Two questions about radial Hörmander algebras of entire functions

José Bonet (IUMPA, UPV)

Poznań, DMV-PTM Mathematical Meeting, September, 2014

On joint work with María José Beltrán and Carmen Fernández

Project Prometeo II/2013/013
Two questions on radial Hörmander algebras $A_p(\mathbb{C})$ and $A_p^0(\mathbb{C})$ of entire functions on the complex plane:

- Let $q \leq p$ radial, subharmonic weights with the doubling property. Investigate conditions to ensure that the sequence space canonically associated with the interpolation for $A_q(\mathbb{C})$ (resp. $A_q^0(\mathbb{C})$) is contained in the range of the restriction map defined on the bigger space $A_p(\mathbb{C})$ (resp. $A_p^0(\mathbb{C})$).

- Investigate the dynamics of the integration operator $Jf(z) = \int_0^z f(\zeta) d\zeta$ and the differentiation operator $Df(z) = f'(z)$ on the spaces $A_p(\mathbb{C})$ and $A_p^0(\mathbb{C})$. 

José Bonet

Two questions about radial Hörmander algebras of entire functions
A function $p : \mathbb{C} \to ]0, \infty[$ is called a **weight function** if it satisfies:

1. (w1) $p$ is continuous and subharmonic.
2. (w2) $p$ is radial, that is, $p(z) = p(|z|), z \in \mathbb{C}$.
3. (w3) $\log(1 + |z|^2) = o(p(z))$ as $|z| \to \infty$.
4. (w4) $p$ is doubling, i.e. $p(2z) = O(p(z))$ as $|z| \to \infty$.

Most important example: $p(z) = |z|^s, s > 0$.

We use Landau’s notation of little $o$-growth and capital $O$-growth.

The space of entire functions is denoted by $\mathcal{H}(\mathbb{C})$. 
Given a weight $p$, we define the following weighted (LB)-space of entire functions.

\[ A_p(\mathbb{C}) := \{ f \in \mathcal{H}(\mathbb{C}) : \text{there is } A > 0 : \sup_{z \in \mathbb{C}} |f(z)| \exp(-Ap(z)) < \infty \}, \]

endowed with the inductive limit topology, for which it is a (DFN)-algebra.
Given a weight $p$, we define the following weighted Fréchet space of entire functions.

$$A^0_p(\mathbb{C}) := \{ f \in \mathcal{H}(\mathbb{C}) : \text{for all } \varepsilon > 0 : \sup_{z \in \mathbb{C}} |f(z)| \exp(-\varepsilon p(z)) < \infty \},$$

endowed with the projective topology, for which it is a nuclear Fréchet algebra.
Radial Hörmander algebras. Definition.

For a weight $p$ and $\alpha > 0$, setting

$$A(\alpha p) := \{f \in \mathcal{H}(\mathbb{C}) : \sup_{z \in \mathbb{C}} |f(z)| \exp(-\alpha p(z)) < \infty\},$$

we have

$$A_p(\mathbb{C}) = \bigcup_{n \in \mathbb{N}} A(np),$$

and

$$A_p^0(\mathbb{C}) = \bigcap_{n \in \mathbb{N}} A((1/n)p),$$
Radial Hörmander algebras. Comments.

- $A^0_p(\mathbb{C}) \subset A_p(\mathbb{C})$.

- Condition (w3) implies that $A^0_p(\mathbb{C})$ contains the polynomials.

- Condition (w4) implies that the spaces are stable under differentiation.

- **Braun, Meise and Taylor** studied in 1987 the structure of (complemented) ideals in these algebras.
Examples:

- When $p(z) = |z|^s$, then $A_p(\mathbb{C})$ consists of all entire functions of order $s$ and finite type or order less than $s$.

- When $p(z) = |z|^s$, then $A^0_p(\mathbb{C})$ is the space of all entire functions of order at most $s$ and type 0.

- For $s = 1$, $p(z) = |z|$, $A_p(\mathbb{C})$ is the space of all entire functions of exponential type, and $A^0_p(\mathbb{C})$ is the space of entire functions of infraexponential type.
PART 1
A multiplicity variety $V = \{(z_k, m_k) | k \in \mathbb{N}\}$ is a sequence of different points $(z_k)_k$ with $\lim_{k \to \infty} |z_k| = \infty$ and a sequence $(m_k)_k$ of positive integers corresponding to the multiplicities at the points $z_k$.

By Weierstrass interpolation theorem, the restriction map $R_V : \mathcal{H}(\mathbb{C}) \to \prod_{k \in \mathbb{N}} \mathbb{C}^{m_k}$, $R_V(g) := \left(\left(\frac{g^{(l)}(z_k)}{l!}\right)_{0 \leq l < m_k}\right)_k$, is surjective.
Given $V = \{(z_k, m_k) | k \in \mathbb{N}\}$ and $p$, define

$$A_p(V) := \{a = (a_k, l) | \text{there is } B > 0 : \sup_{k \in \mathbb{N}} \sum_{l=0}^{m_k-1} |a_k, l| \exp(-Bp(z_k)) < \infty\},$$

endowed with the inductive limit topology; and

$$A_p^0(V) := \{a = (a_k, l) | \text{for all } \varepsilon > 0 : \sup_{k \in \mathbb{N}} \sum_{l=0}^{m_k-1} |a_k, l| \exp(-\varepsilon p(z_k)) < \infty\},$$

endowed with the projective topology, for which it is a Fréchet space.
R_V(A_p(\mathbb{C})) \subset A_p(V) and R_V(A^0_p(\mathbb{C})) \subset A^0_p(V).

A multiplicity variety is called interpolating for $A_p(\mathbb{C})$ (resp. for $A^0_p(\mathbb{C})$) if $R_V(A_p(\mathbb{C})) = A_p(V)$ (resp. $R_V(A^0_p(\mathbb{C})) = A^0_p(V)$).

After the seminal work by Berenstein and Taylor, a geometric characterization of the interpolating varieties for $A_p(\mathbb{C})$ (resp. for $A^0_p(\mathbb{C})$) was obtained by Berenstein and Li (resp. Berenstein, Li and Vidras) in 1995.
The geometric characterizations were formulated in terms of the counting function and the integrated counting function of the multiplicity variety $V$, that are defined as follows:

For $z \in \mathbb{C}$ and $r > 0$, we set

$$n_V(z, r) := \sum_{|z-z_k| \leq r} m_k,$$

and

$$N_V(z, r) := \int_0^r \frac{n_V(z, t) - n_V(z, 0)}{t} dt + n_V(z, 0) \log r.$$
Characterizations


\( V \) is interpolating for \( A_p(\mathbb{C}) \) if and only if

(i) \( N_V(r, 0) = O(p(r)) \) as \( r \to \infty \), and

(ii) \( N_V(z_k, |z_k|) = O(p(z_k)) \) as \( k \to \infty \).
Characterizations


$V$ is interpolating for $A^0_p(\mathbb{C})$ if and only if

(i) $N_V(r, 0) = o(p(r))$ as $r \to \infty$, and

(ii) $N_V(z_k, |z_k|) = o(p(z_k))$ as $k \to \infty$. 
Massaneda, Ortega-Cerdà and Ounaïes 2006 gave a geometric description of the interpolating varieties for the algebra of Fourier transforms of distributions and Beurling ultradistributions with compact support on the real line, improving earlier results by Ehrenpreis, Malliavin and Squires. This corresponds to spaces of type $A_p(\mathbb{C})$ for non radial weights $p$. The case of Roumieu ultradistributions was studied by Ziołko in 2011.

The arguments of Berenstein and Li were simplified by Hartmann and Massaneda in 2000 and by Ounaïes in 2007 using Hörmander’s $L^2$ estimates for the $\overline{\partial}$ equation, treating also weights that are radial but not doubling.
Let $q$ and $p$ two weights such that $q(z) = O(p(z))$ as $|z| \to \infty$. In this case,

\[ A_q(\mathbb{C}) \subset A_p(\mathbb{C}), \quad A_q(V) \subset A_p(V). \]

By Berenstein, Li's theorem, if $V$ is interpolating for $A_q(\mathbb{C})$ (i.e. $R_V : A_q(\mathbb{C}) \to A_q(V)$ is surjective), then it is interpolating for $A_p(\mathbb{C})$ (i.e. $R_V : A_p(\mathbb{C}) \to A_p(V)$ is also surjective).
Problem to be considered

Questions

Assume that \( q(z) = o(p(z)) \) as \( |z| \to \infty \). Is there a multiplicity variety \( V \) such that \( V \) is interpolating for \( A_p(\mathbb{C}) \), but not for \( A_q(\mathbb{C}) \)?

Assume that the range \( R_V(A_p(\mathbb{C})) \) contains the sequence space \( A_q(V) \) associated with the weight \( q \). Is \( V \) interpolating for \( A_p(\mathbb{C}) \)?

In other words, is it true that if every sequence in the space \( A_q(V) \) can be interpolated by a function in \( A_p(\mathbb{C}) \), then every sequence in the larger space \( A_p(V) \) can be interpolated by a function in \( A_p(\mathbb{C}) \)?
Theorem

Let $V = \{(z_k, m_k) | k \in \mathbb{N}\}$ a multiplicity variety and let $q$ and $p$ be weights such that $q(z) = O(p(z))$ as $|z| \to \infty$.

(1) If the restriction map $R_V$ satisfies $A_q(V) \subset R_V(A_p(\mathbb{C}))$, then $R_V(A_p(\mathbb{C})) = A_p(V)$, i.e. $V$ is interpolating for $A_p(V)$.

(2) If the restriction map $R_V$ satisfies $A_0^0(V) \subset R_V(A_0^0(\mathbb{C}))$, then $R_V(A_0^0(\mathbb{C})) = A_0^0(V)$, i.e. $V$ is interpolating for $A_0^0(V)$.

The proof uses Grothendieck’s factorization theorem in the (LB)-space case, the open mapping theorem and the lifting of compact sets from a quotient in the Fréchet space case, and an argument of Ounaïes that uses Jensen’s formula.
(1) **Bonet, Meise and Taylor (1992)** investigated when the range of the Borel map on a non-quasianalytic class contains a sequence space associated with a quasianalytic weight.

(2) **Bonet, Galbis and Meise (1996, 1998)** investigated the range of non-surjective convolution operators on spaces of non-quasianalytic ultradifferentiable functions.

(3) **Frerick and Wengenroth (2003, 2004)** proved that a convolution operator is surjective in a class of Beurling ultradifferentiable functions if the range contains both the space of real analytic functions and the space of smooth functions with compact support.
Theorem

If $q$ and $p$ are weights such that $q(z) = o(p(z))$ as $|z| \to \infty$, then there are multiplicity varieties $V$ and $W$ such that $V$ is interpolating for $A_p(\mathbb{C})$, but not for $A_q(\mathbb{C})$ and $W$ is interpolating for $A^0_p(\mathbb{C})$, but not for $A^0_q(\mathbb{C})$.

The construction of $V$ and $W$ proceeds by induction and it uses the following result by Berenstein and Li 1995:

Proposition

If a multiplicity variety $V = \{(z_k, m_k) | k \in \mathbb{N}\}$ satisfies $|z_{k+1}| \geq L|z_k|$, $k \in \mathbb{N}$, for some constant $L > 1$, then $V$ is interpolating for $A_p(\mathbb{C})$ if and only if $N_V(r, 0) = O(p(r))$ as $r \to \infty$ and $m_k \log |z_k| = O(p(z_k))$ as $k \to \infty$. 
Further Results

- **Ounaïes** in 2008 used divided differences to characterize those sequences that are in the range $R_V(\mathbb{A}_p(\mathbb{C}))$ of the restriction map $R_V$ when the multiplicity variety satisfies the assumption (a) (i) in the Theorem Berenstein and Li.

- This work was continued in 2009 by **Massaneda, Ortega-Cerdà and Ounaïes**. Traces of functions in Bargmann-Fock spaces on lattices of critical density are investigated by **Buckley, Massaneda and Ortega-Cerdà** in 2012.

- A reduction argument of Meise and Taylor can be used to obtain the description of the range $R_V(A_0^0(\mathbb{C}))$ of the restriction operator on $A_0^0(\mathbb{C})$ in terms of divided differences, as a consequence of a theorem of Ounaïes.
PART 2

We need not assume in this part that the weight $p$ is subharmonic.
$X$ is a Hausdorff locally convex space (lcs).

$L(X)$ is the space of all continuous linear operators on $X$.

**Power bounded operators**

An operator $T \in L(X)$ is said to be *power bounded* if $\{T^m\}_{m=1}^{\infty}$ is an equicontinuous subset of $L(X)$.

If $X$ is a Fréchet space, or more generally if the uniform boundedness principle is valid for operators defined on $X$, then $T$ is power bounded if and only if the orbits $\{T^m(x)\}_{m=1}^{\infty}$ of all the elements $x \in X$ under $T$ are bounded.
Mean ergodic operators

An operator $T \in \mathcal{L}(X)$ is said to be *mean ergodic* if the limits

$$Px := \lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} T^m x, \quad x \in X,$$

exist in $X$.

If $\frac{1}{n} \sum_{m=1}^{n} T^m$ converges to $P$ uniformly on the bounded sets, we say that $T$ is *uniformly mean ergodic*.

A power bounded operator $T$ is mean ergodic precisely when

$$X = \ker(I - T) \oplus \overline{\text{Im}(I - T)},$$

where $I$ is the identity operator, $\text{Im}(I - T)$ denotes the range of $(I - T)$ and the bar denotes the “closure in $X$”.
Definitions

**Hypercyclic operators**

An operator \( T \in \mathcal{L}(X) \) is said to be hypercyclic if there is a vector \( x \in X \) whose orbit \( \left\{ T^m(x) \right\}_{m=1}^{\infty} \) is dense in \( X \).

**Transitive operators**

An operator \( T \in \mathcal{L}(X) \) is said to be topologically transitive if for every pair of non-empty open subsets \( U, V \) in \( X \) there is \( n \) such that \( T^n(U) \cap V \neq \emptyset \).

**Proposition**

If \( T \in \mathcal{L}(X) \) is an operator on a separable complete metrizable lcs \( X \), \( T \) is hypercyclic if and only if it is topologically transitive. This is a consequence of Baire category theorem.
For $T \in \mathcal{L}(X)$, we say

**Definitions**

- $T$ *topologically mixing* $\iff \forall U, V \neq \emptyset$ open, $\exists n_0 : T^n U \cap V \neq \emptyset \forall n \geq n_0$.

Every topologically mixing operator is transitive.

**Definition (Godefroy, Shapiro)**

$T$ is *chaotic* if

- $T$ has a dense set of periodic points,
- $T$ is hypercyclic.
If an operator $T$ is power bounded, then it is not transitive.

Every power bounded operator on a Fréchet or (LB)-space in which the bounded sets are compact is uniformly mean ergodic, as a consequence of Yosida’s mean ergodic theorem.

A. Peris has constructed mixing, hence hypercyclic, operators on a Banach space that are even uniformly mean ergodic. The example is a variation of a recent example due to F. Martínez, Oprocha and Peris that appeared in 2013 in Math. Z.
A description of $A_p(\mathbb{C})$ and $A^0_p(\mathbb{C})$

Let $v : \mathbb{C} \rightarrow ]0, 1]$ be a continuous function such that $v(z) = v(|z|)$, $z \in \mathbb{C}$, $v(r)$ is non-increasing and $\lim_{r \to \infty} v(r)r^k = 0 \ \forall k \in \mathbb{N}$.

The weighted Banach space of entire functions

$$H^0_v(\mathbb{C}) = \{ f \in H(\mathbb{C}) : v|f| \text{ vanishes at infinity} \}$$

endowed by the norm $\|f\|_v = \sup_{z \in \mathbb{C}} v(z)|f(z)|$, $f \in H^0_v(\mathbb{C})$.

For $v(r) = e^{-p(r)}$, $r \geq 0$, $p$ a weight function,

$$A_p(\mathbb{C}) = \text{ind}_n H^0_{v^n}(\mathbb{C}) \text{ and } A^0_p(\mathbb{C}) = \text{proj}_n H^0_{v^{1/n}}(\mathbb{C})$$
Continuity of $D$ and $J$ on $A_p(C)$ and $A_0^p(C)$

**Proposition**

$D$ and $J$ are continuous on $A_p(C)$ and $A_0^p(C)$.

**Remark:**

Given $v(r) = e^{-\alpha p(r)}$, $p(r) = r^a$, $a > 0$, $\alpha > 0$:

- $D$ is not continuous on $H_v^0(C)$ for $a < 1$.
- $J$ is not continuous on $H_v^0(C)$ for $a > 1$. 
The differentiation operator

Lemma
Let $E$ be a locally convex space continuously included in $H(\mathbb{C})$ and assume that there is $a > 1$ such that $e^{az} \in E$. If $D : E \to E$ is continuous, then it is not mean ergodic.

Theorem
(i) If $r = O(p(r))$ as $r \to \infty$, then $D$ is not mean ergodic on $A_p(\mathbb{C})$.
(ii) If $r = o(p(r))$ as $r \to \infty$, then $D$ is not mean ergodic on $A^0_p(\mathbb{C})$.
(iii) If $p(r) = o(r)$ as $r \to \infty$, then $D$ is power bounded, hence uniformly mean ergodic and not hypercyclic on $A_p(\mathbb{C})$ and on $A^0_p(\mathbb{C})$.
Idea of the **Proof** of (iii) for $A_p(\mathbb{C})$: Assume $p(r) = o(r)$

The Cauchy inequalities imply

$$\forall m \quad \forall 0 < \alpha < 1 \quad \exists k \quad \exists C > 0 \quad \forall n \quad \forall z \in \mathbb{C}:$$

$$e^{-kp(|z|)}|f^{(n)}(z)| \leq C\alpha^n \sup_{z \in \mathbb{C}} e^{-mp(|z|)}|f(z)|.$$

This implies $(D^n(f))_n$ converges to 0 in $A_p(\mathbb{C})$ for each $f \in A_p(\mathbb{C})$. 
The differentiation operator

**Lemma**

For $v(r) = e^{-\alpha r}$, $\alpha > 1$, $D$ is topologically mixing and chaotic on $H^0_v(\mathbb{C})$.

**Theorem**

(i) If $p(r) = o(r - \frac{1}{2} \log(r))$ as $r \to \infty$, then $D$ is not hypercyclic on $A_p(\mathbb{C})$.

(ii) If $r = O(p(r))$ as $r \to \infty$, then $D$ is topologically mixing and has a dense set of periodic points on $A_p(\mathbb{C})$.

(iii) If $r = o(p(r))$ as $r \to \infty$, then $D$ is topologically mixing and has a dense set of periodic points on $A^0_p(\mathbb{C})$. 

José Bonet

Two questions about radial Hörmander algebras of entire functions
Corollary

Let \( p_a(r) = r^a, \ a > 0 \):

(i) If \( a > 1 \), then \( D \) is topologically mixing, chaotic and not mean ergodic on \( A_{p_a}(\mathbb{C}) \) and on \( A^0_{p_a}(\mathbb{C}) \).

(ii) If \( a < 1 \), then \( D \) is power bounded, hence uniformly mean ergodic on \( A_{p_a}(\mathbb{C}) \) and \( A^0_{p_a}(\mathbb{C}) \).

(iii) If \( a = 1 \), then \( D \) is topologically mixing, chaotic and not mean ergodic on \( A_{p_1}(\mathbb{C}) \), and it is power bounded on \( A^0_{p_1}(\mathbb{C}) \).
The integration operator

**Proposition**

\[ J \] is not hypercyclic on the Hörmander algebras \( A_p(\mathbb{C}) \) nor \( A_p^0(\mathbb{C}) \).

**Theorem**

(i) The operator of integration is power bounded and hence uniformly mean ergodic on \( A_p(\mathbb{C}) \), provided that \( r = O(p(r)) \) as \( r \to \infty \).

(ii) If \( p(r) = o(r) \) as \( r \to \infty \), then \( J \) is not mean ergodic on \( A_p(\mathbb{C}) \).

(iii) \( J \) is power bounded and hence uniformly mean ergodic on \( A_p^0(\mathbb{C}) \) provided that \( r = o(p(r)) \) as \( r \to \infty \).

(iv) If \( p(r) = O(r) \) as \( r \to \infty \), then \( J \) is not mean ergodic on \( A_p^0(\mathbb{C}) \).
The integration operator

Corollary

Let \( p_a(r) = r^a, \ a > 0 \):

(i) \( J \) is power bounded on \( A_{p_a}(\mathbb{C}) \) for \( a \geq 1 \), and it is not mean ergodic for \( a < 1 \).

(ii) \( J \) is power bounded on \( A_{p_a}^0(\mathbb{C}) \) for \( a > 1 \) and it is not mean ergodic for \( a \leq 1 \).
Fix $T \in \mathcal{L}(X)$.

The **resolvent set** $\rho(T)$ of $T$ consists of all $\lambda \in \mathbb{C}$ such that $R(\lambda, T) := (\lambda I - T)^{-1}$ exists in $\mathcal{L}(X)$.

The **spectrum** of $T$ is the set $\sigma(T) := \mathbb{C} \setminus \rho(T)$. 
The spectrum of $D$

**Theorem**

(i) If $r = O(p(r))$ (resp. $r = o(p(r))$) as $r \to \infty$, then the spectrum of $D$ in $A_p(\mathbb{C})$ (resp. in $A^0_p(\mathbb{C})$) is $\mathbb{C}$.

(ii) If $p(r) = o(r)$ as $r \to \infty$, then the spectrum of $D$ in $A_p(\mathbb{C})$ and in $A^0_p(\mathbb{C})$ reduces to $\{0\}$. 

José Bonet

Two questions about radial Hörmander algebras of entire functions
The spectrum of $D$

Idea of the proof

(i) If $r = O(p(r))$ (resp. $r = o(p(r))$) as $r \to \infty$, then $e^{az} \in A_p$ (resp. $e^{az} \in A^0_p$) for every $a \in \mathbb{C}$, and each complex number is an eigenvalue of the operator.

(ii) Assume now $p(r) = o(r)$ as $r \to \infty$, and the case of $A_p(\mathbb{C})$. It can be shown that, for each $b \in \mathbb{C}$ and every $f \in A_p$, the series

$$\sum_{n=0}^{\infty} b^n D^n f$$

converges in $A_p$. This implies that $al - D$ in invertible in $A_p$ whenever $a \neq 0$. 

José Bonet
Two questions about radial Hörmander algebras of entire functions
The spectrum of $J$

**Theorem**

(i) If $p(r) = o(r)$ (resp. $p(r) = O(r)$) as $r \to \infty$, the spectrum of $J$ in $A_p(\mathbb{C})$ (resp. in $A^0_p(\mathbb{C})$) is $\mathbb{C}$.

(ii) If $r = O(p(r))$ (resp. $r = o(p(r))$) as $r \to \infty$), the spectrum of $J$ in $A_p(\mathbb{C})$ (resp. in $A^0_p(\mathbb{C})$) reduces to $\{0\}$.

**Idea:**

(i) $Jf - \lambda f = 1$ has no solution for $\lambda \in \mathbb{C}$, since the exponentials do not belong to the space.

(ii) $Jf = 1$ has no solution and for $\lambda \neq 0$, $\sum_{n \geq 0} (\frac{J}{\lambda})^n$ converges, and so, $(\lambda I - J)^{-1}$ exists.