the function ϕ satisfies, for some constant $R > 0$,

$$R\phi'(R) = 1. \tag{3.6}$$

Namely, $\lim_{r\to 0} r\phi'(r) = 0$, since ϕ' is assumed continuous on $[0, \infty[$, and also $\lim_{r\to\infty} r\phi'(r) = \infty$ by the first relation (3.3). Hence, (3.6) must hold for some R.

The result [14, Lemma 1] gives an entire function g with

$$0 < c_1 \leq g(r)e^{-\sigma(\log r)} \leq c_2 \quad \forall r \geq 1, \tag{3.7}$$

whenever $\sigma : [0, \infty[\to \mathbb{R}$ is a C^2-function with $\sigma'(0) = 1$ and $\sigma''(r) \geq c_3 > 0$ for $r \geq 0$. Moreover, the Taylor coefficients of g are positive, so the upper bound in (3.7) can be extended so as $|g(z)|e^{-\sigma(\log |z|)} \leq c_2$ for all $z \in \mathbb{C}$ with $|z| \geq 1$.

In order to use this we define $\sigma(r) = \phi(Re^r)$, where $R > 0$ is as in (3.6). As a consequence, $\sigma'(0) = R\phi'(R) = 1$. Furthermore,

$$\sigma''(r) = Re^r\big(\phi'(Re^r) + Re^r\phi''(Re^r)\big),$$

which is larger than c for all $r \geq 0$, by (3.5). In view of (3.7), the function g has the properties

$$g(r) \geq c_1 e^{\sigma(\log r)} = c_1 e^{\phi(Rr)}, \quad |g(z)| \leq g(|z|) \leq c_2 e^{-\phi(R|z|)}$$

for $r \geq 1$, $|z| \geq 1$. Thus, the entire function $h(z) := g(R^{-1}z)$ satisfies $h(r) \geq c_1 e^{\phi(r)}$, $|h(z)| \leq c_2 e^{\phi(|z|)}$ for $r \geq R$, $|z| \geq R$. This implies that the weight $e^{-\phi(z)}$ is essential, since ϕ is a radially symmetric function. □

Remark 3.11 (i) By Lemma 3.10, all functions $v(r) = \exp(-\alpha r^p)$, where $\alpha, p > 0$, are essential weights: it is easy to see that $\phi(r) = \alpha r^p$ satisfies the conditions in (3.3), (3.5).

(ii) The same is true for the more general functions

$$v(r) = \exp\left(-\alpha r^p + \beta(\log r)^q\right) \quad , \quad r \geq 2,$$

with $\alpha, p, q > 0$, $\beta \in \mathbb{R}$, assuming the function is extended to $[0, 2]$ properly.

(iii) The case of the weights $v(r) = \exp(-(\log r)^p)$ with $p > 1$ and r as in (ii), is more subtle. We have $\phi'(r) = pr^{-1}(\log r)^{p-1}$,

$$\phi''(r) = \frac{p}{r^2}\Big((p-1)(\log r)^{p-2} - (\log r)^{p-1}\Big).$$

It follows that (3.3) holds for all $p > 1$, but (3.5) is valid if and only if $p \geq 2$. So, Lemma 3.10 only permits us to conclude that these weights are essential for $p \geq 2$.

As a consequence of Remark 3.11 and Theorems 3.6, 3.7 we obtain the following results. The next corollary should be compared with [16, Corollary 25].

Corollary 3.12 *Let* $v(r) = \exp(-\phi(r))$, *where* ϕ *satisfies* (3.3) *and* (3.5). *Then,* $V_g : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ *is continuous if and only if there exists a constant* $C > 0$ *such that*

$$|g'(z)| \leq C\phi'(|z|) \quad \forall z \in \mathbb{C}.$$

In particular, if $v(r) = \exp(-\alpha r^p)$, $\alpha > 0, p \geq 1$, *then* $V_g : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ *is continuous if and only if* g *is a polynomial of degree less than or equal to the integer part of* p.

Corollary 3.13 *If* $v(r) = \exp(-\alpha r^p)$, $\alpha > 0, p > 0$, *then* $V_g : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ *is compact if and only if it is weakly compact if and only if* g *is a polynomial of degree less then or equal to the integer part of* $p - 1$.

Proof. Every discrete sequence in $\mathbb{C}$ has a subsequence that is interpolating for $H_v^\infty(\mathbb{C})$ by [7, Proposition 9]. □

References

[1] A. Aleman, A.G. Siskakis, *An integral operator on* H^p, Complex Var. Theory Appl. **28** (1995), 149-158.

[2] A. Aleman, A.G. Siskakis, *Integration operators on Bergman spaces,* Indiana univ. Math. J. **46** (1997), 337-356.

[3] M. Basallote, M.D. Contreras, C. Hernández-Mancera, M.J. martín, P.J. Paúl, *Volterra operators and semigroups in weighted Banach spaces of analytic functions,* Collect. Math. (to appear), DOI: 10.1007/s13348-013-0092-5.

[4] M.J. Beltrán, J. Bonet, C. Fernández, *Classical operators on weighted Banach spaces of entire functions*, Proc. Amer. Math. Soc. **141** (2013), 4293174303.

[5] C. A. Berenstein and R. Gay. *Complex variables*, volume 125 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. An introduction.

[6] C. A. Berenstein and R. Gay. *Complex analysis and special topics in harmonic analysis*. Springer-Verlag, New York, 1995.

[7] K.D. Bierstedt, J. Bonet, A. Galbis, *Weighted (LB)-spaces of holomorphic functions: $\mathcal{V}H(G) = \mathcal{V}_0H(G)$ and completeness of $\mathcal{V}_0H(G)$*, J. Math. Anal. Appl. **323** (2006), 747-767.

[8] K.D. Bierstedt, J. Bonet, A. Galbis, *Weighted spaces of holomorphic functions on balanced domains*, Michigan Math. J. **40** (1993), no. 2, 271-297.

[9] K.D. Bierstedt, J. Bonet, J. Taskinen, *Associated weights and spaces of holomorphic functions*, Studia Math. **127** (1998), 137-168.

[10] K.D. Bierstedt, W. H. Summers, *Biduals of weighted Banach spaces of analytic functions*, J. Austral. Math. Soc. Ser. A **54** (1993), no. 1, 70-79.

[11] O. Blasco, A. Galbis, *On Taylor coefficients of entire functions integrable against exponential weights*, Math. Nachr. **223** (2001), 5-21.

[12] J. Bonet, A. Bonilla , *Chaos of the differentiation operator on weighted Banach spaces of entire functions*, Complex Anal. Oper. Theory **7** (2013), 33-42.

[13] J. Bonet, P. Domański, M. Lindström, *Pointwise multiplication operators on weighted Banach spaces of analytic functions*, Studia Math. **137** (1999), 17717194.

[14] A. Borichev, *The polynomial approximation property in Fock-type spaces*, Math. Scand. 82 (1998), 256-264.

[15] O. Constantin, *A Volterra-type integration operator on Fock spaces,* Proc. Amer. Math. Soc. **140** (2012), 4247-4257.

[16] O. Constantin, J.A. Peláez, *Integral operators, embedding theorems and a Littlewood-Paley formula on weighted Fock spaces,* arXiv:1304.7501v1.

[17] M. Contreras, A.G. Hernández-Díaz, *Weighted composition operators in weighted Banach spacs of analytic functions*, J. Austral. Math. Soc. (Series A) **69** (2000), 41-60.

[18] N. Dunford, J.T. Schwartz, *Linear Operators I*, John Wiley and Sons, New York, 1958.

[19] A. Galbis, *Weighted Banach spaces of entire functions,* Archiv Math. **62**, (1994), 58-64.

[20] A. Harutyunyan, W. Lusky, *On the boundedness of the differentiation operator between weighted spaces of holomorphic functions*, Studia Math. **184** (2008), 233-247.

[21] W. Lusky, *On the Fourier series of unbounded harmonic functions*, J. London Math. Soc. **61** (2000), 568-580.

[22] W. Lusky, *On the isomorphism classes of weighted spaces of harmonic and holomophic functions*, Studia Math. **75** (2006) 19-45.

[23] N. Marco, X. Massaneda, J. Ortega-Cerdà, *Interpolating and sampling sequences of entire functions*, Geom. Funct. Anal. **13** (2003), 862-914.

[24] R. Meise and D. Vogt. *Introduction to functional analysis*, volume 2 of *Oxford Graduate Texts in Mathematics.* The Clarendon Press Oxford University Press, New York, 1997.

[25] J. Pau, J.A. Peláez, *Embedding theorems and integration operators on Bergman spaces with rapidly decreasing weights,* J. Funct. Anal. **259** (2010), 2727-2756.

[26] M. Pavlovic, *On harmonic conjugates with exponential mean growth,* Czech. Math. J. **49** (1999), 733-742.

[27] M. Pavlovic, J.A. Peláez, *An equivalence for weighted integrals of an analytic function and its derivative,* Math. Nachr. **281** (2008), 1612-1623.

[28] J.A. Peláez, J. Rättyä, *Weighted Bergman Spaces induced by rapidly decreasing weights, Memoirs Amer. Math. Soc.* **227**, 2014.

[29] Ch. Pommerenke, *Schlichte Funktionen un analytische Functionen von beschrn̈kter mittlerer Oszilation,* Comment. Math. Helv. **52** (1977), 591-602.

[30] J.H. Shapiro, *Composition Operators and Classical Function Theorey*, Springer, New York, 1993.

[31] K. Zhu, *Analysis on Fock spaces,* Springer, New York, 2012.

Authors' addresses:

José Bonet: Instituto Universitario de Matemática Pura y Aplicada IUMPA, Universitat Politècnica de València, E-46071 Valencia, Spain
email: jbonet@mat.upv.es

Jari Taskinen: Department of Mathematics and Statistics, P.O. Box 68, University of Helsinki, 00014 Helsinki, Finland.
email: jari.taskinen@helsinki.fi