Abel's functional equation and eigenvalues of composition operators on spaces of real analytic functions

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Aim of the talk

- $\varphi : \mathbb{R} \to \mathbb{R}$ is a non-constant real analytic map.
- $\mathscr{A}(\mathbb{R})$ denotes the space of real analytic functions defined on $\mathbb{R}$.
- Each symbol $\varphi : \mathbb{R} \to \mathbb{R}$ defines a composition operator $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ by $C_\varphi(f) := f \circ \varphi, f \in \mathscr{A}(\mathbb{R})$.
- When $\mathscr{A}(\mathbb{R})$ is endowed with its natural locally convex topology (see below), C_φ is a continuous linear operator on $\mathscr{A}(\mathbb{R})$.

Our purpose is to determine the eigenvalues and eigenvectors of composition operators $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$.

Schröder equation

Schröder equation

The problem is to find a real analytic solution $f \in \mathscr{A}(\mathbb{R})$ of the equation

$$C_\varphi(f) = \lambda f \qquad \text{for } \lambda \in \mathbb{C}. \tag{1}$$

The equation appeared probably for the first time already in 1871 in a paper of **Schröder** and was partially solved in 1884 in a paper of **Königs** also for real analytic functions.

Notation:

- $\mathrm{id}\,(x) = x, x \in \mathbb{R}$, and $I : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is the identity operator on $\mathscr{A}(\mathbb{R})$.
- For a map $\varphi : \mathbb{R} \to \mathbb{R}$, we write $\varphi^{[0]} = \mathrm{id}$ and $\varphi^{[n]}$ for the n-times composition of φ, $n \in \mathbb{N}$.

The space $\mathscr{A}(\mathbb{R})$. Martineau

- The space $\mathscr{A}(\mathbb{R})$ is equipped with the unique locally convex topology such that for any $U \subset \mathbb{C}$ open, $\mathbb{R} \subset U$, the restriction map $R : H(U) \longrightarrow \mathscr{A}(\mathbb{R})$ is continuous and for any compact set $K \subset \mathbb{R}$ the restriction map $r : \mathscr{A}(\mathbb{R}) \longrightarrow H(K)$ is continuous. In fact,

$$\mathscr{A}(\mathbb{R}) = \mathrm{proj}_{N\in\mathbb{N}}\ H([-N,N]) = \mathrm{proj}_{N\in\mathbb{N}}\ \mathrm{ind}_{n\in\mathbb{N}}\ H^{\infty}(U_{N,n}).$$

- $(f_n)_{n\in\mathbb{N}}$ in $\mathscr{A}(\mathbb{R})$ tends to f if and only if there is a complex neighbourhood W of $\mathbb{R}$ such that each f_n and f extend to W and $f_n \to f$ uniformly on compact subsets of W.

- $\mathscr{A}(\mathbb{R})$ is complete, separable, bounded sets are relatively compact and it satisfies the assumptions of the open mapping and the closed graph theorems. **Domański, Vogt, 2000,** proved that the space $\mathscr{A}(\mathbb{R})$ has no Schauder basis.

Spectrum and point spectrum

Let T be a continuous linear operator on a locally convex space E

- The kernel and image of T are denoted respectively by $\ker T$ and $\operatorname{im} T$.
- The **point spectrum** $\sigma_p(T)$ of T is the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not injective. Elements of $\sigma_p(T)$ are called **eigenvalues** of T. The **eigenspace** of $\lambda \in \sigma_p(T)$ is $\ker(T - \lambda I)$.
- The **spectrum** $\sigma(T)$ of T is the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not a topological isomorphism from E onto E. By the open mapping theorem, $\lambda \notin \sigma(C_\varphi)$ if and only if $C_\varphi - \lambda I : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is bijective.

A (rather trivial) apéritif

Proposition 1

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a non-constant real analytic map.

- 0 is never an eigenvalue of C_φ. In particular, C_φ is injective.
- 1 is always an eigenvalue of C_φ and the constant functions are eigenvectors.
- C_φ is surjective if and only if it is bijective if and only if $\varphi : \mathbb{R} \to \mathbb{R}$ is bijective and its inverse is real analytic, i.e. φ is a real analytic diffeomorphism.
- $0 \in \sigma(C_\varphi)$ if and only if φ is not a real analytic diffeomorphism.

Another related result

Proposition 2

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic map.

(a) im C_φ is dense in $\mathscr{A}(\mathbb{R})$ if and only if φ has no critical points.

(b) (Domanski, Langenbruch) The following conditions are equivalent for any non-constant φ:

- φ is surjective;
- $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is an isomorphism onto its image;
- im C_φ is closed.

Self map with fixed points

Theorem 1

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic function with a fixed point u and let us consider the map $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$.

(a) If $\varphi'(u) = 1$, then 1 is the only eigenvalue and

- (i) either $\varphi = \text{id}$ and in this case the eigenspace of C_φ is equal to $\mathscr{A}(\mathbb{R})$
- (ii) or $\varphi \neq \text{id}$ and the eigenspace is one dimensional.

(b) If $\varphi'(u) = -1$ then

- (i) either $\varphi^{[2]} = \text{id}$ but $\varphi \neq \text{id}$ and in this case there are two eigenvalues ± 1 and $\mathscr{A}(\mathbb{R})$ is a direct sum of two eigenspaces
- (ii) or $\varphi^{[2]} \neq \text{id}$, 1 is the only eigenvalue and its eigenspace is one-dimensional.

Self map with fixed points

Theorem 1 continued

(c) If $\varphi'(u) = 0$ then 1 is the only eigenvalue and its eigenspace is one-dimensional.

(d) If $0 < |\varphi'(u)| < 1$ then

 (i) either $\varphi^{[2]}$ has at least two fixed points and then 1 is the only eigenvalue and its eigenspace is one-dimensional

 (ii) or $((\varphi'(u))^n)_{n\in\mathbb{N}}$ is the sequence of eigenvalues and all of them have one-dimensional eigenspaces.

(e) If $1 < |\varphi'(u)|$ then

 (i) either $\varphi^{[2]}$ has at least two fixed points or φ has a critical point and then in both cases 1 is the only eigenvalue and its eigenspace is one-dimensional

 (ii) or $((\varphi'(u))^n)_{n\in\mathbb{N}}$ is the sequence of eigenvalues and all of them have one-dimensional eigenspaces.

Self map with fixed points

There is an impressive bibliography related to Theorem 1; in particular books by **Kuczma** 1968 and by **Kuczma, Choczewski and Ger** 1990. There are also many papers about the holomorphic case, e.g. by **Cowen** 1981, by **Shapiro** 1996.

Parts of Theorem 1 are certainly well-known. **Königs** already knew that for a fixed point u the eigenvalues are powers of $\varphi'(u)$, and he also dealt with the local existence of eigenvalues. Further results were proved by **Kneser** 1949 and **Smajdor** 1967.

A full description for real analytic functions, together with the isomorphic classification of the eigenspaces seemed to be missing.

Self map with fixed points. Ingredients of the proof

Lemma 1

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic map with a fixed point $u \in \mathbb{R}$. Then either

(1) $\varphi^{[2]} = \mathrm{id}$ or

(2) There is a convergent sequence $(x_n)_n$ in $\mathbb{R}$ such that for each $n \neq k$ we have $x_n \neq x_k$ and there is m such that $\varphi^{[m]}(x_n) = x_k$ or $\varphi^{[m]}(x_k) = x_n$.

For the proof, one distinguishes cases depending on the value of $\varphi'(u)$.

Self map with fixed points. Ingredients of the proof.

Proposition 3

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic map with a fixed point $u \in \mathbb{R}$.

(1) If -1 is an eigenvalue of C_φ, then $\varphi^{[2]} = \mathrm{id}$.

(2) If $\varphi^{[2]} = \mathrm{id}$ but $\varphi \neq \mathrm{id}$, then C_φ has only two eigenvalues 1 and -1 and $\varphi'(u) = -1$. In this case each $f \in \mathscr{A}(\mathbb{R})$ can be decomposed as $f = f_1 + f_2$, where f_1 (resp. f_2) is an eigenvector with eigenvalue 1 (resp. -1). In this case both eigenspaces are isomorphic to the space of even real analytic functions $\mathscr{A}_+(\mathbb{R})$ on $\mathbb{R}$.

Proof of Part (1).

Let $f \in \mathscr{A}(\mathbb{R})$ be an eigenvector of C_φ for the eigenvalue -1. Then $f(\varphi(x)) = -f(x)$ for each $x \in \mathbb{R}$.

Proceeding by contradiction, if $\varphi^{[2]} \neq \mathrm{id}$, we apply the Lemma above to find a convergent sequence of pairwise different points $(x_n)_n$ such that $f(x_n) = f(x_1)$ or $f(x_n) = -f(x_1)$.

Passing to a subsequence, it follows that the real analytic function f is constant. As $f(\varphi(x)) = -f(x)$, this constant value must be 0; a contradiction.

Proposition 4

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic map with a fixed point $u \in \mathbb{R}$ such that $\varphi^{[2]} \neq \mathrm{id}$. Then the only possible eigenvalues λ of C_φ are of the form $\lambda = (\varphi'(u))^n$ for some $n \in \mathbb{N}$. All of them have at most one dimensional eigenspace consisting of functions f with zero of order n at u.

Proof of the first part. Assume that f is an eigenvector of C_φ with the eigenvalue $\lambda \in \mathbb{C} \setminus \{0, 1\}$. As $f(u) = f(\varphi(u)) = \lambda f(u)$, we get $f(u) = 0$, and on a neighbourhood of u we have

$$f(z) = a_n(z-u)^n + a_{n+1}(z-u)^{n+1} + \ldots, \qquad a_n \neq 0.$$

Thus, on this neighbourhood we have

$$\lambda = \frac{f(\varphi(z))}{f(z)} = \left(\frac{\varphi(z)-u}{z-u}\right)^n \frac{a_n + a_{n+1}(\varphi(z)-u) + \ldots}{a_n + a_{n+1}(z-u) + \ldots}.$$

Letting z tend to u the right hand side converges to $(\varphi'(u))^n$.

Self map with fixed points. Examples

Examples

- $\varphi(x) = ax, a \notin \{0, 1\}$. Eigenvalues $\{a^n, n = 0, 1, 2, ...\}$, with eigenvector x^n of a^n.
- $\varphi(x) = x^n, n = 2, 3, 4, ...$. In this case $\varphi^{[2]}$ has at least two fixed points, and 1 is the only eigenvalue and its eigenspace is one-dimensional.
- $\varphi(x) = \sin(ax), a > 0$. If $0 < a \leq 1$, then 0 is the only fixed point of φ and $\varphi'(0) = a$; hence $(a^n)_{n\in\mathbb{N}}$ is the sequence of eigenvalues and all of them have one-dimensional eigenspaces. If $a > 1$, then φ has 3 fixed points and 1 is the only eigenvalue and its eigenspace is one-dimensional.
- $\varphi(x) = e^x - 1$. In this case 0 is the only fixed point of φ and $\varphi'(0) = 1$, 1 is the only eigenvalue and the eigenspace is one dimensional.

It follows from Theorem 1 that the values $(\varphi'(u))^n$ are sometimes eigenvalues and sometimes they are not. They are always elements of the spectrum by a result that is proved with a technique due to Hammond, 2003.

Proposition 5

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic function and let $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ be the associated composition operator. If u is a fixed point of φ such that $|\varphi'(u)| \neq 1, 0$, then $\varphi'(u)^n \in \sigma(C_\varphi)$ for each $n \in \mathbb{N}_0$.

If $\varphi(x) = x^4$, then $4^n \in \sigma(C_\varphi) \setminus \sigma_p(C_\varphi)$ for all $n \in \mathbb{N}$.

Proof. If $n = 0$, then $\varphi'(u)^n = 1 \in \sigma_p(C_\varphi)$. Fix $n \in \mathbb{N}$. Proceeding by contradiction, assume that there is $f \in \mathscr{A}(\mathbb{R})$ such that

$$f(\varphi(x)) - \varphi'(u)^n f(x) = (x-u)^n, \qquad x \in \mathbb{R}.$$

Since $|\varphi'(u)| \neq 1$, $f(u) = 0$. Suppose by induction that $f^{(k)}(u) = 0, 0 \leq k \leq j-1$. Taking the j-th derivative in the equality above, $j < n$, we get

$$0 = \varphi'(u)^j (1 - \varphi'(u)^{n-j}) f^{(j)}(u),$$

hence $f^{(j)}(u) = 0$. Now taking the n-th derivative we reach a contradiction:

$$0 \neq \frac{d^n}{dx^n}((x-u)^n)\Big|_{x=u} = \frac{d^n}{dx^n}(f(\varphi(x)))\Big|_{x=u} - \varphi'(u)^n f^{(n)}(u) =$$

$$= \varphi'(u)^n f^{(n)}(u) - \varphi'(u)^n f^{(n)}(u) = 0.$$

Self map without fixed points and the Abel equation

Abel equation

$$f(\varphi(x)) = f(x) + 1.$$

Clearly, if φ has a fixed point, there is no solution of the Abel equations.

If φ has no fixed points, then either $\varphi > \mathrm{id}$ or $\varphi < \mathrm{id}$.

The Abel equation is another classical subject. It was probably mentioned for the first time by Abel in a note published posthumously. There is also a broad literature about the equation in various function classes.

Self map without fixed points and the Abel equation

- The Abel equation was solved in real analytic functions globally on $\mathbb{R}$ for $\varphi = \exp$ by **Kneser in 1949**.
- Belitskii and Lyubich obtained in 1999 a characterization of **real analytic diffeomorphisms** φ for which the Abel equation is solvable (iff φ has no fixed point).
- In 1998 they had shown that a necessary condition for real analytic solvability of the Abel equation is that all compact sets $K \subset \mathbb{R}$ are **wandering**, i.e., that there is $\nu \in \mathbb{N}$ such that for $n, m \in \mathbb{N}$, $|n - m| > \nu$ holds $\varphi^{[n]}(K) \cap \varphi^{[m]}(K) = \emptyset$.

A consequence of Belitskii and Lyubich's result

Theorem. Belitskii and Lyubich, 1999

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic diffeomorphism without fixed points. Then φ is real analytic conjugate to the shift $x \to x + 1$.

Corollary

Let $\varphi : I \to I$ be a real analytic diffeomorphism without fixed points on an open interval I in $\mathbb{R}$. Then $C_\varphi : \mathscr{A}(I) \to \mathscr{A}(I)$ is a hypercyclic operator; i.e. there is $f \in \mathscr{A}(I)$ with a dense orbit in $\mathscr{A}(I)$.

In particular, if $\varphi(x) = x^2$ in $I =]0, 1[$, then C_φ is hypercyclic. This solves a problem asked by Bonet and Domański in 2012, that had been answered already by A. Peris in a different way.

Abel equation and eigenvalues of C_φ

Proposition K1. Kneser, 1949

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic map such that the Abel equation $f \circ \varphi = f + 1$ has a real analytic solution f_0. Then each $\lambda \in \mathbb{C} \setminus \{0\}$ is an eigenvalue of $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ and this operator has an infinite dimensional eigenspace for the eigenvalue λ. Moreover, for every $\lambda \neq 0$ there is an eigenvector f which does not vanish at any point.

Proof. Observe first that the function f_0 cannot be constant. Let p be a periodic function with period 1 and define $f := p \circ f_0$. We have

$$C_\varphi(f)(x) = f(\varphi(x)) = (p \circ f_0)(\varphi(x)) = p(f_0(x) + 1) = p(f_0(x)) = f(x).$$

Thus $C_\varphi(f) = f$. The infinite dimensionality follows varying p. This settles the case $\lambda = 1$.

Take now $\lambda \in \mathbb{C} \setminus \{0, 1\}$. Select a complex number μ such that $e^\mu = \lambda$. Set $G(x) := \exp(\mu f_0(x)), x \in \mathbb{R}$. We have

$$C_\varphi(G)(x) = G(\varphi(x)) = \exp(\mu f_0(\varphi(x))) = \exp(\mu(f_0(x) + 1)) = \lambda G(x).$$

Hence G is an eigenvector of C_φ with eigenvalue λ.

If $F \in \mathscr{A}(\mathbb{R})$ is a fixed point of C_φ, we get $C_\varphi(FG) = \lambda FG$. This implies that the eigenspace of the eigenvector λ is also infinite dimensional.

Proposition K2. Kneser, 1949

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic map such that some $\lambda \in \mathbb{C} \setminus \{0, 1\}$ is an eigenvalue of $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ with a never vanishing eigenvector $f_0 \in \mathscr{A}(\mathbb{R})$. Then the Abel equation $f \circ \varphi = f + 1$ has a real analytic solution f.

Proof. Clearly f_0 extends to a non vanishing holomorphic function on some one-connected complex neighbourhood U of $\mathbb{R}$. Thus $f_0(x) = \exp(h(x))$ for some holomorphic function h on U (so the restriction of h to $\mathbb{R}$ is real analytic).

Select a complex number μ such that $e^\mu = \lambda$. Since $f_0(\varphi(x)) = \lambda f_0(x), x \in \mathbb{R}$, we have:

$$\exp(h(\varphi(x)) = \exp(\mu + h(x)).$$

Since $\lambda \neq 1$ we have for some $k \in \mathbb{Z}$

$$h(\varphi(x)) = h(x) + \mu + 2k\pi i, \qquad \text{where } \mu + 2k\pi i \neq 0.$$

Then $f(x) := \frac{1}{\mu+2k\pi i} h(x)$, is the required solution of the Abel equation.

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic map.

- For any $x \in \mathbb{R}$ we denote by $O(x)$ the **full orbit** of x via φ, i.e.,

  $$O(x) := \{y : \ \exists\, k, l \in \mathbb{N} : \varphi^{[k]}(x) = \varphi^{[l]}(y)\}.$$

  The full orbits form a partition of $\mathbb{R}$.

- The quotient topological space with respect to that partition is denoted by $\mathbb{R}/\varphi$ and the corresponding (continuous) canonical quotient map by $\pi_\varphi : \mathbb{R} \to \mathbb{R}/\varphi$.

- Our study of the natural manifold structure on $\mathbb{R}/\varphi$ for non-diffeomorphic φ is inspired by the method presented by Belitskii and Lyubich in 1999, but it requires further analysis and work.

Main Result

MAIN Theorem

The following assertions are equivalent for a real analytic function $\varphi : \mathbb{R} \to \mathbb{R}$:

(a) Every complex $\lambda \neq 0$ is an eigenvalue of C_φ with at least one real analytic eigenvector non-vanishing at any point.

(a') Every complex $\lambda \neq 0$ is an eigenvalue of C_φ with an infinite dimensional eigenspace.

(b) There is a complex eigenvalue $\lambda \neq 1$ for C_φ with at least one real analytic eigenvector non-vanishing at any point.

(b') There is a complex eigenvalue $\lambda \neq 1$ for C_φ and φ has no fixed point.

(c) There is a non-constant eigenvector for the eigenvalue 1 and $\varphi^{[2]} \neq \mathrm{id}$.

Main Result continued

MAIN Theorem continued

The following assertions are equivalent to (a)-(c):

(d) The space $\mathbb{R}/\varphi$ of full orbits of φ is a manifold homeomorphic to $\mathbb{T}$ which has a real analytic structure making the canonical map $\pi_\varphi : \mathbb{R} \to \mathbb{R}/\varphi$ real analytic (and, of course, then $\mathbb{R}/\varphi$ is real analytic diffeomorphic to $\mathbb{T}$).

(e) Either $\varphi > \mathrm{id}$ and the set of critical points of φ is bounded from above or $\varphi < \mathrm{id}$ and the set of critical points of φ is bounded from below.

(f) The Abel equation $f \circ \varphi = f + 1$ has a real analytic solution f.

Main Result continued again

MAIN Theorem continued

If these conditions hold then for $\lambda > 0$ there is at least one strictly positive eigenvector. Moreover, there is a real analytic solution f_0 of the Abel equation with real values such that the set of critical points is bounded from above (in case $\varphi > \mathrm{id}$) or bounded from below (in case $\varphi < \mathrm{id}$).

In that case for every complex $\lambda \neq 0$, $e^{\mu} = \lambda$, the map

$$T_\lambda : \mathscr{A}(\mathbb{T}) \to \ker(C_\varphi - \lambda I), \qquad T_\lambda(g) := [\exp \circ(\mu f_0)] \cdot [g \circ q \circ f_0]\,,$$

is a topological isomorphism of $\mathscr{A}(\mathbb{T})$ onto the eigenspace of C_φ for λ (here $q : \mathbb{R} \to \mathbb{T}$, $q(x) := \exp(2\pi i x)$).

Self map without fixed points. Examples

Examples

- $\varphi(x) = e^{ax}, a > 1/e$ has no fixed points. Abel equation $f(\varphi(x)) = f(x) + 1$ has a real analytic solution $f \in \mathscr{A}(\mathbb{R})$. In this case $\sigma_p(C_\varphi) = \mathbb{C} \setminus \{0\}$ and $\sigma(C_\varphi) = \mathbb{C}$.

- $\varphi(x) = x + 1 + a\sin(a^{-1}x), 0 < a < 1$, is real analytic, it is a (continuous) homeomorphism on $\mathbb{R}$, has no fixed points, but it has an unbounded sequence of critical points, namely $\varphi'(x) = 0$ if and only if $x = 2s\pi a, s \in \mathbb{Z}$. By our main Theorem, Abel equation $f(\varphi(x)) = f(x) + 1$ has no real analytic solution $f \in \mathscr{A}(\mathbb{R})$. The only eigenvalue of C_φ is 1.

- (Belitskii, Lyubich, 1999) If φ is a real analytic diffeomorphism without fixed points, then Abel equation $f(\varphi(x)) = f(x) + 1$ has a real analytic solution $f \in \mathscr{A}(\mathbb{R})$, and $\sigma_p(C_\varphi) = \sigma(C_\varphi) = \mathbb{C} \setminus \{0\}$.

Main Result. Ingredients of the proof

About the equivalences in the proof of the main Theorem:

- (a)$\Rightarrow$(b) is obvious.
- The implication (b)$\Rightarrow$(f) follows from Proposition K2.
- (f)$\Rightarrow$(a) follows from Proposition K1.
- This completes the proof of (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (f).

It remains to prove that (f) is equivalent to (d) and (e).

Main Result (d), (e), (f)

MAIN Theorem

(d) The space $\mathbb{R}/\varphi$ of full orbits of φ is a manifold homeomorphic to $\mathbb{T}$ which has a real analytic structure making the canonical map $\pi_\varphi : \mathbb{R} \to \mathbb{R}/\varphi$ real analytic (and, of course, then $\mathbb{R}/\varphi$ is real analytic diffeomorphic to $\mathbb{T}$).

(e) Either $\varphi > \mathrm{id}$ and the set of critical points of φ is bounded from above or $\varphi < \mathrm{id}$ and the set of critical points of φ is bounded from below.

(f) The Abel equation $f \circ \varphi = f + 1$ has a real analytic solution f.

(f)$\Rightarrow$(e) is a consequence of Corollary 1 or Lemma 2 below.

Lemma 2

Let $\lambda \in \mathbb{C} \setminus \{0\}$, $r \in \mathbb{C}$, $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic function satisfying $\varphi > \mathrm{id}$. If there is a real analytic non-constant function $f : \mathbb{R} \to \mathbb{C}$ which solves the equation:

$$f \circ \varphi = \lambda f + r,$$

then the set of critical points of φ is bounded from above; in particular, φ is strictly increasing from some point on.

Main Result. Ingredients of the proof. (f)$\Rightarrow$(e)

Corollary 1

If φ has no fixed point but the set of critical points is unbounded from above (if $\varphi > \text{id}$) or from below (if $\varphi < \text{id}$), then the only eigenvalue of $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is 1 and the corresponding eigenspace consists of constant functions only. Moreover, the Abel equation $f \circ \varphi = f + 1$ has no real analytic solution $f \in \mathscr{A}(\mathbb{R})$.

Main Result. Ingredients of the proof

Recall that we define, for a real analytic map $\varphi : \mathbb{R} \to \mathbb{R}$,

For $x \in \mathbb{R}$ we denote by $O(x)$ the **full orbit** of x via φ, i.e.,

$$O(x) := \{y : \ \exists\ k, l \in \mathbb{N} : \varphi^{[k]}(x) = \varphi^{[l]}(y)\}.$$

The full orbits form a partition of $\mathbb{R}$.

The quotient topological space with respect to that partition is denoted by $\mathbb{R}/\varphi$ and the corresponding (continuous) canonical quotient map by $\pi_\varphi : \mathbb{R} \to \mathbb{R}/\varphi$. The space $\mathbb{R}/\varphi$ need not be Hausdorff. It is compact if φ has no fixed points.

Main Result (d), (e), (f)

MAIN Theorem

(d) The space $\mathbb{R}/\varphi$ of full orbits of φ is a manifold homeomorphic to $\mathbb{T}$ which has a real analytic structure making the canonical map $\pi_\varphi : \mathbb{R} \to \mathbb{R}/\varphi$ real analytic (and, of course, then $\mathbb{R}/\varphi$ is real analytic diffeomorphic to $\mathbb{T}$).

(e) Either $\varphi > \mathrm{id}$ and the set of critical points of φ is bounded from above or $\varphi < \mathrm{id}$ and the set of critical points of φ is bounded from below.

(f) The Abel equation $f \circ \varphi = f + 1$ has a real analytic solution f.

(e)⇒(d) is a consequence of the following Lemma.

Lemma 3

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be real analytic and $\varphi > \mathrm{id}$. If the set of critical points of φ is bounded from above, then $\mathbb{R}/\varphi$ is homeomorphic to the circle $\mathbb{T}$ and there is a real analytic structure on $\mathbb{R}/\varphi$ which makes it diffeomorphic to $\mathbb{T}$ and makes the canonical map $\pi_\varphi : \mathbb{R} \to \mathbb{R}/\varphi$ real analytic, such that its set of critical points coincides with the set of critical points of all the maps $\varphi^{[n]}$ for $n \in \mathbb{N}$.

The proof relies on some results about 1-manifolds and algebraic topology.

MAIN Theorem

(d) The space $\mathbb{R}/\varphi$ of full orbits of φ is a manifold homeomorphic to $\mathbb{T}$ which has a real analytic structure making the canonical map $\pi_\varphi : \mathbb{R} \to \mathbb{R}/\varphi$ real analytic (and, of course, then $\mathbb{R}/\varphi$ is real analytic diffeomorphic to $\mathbb{T}$).

(e) Either $\varphi > \mathrm{id}$ and the set of critical points of φ is bounded from above or $\varphi < \mathrm{id}$ and the set of critical points of φ is bounded from below.

(f) The Abel equation $f \circ \varphi = f + 1$ has a real analytic solution f.

(d)⇒(f) follows from a result on 1-manifolds and Lemma 4.

Lemma 4

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic map such that $\mathbb{R}/\varphi$ is a real analytic manifold with $\pi_\varphi : \mathbb{R} \to \mathbb{R}/\varphi$ real analytic. If $d : \mathbb{R}/\varphi \to \mathbb{T}$ is a real analytic map non-homotopic to a constant map, then the Abel equation $f \circ \varphi = f + 1$ has a real analytic solution f on $\mathbb{R}$ with real values. The solution f has critical points exactly in the critical points of $d \circ \pi_\varphi$ (i.e., f has critical points if and only if d or π_φ has critical points).

The motivation of Kneser for solving the Abel equation comes form the problem of finding an iteration root of the exponential map exp, i.e., of a real analytic function r such that $r^{[2]} = \exp$.

If f is an invertible solution of the Abel or Schröder equations ($\lambda > 0$), then $G(t, x) = f^{-1}(f(x) + t)$ or $G(t, x) = f^{-1}(\lambda^t f(x))$, respectively, is a so-called **real analytic iteration semigroup** in which φ embeds.

We say that φ **embeds in the real analytic iteration semigroup** G if G is real analytic satisfying the following conditions

$$G : (\mathbb{R}_+ \cup \{0\}) \times \mathbb{R} \to \mathbb{R},$$
$$G(t+s,x) = G(t, G(s,x)), \quad G(n,x) = \varphi^{[n]}(x), \text{for } n = 0, 1, \dots.$$

Clearly, in this case $r(x) = G(1/2, x)$ is the required root of φ.

Abel equation and iteration semigroups

Theorem 2

A real analytic map $\varphi : \mathbb{R} \to \mathbb{R}$ embeds into a real analytic iteration semigroup whenever φ has no critical points and either φ has no fixed points or $\varphi^{[2]}$ has exactly one fixed point u and $0 < \varphi'(u) \neq 1$. In particular in that case there exist roots of the operator C_φ of arbitrary order.

It is known that there is no real analytic iteration root for $\varphi(x) = \exp(x) - 1$.

Conjecture

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A real analytic map $\varphi : \mathbb{R} \to \mathbb{R}$ embeds into a real analytic iteration semigroup if and only if it has no critical point, has at most one fixed point u and in that case $0 < \varphi'(u) \neq 1$.

The case we cannot decide is when there is a fixed point u such that $\varphi'(u) = 1$. Important work related to this topic is due e.g. to I.N. Baker, M. Kuczma, G. Szekeres and P.L. Walker.

References

- **G. Belitskii and Yu Lyubich,** The real analytic solutions of the Abel functional equation, Studia Math. 127 (1998), 81-97.
- **J. Bonet and P. Domański,** Abel's functional equation and eigenvalues of composition operators on spaces of real analytic functions, Integral Equations and Operator Theory (to appear).