SPECTRUM AND COMPACTNESS OF THE CESÀRO OPERATOR ON WEIGHTED ℓ_p SPACES

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ABSTRACT. An investigation is made of the continuity, the compactness and the spectrum of the Cesàro operator C when acting on the weighted Banach sequence spaces $\ell_p(w)$, $1 < p < \infty$, for a positive, decreasing weight w, thereby extending known results for C when acting on the classical spaces ℓ_p.

1. INTRODUCTION

The discrete Cesàro operator C is defined on the linear space $\mathbb{C}^{\mathbb{N}}$ (consisting of all scalar sequences) by

$$\mathsf{C}x := \left(x_1, \frac{x_1 + x_2}{2}, \ldots, \frac{x_1 + \ldots + x_n}{n}, \ldots\right), \quad x = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}. \tag{1.1}$$

The operator C is said to act in a vector subspace $X \subseteq \mathbb{C}^{\mathbb{N}}$ if it maps X into itself. Of particular interest is the situation when X is a Banach space. The fundamental questions in this case are: Is $\mathsf{C}\colon X \to X$ continuous and, if so, what is the spectrum of $\mathsf{C}\colon X \to X$? Amongst the classical Banach spaces $X \subseteq \mathbb{C}^{\mathbb{N}}$ where answers are known we mention ℓ_p $(1 < p < \infty)$, [6], [14], and c_0, [14], [18], and both c, ℓ_∞, [1], [14], as well as ces_p, $p \in \{0\} \cup (1, \infty)$, [8], and the spaces of bounded variation bv_0, [17], and bv_p, $1 \leq p < \infty$, [2]. There is no claim that this list of spaces (and references) is complete.

The aim of this paper is to investigate the two questions mentioned above for C acting on the weighted Banach spaces $\ell_p(w)$. To be precise, let $w = (w(n))_{n=1}^\infty$ be a bounded sequence, always assumed to be *strictly* positive. Define the space

$$\ell_p(w) := \left\{ x = (x_n)_{n\in N} \in \mathbb{C}^{\mathbb{N}} \colon \|x\|_{p,w} := \left(\sum_{n=1}^\infty |x_n|^p w(n)\right)^{1/p} < \infty \right\},$$

for each $1 < p < \infty$, equipped with the norm $\|\cdot\|_{p,w}$. Observe that $\ell_p(w)$ is isometric to ℓ_p via the linear multiplication operator

$$\Phi_w \colon \ell_p(w) \to \ell_p, \quad x = (x_n)_{n\in\mathbb{N}} \to \Phi_w(x) := (w(n)^{1/p} x_n)_{n\in\mathbb{N}}.$$

Therefore, the $\ell_p(w)$ are Banach spaces. The dual space $(\ell_p(w))'$ of $\ell_p(w)$ is the Banach space $\ell_{p'}(v)$, where $\frac{1}{p} + \frac{1}{p'} = 1$ (i.e., p' is the conjugate exponent of p) and $v(n) = w(n)^{-p'/p}$ for $n \in \mathbb{N}$. In particular, $\ell_p(w)$ is reflexive and separable for $1 < p < \infty$. Moreover, the canonical unit vectors $e_k := (\delta_{kn})_{n\in\mathbb{N}}$, for $k \in \mathbb{N}$, form an unconditional basis in $\ell_p(w)$ for $1 < p < \infty$. If $\inf_{n\in\mathbb{N}} w(n) > 0$,

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then $\ell_p(w) = \ell_p$ with equivalent norms and we are in the standard situation. Accordingly, we are mainly interested in the case when $\inf_{n\in\mathbb{N}} w(n) = 0$.

By Hardy's inequality, [13, Theorem 326, p.239], for every $1 < p < \infty$ the restriction of the Cesàro operator $\mathsf{C}\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ as given in (1.1) defines a bounded linear operator from ℓ_p into itself with operator norm equal to p'. Denote these operators by $\mathsf{C}^{(p)}$ so that $\|\mathsf{C}^{(p)}\| = p'$. In Section 2, where the papers [5], [10], [11] are relevant, we discuss various aspects of the continuity of C when restricted to $\ell_p(w)$, $1 < p < \infty$; denote this operator by $\mathsf{C}^{(p,w)}$ whenever it is continuous.

For any Banach space X, let I denote the identity operator on X and $\mathcal{L}(X)$ denote the space of all continuous linear operators from X into itself. The *spectrum* and the *resolvent set* of $T \in \mathcal{L}(X)$ are denoted by $\sigma(T)$ and $\rho(T)$, respectively; see [9, Ch. VII], for example. The set of all *eigenvalues* of T, also called the *point spectrum* of T, is denoted by $\sigma_{pt}(T)$. The *spectral radius* $r(T) := \sup\{|\lambda|\colon \lambda \in \sigma(T)\}$ of T always satisfies $r(T) \leq \|T\|$, [9, p.567].

Section 3 is devoted to an analysis of the spectrum of C when acting in $\ell_p(w)$. The main result is Theorem 3.3; it is complemented by Examples 3.5 which clarify the scope of this theorem. Unlike for $\mathsf{C}^{(p)}$, it can happen that $\sigma_{pt}(\mathsf{C}^{(p,w)}) \neq \emptyset$. Actually, $\mathsf{C}^{(p,w)}$ can even have infinitely many eigenvalues; see Proposition 3.6. The final section deals with the *compactness* of $\mathsf{C}^{(p,w)}$. Relevant is how fast w decreases to 0; see Proposition 4.1, Theorem 4.2, Corollary 4.3 and Proposition 4.6.

2. Continuity of C in weighted ℓ_p spaces

Some of the concepts and results from [11] that are quoted in this section actually have their origins in the paper [10]. We begin with the following fact.

Lemma 2.1. *Let $w = (w(n))_{n=1}^{\infty}$ be a positive sequence and $1 < p < \infty$. Then the Cesàro operator C maps $\ell_p(w)$ continuously into itself if, and only if,*

$$\sup_{m\in\mathbb{N}} \left(\sum_{k=1}^{m} w(k)^{-p'/p}\right)^{-1} \left(\sum_{n=1}^{m} \frac{w(n)}{n^p}\left(\sum_{k=1}^{n} w(k)^{-p'/p}\right)^{p}\right) < \infty,$$

i.e., if, and only if, there exists $K > 0$ such that

$$\sum_{n=1}^{m} \frac{w(n)}{n^p}\left(\sum_{k=1}^{n} w(k)^{-p'/p}\right)^{p} \leq K\left(\sum_{k=1}^{m} w(k)^{-p'/p}\right), \quad m \in \mathbb{N}. \tag{2.1}$$

Moreover, if the constant K satisfying (2.1) *is chosen as small as possible, then the operator norm of C is at most $p'K^{1/p}$.*

Proof. Let $T_w\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ denote the linear operator defined by

$$T_w x := \left(\frac{w(n)^{1/p}}{n}\sum_{k=1}^{n} w(k)^{-1/p} x_k\right)_{n\in\mathbb{N}}, \quad x = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}. \tag{2.2}$$

Then $\Phi_w C = T_w \Phi_w$. Since Φ_w is isometric from $\ell_p(w)$ onto ℓ_p, it follows that C maps $\ell_p(w)$ continuously into itself if, and only if, T_w maps ℓ_p continuously into itself. But, the matrix of T_w is factorable (cf. [5, §4] with $a_n = w(n)^{1/p}/n$ and $b_k = w(k)^{-1/p}$ for $1 \leq k \leq n$) and so it follows from [5, Theorem 2] that $T_w \in \mathcal{L}(\ell^p)$ if, and only if, (2.1) holds.

The proof of Theorem 2 in [5] yields that the operator norm of C is at most $p'K^{1/p}$. $\square$

Proposition 2.2. *Let $w = (w(n))_{n=1}^{\infty}$ be a decreasing, positive sequence and $1 < p < \infty$. Then the Cesàro operator $\mathsf{C}^{(p,w)} \in \mathcal{L}(\ell_p(w))$ and satisfies*

$$1 < \left(\frac{1}{w(1)} \sum_{n=1}^{\infty} \frac{w(n)}{n^p} \right)^{1/p} \leq \|\mathsf{C}^{(p,w)}\| \leq p'. \tag{2.3}$$

Proof. Fix $m \in \mathbb{N}$. Because w is decreasing, we have

$$\begin{aligned} \sum_{n=1}^{m} \frac{w(n)}{n^p} \left(\sum_{k=1}^{n} w(k)^{-p'/p} \right)^p &= \sum_{n=1}^{m} \left(\frac{w(n)^{1/p}}{n} \sum_{k=1}^{n} w(k)^{-p'/p} \right)^p \\ &\leq \sum_{n=1}^{m} \left(\frac{w(n)^{1/p}}{n} \cdot \frac{n}{w(n)^{p'/p}} \right)^p = \sum_{n=1}^{m} w(n)^{-p'/p}, \end{aligned}$$

which is precisely (2.1) with $K = 1$. So, Lemma 2.1 implies that C is continuous on $\ell_p(w)$ with $\|\mathsf{C}^{(p,w)}\| \leq p'$.

For an alternate proof of the continuity of $\mathsf{C}^{(p,w)}$, based directly on Hardy's inequality in ℓ_p, see [11, Proposition 5.1].

Since $T_w = \Phi_w \mathsf{C}^{(p,w)} \Phi_w^{-1}$, with Φ_w mapping the closed unit ball of $\ell_p(w)$ onto that of ℓ_p and Φ_w^{-1} mapping the closed unit ball of ℓ_p onto that of $\ell_p(w)$, it follows that $\|T_w\| = \|\mathsf{C}^{(p,w)}\|$. Of course,

$$\Phi_w^{-1} x = (w(n)^{-1/p} x_n)_{n \in \mathbb{N}}, \quad x \in \ell_p.$$

Substituting $x = e_1$ into (2.2) it follows that

$$\|\mathsf{C}^{(p,w)}\| = \|T_w\| \geq \|T_w e_1\|_p = \left(\frac{1}{w(1)} \sum_{n=1}^{\infty} \frac{w(n)}{n^p} \right)^{1/p} \geq \left(1 + \frac{w(2)}{w(1)2^p} \right)^{1/p} > 1.$$

See also [11, Proposition 5.5]. $\square$

Some comments regarding Proposition 2.2 are in order. As noted above, for each $1 < p < \infty$ we have $\|\mathsf{C}^{(p)}\| = p'$ and, for a positive, decreasing weight w, that (2.3) holds. These estimates are not the best possible in general. Denote by $\delta_p(w)$ the set of all decreasing, non-negative sequences in $\ell_p(w)$ and define

$$\Delta_{p,w}(\mathsf{C}^{(p,w)}) := \sup\{\|\mathsf{C}^{(p,w)} x\|_{p,w} \colon x \in \delta_p(w),\ \|x\|_{p,w} = 1\} \leq \|\mathsf{C}^{(p,w)}\|.$$

The following result follows from Propositions 6.3, 6.5 and 6.6 of [11].

Proposition 2.3. *Let $1 < p < \infty$ and $w(n) = 1/n^{\alpha}$, $n \in \mathbb{N}$, for a fixed $\alpha > 0$. Then*

$$\max\{m_1, m_2\} \leq \Delta_{p,w}(\mathsf{C}^{(p,w)}) \leq \|\mathsf{C}^{(p,w)}\| \leq M_2(r) := [r\zeta(r+\alpha)]^{r/p}, \tag{2.4}$$

for $1 \leq r \leq p$, where $m_1 := p/(p+\alpha-1)$ and $m_2 := \zeta(p+\alpha)^{1/p}$, with ζ denoting the Riemann zeta function. Moreover, for $1 \leq r \leq p$, it is also the case that

$$\|\mathsf{C}^{(p,w)}\| \leq M_3(r) := \left(\frac{p}{p+\alpha-r} \right)^{1/p'} \zeta \left(1 + \frac{r}{p'} + \frac{\alpha}{p} \right)^{1/p}. \tag{2.5}$$

We provide some relevant examples.

Example 2.4. (i) For $w(n) = 1/n^\alpha$, if $\alpha = 0.9$ and $p = 1.1$, then $\max\{m_1, m_2\} \simeq 1.572$ and $M_2(1) = M_3(0.9) \simeq 1.663$ (see pp.15-16 of [11]) and so Proposition 2.3 shows that

$$1.572 \le \|\mathsf{C}^{(p,w)}\| \le 1.663.$$

On the other hand, $p' = 11$ and so Proposition 2.2 only yields $\|\mathsf{C}^{(p,w)}\| \le 11$.

(ii) Still for $w(n) = 1/n^\alpha$, but now with $\alpha = 0.5$ and $p = 2$, we have $m_1 = 4/3$ and $M_3(3/4) \simeq 1.593$ (see p.16 of [11]) so that Proposition 2.3 reveals that

$$\frac{4}{3} \le \|\mathsf{C}^{(p,w)}\| \le 1.593.$$

In this case, $p' = 2$ and so Proposition 2.2 only yields $\|\mathsf{C}^{(p,w)}\| \le 2$.

(iii) Again for $w(n) = 1/n^\alpha$, with $\alpha > 0$, it follows (in the notation of Proposition 2.3) that

$$\left(\frac{1}{w(1)}\sum_{n=1}^{\infty}\frac{w(n)}{n^p}\right)^{1/p} = \left(\sum_{n=1}^{\infty}\frac{1}{n^{p+\alpha}}\right)^{1/p} = \zeta(p+\alpha)^{1/p} = m_2.$$

Hence, the lower bound in (2.3) reduces to $m_2 \le \|\mathsf{C}^{(p,w)}\|$ whereas (2.4) yields $\max\{m_1, m_2\} \le \|\mathsf{C}^{(p,w)}\|$. Of course, (2.3) applies to more general weights w.

The following example is not a consequence of Proposition 2.3.

Example 2.5. Let $p = 2$ and set $w(n) = 2^{-n}$ for $n \in \mathbb{N}$. The proof of Proposition 2.2 yields that $\|\mathsf{C}^{(2,w)}\| = \|T_w\|$. Recall, via (2.2), that

$$T_w x = \left(\frac{1}{n 2^{n/2}}\sum_{k=1}^{n} 2^{k/2} x_k\right)_{n\in\mathbb{N}}, \quad x = (x_n)_{n\in\mathbb{N}} \in \ell_2.$$

For every $x \in \ell_2$, it follows via the Cauchy-Schwarz inequality and the identity $\sum_{k=1}^{n} r^k = (r - r^{n+1})/(1-r)$, for $r \ne 1$, that

$$\begin{aligned} \|T_w x\|_2^2 &= \sum_{n=1}^{\infty}\frac{1}{n^2 2^n}\left|\sum_{k=1}^{n} 2^{k/2} x_k\right|^2 \le \sum_{n=1}^{\infty}\frac{1}{n^2 2^n}\Big(\sum_{k=1}^{n} 2^k\Big)\Big(\sum_{k=1}^{n}|x_k|^2\Big) \\ &\le \|x\|_2^2 \sum_{n=1}^{\infty}\frac{1}{n^2 2^n}(2^{n+1} - 2) = \|x\|_2^2 \sum_{n=1}^{\infty}\frac{2(1-2^{-n})}{n^2}. \end{aligned}$$

Accordingly, $\|T_w\| \le \left(\sum_{n=1}^{\infty}\frac{2(1-2^{-n})}{n^2}\right)^{1/2}$. Observe that

$$\sum_{n=1}^{\infty}\frac{(1-2^{-n})}{n^2} = \frac{\pi^2}{6} - \int_0^{1/2}\frac{-\log(1-t)}{t}\,dt,$$

because of the fact that $\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$ and the identity

$$\int_0^{1/2}\frac{-\log(1-t)}{t}\,dt = \int_0^{1/2}\sum_{n=0}^{\infty}\frac{t^n}{(n+1)} = \sum_{n=1}^{\infty}\frac{1}{n^2 2^n}.$$

The function $f(t) = \frac{-\log(1-t)}{t}$ for $t \in (0,1]$, with $f(0) := 1$, is positive, continuous and increasing on $[0,1)$ and so

$$1 = f(0) \le f(t) \le f\left(\frac{1}{2}\right) = 2\log 2, \quad t \in [0, 1/2],$$

which implies that $-\log 2 \le -\int_0^{1/2} \frac{-\log(1-t)}{t}\, dt \le -\frac{1}{2}$. Consequently,

$$\sum_{n=1}^{\infty} \frac{2(1-2^{-n})}{n^2} \le 2(\frac{\pi^2}{6} - \frac{1}{2}) \simeq 2.2898$$

and so

$$\|\mathsf{C}^{(2,w)}\| = \|T_w\| \le \sqrt{\left(\frac{\pi^2}{3} - 1\right)} \simeq 1.513 < p' = 2.$$

Direct calculation yields

$$\|T_w e_1\|_2 = \left(2\sum_{n=1}^{\infty} \frac{1}{n^2 2^n}\right)^{1/2} \ge \left(2\sum_{n=1}^{3} \frac{1}{n^2 2^n}\right)^{1/2} \simeq 1.073$$

and so we have

$$1.073 \le \|\mathsf{C}^{(2,w)}\| \le \sqrt{\left(\frac{\pi^2}{3} - 1\right)} \simeq 1.513;$$

see also Proposition 2.2.

3. Spectrum of $\mathsf{C}^{(p,w)}$

The aim of this section is to provide some detailed knowledge of the spectrum of $\mathsf{C}^{(p,w)}$. Unlike for the classical Cesàro operators $\mathsf{C}^{(p)} \in \mathcal{L}(\ell_p)$, for $1 < p < \infty$, it can now happen that eigenvalues appear.

Given a (strictly) positive, bounded sequence $w = (w(n))_{n\in\mathbb{N}}$ and $1 < p < \infty$, let $S_w(p) := \{s \in \mathbb{R} \colon \sum_{n=1}^{\infty} \frac{1}{n^s w(n)^{p'/p}} < \infty\}$. In case $S_w(p) \ne \emptyset$ we define $s_p := \inf S_w(p)$. We point out that $\frac{p'}{p} = \frac{1}{p-1}$, for every $1 < p < \infty$. Moreover, let $R_w := \{t \in \mathbb{R} \colon \sum_{n=1}^{\infty} n^t w(n) < \infty\}$. In case $R_w \ne \mathbb{R}$ we define $t_0 := \sup R_w$.

Fix $1 < p < \infty$ and let $w(n) = 2^{-np/p'}$ for $n \in \mathbb{N}$. Then $S_w(p) = \emptyset$, i.e., it can happen that $S_w(p)$ is empty. However, in the event that $S_w(p) \ne \emptyset$, then $s_p \ge 1$. Indeed, for any fixed $s \in \mathbb{R}$ we have

$$\sum_{n=1}^{\infty} \frac{1}{n^s w(n)^{p'/p}} \ge \|w\|_\infty^{-p'/p} \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{3.1}$$

So, whenever $s \in S_w(p)$ it follows that $\sum_{n=1}^{\infty} \frac{1}{n^s} < \infty$, that is, $s > 1$. Hence, $S_w(p) \subseteq (1,\infty)$ which implies that $s_p \ge 1$. Moreover, for any $r > s \in S_w(p)$ we have

$$\sum_{n=1}^{\infty} \frac{1}{n^r w(n)^{p'/p}} < \sum_{n=1}^{\infty} \frac{1}{n^s w(n)^{p'/p}}$$

and so also $r \in S_w(p)$. Accordingly, whenever $S_w(p) \ne \emptyset$, then it is an infinite interval, i.e., $S_w(p) = [s_p, \infty)$ or $S_w(p) = (s_p, \infty)$ with $s_p \ge 1$. It is a consequence of (3.1) that $1 \notin S_w(p)$, for all $1 < p < \infty$ and all positive, bounded sequences w.

In the event that $a_w := \inf_{n\in\mathbb{N}} w(n) > 0$ it follows that necessarily $s_p = 1$. Indeed, in this case $w(n)^{-p'/p} \le a_w^{-p'/p}$, $n \in \mathbb{N}$, which implies that $\frac{1}{n^s w(n)^{p'/p}} \le \frac{a_w^{-p'/p}}{n^s}$, for all $n \in \mathbb{N}$ and $s \in \mathbb{R}$. Hence, $(1,\infty) \subseteq S_w(p)$ and so $s_p \le 1$. Since we are assuming that $S_w(p) \ne \emptyset$, we already know that $s_p \ge 1$. Accordingly, $s_p = 1$.

Let $1 < p < \infty$ and fix $\alpha > 0$. For $w(n) = 1/n^{\alpha p/p'}$ and any $s \in \mathbb{R}$ it follows that $\sum_{n=1}^{\infty} \frac{1}{n^s w(n)^{p'/p}} = \sum_{n=1}^{\infty} \frac{1}{n^{s-\alpha}} < \infty$ precisely when $s > (1+\alpha)$ and so $s_p = 1+\alpha$. Hence, given any $\beta > 1$ and $1 < p < \infty$, there exists a positive, decreasing weight $w \downarrow 0$ such that $S_w(p) = (\beta, \infty)$, i.e., $s_p = \beta$.

Concerning the set R_w, a similar discussion applies. For $w(n) = 2^{-n}$ it turns out that $R_w = \mathbb{R}$ with $t_0 = \infty$. However, if $R_w \neq \mathbb{R}$, then t_0 is finite with $t_0 \geq -1$ and $R_w = (-\infty, t_0)$ or $R_w = (-\infty, t_0]$. Moreover, $R_w = \emptyset$ is not possible as $\sum_{n=1}^{\infty} n^t w(n) \leq \|w\|_\infty \sum_{n=1}^{\infty} n^t < \infty$ whenever $t < -1$. If $a_w > 0$, then necessarily $t_0 = -1$ but, $-1 \notin R_w$ as $\sum_{n=1}^{\infty} n^t w(n) \geq a_w \sum_{n=1}^{\infty} n^t$ for all $t \in \mathbb{R}$.

The following result clarifies the connection between s_p and t_0.

Proposition 3.1. *Let $w = (w(n))_{n\in\mathbb{N}}$ be a bounded, strictly positive sequence.*

(i) *For each $1 < p < \infty$ such that $S_w(p) \neq \emptyset$ we have*

$$t_0 \leq \frac{s_p p}{p'} = (p-1)s_p.$$

In particular, $R_w \neq \mathbb{R}$ whenever there exists $p \in (1,\infty)$ with $S_w(p) \neq \emptyset$.

(ii) *If $R_w \neq \mathbb{R}$, then $S_w(p) \subseteq [1 + \frac{t_0}{(p-1)}, \infty)$, for every $1 < p < \infty$.*

(iii) *Suppose that $1 < p < \infty$ satisfies $S_w(p) \neq \emptyset$. Then*

$$S_w(p) \subseteq S_w(q), \quad q \in [p, \infty).$$

In particular, $S_w(q) \neq \emptyset$ and $s_q \leq s_p$ whenever $q \geq p$.

(iv) *If $S_w(p) = \emptyset$ for some $1 < p < \infty$, then $S_w(q) = \emptyset$ for all $1 < q \leq p$.*

Proof. (i) Suppose that $S_w(p) \neq \emptyset$. Fix $s > s_p$. Since $\sum_{n=1}^{\infty} \frac{1}{n^s w(n)^{p'/p}} < \infty$, there exists $N \in \mathbb{N}$ such that $\frac{1}{n^s w(n)^{p'/p}} \leq 1$ for $n \geq N$ and hence, $n^{sp/p'} w(n) \geq 1$ for $n \geq N$. So, the series $\sum_{n=1}^{\infty} n^{sp/p'} w(n)$ diverges which yields that $t_0 \leq \frac{sp}{p'}$. Accordingly, $t_0 \leq \frac{s_p p}{p'}$. In particular, $R_w \neq \mathbb{R}$.

(ii) Fix $p \in (1,\infty)$ and any $t < t_0$, in which case $\sum_{n=1}^{\infty} n^t w(n) < \infty$. Hence, there exists $K \in \mathbb{N}$ such that $n^t \leq \frac{1}{w(n)}$ for $n \geq K$, that is, $n^{tp'/p} \leq \frac{1}{w(n)^{p'/p}}$ for $n \geq K$. So, for any $s \in \mathbb{R}$ we have (as $\frac{1}{n^s} > 0$ for $n \in \mathbb{N}$) that

$$\frac{1}{n^{s-(tp'/p)}} = \frac{n^{tp'/p}}{n^s} \leq \frac{1}{n^s w(n)^{p'/p}}, \quad n \geq K.$$

Choose now $s \leq 1+\frac{tp'}{p}$. It follows from the previous inequality that $\sum_{n=1}^{\infty} \frac{1}{n^s w(n)^{p'/p}}$ diverges. Hence, $\sum_{n=1}^{\infty} \frac{1}{n^s w(n)^{p'/p}}$ diverges whenever $s \leq 1 + \frac{tp'}{p}$, for some $t < t_0$, that is, whenever $s \in (-\infty, 1 + \frac{t_0 p'}{p})$. So, $S_w(p) \subseteq [1 + \frac{t_0 p'}{p}, \infty) = [1 + \frac{t_0}{(p-1)}, \infty)$.

(iii) Fix $s \in S_w(p)$, i.e., $\sum_{n=1}^{\infty} \frac{1}{n^s w(n)^{p'/p}} < \infty$. For every $1 < q < \infty$ we have

$$\sum_{n=1}^{\infty} \frac{1}{n^s w(n)^{q'/q}} = \sum_{n=1}^{\infty} \frac{1}{n^s w(n)^{p'/p}} \cdot w(n)^{\frac{p'}{p} - \frac{q'}{q}} \leq \|w\|_\infty^{\frac{p'}{p} - \frac{q'}{q}} \sum_{n=1}^{\infty} \frac{1}{n^s w(n)^{p'/p}},$$

which is finite provided that $\frac{p'}{p} \geq \frac{q'}{q}$. This is equivalent to $(p'-1) \geq (q'-1)$, that is, to $q \geq p$. Hence, whenever $q \geq p$ we have $S_w(p) \subseteq S_w(q)$ which clearly implies $S_w(q) \neq \emptyset$ and $s_q \leq s_p$.

(iv) Follows immediately from part (iii). ☐

Define $\Sigma := \{\frac{1}{n} \colon n \in \mathbb{N}\}$ and let $\Sigma_0 := \{0\} \cup \{\frac{1}{n} \colon n \in \mathbb{N}\}$ be its closure. The following inequalities will be needed later.

Lemma 3.2. (i) *Let $\lambda \in \mathbb{C} \setminus \Sigma_0$ and set $\alpha := \operatorname{Re}\left(\frac{1}{\lambda}\right)$. Then there exist constants $d > 0$ and $D > 0$ (depending on α) such that*

$$\frac{d}{n^\alpha} \le \prod_{k=1}^{n} \left|1 - \frac{1}{k\lambda}\right| \le \frac{D}{n^\alpha}, \quad n \in \mathbb{N}. \tag{3.2}$$

(ii) *For each $m \in \mathbb{N}$ we have that*

$$\frac{(n-1)!}{(n-m)!} \simeq n^{m-1}, \quad \textit{for all large } n \in \mathbb{N}. \tag{3.3}$$

(iii) Let $1 < p < \infty$ and $w = (w(n))_{n\in\mathbb{N}}$ be a positive, decreasing sequence. Then

$$(n^m w(n))_{n\in\mathbb{N}} \in \ell_p, \quad \forall m \in \mathbb{N}, \tag{3.4}$$

if and only if

$$(n^m w(n)^{1/p})_{n\in\mathbb{N}} \in \ell_p, \quad \forall m \in \mathbb{N}, \tag{3.5}$$

Proof. (i) The inequalities in (3.2) follow as in the proof of Lemma 7 in [18], where the restriction $\alpha < 1$ is assumed. Indeed, with $\frac{1}{\lambda} = \alpha + i\beta$ (for $\alpha,\ \beta \in \mathbb{R}$) and using $1 + x \le e^x$ for $x > 0$, we have

$$\begin{aligned}\prod_{k=1}^{n} \left|1 - \frac{1}{k\lambda}\right| &= \prod_{k=1}^{n} \left(1 - \frac{2\alpha}{k} + \frac{\alpha^2+\beta^2}{k^2}\right)^{1/2} \\ &\le \exp \sum_{k=1}^{n} \left(-\frac{\alpha}{k} + \frac{C}{k^2}\right) \le \exp\left(-\alpha \log(n) + v\right) \le \frac{D}{n^\alpha}.\end{aligned}$$

An application of Taylor's formula for $x \mapsto (1+x)^{-1/2}$, for $x > -1$, yields

$$\begin{aligned}\prod_{k=1}^{n} \left|1 - \frac{1}{k\lambda}\right|^{-1} &= \prod_{k=1}^{n} \left(1 - \frac{2\alpha}{k} + \frac{\alpha^2+\beta^2}{k^2}\right)^{-1/2} \le \prod_{k=1}^{n} \left(1 + \frac{\alpha}{k} + \frac{C'}{k^2}\right) \\ &\le \exp \sum_{k=1}^{n} \left(\frac{\alpha}{k} + \frac{C'}{k^2}\right) \le \exp\left(\alpha \log(n) + v'\right) = d^{-1} n^\alpha.\end{aligned}$$

(ii) Fix $m \in \mathbb{N}$. Then, for all large $n > m$, we have

$$\frac{(n-1)!}{(n-m)!} = (n-1)\cdot \ldots \cdot (n-m+1) = n^{m-1} \cdot \left(1 - \frac{1}{n}\right) \cdot \ldots \cdot \left(1 - \frac{m-1}{n}\right) \simeq n^{m-1}.$$

(iii) Suppose that (3.4) holds. Fix $m \in \mathbb{N}$. Let $k \in \mathbb{N}$ satisfy $k \ge (2 + mp)$. Since $(n^k w(n))_{n\in\mathbb{N}} \in \ell_p$, there exists $N \in \mathbb{N}$ such that

$$w(n) \le \frac{1}{n^k} \le \frac{1}{n^{2+mp}}, \quad n > N.$$

It follows that

$$\sum_{n=1}^{\infty} n^{mp} w(n) \le \sum_{n=1}^{N} n^{mp} w(n) + \sum_{n=N+1}^{\infty} n^{mp} \left(\frac{1}{n^{2+mp}}\right) < \infty,$$

that is, $(n^m w(n)^{1/p})_{n\in\mathbb{N}} \in \ell_p$. Accordingly, (3.5) is satisfied.

Conversely, suppose that (3.5) holds. Since $(nw(n)^{1/p})_{n\in\mathbb{N}} \in \ell_p$, there exists $K \in \mathbb{N}$ such that $w(n) \leq 1$ for $n \geq K$ and hence, $w(n) \leq w(n)^{1/p}$ for $n \geq K$. Fix $m \in \mathbb{N}$. Then $n^m w(n) \leq n^m w(n)^{1/p}$ for $n \geq K$. Since $(n^m w(n)^{1/p})_{n\in\mathbb{N}} \in \ell_p$, we can conclude that also $(n^m w(n))_{n\in\mathbb{N}} \in \ell_p$. Hence, (3.4) is satisfied. □

If $S_w(p) \neq \emptyset$, then $s_p \geq 1$ and so $\frac{p'}{2s_p} \leq \frac{p'}{2}$, which is relevant for the following results. Also relevant is that $\|\mathsf{C}^{(p,w}\| < p'$ is possible; see Section 2.

We now come to the main result of this section.

Theorem 3.3. *Let $w = (w(n))_{n\in\mathbb{N}}$ be a positive, decreasing sequence.*

(i) *Suppose that $S_w(p) \neq \emptyset$ for some $1 < p < \infty$. Then for the dual operator $(\mathsf{C}^{(p,w)})' \in \mathcal{L}((\ell_p(w))')$ of $\mathsf{C}^{(p,w)}$ we have*

$$\left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{p'}{2s_p}\right| < \frac{p'}{2s_p}\right\} \cup \Sigma \subseteq \sigma_{pt}((\mathsf{C}^{(p,w)})') \tag{3.6}$$

and

$$\sigma_{pt}((\mathsf{C}^{(p,w)})') \setminus \Sigma \subseteq \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{p'}{2s_p}\right| \leq \frac{p'}{2s_p}\right\}. \tag{3.7}$$

For the Cesàro operator $\mathsf{C}^{(p,w)}$ itself we have

$$\left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{p'}{2s_p}\right| \leq \frac{p'}{2s_p}\right\} \cup \Sigma \subseteq \sigma(\mathsf{C}^{(p,w)}) \tag{3.8}$$

and

$$\sigma(\mathsf{C}^{(p,w)}) \subseteq \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{p'}{2}\right| \leq \frac{p'}{2}\right\} \cap \left\{\lambda \in \mathbb{C}\colon |\lambda| \leq \|\mathsf{C}^{(p,w)}\|\right\}. \tag{3.9}$$

(ii) *Suppose that $R_w \neq \mathbb{R}$, i.e., $t_0 < \infty$. Then, for every $1 < p < \infty$, we have*

$$\left\{\frac{1}{m}\colon m \in \mathbb{N},\ 1 \leq m < \frac{t_0}{p} + 1\right\} \subseteq \sigma_{pt}(\mathsf{C}^{(p,w)}) \subseteq \left\{\frac{1}{m}\colon m \in \mathbb{N},\ 1 \leq m \leq \frac{t_0}{p} + 1\right\}. \tag{3.10}$$

If $R_w = \mathbb{R}$, then

$$\sigma_{pt}(\mathsf{C}^{(p,w)}) = \Sigma, \quad \forall\ 1 < p < \infty. \tag{3.11}$$

Proof. The proof is via a series of steps.

(i) By Proposition 2.2 we have $\mathsf{C}^{(p,w)} \in \mathcal{L}(\ell_p(w))$ with $\|\mathsf{C}^{(p,w)}\| \leq p'$. The dual operator $A := (\mathsf{C}^{p,w})' \in \mathcal{L}(\ell_{p'}(w^{-p'/p}))$ also satisfies $\|A\| \leq p'$ and is given by

$$Ay = \left(\sum_{k=n}^{\infty} \frac{y_k}{k}\right)_{n\in\mathbb{N}}, \quad y = (y_n)_{n\in\mathbb{N}} \in \ell_{p'}(w^{-p'/p}). \tag{3.12}$$

Step 1. $0 \notin \sigma_{pt}(A)$.

Observe that $Ay = 0$, for some $y \in \ell_{p'}(w^{-p'/p})$, implies that $z_n := \sum_{k=n}^{\infty} \frac{y_k}{k} = 0$ for all $n \in \mathbb{N}$. Hence, $y_n = n(z_n - z_{n+1}) = 0$, for $n \in \mathbb{N}$, and so A is injective.

Step 2. $\Sigma \subseteq \sigma_{pt}(A)$.

Let $\lambda \in \Sigma$, i.e., $\lambda = \frac{1}{m}$ for some $m \in \mathbb{N}$. Via (3.13) below, the non-zero vector $y = (y_n)_{n\in\mathbb{N}}$ defined via $y_1 \in \mathbb{C} \setminus \{0\}$ arbitrary, $y_n := y_1 \prod_{k=1}^{n-1} \left(1 - \frac{1}{\lambda k}\right)$ for $1 < n \leq m$ and $y_n := 0$ for $n > m$, which belongs to $\ell_{p'}(w^{-p'/p})$, satisfies $Ay = \lambda y$.

Step 3. $\left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{p'}{2s_p}\right| < \frac{p'}{2s_p}\right\} \subseteq \sigma_{pt}(A)$.

Let $\lambda \in \mathbb{C}\backslash\{0\}$. Then $Ay = \lambda y$ for some non-zero $y \in \ell_{p'}(w^{-p'/p})$ if, and only if, $\lambda y_n = \sum_{k=n}^{\infty} \frac{y_k}{k}$ for all $n \in \mathbb{N}$. This yields, for every $n \in \mathbb{N}$, that $\lambda(y_n - y_{n+1}) = \frac{y_n}{n}$ and so $y_{n+1} = \left(1 - \frac{1}{\lambda n}\right) y_n$. It follows that

$$y_{n+1} = y_1 \prod_{k=1}^{n} \left(1 - \frac{1}{\lambda k}\right), \quad n \in \mathbb{N}, \tag{3.13}$$

with $y_1 \neq 0$. In particular, each eigenvalue of A is simple.

Let now $\lambda \in \mathbb{C} \setminus \Sigma$ satisfy $\left|\lambda - \frac{p'}{2s_p}\right| < \frac{p'}{2s_p}$ (equivalently, $\alpha := \mathrm{Re}\left(\frac{1}{\lambda}\right) > \frac{s_p}{p'}$, i.e., $\alpha p' = \mathrm{Re}\left(\frac{p'}{\lambda}\right) > s_p$). For such a λ the vector $y = (y_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ defined by (3.13) actually belongs to $\ell_{p'}(w^{-p'/p})$. Indeed, via Lemma 3.2(i) there exists $c = c(\lambda) > 0$ such that

$$\prod_{k=1}^{n} \left|1 - \frac{1}{\lambda k}\right|^{p'} \leq c n^{-\mathrm{Re}(p'/\lambda)}, \quad n \in \mathbb{N}.$$

It then follows from (3.13) that

$$\begin{aligned} \sum_{n=1}^{\infty} |y_n|^{p'} w(n)^{-p'/p} &= |y_1|^{p'} w(1)^{-p'/p} + |y_1|^{p'} \sum_{n=2}^{\infty} \prod_{k=1}^{n} \left|1 - \frac{1}{\lambda k}\right|^{p'} w(n)^{-p'/p} \\ &\leq |y_1|^{p'} w(1)^{-p'/p} + c|y_1|^{p'} \sum_{n=2}^{\infty} n^{-\mathrm{Re}(p'/\lambda)} w(n)^{-p'/p}, \end{aligned}$$

where the series $\sum_{n=2}^{\infty} n^{-\mathrm{Re}(p'/\lambda)} w(n)^{-p'/p}$ converges because $\mathrm{Re}(p'/\lambda) \in S_w(p)$, that is, $y \in \ell_{p'}(w^{-p'/p})$. Hence, $\lambda \in \sigma_{pt}(A)$.

Step 4. $\sigma_{pt}(A) \setminus \Sigma_0 \subseteq \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{p'}{2s_p}\right| \leq \frac{p'}{2s_p}\right\}$.

Fix $\lambda \in \sigma_{pt}(A) \setminus \Sigma_0$. According to (3.2) there exists $\beta = \beta(\lambda) > 0$ such that

$$\prod_{k=1}^{n} \left|1 - \frac{1}{\lambda k}\right|^{p'} \geq \beta \cdot n^{-\mathrm{Re}(p'/\lambda)}, \quad n \in \mathbb{N}. \tag{3.14}$$

But, as argued in Step 2 (for any $y_1 \in \mathbb{C} \setminus \{0\}$) the eigenvector $y = (y_n)_{n\in\mathbb{N}}$ corresponding to the eigenvalue λ of A, which necessarily belongs to $\ell_{p'}(w^{-p'/p})$, i.e., $\sum_{n=1}^{\infty} |y_n|^{p'} w(n)^{-p'/p} < \infty$, is given by (3.13). Then (3.14) implies that also $\sum_{n=1}^{\infty} \frac{1}{n^{\mathrm{Re}(p'/\lambda)} w(n)^{p'/p}} < \infty$, i.e., $\mathrm{Re}\left(\frac{p'}{\lambda}\right) \in S_w(p)$ and so $\mathrm{Re}\left(\frac{p'}{\lambda}\right) \geq s_p$. Equivalently, $\mathrm{Re}\left(\frac{1}{\lambda}\right) \geq \frac{s_p}{p'}$, i.e., $\lambda \in \left\{\mu \in \mathbb{C}\colon \left|\mu - \frac{p'}{2s_p}\right| \leq \frac{p'}{2s_p}\right\}$.

It is clear that Steps 1-4 establish the two containments in (3.6) and (3.7).

For every $T \in \mathcal{L}(X)$ with X a Banach space, it is known that $\sigma_{pt}(T') \subseteq \sigma(T)$, [9, p.581], with $\sigma(T)$ a closed subset of $\mathbb{C}$. Accordingly, (3.8) follows from (3.6).

Step 5. $\sigma(\mathsf{C}^{(p,w)}) \subseteq \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{p'}{2}\right| \leq \frac{p'}{2}\right\}$.

It suffices to show that every $\lambda \in \mathbb{C}$ with $\left|\lambda - \frac{p'}{2}\right| > \frac{p'}{2}$ belongs to $\rho(\mathsf{C}^{(p,w)})$. To do this we argue as in [7]. We recall the formula for $(\mathsf{C} - \lambda I)^{-1}\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ whenever $\lambda \notin \Sigma_0$, [18, p.266]. For $n \in \mathbb{N}$ the n–th row of the matrix for $(\mathsf{C} - \lambda I)^{-1}$

has the entries

$$\frac{-1}{n\lambda^2 \prod_{k=m}^n \left(1-\frac{1}{\lambda k}\right)}, \quad 1 \le m < n,$$
$$\frac{n}{1-n\lambda} = \frac{1}{\frac{1}{n}-\lambda}, \quad m = n,$$

and all the other entries in row n are equal to 0. So, we can write

$$(\mathsf{C} - \lambda I)^{-1} = D_\lambda - \frac{1}{\lambda^2} E_\lambda, \tag{3.15}$$

where the diagonal operator $D_\lambda = (d_{nm})_{n,m\in\mathbb{N}}$ is given by $d_{nn} := \frac{1}{\frac{1}{n}-\lambda}$ and $d_{nm} := 0$ if $n \neq m$. The operator $E_\lambda = (e_{nm})_{n,m\in\mathbb{N}}$ is then the lower triangular matrix with $e_{1m} = 0$ for all $m \in \mathbb{N}$, and for every $n \ge 2$ with $e_{nm} := \frac{1}{n\prod_{k=m}^n (1-\frac{1}{\lambda k})}$ if $1 \le m < n$ and $e_{nm} := 0$ if $m \ge n$.

If $\lambda \notin \Sigma_0$, then $d(\lambda) := \mathrm{dist}(\lambda, \Sigma_0) > 0$ and $|d_{nn}| \le \frac{1}{d(\lambda)}$ for $n \in \mathbb{N}$. Hence, for every $x \in \ell_p(w)$, we have

$$\|D_\lambda(x)\|_{p,w} = \left(\sum_{n=1}^\infty |d_{nn}x_n|^p w(n)\right)^{1/p} \le \frac{1}{d(\lambda)} \left(\sum_{n=1}^\infty |x_n|^p w(n)\right)^{1/p} = \|x\|_{p,w}.$$

This means that $D_\lambda \in \mathcal{L}(\ell_p(w))$. So, by (3.15) it remains to show that $E_\lambda \in \mathcal{L}(\ell_p(w))$ whenever $\lambda \in \mathbb{C}$ satisfies $\left|\lambda - \frac{p'}{2}\right| > \frac{p'}{2}$. To this end, we note that if $\lambda \in \mathbb{C} \setminus \Sigma_0$ then, with $\alpha := \mathrm{Re}\left(\frac{1}{\lambda}\right)$, it follows from (3.2) that

$$|e_{n1}| \le \frac{d^{-1}}{n^{1-\alpha}}, \quad n \ge 2,$$
$$|e_{nm}| \le \frac{d^{-1}D'}{n^{1-\alpha}m^\alpha}, \quad 2 \le m < n,$$

for some constants $d > 0$ and $D' > 0$ depending on λ. So, for every $\lambda \in \mathbb{C} \setminus \Sigma_0$ there exists $c = c(\lambda) > 0$ such that

$$|(E_\lambda(x))_n| \le c(G_\lambda(|x|))_n, \quad x \in \mathbb{C}^\mathbb{N},\ n \in \mathbb{N}, \tag{3.16}$$

where $(G_\lambda(x))_n := \sum_{k=1}^n \frac{x_k}{n^{1-\alpha}k^\alpha}$ with $\alpha := \mathrm{Re}\left(\frac{1}{\lambda}\right)$ and for all $x \in \mathbb{C}^\mathbb{N}$ and $n \in \mathbb{N}$. Then (3.16) implies that $E_\lambda \in \mathcal{L}(\ell_p(w))$ whenever $G_\lambda \in \mathcal{L}(\ell_p(w))$.

Claim: *$G_\lambda \in \mathcal{L}(\ell_p(w))$ whenever $\lambda \in \mathbb{C}$ satisfies $\left|\lambda - \frac{p'}{2}\right| > \frac{p'}{2}$.*

To establish this claim fix $\lambda \in \mathbb{C}$ with $\left|\lambda - \frac{p'}{2}\right| > \frac{p'}{2}$. Then necessarily $\lambda \notin \Sigma_0$ with $\alpha := \mathrm{Re}\left(\frac{1}{\lambda}\right) < \frac{1}{p'}$ and so $(1-\alpha)p > 1$. This implies that $\alpha < 1$. Observe that $G_\lambda \in \mathcal{L}(\ell_p(w))$ if, and only if, the operator $\tilde{G}_\lambda \colon \mathbb{C}^\mathbb{N} \to \mathbb{C}^\mathbb{N}$ given by

$$(\tilde{G}_\lambda(x))_n = w(n)^{1/p} \sum_{k=1}^n \frac{w(k)^{-1/p}}{n^{1-\alpha}k^\alpha} x_k, \quad x \in \mathbb{C}^\mathbb{N},\ n \in \mathbb{N},$$

defines a continuous linear operator on ℓ_p (the proof of this is along the lines of that of Lemma 2.1). To prove that indeed $\tilde{G}_\lambda \in \mathcal{L}(\ell_p)$ we need to distinguish the three cases; a) $\alpha = 0$; b) $\alpha < 0$ and c) $0 < \alpha < 1$ and establish relevant inequalities in each case.

Case a). Since w is decreasing, we have, for every $n \in \mathbb{N}$, that

$$\sum_{k=1}^{n} \frac{1}{w(k)^{1/(p-1)} k^{\alpha p/(p-1)}} = \sum_{k=1}^{n} \frac{1}{w(k)^{1/(p-1)}} \leq \frac{n}{w(n)^{1/(p-1)}}$$

and hence, for every $m \in \mathbb{N}$, that

$$\sum_{n=1}^{m} \left(\frac{w(n)^{1/p}}{n} \sum_{k=1}^{n} \frac{1}{w(k)^{1/(p-1)}} \right)^p \leq \sum_{n=1}^{m} \frac{1}{w(n)^{1/(p-1)}}. \tag{3.17}$$

Case b). Observe, for every $n \in \mathbb{N}$, that

$$\begin{aligned} \sum_{k=1}^{n} \frac{1}{w(k)^{1/(p-1)} k^{\alpha p/(p-1)}} &\leq \frac{1}{w(n)^{1/(p-1)}} \int_1^{n+1} x^{-\alpha p/(p-1)}\, dx \\ = \frac{1}{w(n)^{1/(p-1)}} \frac{((n+1)^{-\frac{\alpha p}{p-1}+1} - 1)}{-\frac{\alpha p}{p-1} + 1} &\leq \frac{(p-1)}{(p(1-\alpha)-1)} \frac{(n+1)^{\frac{p(1-\alpha)-1}{p-1}}}{w(n)^{1/(p-1)}}. \end{aligned}$$

Setting $c := \frac{p-1}{p(1-\alpha)-1} > 0$ it follows, for every $m \in \mathbb{N}$, that

$$\begin{aligned} \sum_{n=1}^{m} \left(\frac{w(n)^{1/p}}{n^{1-\alpha}} \sum_{k=1}^{n} \frac{1}{w(k)^{1/(p-1)} k^{\alpha p/(p-1)}} \right)^p &\leq c^p \sum_{n=1}^{m} \frac{(n+1)^{\frac{p[p(1-\alpha)-1]}{p-1}}}{w(n)^{1/(p-1)} n^{(1-\alpha)p}} \\ \leq 2^{\frac{p(p(1-\alpha)-1)}{p-1}} c^p \sum_{n=1}^{m} \frac{1}{w(n)^{1/(p-1)} n^{\alpha p/(p-1)}}. \end{aligned} \tag{3.18}$$

Case c). We have, for every $n \in \mathbb{N}$, still with $c = \frac{p-1}{p(1-\alpha)-1}$, that

$$\begin{aligned} \sum_{k=2}^{n} \frac{1}{w(k)^{1/(p-1)} k^{\alpha p/(p-1)}} &\leq \frac{1}{w(n)^{1/(p-1)}} \int_1^{n} \frac{1}{x^{\alpha p/(p-1)}}\, dx \\ = \frac{c}{w(n)^{1/(p-1)}} (n^{\frac{p(1-\alpha)-1}{p-1}} - 1). \end{aligned}$$

Since $(1-\alpha)p > 1$ (i.e., $(1-\alpha)p - 1 > 0$) and $\alpha p > 0$ with $\frac{1}{w(1)} \leq \frac{1}{w(n)}$, this implies, for every $n \in \mathbb{N}$, that

$$\begin{aligned}
&\left(\frac{w(n)^{1/p}}{n^{1-\alpha}}\sum_{k=1}^{n}\frac{1}{w(k)^{1/(p-1)}k^{\alpha p/(p-1)}}\right)^p \\
&\leq \left[\frac{w(n)^{1/p}}{n^{1-\alpha}w(1)^{1/(p-1)}} + \frac{w(n)^{1/p}c}{n^{1-\alpha}w(n)^{1/(p-1)}}(n^{\frac{p(1-\alpha)-1}{p-1}}-1)\right]^p \\
&\leq \left[\frac{w(n)^{1/p}}{n^{1-\alpha}w(n)^{1/(p-1)}} + \frac{w(n)^{1/p}c}{n^{1-\alpha}w(n)^{1/(p-1)}}(n^{\frac{p(1-\alpha)-1}{p-1}}-1)\right]^p \\
&= \left[(1-c)\frac{w(n)^{1/p}}{n^{1-\alpha}w(n)^{1/(p-1)}} + \frac{w(n)^{1/p}c}{n^{1-\alpha}w(n)^{1/(p-1)}}n^{\frac{p(1-\alpha)-1}{p-1}}\right]^p \\
&= \left(\frac{-\alpha p}{p(1-\alpha)-1}\frac{w(n)^{1/p}}{n^{1-\alpha}w(n)^{1/(p-1)}} + \frac{w(n)^{-1/p(p-1)}c}{n^{1-\alpha}}n^{\frac{p(1-\alpha)-1}{p-1}}\right)^p \\
&\leq \left(\frac{w(n)^{-1/p(p-1)}c}{n^{1-\alpha}}n^{\frac{p(1-\alpha)-1}{p-1}}\right)^p \\
&= c^p w(n)^{-1/(p-1)}n^{-\alpha p/(p-1)}.
\end{aligned}$$

Hence, for every $m \in \mathbb{N}$, we have that

$$\sum_{n=1}^{m}\left(\frac{w(n)^{1/p}}{n^{1-\alpha}}\sum_{k=1}^{n}\frac{1}{w(k)^{1/(p-1)}k^{\alpha p/(p-1)}}\right)^p \leq c^p\sum_{n=1}^{m}\frac{1}{w(n)^{1/(p-1)}n^{\alpha p/(p-1)}}. \quad (3.19)$$

The inequalities (3.17), (3.18) and (3.19) imply that $\tilde{G}_\lambda \in \mathcal{L}(\ell_p)$; indeed, in each case, suitable choices of a_n and b_k (with $p = q$) allow us to apply Theorem 2(ii) of [5]. This establishes the claim and hence, also Step 5.

Step 6. $\sigma(\mathsf{C}^{(p,w)}) \subseteq \{\lambda \in \mathbb{C} \colon |\lambda| \leq \|\mathsf{C}^{(p,w)}\|\}$.

This is well known, [9, Ch.VII Lemma 3.4].

Steps 5 and 6 clearly yield (3.9). The proof of part (i) is thereby complete.

(ii) Suppose first that $R_w \neq \mathbb{R}$. Fix any $1 < p < \infty$.

Step 7. *Both of the inclusions in* (3.10) *are valid.*

The Cesàro operator $\mathsf{C}^{(p,w)}$ is clearly injective. So, $0 \notin \sigma_{pt}(\mathsf{C}^{(p,w)})$. Let $\lambda \in \mathbb{C} \setminus \{0\}$. Consider the equation $(\lambda I - \mathsf{C})x = 0$ with $x = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$. Then $x_1 = \lambda x_1$ and $(2\lambda - 1)x_2 = x_1$ and $(n\lambda - 1)x_n = \lambda(n-1)x_{n-1}$ for all $n \geq 3$. If $m \in \mathbb{N}$ denotes the smallest positive integer such that $x_m \neq 0$, then it follows that $\lambda = \frac{1}{m}$ and so $x_n = \frac{n-1}{n-m}x_{n-1}$ for all $n > m$. Thus, we deduce that

$$x_n = x_{m+(n-m)} = \frac{(n-1)!}{(m-1)!(n-m)!}x_m, \quad n > m. \quad (3.20)$$

According to (3.3) we have $\frac{(n-1)!}{(m-1)!(n-m)!} \simeq \frac{1}{(m-1)!} \cdot n^{m-1}$, for each $m \in \mathbb{N}$. So, $x \in \ell_p(w)$ if, and only if, the series $\sum_{n=m+1}^{\infty} n^{(m-1)p}w(n)$ converges. But, the series $\sum_{n=m+1}^{\infty} n^{(m-1)p}w(n)$ converges precisely when $(m-1)p \in R_w$. In this case, $(m-1)p \leq t_0$, i.e., $m \leq \frac{t_0}{p} + 1$. So, $\sigma_{pt}(\mathsf{C}^{(p,w)}) \subseteq \{\frac{1}{m} \colon m \in \mathbb{N},\ 1 \leq m \leq \frac{t_0}{p} + 1\}$.

Conversely, if $m < \frac{t_0}{p}+1$ for some $m \in \mathbb{N}$, i.e., $(m-1)p < t_0$, then $(m-1)p \in R_w$ as $t_0 = \sup R_w$. Then the vector $x \in \mathbb{C}^{\mathbb{N}}$ defined according to (3.20), with $x_1 = \ldots = x_{m-1} = 0$ and for any arbitrary $x_m \neq 0$, belongs to $\ell_p(w)$. Therefore, $\frac{1}{m} \in \sigma_{pt}(\mathsf{C}^{(p,w)})$.

Step 8. *Assume now that* $R_w = \mathbb{R}$. *Then* (3.11) *is valid.*

Fix $1 < p < \infty$. As argued in Step 7, the point $\frac{1}{m} \in \sigma_{pt}(\mathsf{C}^{(p,w)})$ if and only if $(m-1)p \in R_w$. But, for $R_w = \mathbb{R}$, this is satisfied for every $m \in \mathbb{N}$ and so $\Sigma \subseteq \sigma_{pt}(\mathsf{C}^{(p,w)})$. On the other hand, it is also shown in the proof of Step 7 that *every* eigenvalue λ of $\mathsf{C}\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ must have the form $\lambda = \frac{1}{m}$ for some $m \in \mathbb{N}$. Since every eigenvalue of $\mathsf{C}^{(p,w)}$ is also an eigenvalue of C (as $\ell_p(w) \subseteq \mathbb{C}^{\mathbb{N}}$), it follows that $\sigma_{pt}(\mathsf{C}^{(p,w)}) \subseteq \Sigma$. □

Remark 3.4. (i) If $s_p \notin S_w(p)$, for some $1 < p < \infty$, then the argument of Step 4 in the proof of Theorem 3.3 implies that (3.6) reduces to the equality

$$\sigma_{pt}((\mathsf{C}^{(p,w)})') = \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{p'}{2s_p}\right| < \frac{p'}{2s_p}\right\} \cup \Sigma.$$

Also, if $t_0 \notin R_w$, then (3.10) reduces to the equality

$$\sigma_{pt}(\mathsf{C}^{(p,w)}) = \left\{\frac{1}{m}\colon m \in \mathbb{N},\ 1 \leq m < \frac{t_0}{p} + 1\right\},\ 1 < p < \infty.$$

(ii) For $w(n) = 1$ for all $n \in \mathbb{N}$, in which case $\ell_p(w) = \ell_p$ and $s_p = 1$, we have that $\mathsf{C}^{(p,w)} = \mathsf{C}^{(p)}$ for all $1 < p < \infty$ with $\|\mathsf{C}^{(p,w)}\| = \|\mathsf{C}^{(p)}\| = p'$. Then (3.8) and (3.9) imply the known fact that

$$\sigma(\mathsf{C}^{(p)}) = \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{p'}{2}\right| \leq \frac{p'}{2}\right\}. \tag{3.21}$$

Since $t_0 = -1$, we also recover from (3.10) the known fact that $\sigma_{pt}(\mathsf{C}^{(p)}) = \emptyset$.

(iii) According to (3.8), for w positive, decreasing and with $S_w(p) \neq \emptyset$ we have

$$\max\{1, \frac{p'}{s_p}\} \leq \|\mathsf{C}^{(p,w)}\| \leq p'. \tag{3.22}$$

In particular, whenever $s_p = 1$ (see e.g., Example 3.5(i) below), the inequalities in (3.22) imply that necessarily $\|\mathsf{C}^{(p,w)}\| = p'$ is as large as possible.

For the special case when $w(n) = \frac{1}{n^\alpha}$, $n \in \mathbb{N}$, for some $\alpha > 0$, direct calculation yields that $s_p = 1 + \frac{\alpha p'}{p}$ and so $S_w(p) \neq \emptyset$ for all $1 < p < \infty$. It follows that

$$\frac{p'}{s_p} = \frac{p}{\alpha + p - 1} = m_1,$$

where m_1 occurs in the lower bound for $\|\mathsf{C}^{(p,w)}\|$ as given in (2.4); see Proposition 2.3. Hence, (3.22) yields that $m_1 \leq \|\mathsf{C}^{(p,w)}\|$. Combined with Example 2.4(iii) we can conclude that

$$\max\{m_1, m_2\} \leq \|\mathsf{C}^{(p,w)}\|.$$

This provides an alternate proof, to that in [11], of the same estimate in (2.4).

(iv) An examination of the argument for Step 2 in the proof of Theorem 3.3(i) shows that the assumption $S_w(p) \neq \emptyset$ is not used there, i.e., we always have

$$\Sigma \subseteq \sigma_{pt}((\mathsf{C}^{(p,w)})')$$

for every $1 < p < \infty$ and every positive, decreasing weight w.

We now present some relevant examples.

Examples 3.5. (i) Suppose that $w(n) = \frac{1}{(\log(n+1))^\gamma}$ for $n \in \mathbb{N}$ with $\gamma \geq 0$. Then $\sum_{n=1}^\infty \frac{1}{n^s w(n)^{p'/p}} < \infty$ if and only if $s > 1$ and hence, $s_p = 1$ for every $1 < p < \infty$. In view of Remark 3.4(iii) we have that $\|\mathsf{C}^{(p,w)}\| = p'$. Moreover, $\sum_{n=1}^\infty n^t w(n) < \infty$ if and only if $t < -1$ or $t \leq -1$ in case $\gamma > 1$. Hence, $t_0 = -1$. According to Theorem 3.3 we have, for each $1 < p < \infty$, that

$$\sigma(\mathsf{C}^{(p,w)}) = \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{p'}{2}\right| \leq \frac{p'}{2}\right\}, \quad \sigma_{pt}(\mathsf{C}^{(p,w)}) = \emptyset.$$

In particular, equality may occur in (3.9). For the case when $\gamma = 0$ (so that $w(n) = 1$ for $n \in \mathbb{N}$), we recover the known result about the spectrum of $\mathsf{C}^{(p)} \in \mathcal{L}(\ell_p)$, for $1 < p < \infty$, [6], [14].

(ii) Suppose that $w(n) = \frac{1}{n^\beta (\log(n+1))^\gamma}$ for $n \in \mathbb{N}$ with $\beta \geq 0$ and $\gamma \geq 0$. Then $\sum_{n=1}^\infty \frac{1}{n^s w(n)^{p'/p}} < \infty$ if and only if $s > \beta\frac{p'}{p} + 1$ and so $s_p = \beta\frac{p'}{p} + 1$ for every $1 < p < \infty$. Moreover, $\sum_{n=1}^\infty n^t w(n) < \infty$ if and only if $t < (\beta - 1)$ or $t \leq (\beta - 1)$ in case $\gamma > 1$. Hence, $t_0 = \beta - 1$. According to Theorem 3.3 we have, for each $1 < p < \infty$, that

$$\left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{p'}{2((\beta p'/p) + 1)}\right| \leq \frac{p'}{2((\beta p'/p) + 1)}\right\} \cup \Sigma \subseteq \sigma(\mathsf{C}^{(p,w)})$$

and

$$\sigma_{pt}(\mathsf{C}^{(p,w)}) = \left\{\frac{1}{m} \colon m \in \mathbb{N},\ 1 \leq m < \frac{\beta - 1}{p} + 1\right\}.$$

In particular, $\sigma_{pt}(\mathsf{C}^{(p,w)}) = \emptyset$ whenever $\beta \in [0, 1]$. We claim that actually

$$\left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{p'}{2((\beta p'/p) + 1)}\right| \leq \frac{p'}{2((\beta p'/p) + 1)}\right\} \cup \Sigma = \sigma(\mathsf{C}^{(p,w)}),$$

which shows that equality may occur in (3.8).

Keeping in mind the argument for Step 5 in the proof of Theorem 3.3, to verify this identity it suffices to prove that every $\lambda \in \mathbb{C}\setminus\{0\}$ satisfying $\left|\lambda - \frac{p'}{2((\beta p'/p)+1)}\right| > \frac{p'}{2((\beta p'/p)+1)}$ belongs to $\rho(\mathsf{C}^{(p,w)})$, i.e., that the operator $\tilde{G}_\lambda \in \mathcal{L}(\ell_p)$. So, fix such a λ and note that $\alpha := \mathrm{Re}\left(\frac{1}{\lambda}\right) < \left(\beta\frac{p'}{p} + 1\right)/p' = \frac{\beta}{p} + \frac{1}{p'}$. We also observe, for our particular w, that the operator $\tilde{G}_\lambda$ is given by

$$(\tilde{G}_\lambda(x))_n = \frac{1}{n^{1-\alpha+(\beta/p)} \log^{\gamma/p}(n+1)} \sum_{k=1}^n \frac{x_k}{k^{\alpha-(\beta/p)} \log^{-\gamma/p}(k+1)},\ x = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}.$$

So, $\tilde{G}_\lambda$ is given by the factorable matrix with $a_n := n^{-(1-\alpha+(\beta/p))} \log^{-\gamma/p}(n+1)$ and $b_k := k^{-(\alpha-(\beta/p))} \log^{\gamma/p}(k+1)$, where $\alpha < \frac{\beta}{p} + \frac{1}{p'} = \frac{\beta}{p} + 1 - \frac{1}{p}$ implies that $1 - \alpha + \frac{\beta}{p} > \frac{1}{p}$ and we have that $\left(1 - \alpha + \frac{\beta}{p}\right) + \left(\alpha - \frac{\beta}{p}\right) = 1 = \frac{1}{p} + \frac{1}{p'}$ and also that $\left(\frac{\gamma}{p}\right) + \left(-\frac{\gamma}{p}\right) = 0$. According to Corollary 9(ii) of [5] it follows that $\tilde{G}_\lambda \in \mathcal{L}(\ell_p)$ and the claim is proved.

It is clear from (3.10) that $\mathsf{C}^{(p,w)}$ has at most finitely many eigenvalues whenever $t_0 \in \mathbb{R}$. The following result characterizes when $\sigma_{pt}(\mathsf{C}^{(p,w)})$ is an *infinite set*; see also Remark 3.8(i) below. Recall that a sequence $u = (u_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ is *rapidly decreasing* if $(n^m u_n)_{n\in\mathbb{N}} \in \ell_1$ for every $m \in \mathbb{N}$. The space of all rapidly decreasing, $\mathbb{C}$-valued sequences is usually denoted by s.

Proposition 3.6. *Let $w = (w(n))_{n\in\mathbb{N}}$ be a positive, decreasing sequence.*

(i) *The following assertions are equivalent.*

(1) $R_w = \mathbb{R}$.
(2) $(n^m w(n))_{n\in\mathbb{N}} \in \ell_1$ *for all* $m \in \mathbb{N}$.
(3) $(n^m w(n))_{n\in\mathbb{N}} \in c_0$ *for all* $m \in \mathbb{N}$.
(4) $w \in s$.

(ii) *For each $1 < p < \infty$, the following assertions are equivalent.*

(5) $\Sigma \subseteq \sigma_{pt}(\mathsf{C}^{(p,w)})$.
(6) $(n^m w(n))_{n\in\mathbb{N}} \in \ell_p$ *for all* $m \in \mathbb{N}$.

(iii) *Any one of the equivalent assertions* (1)-(4) *implies that both* (5) *and* (6) *are valid, for every $1 < p < \infty$.*

(iv) *If* (6) *holds for some $1 < p < \infty$, then each assertion* (1)-(4) *is satisfied.*

Proof. (i) (1)$\Leftrightarrow$(2) follows from the definition of R_w.

(2)$\Leftrightarrow$(3). That (2)$\Rightarrow$(3) is immediate from $\ell_1 \subseteq c_0$.

Assume (3). Fix $t \in \mathbb{N}$ and set $m = t + 2$. Then $(n^m w(n))_{n\in\mathbb{N}} \in c_0$ implies that $\sup_{n\in\mathbb{N}} n^m w(n) < \infty$. Accordingly,

$$\sum_{n=1}^{\infty} n^t w(n) = \sum_{n=1}^{\infty} \frac{1}{n^2} n^m w(n) \leq \frac{\pi^2}{6} \sup_{n\in\mathbb{N}} n^m w(n) < \infty.$$

Since t is arbitrary, we can conclude that (2) holds.

(2)$\Leftrightarrow$(4). Clear from the definition of the space s.

(ii) Since $\mathsf{C}^{(p,w)}$ is injective, $0 \notin \sigma_{pt}(\mathsf{C}^{(p,w)})$. By (3.3) and (3.20), $\lambda \in \mathbb{C} \setminus \{0\}$ is an eigenvalue of $\mathsf{C}^{(p,w)}$ if and only if $\lambda = \frac{1}{m}$ for some $m \in \mathbb{N}$ with the corresponding 1-dimensional eigenspace generated by a vector $x^{[m]} = (x_n^{[m]})_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ satisfying $x_n^{[m]} \simeq n^{m-1}$. So, $\Sigma \subseteq \sigma_{pt}(\mathsf{C}^{(p,w)})$ if and only if $(n^{m-1})_{n\in\mathbb{N}} \in \ell_p(w)$ for all $m \in \mathbb{N}$, that is, if and only if $(n^m w(n)^{1/p})_{n\in\mathbb{N}} \in \ell_p$ for all $m \in \mathbb{N}$, which is equivalent to (6) via Lemma 3.2(iii).

(iii) Follows immediately from parts (i) and (ii) and the fact that (2)$\Rightarrow$(6) since $\ell_1 \subseteq \ell_p$ for every $1 < p < \infty$.

(iv) Immediate from $\ell_p \subseteq c_0$ for every $1 < p < \infty$. $\square$

Given a decreasing sequence $w = (w(n))_{n\in\mathbb{N}}$ of positive real numbers, set $\alpha_n := -\log w(n)$, for $n \in \mathbb{N}$. Then $w(n) = e^{-\alpha_n}$, for $n \in \mathbb{N}$. Moreover, $\alpha_n \to \infty$ for $n \to \infty$ if and only if $w(n) \to 0$ for $n \to \infty$.

Corollary 3.7. *Let $w = (w(n))_{n\in\mathbb{N}}$ be a decreasing, positive sequence.*

(i) *If $w \in s$, then* $\lim_{n\to\infty} \frac{\log n}{\alpha_n} = 0$.
(ii) *If* $\lim_{n\to\infty} \frac{\log n}{\alpha_n} = 0$ *and $w(N) < 1$ for some N, then $w \in s$.*

Proof. (i) Since $w \in s$, condition (3) in Proposition 3.6 implies that

$$\forall m \in \mathbb{N}\ \exists n_m \in \mathbb{N}\ \forall n \geq n_m \colon\ n^m w(n) = \frac{n^m}{e^{\alpha_n}} < 1,$$

i.e., that

$$\forall m \in \mathbb{N}\ \exists n_m \in \mathbb{N}\ \forall n \geq n_m \colon\ n^m < e^{\alpha_n}.$$

It follows that

$$\forall m \in \mathbb{N}\ \exists n_m \in \mathbb{N}\ \forall n \geq n_m \colon\ m \log n < \alpha_n.$$

This implies that necessarily $\alpha_n > 0$ for all $n \geq n_m$ and so

$$\forall m \in \mathbb{N}\ \exists n_m \in \mathbb{N}\ \forall n \geq n_m \colon\ \frac{\log n}{\alpha_n} < \frac{1}{m}.$$

This means precisely that $\lim_{n\to\infty} \frac{\log n}{\alpha_n} = 0$.

(ii) Fix $m \in \mathbb{N}$. Then there is $n_0 \in \mathbb{N}$ with $n_0 \geq N$ such that $\frac{\log n}{\alpha_n} < \frac{1}{m+1}$ for all $n \geq n_0$. Since $w(N) < 1$ implies that $\alpha_n = -\log w(n) > 0$ for all $n \geq n_0$, we can conclude that $(m+1)\log n < \alpha_n$, i.e., $n^{m+1}w(n) < 1$ for all $n \geq n_0$. So, $\sup_{n\in\mathbb{N}} n^{m+1}w(n) < \infty$. It follows that

$$n^m w(n) \leq \frac{1}{n} \sup_{r\in\mathbb{N}} r^{m+1} w(r), \quad n \in \mathbb{N},$$

with $\frac{1}{n}\sup_{r\in\mathbb{N}} r^{m+1}w(r) \to 0$ as $n \to \infty$. By (3)⇔(4) in Proposition 3.6(i) it follows that $w \in s$. □

Remark 3.8. (i) Concerning condition (5) in Proposition 3.6 (for any given $1 < p < \infty$), we claim that the *entire* set $\Sigma \subseteq \sigma_{pt}(\mathsf{C}^{(p,w)})$ whenever $\sigma_{pt}(C^{(p,w)})$ is an infinite set. To see this, suppose that $\frac{1}{m} \in \sigma_{pt}(C^{(p,w)})$ for some $m \in \mathbb{N}$. According to the argument in Step 7 of the proof of Theorem 3.3, we can conclude that $(n^{m-1})_{n\in\mathbb{N}} \in \ell_p(w)$. So, for all $1 \leq k < m$, it follows that

$$\sum_{n=1}^{\infty} (n^k)^p w(n) \leq \sum_{n=1}^{\infty} (n^{m-1})^p w(n) < \infty$$

and hence, via (3.3), that the vector $(x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ given by (3.20), with k in place of m, also belongs to $\ell_p(w)$, i.e., it is an eigenvector of $\mathsf{C}^{(p,w)}$ corresponding to $\lambda = \frac{1}{k}$. This shows that $\left\{\frac{1}{k}\right\}_{k=1}^{m} \subseteq \sigma_{pt}(\mathsf{C}^{(p,w)})$ whenever $\frac{1}{m} \in \sigma_{pt}(\mathsf{C}^{(p,w)})$, which clearly implies the stated claim.

(ii) Let $1 < p_0 < \infty$. The constant vector $\mathbf{1} := (1, 1, \ldots) \in \mathbb{C}^{\mathbb{N}}$ satisfies $\mathsf{C}\mathbf{1} = \mathbf{1}$ and so $1 \in \sigma_{pt}(C^{(p_0,w)})$ if and only if, $\mathbf{1} \in \ell_{p_0}(w)$, i.e., if, and only if, $w \in \ell_1$. In this case, $1 \in \sigma_{pt}(C^{(p,w)})$ for every $1 < p < \infty$. Then Theorem 3.3(ii) implies that necessarily $t_0 \in (0, \infty]$.

(iii) Let $w(n) = \frac{1}{n^\alpha}$, for all $n \in \mathbb{N}$ and some $\alpha > 0$. Then $\sum_{n=1}^{\infty} n^t w(n) < \infty$ if, and only if, $t < (\alpha - 1)$ and so $t_0 = (\alpha - 1)$. In particular, $R_w \neq \mathbb{R}$. Moreover, for any $1 < p < \infty$, we have

$$\left\{\frac{1}{m} \colon m \in \mathbb{N},\ 1 \leq m < \frac{t_0}{p} + 1\right\} = \left\{\frac{1}{m} \colon m \in \mathbb{N},\ 1 \leq m < \frac{(\alpha - 1)}{p} + 1\right\}.$$

So, given any $1 < p < \infty$, it is possible to choose an appropriate $\alpha > 0$ such that $\sigma_{pt}(\mathsf{C}^{(p,w)})$ is a *finite* set with any pre-assigned cardinality; see (3.10).

(iv) Condition (1) of Proposition 3.6, i.e., $R_w = \mathbb{R}$, implies that necessarily $S_w(p) = \emptyset$ for every $1 < p < \infty$; see Proposition 3.1(i).

Let $w = (w(n))_{n\in\mathbb{N}}$ be any decreasing, (strictly) positive sequence and let $1 < p < \infty$. The Cesàro operator $\mathsf{C}^{(p,w)}$ is similar (via an isometry) to a continuous linear operator T_w acting on ℓ_p which is defined by the factorable matrix $A(w) = (a_{nk})_{n,k\in\mathbb{N}}$ with entries $a_{nk} = a_n b_k = \frac{w(n)^{1/p}}{n} \cdot w(k)^{-1/p}$ for $1 \le k \le n$ and $a_{nk} = 0$ for $k > n$ (see the proof of Lemma 2.1). In particular, $\sigma(\mathsf{C}^{(p,w)}) = \sigma(T_w)$. Moreover, the matrix $A(w)$ satisfies the following two conditions:

(i) $\sup_{n\in\mathbb{N}} \sum_{k=1}^\infty |a_{nk}| = \sup_{n\in\mathbb{N}} \frac{w(n)^{1/p}}{n} \sum_{k=1}^n w(k)^{-1/p} \le 1$, because w decreasing implies that $\sum_{k=1}^n w(k)^{-1/p} \le nw(n)^{-1/p}$, $n \in \mathbb{N}$, and

(ii) $f_k := \lim_{n\to\infty} a_{nk} = w(k)^{-1/p} \lim_{n\to\infty} \frac{w(n)^{1/p}}{n} = 0, \ \ k \in \mathbb{N}$, because $w \in \ell_\infty$.

If, in addition, the matrix $A(w)$ also satisfies the condition

(iii) $\alpha := \lim_{n\to\infty} \sum_{k=1}^\infty a_{nk} = \lim_{n\to\infty} \frac{w(n)^{1/p}}{n} \sum_{k=1}^n w(k)^{-1/p}$ exists,

then the linear operator corresponding to $A(w)$ is a selfmap of c, the space of all convergent sequences, that is, $A(w)$ is conservative, [19, p.112].

Suppose now that the matrix $A(w)$ satisfies condition (iii) with $\alpha = 1$. Then $A(w)$ is *regular* and the linear operator corresponding to $A(w)$ is limit preserving over c, [19, p.114]. Define $\eta := \limsup_{n\to\infty} a_n b_n$. For the operator T_w (which is similar to the Cesàro operator $\mathsf{C}^{(p,w)}$) it turns out that $\eta = 0$ and so a result of Rhoades and Yildirim [19, Theorem 3] yields that

$$\left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{1}{2}\right| \le \frac{1}{2}\right\} \subseteq \sigma(\mathsf{C}^{(p,w)}), \tag{3.23}$$

after noting that $S := \overline{\{a_n b_n \colon n \in \mathbb{N}\}} = \Sigma_0 \subseteq \{\lambda \in \mathbb{C}\colon |\lambda - \frac{1}{2}| \le \frac{1}{2}\}$.

It is worthwhile to compare (3.8) with (3.23). So, let $1 < p < \infty$ and w be a positive, decreasing sequence such that $S_w(p) \neq \emptyset$. Then

$$\left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{1}{2}\right| \le \frac{1}{2}\right\} \subseteq \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{p'}{2s_p}\right| \le \frac{p'}{2s_p}\right\} \subseteq \sigma(\mathsf{C}^{(p,w)})$$

with the first inclusion holding if and only if $s_p \le p'$. Observe that if $\left(\frac{w(n)^{-1/p}}{n}\right)_{n\in\mathbb{N}} \in \ell_{p'}$, then $s_p \le p'$ is valid and conversely, if $s_p < p'$, then $\left(\frac{w(n)^{-1/p}}{n}\right)_{n\in\mathbb{N}} \in \ell_{p'}$. In this case, (3.8) is a better inclusion than (3.23). For instance, if $w(n) := \frac{1}{n^r}$ for all $n \in \mathbb{N}$ and some $r > 0$, then $\left(\frac{w(n)^{-1/p}}{n}\right)_{n\in\mathbb{N}} \in \ell_{p'}$ if, and only if, $r < 1$. On the other hand, the reverse inclusion

$$\left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{p'}{2s_p}\right| \le \frac{p'}{2s_p}\right\} \subseteq \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{1}{2}\right| \le \frac{1}{2}\right\}$$

holds if and only if $p' \le s_p$. Observe that if $\left(\frac{w(n)^{-1/p}}{n}\right)_{n\in\mathbb{N}} \notin \ell_{p'}$, then $p' \le s_p$ is valid and conversely, if $p' < s_p$, then $\left(\frac{w(n)^{-1/p}}{n}\right)_{n\in\mathbb{N}} \notin \ell_{p'}$. In this case, modulo the additional requirement that $\alpha = 1$ (see condition (iii)), in which case (3.23) is actually valid, we see that (3.23) is a better inclusion than (3.8).

The following example shows that condition (iii) above and the property $S_w(p) \neq \emptyset$ can be compatible.

Example 3.9. Fix $1 < p < \infty$. For each $n \in \mathbb{N}$ set $w(n) = \frac{1}{(\log(n+1))^p}$, in which case $w(n) \downarrow 0$. Then $S_w(p) = (1, \infty)$ and

$$\left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{p'}{2}\right| \leq \frac{p'}{2}\right\} = \sigma(C^{(p,w)}) \quad \text{with} \quad \sigma_{pt}(C^{(p,w)}) = \emptyset;$$

see Example 3.5(i) with $\gamma = p$. Moreover, concerning condition (iii) observe that

$$\frac{w(n)^{1/p}}{n} \sum_{k=1}^{n} w(k)^{-1/p} = \frac{1}{n \log(n+1)} \sum_{k=1}^{n} \log(k+1), \quad n \in \mathbb{N}.$$

Then the inequalities

$$[(n+1)\log(n+1) - n] \leq \sum_{k=1}^{n} \log(k+1) \leq [(n+2)\log(n+2) - n - 2\log 2], \quad n \in \mathbb{N},$$

imply that

$$\alpha = \lim_{n \to \infty} \frac{w(n)^{1/p}}{n} \sum_{k=1}^{n} w(k)^{-1/p} = 1.$$

We also note that $\left(\frac{w(n)^{-1/p}}{n}\right)_{n\in\mathbb{N}} = \left(\frac{\log(n+1)}{n}\right)_{n\in\mathbb{N}} \in \ell_{p'}$.

We conclude this section with some comments about the mean ergodicity and the linear dynamics of $\mathsf{C}^{(p,w)}$. For X a Banach space, recall that $T \in \mathcal{L}(X)$ is *mean ergodic* if its sequence of Cesàro averages $T_{[n]} := \frac{1}{n}\sum_{m=1}^{n} T^m$, for $n \in \mathbb{N}$, converges to some operator $P \in \mathcal{L}(X)$ for the strong operator topology, i.e., $\lim_{n\to\infty} T_{[n]}x = Px$ for each $x \in X$, [9, Ch.VIII]. Since $\frac{1}{n}T^n = T_{[n]} - \frac{n-1}{n}T_{[n-1]}$, for $n \in \mathbb{N}$ (with $T_{[0]} := I$), a necessary condition for T to be mean ergodic is that $\lim_{n\to\infty} \frac{1}{n}T^n = 0$ (in the strong operator topology).

Let w be a positive, decreasing sequence and $1 < p < \infty$ with $S_p(w) \neq \emptyset$. If $s_p < p'$, then it follows from (3.6) that $\mu := \frac{1}{2}\left(1 + \frac{p'}{s_p}\right) \in \sigma_{pt}((\mathsf{C}^{(p,w)})')$ and so there exists a non-zero vector $x' \in \ell_{p'}(w^{-p'/p})$ such that $(\mathsf{C}^{(p,w)})'x' = \mu x'$. Choose any $x \in \ell_p(w) \setminus \{0\}$ satisfying $\langle x, x'\rangle \neq 0$. Then

$$\langle \frac{1}{n}(\mathsf{C}^{(p,w)})^n x, x'\rangle = \frac{1}{n}\langle x, ((\mathsf{C}^{(p,w)})')^n x'\rangle = \frac{\mu^n}{n}\langle x, x'\rangle, \quad n \in \mathbb{N},$$

with $\mu > 1$ and so the set $\left\{\frac{1}{n}(\mathsf{C}^{(p,w)})^n x\colon n \in \mathbb{N}\right\}$ is unbounded in $\ell_p(w)$. In particular, the sequence $\left\{\frac{1}{n}(\mathsf{C}^{(p,w)})^n\right\}_{n\in\mathbb{N}}$ cannot converge to 0 for the strong operator topology in $\mathcal{L}(\ell_p(w))$. Accordingly, $\mathsf{C}^{(p,w)}$ *fails to be mean ergodic whenever* $s_p < p'$. This is the case when $w(n) = 1$, for all $n \in \mathbb{N}$, in which case $s_p = 1$, and we recover the known fact that the classical Cesàro operator $\mathsf{C}^{(p)}$ fails to be mean ergodic for every $1 < p < \infty$; see [3, Section 4], where it is also shown that the Cesàro operator fails to be mean ergodic in the classical Banach sequence spaces c_0, c, ℓ_p $(1 < p \leq \infty)$, bv_0 and bv but, that it is mean ergodic in bv_p $(1 < p < \infty)$.

Concerning the dynamics of a continuous linear operator T defined on a separable Banach space X, recall that T is *hypercyclic* if there exists $x \in X$ such that the orbit $\{T^n x \colon n \in \mathbb{N}_0\}$ is dense in X. If, for some $x \in X$, the projective orbit $\{\lambda T^n x \colon \lambda \in \mathbb{C},\ n \in \mathbb{N}_0\}$ is dense in X, then X is called *supercyclic.* Clearly, hypercyclicity always implies supercyclicity.

Let now w be a positive, decreasing sequence and $1 < p < \infty$. According to Remark 3.4(iv) the *infinite* set $\Sigma \subseteq \sigma_{pt}((\mathsf{C}^{(p,w)})')$. Then, by a result of Ansari and Bourdon [4, Theorem 3.2], $\mathsf{C}^{(p,w)}$ *is not supercyclic and hence, also not hypercyclic.*

4. Compactness of $\mathsf{C}^{(p,w)}$

According to (3.21), for each $1 < p < \infty$ the classical Cesàro operator $\mathsf{C}^{(p)} \in \mathcal{L}(\ell_p)$ is surely not compact. However, in the presence of a positive weight $w \downarrow 0$, this may no longer be the case for $\mathsf{C}^{(p,w)}$ acting on $\ell_p(w)$. We begin with the following fact.

Proposition 4.1. *Let w be a positive, decreasing weight.*

(i) *For every $1 < p < \infty$ we have $\Sigma \subseteq \sigma(\mathsf{C}^{(p,w)})$.*

(ii) *Suppose that $\mathsf{C}^{(p,w)}$ is a compact operator, for some $1 < p < \infty$. Then*

$$\sigma(\mathsf{C}^{(p,w)}) = \Sigma_0 \quad \text{and} \quad \sigma_{pt}(\mathsf{C}^{(p,w)}) = \Sigma. \tag{4.1}$$

Moreover, $w \in s$ and $r(\mathsf{C}^{(p,w)}) < \|\mathsf{C}^{(p,w)}\|$.

Proof. (i) According to Remark 3.4(iv) we have $\Sigma \subseteq \sigma_{pt}((\mathsf{C}^{(p,w)})')$. But, always $\sigma_{pt}((\mathsf{C}^{(p,w)})') \subseteq \sigma(\mathsf{C}^{(p,w)})$, [9, p. 581], and so $\Sigma \subseteq \sigma(\mathsf{C}^{(p,w)})$.

(ii) Since $\mathsf{C}^{(p,w)}$ is injective, $0 \notin \sigma_{pt}(\mathsf{C}^{(p,w)})$. The compactness of $\mathsf{C}^{(p,w)}$ then implies that $\sigma_{pt}(\mathsf{C}^{(p,w)}) = \sigma(\mathsf{C}^{(p,w)}) \setminus \{0\}$, [15, Theorem 3.4.23]. According to the proof of Step 8 for Theorem 3.3 we also have that $\sigma_{pt}(\mathsf{C}^{(p,w)}) \subseteq \Sigma$. In view of part (i), the equalities in (4.1) follow.

By Theorem 3.3(ii) we must have $R_w = \mathbb{R}$ (if not, then t_0 is finite and so (3.10) would imply that $\sigma_{pt}(\mathsf{C}^{(p,w)})$ is finite which is a contradiction to (4.1)). Then, via Proposition 3.6(i), we can conclude that $w \in s$.

It follows from (2.3) and the equality $r(\mathsf{C}^{(p,w)}) = 1$ (see (4.1)) that $r(\mathsf{C}^{(p,w)}) < \|\mathsf{C}^{(p,w)}\|$. □

To decide when $\mathsf{C}^{(p,w)}$ is compact, first observe that $\mathsf{C}^{(p,w)} = \Phi_w^{-1} T_w \Phi_w$ (see Lemma 2.1 and its proof), where $T_w \in \mathcal{L}(\ell_p)$ is given by (2.2). Given any $x \in B_p := \{x \in \ell_p \colon \|x\| \leq 1\}$ and $n \in \mathbb{N}$, it follows from Hölder's inequality that

$$\begin{aligned} \sum_{n=i}^{\infty} |(T_w x)_n|^p &= \sum_{n=i}^{\infty} \frac{w(n)}{n^p} \left| \sum_{k=1}^{n} \frac{1}{w(k)^{1/p}} \cdot x_k \right|^p \\ &\leq \sum_{n=i}^{\infty} \frac{w(n)}{n^p} \left(\sum_{k=1}^{n} \frac{1}{w(k)^{p'/p}} \right)^{p/p'}. \end{aligned}$$

So, T_w (hence, also $\mathsf{C}^{(p,w)}$) will be compact whenever w satisfies the following

$$\text{Compactness criterion:} \qquad \sum_{n=1}^{\infty} \frac{w(n)}{n^p} \left(\sum_{k=1}^{n} \frac{1}{w(k)^{p'/p}} \right)^{p/p'} < \infty. \tag{4.2}$$

Indeed, (4.2) implies that $\lim_{n\to\infty} \sum_{i=n}^{\infty} |(T_w x)_i|^p = 0$ *uniformly* with respect to $x \in B_p$, from which the relative compactness in ℓ_p of the bounded set $T_w(B_p) \subseteq \ell_p$ follows, [9, pp.338-339].

We introduce some notation. Let w be a positive, decreasing sequence. Define

$$A_n(p,w) := w(n)^{p'/p} \sum_{k=1}^{n} \frac{1}{w(k)^{p'/p}}, \quad n \in \mathbb{N},\ 1 < p < \infty.$$

The compactness criterion (4.2) then states that $\mathsf{C}^{(p,w)}$ is a compact operator if $\sum_{n=1}^{\infty} (A_n(p,w))^{p/p'}/n^p < \infty$.

Theorem 4.2. *Suppose, for some $1 < p < \infty$, that there exist constants $M > 0$ and $0 \le \alpha < 1$ such that*

$$A_n(p,w) \le M n^{\alpha}, \quad n \in \mathbb{N}. \tag{4.3}$$

Then $\mathsf{C}^{(q,w)}$ is a compact operator for every $1 < q \le p$. In particular, $w \in s$.

Proof. Observe, for fixed $1 < q \le p$, that

$$\gamma := \frac{q'}{q} - \frac{p'}{p} = \frac{1}{q-1} - \frac{1}{p-1} = \frac{p-q}{(q-1)(p-1)} \ge 0.$$

For each $n \in \mathbb{N}$ we have

$$\sum_{k=1}^{n} \frac{1}{w(k)^{q'/q}} = \sum_{k=1}^{n} \frac{1}{w(k)^{p'/p}} \cdot w(k)^{-\gamma}.$$

Accordingly, for each $n \in \mathbb{N}$,

$$\begin{aligned} A_n(q,w) &= \frac{w(n)^{q'/q}}{w(n)^{p'/p}} \cdot w(n)^{p'/p} \sum_{k=1}^{n} \frac{1}{w(k)^{p'/p}} \cdot w(k)^{-\gamma} \\ &= w(n)^{p'/p} \sum_{k=1}^{n} \frac{1}{w(k)^{p'/p}} \cdot \left(\frac{w(n)}{w(k)}\right)^{\gamma}. \end{aligned}$$

Since w is decreasing, $\frac{w(n)}{w(k)} \le 1$ for all $1 \le k \le n$ and so

$$A_n(q,w) \le w(n)^{p'/p} \sum_{k=1}^{n} \frac{1}{w(k)^{p'/p}} = A_n(p,w) \le M n^{\alpha}.$$

Accordingly,

$$\sum_{n=1}^{\infty} \frac{(A_n(q,w))^{q/q'}}{n^q} \le M^{q/q'} \sum_{n=1}^{\infty} \frac{n^{\alpha q/q'}}{n^q} = M^{q/q'} \sum_{n=1}^{\infty} \frac{1}{n^{q-(\alpha q/q')}}.$$

But, $q - \frac{\alpha q}{q'} = q - \alpha(q-1) = q(1-\alpha) + \alpha > (1-\alpha) + \alpha = 1$ and so

$$\sum_{n=1}^{\infty} \frac{(A_n(q,w))^{q/q'}}{n^q} < \infty.$$

Then the compactness criterion yields that $\mathsf{C}^{(q,w)}$ is a compact operator.

That $w \in s$ is a consequence of Proposition 4.1(ii). □

The following consequence of Theorem 4.2 leads to a rich supply of weights w for which $\mathsf{C}^{(p,w)}$ is compact.

Corollary 4.3. *Let w be a positive weight with $w \downarrow 0$. If the limit*

$$l := \lim_{n\to\infty} \frac{w(n)}{w(n-1)} \tag{4.4}$$

exists in $\mathbb{R} \setminus \{1\}$, then $\mathsf{C}^{(p,w)}$ is compact for every $1 < p < \infty$.

Proof. Fix $1 < p < \infty$. According to Theorem 4.2 (with $\alpha = 0$) it suffices to prove that $\sup_{n\in\mathbb{N}} A_n(p,w) < \infty$. Set $a_n := \sum_{k=1}^n w(k)^{-p'/p}$ and $b_n := w(n)^{-p'/p}$ for $n \in \mathbb{N}$. Since $w \downarrow 0$, we have $b_n \uparrow \infty$. Moreover, the limit

$$\lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = \lim_{n\to\infty} \frac{w(n)^{-p'/p}}{w(n)^{-p'/p} - w(n-1)^{-p'/p}}$$
$$= \lim_{n\to\infty} \frac{1}{1 - (w(n)/w(n-1))^{p'/p}} = \frac{1}{1 - l^{p'/p}}$$

exists in $\mathbb{R}$ as $l \neq 1$. According to the Stolz-Cesàro criterion, [16, Theorem 1.22], it follows that also $\lim_{n\to\infty} \frac{a_n}{b_n} = 1/(1 - l^{p'/p}) \in \mathbb{R}$, i.e., $\lim_{n\to\infty} A_n(p,w) = 1/(1 - l^{p'/p}) \in \mathbb{R}$. In particular, $\sup_{n\in\mathbb{N}} A_n(p,w) < \infty$ is indeed satisfied. □

Remark 4.4. (i) Let w be a positive, decreasing weight.

(a) According to (3.8), if $\mathsf{C}^{(p,w)}$ is a compact operator for some $1 < p < \infty$, then $S_w(p) = \emptyset$.

(b) The condition $w \downarrow 0$ by itself need not imply that $S_w(p) = \emptyset$ (see Examples 3.5, for instance).

(ii) Suppose $S_w(p) \neq \emptyset$ for some $1 < p < \infty$. Then $\mathsf{C}^{(q,w)}$ fails to be compact for every $q \in [p, \infty)$. This follows from part (i)(a) and Proposition 3.1(iii).

(iii) The following examples (a)-(c) all fall within the scope of Corollary 4.3. So, in each case $w \in s$ and the identities in (4.1) hold; see Proposition 4.1.

(a) For any fixed $a > 1$ and $r \geq 0$ set $w(n) := n^r/a^n$ for $n \in \mathbb{N}$. Then

$$\lim_{n\to\infty} \frac{w(n)}{w(n-1)} = a^{-1} \neq 1.$$

(b) For any fixed $a \geq 1$, the weight $w(n) := a^n/n!$ for $n \in \mathbb{N}$ satisfies

$$\lim_{n\to\infty} \frac{w(n)}{w(n-1)} = 0 \neq 1. \tag{4.5}$$

(c) The weight $w(n) := 1/n^n$ for $n \in \mathbb{N}$ also satisfies (4.5).

We point out, since w is decreasing, that $\frac{w(n)}{w(n-1)} \in (0,1]$ for all $n \in \mathbb{N}$. Hence, whenever the limit (4.4) exists, then necessarily $l \in [0,1]$.

As an application, suppose that the positive, decreasing weight w has the property that $l := \lim_{n\to\infty} \frac{w(n)}{w(n-1)}$ exists in $[0,1)$. Then, for each $r > 0$, the positive, decreasing weight $w^r \colon n \mapsto w(n)^r$, for $n \in \mathbb{N}$, satisfies $\lim_{n\to\infty} \frac{w(n)^r}{w(n-1)^r} = l^r \in [0,1)$. Hence, $\mathsf{C}^{(p,w^r)}$ is a compact operator for every $1 < p < \infty$.

(iv) The following criterion is sufficient to ensure that the limit (4.4) exists in $\mathbb{R} \setminus \{1\}$. Hence, both Proposition 4.1 and Corollary 4.3 are applicable to such a weight w. In particular, $w \in s$.

Let $\beta = (\beta_n)_{n\in\mathbb{N}}$ be a positive, increasing sequence with $\beta \uparrow \infty$ such that $\lim_{n\to\infty}(\beta_n - \beta_{n-1}) = \infty$. Then the weight $w(n) := e^{-\beta_n}$, for $n \in \mathbb{N}$, satisfies $l := \lim_{n\to\infty} \frac{w(n)}{w(n-1)} = 0 \neq 1$.

It is routine to verify that $\lim_{n\to\infty} \frac{w(n)}{w(n-1)} = 0$.

For the weight $w(n) := a^{-n}$ for $n \in \mathbb{N}$ (with $a > 1$) we have that $\beta_n := -\log w(n) = n\log(a) \uparrow \infty$ but, $(\beta_n - \beta_{n-1})\log(a) \not\to \infty$ for $n \to \infty$. So, the above criterion is *not* applicable to this weight. However, according to part (iii)(a) of this remark (with $r = 0$) the weight w is admissible for Corollary 4.3.

The following examples illustrate that Theorem 4.2 is more general than Corollary 4.3.

Examples 4.5. (i) Fix $0 < \beta < 1$ and set $w_\beta(n) := e^{-n^\beta}$ for $n \in \mathbb{N}$, in which case $w \downarrow 0$, but

$$\lim_{n\to\infty} \frac{w_\beta(n)}{w_\beta(n-1)} = \lim_{n\to\infty} e^{(n-1)^\beta - n^\beta} = \lim_{n\to\infty} e^{-\beta/n^{(1-\beta)}} = 1,$$

as $(n-1)^\beta - n^\beta = n^\beta\left[\left(1-\frac{1}{n}\right)^\beta - 1\right] = n^\beta\left[1 - \frac{\beta}{n} + o\left(\frac{1}{n}\right) - 1\right] \simeq -\frac{\beta}{n^{1-\beta}}$ for $n \to \infty$. So, Corollary 4.3 is not applicable. We show that Theorem 4.2 does apply.

Fix $1 < p < \infty$ and set $\gamma := \frac{p'}{p}$. Then, for each $n \in \mathbb{N}$, we have that

$$\begin{aligned} A_n(p,w) &= e^{-\gamma n^\beta}\sum_{k=1}^{n} e^{\gamma k^\beta} \leq e^{-\gamma n^\beta}\int_1^{n+1} e^{\gamma x^\beta}\,dx \\ &= \frac{e^{-\gamma n^\beta}}{\beta}\int_1^{(n+1)^\beta} e^{\gamma t} t^{\frac{1}{\beta}-1}\,dt \leq \frac{e^{-\gamma n^\beta}}{\beta}\int_1^{(n+1)^\beta} e^{\gamma t} t^m\,dt, \end{aligned}$$

where $m \in \mathbb{N}$ is chosen minimal such that $(m-1) < \frac{1}{\beta} - 1 \leq m$. An integration by parts $(m+1)$-times yields that

$$\int_1^{(n+1)^\beta} e^{\gamma t} t^m\,dt \leq a_0 + a_1(n+1)^\beta e^{\gamma(n+1)^\beta} + a_2(n+1)^{2\beta} e^{\gamma(n+1)^\beta} + \ldots + a_m(n+1)^{m\beta} e^{\gamma(n+1)^\beta}$$

for positive constants $a_0,\ a_1, \ldots,\ a_m$. It follows that

$$\int_1^{(n+1)^\beta} e^{\gamma t} t^m\,dt \leq M(1+n)^{m\beta} e^{\gamma(1+n)^\beta}, \quad n \in \mathbb{N},$$

for some constant $M > 0$. Accordingly,

$$A_n(p,w) \leq \frac{M}{\beta}(1+n)^{m\beta} e^{\gamma((1+n)^\beta - n^\beta)}, \quad n \in \mathbb{N}.$$

Since $(n+1)^\beta - n^\beta \simeq \frac{\beta}{n^{1-\beta}}$ and $(1+n)^{m\beta} \simeq n^{m\beta}$ for $n \to \infty$, there exists $K > 0$ (independent of n) such that

$$A_n(p,w) \leq K n^{m\beta}, \quad i \in \mathbb{N}.$$

Since $(m-1) < \frac{1}{\beta} - 1$ implies that $\alpha := m\beta \in (0,1)$, Theorem 4.2 yields that $\mathsf{C}^{(p,w_\beta)}$ is compact.

For $\beta \geq 1$ the compactness of $\mathsf{C}^{(p,w_\beta)}$ follows from Corollary 4.3. Indeed, if $\beta = 1$, then $w(n) = e^{-n}$ for $n \in \mathbb{N}$ and so Remark 4.4(iii)(a) implies the compactness of $\mathsf{C}^{(p,w_\beta)}$. For $\beta > 1$, observe from above that

$$\lim_{n\to\infty} \frac{w_\beta(n)}{w_\beta(n-1)} = \lim_{n\to\infty} e^{(n-1)^\beta - n^\beta} = \lim_{n\to\infty} e^{-\beta n^{\beta-1}} = 0$$

and so the compactness of $\mathsf{C}^{(p,w_\beta)}$ follows again from Corollary 4.3.

(ii) There also exist positive, decreasing weights $w \in s$ such that the sequence $\{\frac{w(n)}{w(n-1)}\}_{n\in\mathbb{N}}$ *fails to converge* at all, yet $\mathsf{C}^{(p,w)}$ is a compact operator for every $1 < p < \infty$.

Define $w(n) := \frac{1}{j^j}$, $n = 2j-1$, and $w(n) := \frac{1}{2j^j}$, $n = 2j$, for each $j \in \mathbb{N}$. Then w is (strictly) decreasing to 0. For $n_j := 2j$, $j \in \mathbb{N}$, we have $\frac{w(n_j)}{w(n_j-1)} = \frac{1}{2}$ for all $j \in \mathbb{N}$ and so $\lim_{j\to\infty} \frac{w(n_j)}{w(n_j-1)} = \frac{1}{2}$, whereas for $n_r := 2r+1$, $r \in \mathbb{N}$, the subsequence $\{\frac{w(n_r)}{w(n_r-1)}\}_{r\in\mathbb{N}}$ of $\{\frac{w(n)}{w(n-1)}\}_{n\in\mathbb{N}}$ converges to 0. Accordingly, the sequence $\{\frac{w(n)}{w(n-1)}\}_{n\in\mathbb{N}}$ is not convergent and so Corollary 4.3 is not applicable.

Fix $1 < p < \infty$ and set $\gamma := \frac{p'}{p} > 0$. To establish the compactness of $\mathsf{C}^{(p,w)}$ observe, for every $j \in \mathbb{N}$, that

$$A_{2j}(p,w) = \frac{1}{(2j^j)^\gamma}\left(\sum_{k=1}^{j}(k^k)^\gamma + \sum_{k=1}^{j}(2k^k)^\gamma\right) = \frac{1+2^\gamma}{2^\gamma}\frac{1}{(j^j)^\gamma}\sum_{k=1}^{j}(k^k)^\gamma, \quad (4.6)$$

and that

$$A_{2j-1}(p,w) = 1 + \frac{1}{(j^j)^\gamma}\sum_{k=1}^{2(j-1)} w(k)^{-\gamma} = 1 + \frac{(j-1)^{(j-1)\gamma}}{(j^j)^\gamma} A_{2(j-1)}(p,w), \quad (4.7)$$

with $\lim_{j\to\infty} \frac{(j-1)^{(j-1)\gamma}}{(j^j)^\gamma} = 0$. Set $a_j := \sum_{k=1}^{j}(k^k)^\gamma$ and $b_j := (j^j)^\gamma$ for $j \in \mathbb{N}$. Then $b_j \uparrow \infty$. Moreover,

$$\lim_{j\to\infty} \frac{a_j - a_{j-1}}{b_j - b_{j-1}} = \lim_{j\to\infty} \frac{(j^j)^\gamma}{(j^j)^\gamma - ((j-1)^{j-1})^\gamma} = \lim_{j\to\infty} \frac{1}{1 - \frac{(j-1)^{(j-1)\gamma}}{(j^j)^\gamma}} = 1.$$

According to the Stolz-Cesàro criterion, [16, Theorem 1.22], it follows that also $\lim_{j\to\infty} \frac{a_j}{b_j} = 1$. So, via (4.6) and (4.7) we can conclude that $\lim_{j\to\infty} A_{2j}(p,w) = \frac{1+2^\gamma}{2^\gamma}$ and $\lim_{j\to\infty} A_{2j-1}(p,w) = 1$. In particular, $\sup_{i\in\mathbb{N}} A_i(p,w) < \infty$ and so Theorem 4.2 applies (with $\alpha = 0$). Hence, $\mathsf{C}^{(p,w)}$ is compact and $w \in s$.

The following result is a comparison type criterion for compactness. One knows something about the compactness of $\mathsf{C}^{(p,w)}$ for a certain weight w and $1 < p < \infty$ and one has a second weight v whose growth relative to w is *controlled*. Then also $\mathsf{C}^{(p,v)}$ is compact.

Proposition 4.6. *Let w be a positive, decreasing sequence. Suppose, for some $1 < p < \infty$, that there exists $0 \leq \alpha < 1$ such that*

$$A_n(p,w) \leq M n^\alpha, \quad n \in \mathbb{N}, \quad (4.8)$$

for some constant $M > 0$.

Let v be any positive, decreasing sequence such that $\{\frac{v(n)}{w(n)}\}_{n\in\mathbb{N}} \in \ell_\infty$ and satisfying

$$w(n) \leq Kn^\beta v(n), \quad n \in \mathbb{N}, \tag{4.9}$$

for some $0 \leq \beta < (p-1)(1-\alpha)$ and some constant $K > 0$. Then $\mathsf{C}^{(q,v)}$ is a compact operator for every $1 < q \leq p$.

Proof. Let $L := \sup_{n\in\mathbb{N}} \frac{v(n)}{w(n)}$. Then, for each $n \in \mathbb{N}$, we have via (4.8) and(4.9) that

$$\begin{aligned} A_n(p,v) &= v(n)^{p'/p}\sum_{k=1}^n \frac{1}{v(k)^{p'/p}} = \left(\frac{v(n)}{w(n)}\right)^{p'/p} w(n)^{p'/p}\sum_{k=1}^n \frac{1}{w(k)^{p'/p}}\cdot\left(\frac{w(k)}{v(k)}\right)^{p'/p} \\ &\leq L^{p'/p}w(n)^{p'/p}\sum_{k=1}^n \frac{1}{w(k)^{p'/p}}(Kk^\beta)^{p'/p} \leq (LK)^{p'/p}w(n)^{p'/p}\sum_{k=1}^n \frac{1}{w(k)^{p'/p}}n^{\beta p'/p} \\ &= (LK)^{p'/p}n^{\beta p'/p}A_n(p,w) \leq M(LK)^{p'/p}n^{\alpha+(\beta p'/p)}. \end{aligned}$$

Moreover, $\alpha + \frac{\beta p'}{p} = \alpha + \frac{\beta}{(p-1)} < 1$ because $0 \leq \beta < (p-1)(1-\alpha)$ implies $\frac{\beta}{(p-1)} < (1-\alpha)$ which implies $\alpha + \frac{\beta}{(p-1)} < 1$. So, Theorem 4.2 applied to v (with $\alpha + \frac{\beta}{(p-1)}$ in place of α) implies that $\mathsf{C}^{(q,v)}$ is compact for all $1 < q \leq p$. □

Example 4.7. Let $v(n) := \frac{1}{e^{n^\beta}\log^\gamma(n+1)}$ for $n \in \mathbb{N}$, where $0 < \beta < 1$ and $\gamma > 0$. Then $\mathsf{C}^{(p,v)}$ is compact for every $1 < p < \infty$. Observe that $\lim_{n\to\infty} \frac{v(n)}{v(n-1)} = 1$ and so Corollary 4.3 is not applicable.

So, fix $1 < p < \infty$. Define $w(n) := e^{-n^\beta}$ for $n \in \mathbb{N}$. According to Example 4.5(i), there exist constants $M > 0$ and $0 < \alpha < 1$ such that

$$A_n(p,w) \leq Mn^\alpha, \quad n \in \mathbb{N}.$$

Since $v(n) \leq w(n)$ for $n \in \mathbb{N}$, it is clear that $\left\{\frac{v(n)}{w(n)}\right\}_{n\in\mathbb{N}} \in \ell_\infty$. Choose any $r \in (0, (p-1)(1-\alpha))$. Then

$$\frac{w(n)}{v(n)} = \log^\gamma(n+1) = \frac{\log^\gamma(n+1)}{n^r}\cdot n^r \leq Kn^r, \quad n \in \mathbb{N},$$

for some $K > 0$ (as $\lim_{n\to\infty} \frac{\log^\gamma(n+1)}{n^r} = 0$). According to Proposition 4.6, we can conclude that $\mathsf{C}^{(p,v)}$ is compact.

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