SPECTRUM AND COMPACTNESS OF THE CESÀRO OPERATOR ON WEIGHTED $\ell_p$ SPACES

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Abstract. An investigation is made of the continuity, the compactness and the spectrum of the Cesàro operator $C$ when acting on the weighted Banach sequence spaces $\ell_p(w)$, $1 < p < \infty$, for a positive, decreasing weight $w$, thereby extending known results for $C$ when acting on the classical spaces $\ell_p$.

1. Introduction

The discrete Cesàro operator $C$ is defined on the linear space $\mathbb{C}^N$ (consisting of all scalar sequences) by

$$Cx := \left( x_1, \frac{x_1 + x_2}{2}, \ldots, \frac{x_1 + \ldots + x_n}{n}, \ldots \right), \quad x = (x_n)_{n \in \mathbb{N}} \in \mathbb{C}^N. \quad (1.1)$$

The operator $C$ is said to act in a vector subspace $X \subseteq \mathbb{C}^N$ if it maps $X$ into itself. Of particular interest is the situation when $X$ is a Banach space. The fundamental questions in this case are: Is $C: X \to X$ continuous and, if so, what is the spectrum of $C$: $X \to X$? Amongst the classical Banach spaces $X \subseteq \mathbb{C}^N$ where answers are known we mention $\ell_p$ (1 < $p$ < $\infty$), [6], [14], and $c_0$, [14], [18], and both $c$, $\ell_\infty$, [1], [14], as well as $ces_p$, $p \in \{0\} \cup (1, \infty)$, [8], and the spaces of bounded variation $bv_0$, [17], and $bv_p$, 1 ≤ $p$ < $\infty$, [2]. There is no claim that this list of spaces (and references) is complete.

The aim of this paper is to investigate the two questions mentioned above for $C$ acting on the weighted Banach spaces $\ell_p(w)$. To be precise, let $w = (w(n))_{n=1}^\infty$ be a bounded sequence, always assumed to be strictly positive. Define the space

$$\ell_p(w) := \left\{ x = (x_n)_{n \in \mathbb{N}} \in \mathbb{C}^N : \|x\|_{p,w} := \left( \sum_{n=1}^\infty |x_n|^{p,w(n)} \right)^{1/p} < \infty \right\},$$

for each 1 < $p$ < $\infty$, equipped with the norm $\| \cdot \|_{p,w}$. Observe that $\ell_p(w)$ is isometric to $\ell_p$ via the linear multiplication operator

$$\Phi_w: \ell_p(w) \to \ell_p, \quad x = (x_n)_{n \in \mathbb{N}} \to \Phi_w(x) := (w(n)^{1/p}x_n)_{n \in \mathbb{N}}.$$

Therefore, the $\ell_p(w)$ are Banach spaces. The dual space $(\ell_p(w))'$ of $\ell_p(w)$ is the Banach space $\ell_p'(w)$, where $\frac{1}{p} + \frac{1}{p'} = 1$ (i.e., $p'$ is the conjugate exponent of $p$) and $v(n) = w(n)^{-1/p'}$ for $n \in \mathbb{N}$. In particular, $\ell_p(w)$ is reflexive and separable for 1 < $p$ < $\infty$. Moreover, the canonical unit vectors $e_k := (\delta_{kn})_{n \in \mathbb{N}}$, for $k \in \mathbb{N}$, form an unconditional basis in $\ell_p(w)$ for 1 < $p$ < $\infty$. If $\inf_{n \in \mathbb{N}} w(n) > 0$, $w$ is separable for $\ell_p$.

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then $\ell_p(w) = \ell_p$ with equivalent norms and we are in the standard situation. Accordingly, we are mainly interested in the case when $\inf_{n \in \mathbb{N}} w(n) = 0$.

By Hardy’s inequality, [13, Theorem 326, p.239], for every $1 < p < \infty$ the restriction of the Cesàro operator $C: \mathbb{C}^N \to \mathbb{C}^N$ as given in (1.1) defines a bounded linear operator from $\ell_p$ into itself with operator norm equal to $p'$. Denote these operators by $C^{(p)}$ so that $\|C^{(p)}\| = p'$. In Section 2, where the papers [5], [10], [11] are relevant, we discuss various aspects of the continuity of $C$ when restricted to $\ell_p(w), 1 < p < \infty$; denote this operator by $C^{(p,w)}$ whenever it is continuous.

For any Banach space $X$, let $I$ denote the identity operator on $X$ and $L(X)$ denote the space of all continuous linear operators from $X$ into itself. The spectrum and the resolvent set of $T \in L(X)$ are denoted by $\sigma(T)$ and $\rho(T)$, respectively; see [9, Ch. VII], for example. The set of all eigenvalues of $T$, also called the point spectrum of $T$, is denoted by $\sigma_p(T)$. The spectral radius $r(T) := \sup\{\{\lambda: \lambda \in \sigma(T)\}$ of $T$ always satisfies $r(T) \leq \|T\|$, [9, p.567].

Section 3 is devoted to an analysis of the spectrum of $C$. The main result is Theorem 3.3; it is complemented by Examples 3.5 which clarify the scope of this theorem. Unlike for $C^{(p)}$, it can happen that $\sigma_p(C^{(p,w)}) \neq \emptyset$. Actually, $C^{(p,w)}$ can even have infinitely many eigenvalues; see Proposition 3.6.

The final section deals with the compactness of $C^{(p,w)}$. Relevant is how fast $w$ decreases to 0; see Proposition 4.1, Theorem 4.2, Corollary 4.3 and Proposition 4.6.

## 2. Continuity of $C$ in weighted $\ell_p$ spaces

Some of the concepts and results from [11] that are quoted in this section actually have their origins in the paper [10]. We begin with the following fact.

**Lemma 2.1.** Let $w = (w(n))_{n=1}^{\infty}$ be a positive sequence and $1 < p < \infty$. Then the Cesàro operator $C$ maps $\ell_p(w)$ continuously into itself if, and only if,

$$\sup_{m \in \mathbb{N}} \left(\sum_{k=1}^{m} w(k)^{-p'/p} \right)^{-1/\left(\sum_{k=1}^{m} w(k)^{-p'/p} \right)} \left(\sum_{n=1}^{m} \frac{w(n)}{n^p} \left(\sum_{k=1}^{n} w(k)^{-p'/p} \right)^{p'}\right) < \infty,$$

i.e., if, and only if, there exists $K > 0$ such that

$$\sum_{n=1}^{m} \frac{w(n)}{n^p} \left(\sum_{k=1}^{n} w(k)^{-p'/p} \right)^{p'} \leq K \left(\sum_{k=1}^{m} w(k)^{-p'/p} \right), \quad m \in \mathbb{N}. \quad (2.1)$$

Moreover, if the constant $K$ satisfying (2.1) is chosen as small as possible, then the operator norm of $C$ is at most $p/K^{1/p}$.

**Proof.** Let $T_w: \mathbb{C}^N \to \mathbb{C}^N$ denote the linear operator defined by

$$T_w x := \frac{(w(n))_{1}^{1/p}}{n} \sum_{k=1}^{n} w(k)^{-1/p} x_k, \quad x = (x_n)_{n \in \mathbb{N}} \in \mathbb{C}^N. \quad (2.2)$$

Then $\Phi_w C = T_w \Phi_w$. Since $\Phi_w$ is isometric from $\ell_p(w)$ onto $\ell_p$, it follows that $C$ maps $\ell_p(w)$ continuously into itself if, and only if, $T_w$ maps $\ell_p$ continuously into itself. But, the matrix of $T_w$ is factorable (cf. [5, §4] with $a_n = w(n)^{1/p}/n$ and $b_k = w(k)^{-1/p}$ for $1 \leq k \leq n$) and so it follows from [5, Theorem 2] that $T_w \in L(p^p)$ if, and only if, (2.1) holds.
Let $w = (w(n))_{n=1}^\infty$ be a decreasing, positive sequence and $1 < p < \infty$. Then the Cesàro operator $C^{(p,w)} \in \mathcal{L}(\ell_p(w))$ and satisfies
\[ 1 < \left( \frac{1}{w(1)} \sum_{n=1}^\infty \frac{w(n)}{n^p} \right)^{1/p} \leq \|C^{(p,w)}\| \leq p'. \tag{2.3} \]

**Proof.** Fix $m \in \mathbb{N}$. Because $w$ is decreasing, we have
\[ \sum_{n=m}^\infty \frac{w(n)}{n^p} \left( \sum_{k=1}^n w(k)^{-p'/p} \right)^p \leq \sum_{n=m}^\infty \left( \frac{w(n)^{1/p}}{n} \sum_{k=1}^n \frac{w(k)^{-p'/p}}{n} \right)^p = \sum_{n=m}^\infty \frac{w(n)^{1/p}}{n} \cdot \frac{n}{w(n)^{p'/p}} = \sum_{n=1}^\infty \frac{w(n)^{1/p}}{n}, \]
which is precisely (2.1) with $K = 1$. So, Lemma 2.1 implies that $C$ is continuous on $\ell_p(w)$ with $\|C^{(p,w)}\| \leq p'$.

For an alternate proof of the continuity of $C^{(p,w)}$, based directly on Hardy’s inequality in $\ell_p$, see [11, Proposition 5.1].

Since $T_w = \Phi_w C^{(p,w)} \Phi_w^{-1}$, with $\Phi_w$ mapping the closed unit ball of $\ell_p(w)$ onto that of $\ell_p$ and $\Phi_w^{-1}$ mapping the closed unit ball of $\ell_p$ onto that of $\ell_p(w)$, it follows that $\|T_w\| = \|C^{(p,w)}\|$. Of course,
\[ \Phi_w^{-1} x = (w(n)^{-1/p} x_n)_{n \in \mathbb{N}}, \quad x \in \ell_p. \]
Substituting $x = e_1$ into (2.2) it follows that
\[ \|C^{(p,w)}\| = \|T_w\| \geq \|T_w e_1\|_p = \left( \frac{1}{w(1)} \sum_{n=1}^\infty \frac{w(n)}{n^p} \right)^{1/p} \geq \left( 1 + \frac{w(2)}{w(1)2^p} \right)^{1/p} > 1. \]

See also [11, Proposition 5.5].

Some comments regarding Proposition 2.2 are in order. As noted above, for each $1 < p < \infty$ we have $\|C^{(p)}\| = p'$ and, for a positive, decreasing weight $w$, that (2.3) holds. These estimates are not the best possible in general. Denote by $\delta_p(w)$ the set of all decreasing, non-negative sequences in $\ell_p(w)$ and define
\[ \Delta_{p,w}(C^{(p,w)}) := \sup \{ \|C^{(p,w)} x\|_{p,w} : x \in \delta_p(w), \|x\|_{p,w} = 1 \} \leq \|C^{(p,w)}\|. \]

The following result follows from Propositions 6.3, 6.5 and 6.6 of [11].

**Proposition 2.3.** Let $1 < p < \infty$ and $w(n) = 1/n^\alpha$, $n \in \mathbb{N}$, for a fixed $\alpha > 0$. Then
\[ \max\{m_1, m_2\} \leq \Delta_{p,w}(C^{(p,w)}) \leq \|C^{(p,w)}\| \leq M_2(r) := [r \zeta(r + \alpha)]^{r/p}, \tag{2.4} \]
for $1 \leq r \leq p$, where $m_1 := p/(p + \alpha - 1)$ and $m_2 := \zeta(p + \alpha)^{1/p}$, with $\zeta$ denoting the Riemann zeta function. Moreover, for $1 \leq r \leq p$, it is also the case that
\[ \|C^{(p,w)}\| \leq M_3(r) := \left( \frac{p}{p + \alpha - r} \right)^{1/p'} \zeta \left( 1 + \frac{r}{p'} + \frac{\alpha}{p} \right)^{1/p}. \tag{2.5} \]
We provide some relevant examples.
Example 2.4. (i) For \( w(n) = 1/n^\alpha \), if \( \alpha = 0.9 \) and \( p = 1.1 \), then \( \max\{m_1, m_2\} \simeq 1.572 \) and \( M_2(1) = M_3(0.9) \simeq 1.663 \) (see pp.15-16 of [11]) and so Proposition 2.3 shows that
\[
1.572 \leq \|C^{(p,w)}\| \leq 1.663.
\]
On the other hand, \( p' = 11 \) and so Proposition 2.2 only yields \( \|C^{(p,w)}\| \leq 11 \).

(ii) Still for \( w(n) = 1/n^\alpha \), but now with \( \alpha = 0.5 \) and \( p = 2 \), we have \( m_1 = 4/3 \) and \( M_3(3/4) \simeq 1.593 \) (see p.16 of [11]) so that Proposition 2.3 reveals that
\[
\frac{4}{3} \leq \|C^{(p,w)}\| \leq 1.593.
\]
In this case, \( p' = 2 \) and so Proposition 2.2 only yields \( \|C^{(p,w)}\| \leq 2 \).

(iii) Again for \( w(n) = 1/n^\alpha \), with \( \alpha > 0 \), it follows (in the notation of Proposition 2.3) that
\[
\left( \frac{1}{w(1)} \sum_{n=1}^{\infty} \frac{w(n)}{np} \right)^{1/p} = \left( \sum_{n=1}^{\infty} \frac{1}{n^{p+\alpha}} \right)^{1/p} = \zeta(p+\alpha)^{1/p} = m_2.
\]
Hence, the lower bound in (2.3) reduces to \( m_2 \leq \|C^{(p,w)}\| \) whereas (2.4) yields \( \max\{m_1, m_2\} \leq \|C^{(p,w)}\| \). Of course, (2.3) applies to more general weights \( w \).

The following example is not a consequence of Proposition 2.3.

Example 2.5. Let \( p = 2 \) and set \( w(n) = 2^{-n} \) for \( n \in \mathbb{N} \). The proof of Proposition 2.2 yields that \( \|C^{(2,w)}\| = \|T_w\| \). Recall, via (2.2), that
\[
T_w x = \left( \frac{1}{n^{2n/2}} \sum_{k=1}^{n} 2^{k/2} x_k \right)_{n \in \mathbb{N}}, \quad x = (x_n)_{n \in \mathbb{N}} \in \ell_2.
\]
For every \( x \in \ell_2 \), it follows via the Cauchy-Schwarz inequality and the identity \( \sum_{k=1}^{n} r^k = (r-r^{n+1})/(1-r) \), for \( r \neq 1 \), that
\[
\|T_w x\|_2^2 = \sum_{n=1}^{\infty} \frac{1}{n^{2n/2}} \left| \sum_{k=1}^{n} 2^{k/2} x_k \right|^2 \leq \sum_{n=1}^{\infty} \frac{1}{n^{2n/2}} \left( \sum_{k=1}^{n} 2^k \right) \left( \sum_{k=1}^{n} |x_k|^2 \right) \leq \|x\|_2^2 \sum_{n=1}^{\infty} \frac{1}{n^{2n}} \left( 2^{n+1} - 2 \right) = \|x\|_2^2 \sum_{n=1}^{\infty} \frac{2(1-2^{-n})}{n^2}.
\]
Accordingly, \( \|T_w\| \leq \left( \sum_{n=1}^{\infty} \frac{2(1-2^{-n})}{n^2} \right)^{1/2} \). Observe that
\[
\sum_{n=1}^{\infty} \frac{(1-2^{-n})}{n^2} = \frac{\pi^2}{6} - \int_0^{1/2} - \log(1-t) \frac{dt}{t},
\]
because of the fact that \( \frac{n^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} \) and the identity
\[
\int_0^{1/2} - \log(1-t) \frac{dt}{t} = \int_0^{1/2} \sum_{n=0}^{\infty} \frac{t^n}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n^2 2^n}.
\]
The function \( f(t) = \frac{-\log(1-t)}{t} \) for \( t \in (0,1) \), with \( f(0) := 1 \), is positive, continuous and increasing on \([0,1)\) and so
\[
1 = f(0) \leq f(t) \leq f \left( \frac{1}{2} \right) = 2 \log 2, \quad t \in [0,1/2],
\]
which implies that \(- \log 2 \leq - \int_0^{1/2} \frac{\log(1-t)}{t} \, dt \leq -\frac{1}{2}\). Consequently,
\[\sum_{n=1}^{\infty} \frac{2(1 - 2^{-n})}{n^2} \leq 2\left(\frac{\pi^2}{6} - \frac{1}{2}\right) \approx 2.2898\]
and so
\[\|C(2,w)\| = \|T_w\| \leq \sqrt{\left(\frac{\pi^2}{3} - 1\right)} \approx 1.513 < p' = 2.\]

Direct calculation yields
\[\|T_w e_1\|_2 = \left(2 \sum_{n=1}^{\infty} \frac{1}{n^2 2^n}\right)^{1/2} \geq \left(2 \sum_{n=1}^{\infty} \frac{1}{n^2 2^n}\right)^{1/2} \approx 1.073\]
and so we have
\[1.073 \leq \|C(2,w)\| \leq \sqrt{\left(\frac{\pi^2}{3} - 1\right)} \approx 1.513;\]
see also Proposition 2.2.

3. Spectrum of \(C(p,w)\)

The aim of this section is to provide some detailed knowledge of the spectrum of \(C(p,w)\). Unlike for the classical Cesàro operators \(C(p) \in \mathcal{L}(\ell_p)\), for \(1 < p < \infty\), it can now happen that eigenvalues appear.

Given a (strictly) positive, bounded sequence \(w = (w(n))_{n \in \mathbb{N}}\) and \(1 < p < \infty\), let \(S_w(p) := \{s \in \mathbb{R}: \sum_{n=1}^{\infty} \frac{1}{n^p w(n)^{p'}} < \infty\}\). In case \(S_w(p) \neq \emptyset\) we define \(s_p := \inf S_w(p)\). We point out that \(\frac{p'}{p} = \frac{1}{p-1}\), for every \(1 < p < \infty\). Moreover, let \(R_w := \{t \in \mathbb{R}: \sum_{n=1}^{\infty} n^t w(n) < \infty\}\). In case \(R_w \neq \mathbb{R}\) we define \(t_0 := \sup R_w\).

Fix \(1 < p < \infty\) and let \(w(n) = 2^{-np/p'}\) for \(n \in \mathbb{N}\). Then \(S_w(p) = \emptyset\), i.e., it can happen that \(S_w(p)\) is empty. However, in the event that \(S_w(p) \neq \emptyset\), then \(s_p \geq 1\).

Indeed, for any fixed \(s \in \mathbb{R}\) we have
\[\sum_{n=1}^{\infty} \frac{1}{n^s w(n)^{p''/p}} \geq \|w\|_{\infty}^{-p''/p} \sum_{n=1}^{\infty} \frac{1}{n^s}.\]  \hspace{1cm} (3.1)

So, whenever \(s \in S_w(p)\) it follows that \(\sum_{n=1}^{\infty} \frac{1}{n^s} < \infty\), that is, \(s > 1\). Hence, \(S_w(p) \subseteq (1, \infty)\) which implies that \(s_p \geq 1\). Moreover, for any \(r > s \in S_w(p)\) we have
\[\sum_{n=1}^{\infty} \frac{1}{n^r w(n)^{p''/p}} < \sum_{n=1}^{\infty} \frac{1}{n^s w(n)^{p''/p}}\]
and so also \(r \in S_w(p)\). Accordingly, whenever \(S_w(p) \neq \emptyset\), then it is an infinite interval, i.e., \(S_w(p) = [s_p, \infty)\) or \(S_w(p) = (s_p, \infty)\) with \(s_p \geq 1\). It is a consequence of (3.1) that \(1 \notin S_w(p)\), for all \(1 < p < \infty\) and all positive, bounded sequences \(w\).

In the event that \(a_w := \inf_{n \in \mathbb{N}} w(n) > 0\) it follows that necessarily \(s_p = 1\).

Indeed, in this case \(w(n)^{-p''/p} \leq \alpha w^{-p''/p}, \, n \in \mathbb{N}\), which implies that \(\frac{1}{n^s w(n)^{p''/p}} \leq \frac{a_w^{-p''/p}}{n^s}, \, n \in \mathbb{N}\). Hence, \((1, \infty) \subseteq S_w(p)\) and so \(s_p \leq 1\). Since we are assuming that \(S_w(p) \neq \emptyset\), we already know that \(s_p \geq 1\). Accordingly, \(s_p = 1\).
Let $1 < p < \infty$ and fix $\alpha > 0$. For $w(n) = 1/n^{\alpha p/p'}$ and any $s \in \mathbb{R}$ it follows that $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha p/p'} w(n)} = \sum_{n=1}^{\infty} \frac{1}{n^{\alpha + 1} n^{-\alpha}} < \infty$ precisely when $s > (1 + \alpha)$ and so $s_p = 1 + \alpha$. Hence, given any $\beta > 1$ and $1 < p < \infty$, there exists a positive, decreasing weight $w \downarrow 0$ such that $S_w(p) = (\beta, \infty)$, i.e., $s_p = \beta$.

Concerning the set $R_w$, a similar discussion applies. For $w(n) = 2^{-n}$ it turns out that $R_w = \mathbb{R}$ with $t_0 = \infty$. However, if $R_w \neq \mathbb{R}$, then $t_0$ is finite with $t_0 \geq -1$ and $R_w = (-\infty, t_0)$ or $R_w = (-\infty, t_0)$. Moreover, $R_w = \emptyset$ is not possible as $\sum_{n=1}^{\infty} n^t w(n) \leq ||w||_{\infty} \sum_{n=1}^{\infty} n^t < \infty$ whenever $t < -1$. If $a_w > 0$, then necessarily $t_0 = -1$ but, $-1 \notin R_w$ as $\sum_{n=1}^{\infty} n^t w(n) \geq a_w \sum_{n=1}^{\infty} n$ for all $t \in \mathbb{R}$.

The following result clarifies the connection between $s_p$ and $t_0$.

**Proposition 3.1.** Let $w = (w(n))_{n \in \mathbb{N}}$ be a bounded, strictly positive sequence.

(i) For each $1 < p < \infty$ such that $S_w(p) \neq \emptyset$ we have

$$t_0 \leq \frac{s_p}{p'} = (p-1)s_p.$$ 

In particular, $R_w = \mathbb{R}$ whenever there exists $p \in (1, \infty)$ with $S_w(p) \neq \emptyset$.

(ii) If $R_w \neq \mathbb{R}$, then $S_w(p) \subseteq [1 + \frac{t_0}{p-1}, \infty)$, for every $1 < p < \infty$.

(iii) Suppose that $1 < p < \infty$ satisfies $S_w(p) \neq \emptyset$. Then

$$S_w(p) \subseteq S_w(q), \quad q \in [p, \infty).$$

In particular, $S_w(q) \neq \emptyset$ whenever $q \geq p$.

(iv) If $S_w(p) = \emptyset$ for some $1 < p < \infty$, then $S_w(q) = \emptyset$ for all $1 < q \leq p$.

**Proof.** (i) Suppose that $S_w(p) \neq \emptyset$. Fix $s > s_p$. Since $\sum_{n=1}^{\infty} \frac{1}{n^{s w(n)} p'} < \infty$, there exists $N \in \mathbb{N}$ such that $\frac{1}{n^{s w(n)} p'} \leq 1$ for $n \geq N$ and hence, $n^{s w(n)} p' \geq 1$ for $n \geq N$. So, the series $\sum_{n=1}^{\infty} n^{s w(n)} p'$ diverges which yields that $t_0 \leq \frac{s_p}{p'}$. Accordingly, $t_0 \leq \frac{s_p}{p'}$.

(ii) Fix $p \in (1, \infty)$ and any $t < t_0$, in which case $\sum_{n=1}^{\infty} n^t w(n) < \infty$. Hence, there exists $K \in \mathbb{N}$ such that $n^t \leq \frac{1}{w(n)}$ for $n \geq K$, that is, $n^t w(n) \geq \frac{1}{w(n)^{p'}}$ for $n \geq K$. So, for any $s \in \mathbb{R}$ we have (as $\frac{1}{n} > 0$ for $n \in \mathbb{N}$) that

$$\frac{1}{n^{s-(t/p')}} = \frac{n^{t/p'}}{n^s} \leq \frac{n^t w(n)^{p'}}{w(n)^{p'}} = n^{s w(n)p'/p}, \quad n \geq K.$$ 

Choose now $s \leq 1 + t/p'$. It follows from the previous inequality that $\sum_{n=1}^{\infty} \frac{1}{n^{w(n)p'/p'}}$ diverges. Hence $\sum_{n=1}^{\infty} \frac{1}{n^{w(n)p'/p'}}$ diverges whenever $s \leq 1 + \frac{t p'}{p}$, for some $t < t_0$, that is, whenever $s \in (-\infty, 1 + \frac{t p'}{p})$. So, $S_w(p) \subseteq [1 + \frac{t p'}{p}, \infty) = [1 + \frac{t_0}{(p-1)}, \infty)$.

(iii) Fix $s \in S_w(p)$, i.e., $\sum_{n=1}^{\infty} \frac{1}{n^{s w(n)p'/p'}} < \infty$. For every $1 < q < \infty$ we have

$$\sum_{n=1}^{\infty} \frac{1}{n^{s w(n)p'/p'}} w(n)^{p'/q} = \sum_{n=1}^{\infty} \frac{1}{n^{s w(n)p'/p'}} \cdot w(n)^{p'/q} \leq \|w\|_{\infty}^{p'/q} \sum_{n=1}^{\infty} \frac{1}{n^{s w(n)p'/p'}}.$$ 

which is finite provided that $\frac{p'}{q} \geq \frac{q'}{q}$. This is equivalent to $(p'-1) \geq (q'-1)$, that is, to $q \geq p$. Hence, whenever $q \geq p$ we have $S_w(p) \subseteq S_w(q)$ which clearly implies $S_w(q) \neq \emptyset$ and $q_s \leq s_p$.

(iv) Follows immediately from part (iii). \qed
Define $\Sigma := \{ \frac{1}{n} : n \in \mathbb{N} \}$ and let $\Sigma_0 := \{ 0 \} \cup \{ \frac{1}{n} : n \in \mathbb{N} \}$ be its closure. The following inequalities will be needed later.

**Lemma 3.2.** (i) Let $\lambda \in \mathbb{C} \setminus \Sigma_0$ and set $\alpha := \text{Re} \left( \frac{1}{\lambda} \right)$. Then there exist constants $d > 0$ and $D > 0$ (depending on $\alpha$) such that

$$
\frac{d}{n^\alpha} \leq \prod_{k=1}^{n} \left| 1 - \frac{1}{k\lambda} \right| \leq \frac{D}{n^\alpha}, \quad n \in \mathbb{N}. \tag{3.2}
$$

(ii) For each $m \in \mathbb{N}$ we have that

$$
\frac{(n-1)!}{(n-m)!} \simeq n^{m-1}, \quad \text{for all large } n \in \mathbb{N}. \tag{3.3}
$$

(iii) Let $1 < p < \infty$ and $w = (w(n))_{n \in \mathbb{N}}$ be a positive, decreasing sequence. Then

$$
(n^m w(n))_{n \in \mathbb{N}} \in \ell_p, \quad \forall m \in \mathbb{N}, \tag{3.4}
$$

if and only if

$$
(n^m w(n)^{1/p})_{n \in \mathbb{N}} \in \ell_p, \quad \forall m \in \mathbb{N}, \tag{3.5}
$$

**Proof.** (i) The inequalities in (3.2) follow as in the proof of Lemma 7 in [18], where the restriction $\alpha < 1$ is assumed. Indeed, with $\frac{1}{\lambda} = \alpha + i\beta$ (for $\alpha, \beta \in \mathbb{R}$) and using $1 + x \leq e^x$ for $x > 0$, we have

$$
\prod_{k=1}^{n} \left| 1 - \frac{1}{k\lambda} \right| = \prod_{k=1}^{n} \left( 1 - \frac{2\alpha}{k} + \frac{\alpha^2 + \beta^2}{k^2} \right)^{1/2} \leq \exp \sum_{k=1}^{n} \left( -\frac{\alpha}{k} + \frac{C}{k^2} \right) \leq \exp (-\alpha \log(n) + v) \leq \frac{D}{n^\alpha}.
$$

An application of Taylor's formula for $x \mapsto (1 + x)^{-1/2}$, for $x > -1$, yields

$$
\prod_{k=1}^{n} \left| 1 - \frac{1}{k\lambda} \right|^{-1} = \prod_{k=1}^{n} \left( 1 - \frac{2\alpha}{k} + \frac{\alpha^2 + \beta^2}{k^2} \right)^{-1/2} \leq \prod_{k=1}^{n} \left( 1 + \frac{\alpha}{k} + \frac{C'}{k^2} \right) \leq \exp \sum_{k=1}^{n} \left( \frac{\alpha}{k} + \frac{C'}{k^2} \right) \leq \exp (\alpha \log(n) + v') = d^{-1}n^\alpha.
$$

(ii) Fix $m \in \mathbb{N}$. Then, for all large $n > m$, we have

$$
\frac{(n-1)!}{(n-m)!} = (n-1) \ldots (n-m+1) = n^{m-1} \left( 1 - \frac{1}{n} \right) \ldots \left( 1 - \frac{m-1}{n} \right) \simeq n^{m-1}.
$$

(iii) Suppose that (3.4) holds. Fix $m \in \mathbb{N}$. Let $k \in \mathbb{N}$ satisfy $k \geq 2 + mp$. Since $(n^k w(n))_{n \in \mathbb{N}} \in \ell_p$, there exists $N \in \mathbb{N}$ such that

$$
w(n) \leq \frac{1}{n^k} \leq \frac{1}{n^{2+mp}}, \quad n \geq N.
$$

It follows that

$$
\sum_{n=1}^{\infty} n^{mp} w(n) \leq \sum_{n=1}^{N} n^{mp} w(n) + \sum_{n=N+1}^{\infty} n^{mp} \left( \frac{1}{n^{2+mp}} \right) < \infty,
$$

that is, $(n^m w(n)^{1/p})_{n \in \mathbb{N}} \in \ell_p$. Accordingly, (3.5) is satisfied.
Conversely, suppose that \((3.5)\) holds. Since \((nw(n)^{1/p})_{n \in \mathbb{N}} \in \ell_p\), there exists \(K \in \mathbb{N}\) such that \(w(n) \leq 1\) for \(n \geq K\) and hence, \(w(n) \leq w(n)^{1/p}\) for \(n \geq K\). Fix \(m \in \mathbb{N}\). Then \(n^m w(n) \leq n^m (w(n)^{1/p})\) for \(n \geq K\). Since \((n^m w(n)^{1/p})_{n \in \mathbb{N}} \in \ell_p\), we can conclude that also \((n^m w(n))_{n \in \mathbb{N}} \in \ell_p\). Hence, \((3.4)\) is satisfied. \(\square\)

If \(S_w(p) \neq \emptyset\), then \(s_p \geq 1\) and so \(\frac{p'}{2s_p} \leq \frac{p'}{2}\), which is relevant for the following results. Also relevant is that \(\|C_{w,p}\| < p'\) is possible; see Section 2.

We now come to the main result of this section.

**Theorem 3.3.** Let \(w = (w(n))_{n \in \mathbb{N}}\) be a positive, decreasing sequence.

(i) Suppose that \(S_w(p) \neq \emptyset\) for some \(1 < p < \infty\). Then for the dual operator \((C_{w,p})' \in \mathcal{L}((\ell_p(w)))'\) of \(C_{w,p}\) we have

\[
\left\{ \lambda \in \mathbb{C} : \left| \lambda - \frac{p'}{2s_p} \right| < \frac{p'}{2s_p} \right\} \cup \Sigma \subseteq \sigma_{pt}(C_{w,p})'
\]

and

\[
\sigma_{pt}(C_{w,p})' \setminus \Sigma \subseteq \left\{ \lambda \in \mathbb{C} : \left| \lambda - \frac{p'}{2s_p} \right| \leq \frac{p'}{2s_p} \right\}.
\]

For the Cesàro operator \(C_{w,p}\) itself we have

\[
\left\{ \lambda \in \mathbb{C} : \left| \lambda - \frac{p'}{2s_p} \right| \leq \frac{p'}{2s_p} \right\} \cup \Sigma \subseteq \sigma(C_{w,p})
\]

and

\[
\sigma(C_{w,p}) \subseteq \left\{ \lambda \in \mathbb{C} : \left| \lambda - \frac{p'}{2} \right| \leq \frac{p'}{2} \right\} \cap \left\{ \lambda \in \mathbb{C} : |\lambda| \leq \|C_{w,p}\| \right\}.
\]

(ii) Suppose that \(R_w \neq \mathbb{R}\), i.e., \(t_0 < \infty\). Then, for every \(1 < p < \infty\), we have

\[
\left\{ \frac{1}{m} : m \in \mathbb{N}, 1 \leq m < \frac{t_0}{p} + 1 \right\} \subseteq \sigma_{pt}(C_{w,p}) \subseteq \left\{ \frac{1}{m} : m \in \mathbb{N}, 1 \leq m \leq \frac{t_0}{p} + 1 \right\}.
\]

If \(R_w = \mathbb{R}\), then

\[
\sigma_{pt}(C_{w,p}) = \Sigma, \quad \forall 1 < p < \infty.
\]

**Proof.** The proof is via a series of steps.

(i) By Proposition 2.2 we have \(C_{w,p} \in \mathcal{L}(\ell_p(w))\) with \(\|C_{w,p}\| \leq p'\). The dual operator \(A := (C_{w,p})' \in \mathcal{L}(\ell_{p'}(w^{-p'/p}))\) also satisfies \(\|A\| \leq p'\) and is given by

\[
Ay = \left( \sum_{k=n}^{\infty} \frac{y_k}{k} \right)_{n \in \mathbb{N}}, \quad y = (y_n)_{n \in \mathbb{N}} \in \ell_{p'}(w^{-p'/p}).
\]

**Step 1.** \(0 \notin \sigma_{pt}(A)\).

Observe that \(Ay = 0\), for some \(y \in \ell_{p'}(w^{-p'/p})\), implies that \(z_n := \sum_{k=n}^{\infty} \frac{y_k}{k} = 0\) for all \(n \in \mathbb{N}\). Hence, \(y_n = n(z_n - z_{n+1}) = 0\), for \(n \in \mathbb{N}\), and so \(A\) is injective.

**Step 2.** \(\Sigma \subseteq \sigma_{pt}(A)\).

Let \(\lambda \in \Sigma\), i.e., \(\lambda = \frac{1}{m}\) for some \(m \in \mathbb{N}\). Via (3.13) below, the non-zero vector \(y = (y_n)_{n \in \mathbb{N}}\) defined via \(y_1 \in \mathbb{C} \setminus \{0\}\) arbitrary, \(y_n := y_1 \prod_{k=1}^{n-1} (1 - \frac{1}{k})\) for \(1 \leq n \leq m\) and \(y_n := 0\) for \(n > m\), which belongs to \(\ell_{p'}(w^{-p'/p})\), satisfies

\[
Ay = \lambda y.
\]

**Step 3.** \(\left\{ \lambda \in \mathbb{C} : \left| \lambda - \frac{p'}{2s_p} \right| < \frac{p'}{2s_p} \right\} \subseteq \sigma_{pt}(A)\).
Let $\lambda \in \mathbb{C} \setminus \{0\}$. Then $Ay = \lambda y$ for some non-zero $y \in \ell_p(w^{-p'/p})$ if, and only if, 
$\lambda y_n = \sum_{k=n}^{\infty} \frac{y_k}{k}$ for all $n \in \mathbb{N}$. This yields, for every $n \in \mathbb{N}$, that $\lambda (y_n - y_{n+1}) = \frac{y_n}{n}$ and so $y_{n+1} = \left(1 - \frac{1}{\lambda n}\right) y_n$. It follows that 
y_{n+1} = y_1 \prod_{k=1}^{n} \left(1 - \frac{1}{\lambda k}\right), \quad n \in \mathbb{N}, \tag{3.13}
with $y_1 \neq 0$. In particular, each eigenvalue of $A$ is simple.

Let now $\lambda \in \mathbb{C} \setminus \Sigma$ satisfy $\left|\lambda - \frac{p'}{2sp}\right| < \frac{p'}{2sp}$ (equivalently, $\alpha := \text{Re} \left(\frac{1}{\lambda}\right) > \frac{s_p}{p}$, i.e., $\alpha p' = \text{Re} \left(\frac{p'}{\lambda}\right) > s_p$). For such a $\lambda$ the vector $y = (y_n)_{n \in \mathbb{N}} \in \mathbb{C}^\mathbb{N}$ defined by (3.13) actually belongs to $\ell_p'(w^{-p'/p})$. Indeed, via Lemma 3.2(i) there exists $c = c(\lambda) > 0$ such that 

$$
\prod_{k=1}^{n} \left|1 - \frac{1}{\lambda k}\right|^{p'} \leq cn^{-\text{Re}(p'/\lambda)}, \quad n \in \mathbb{N}.
$$

It then follows from (3.13) that 

$$
\sum_{n=1}^{\infty} |y_n|^{p'} w(n)^{-p'/p} = |y_1|^{p'} w(1)^{-p'/p} + |y_1|^{p'} \sum_{n=2}^{\infty} \prod_{k=1}^{n} \left|1 - \frac{1}{\lambda k}\right|^{p'} w(n)^{-p'/p} 
\leq |y_1|^{p'} w(1)^{-p'/p} + c|y_1|^{p'} \sum_{n=2}^{\infty} n^{-\text{Re}(p'/\lambda)} w(n)^{-p'/p},
$$

where the series $\sum_{n=2}^{\infty} n^{-\text{Re}(p'/\lambda)} w(n)^{-p'/p}$ converges because $\text{Re}(p'/\lambda) \in S_w(p)$, that is, $y \in \ell_p'(w^{-p'/p})$. Hence, $\lambda \in \sigma_{\text{pl}}(A)$.

**Step 4.** $\sigma_{\text{pl}}(A) \setminus \Sigma_0 \subseteq \{\lambda \in \mathbb{C} : \left|\lambda - \frac{p'}{2sp}\right| \leq \frac{p'}{2sp}\}$.

Fix $\lambda \in \sigma_{\text{pl}}(A) \setminus \Sigma_0$. According to (3.2) there exists $\beta = \beta(\lambda) > 0$ such that 

$$
\prod_{k=1}^{n} \left|1 - \frac{1}{\lambda k}\right|^{p'} \geq \beta \cdot n^{-\text{Re}(p'/\lambda)}, \quad n \in \mathbb{N}. \tag{3.14}
$$

But, as argued in Step 2 (for any $y_1 \in \mathbb{C} \setminus \{0\}$) the eigenvector $y = (y_n)_{n \in \mathbb{N}}$ corresponding to the eigenvalue $\lambda$ of $A$, which necessarily belongs to $\ell_p'(w^{-p'/p})$, i.e., $\sum_{n=1}^{\infty} |y_n|^{p'} w(n)^{-p'/p} < \infty$, is given by (3.13). Then (3.14) implies that also $\sum_{n=1}^{\infty} \frac{1}{n^{\text{Re}(p'/\lambda)} w(n)^{p'/p}} < \infty$, i.e., $\text{Re} \left(\frac{p'}{\lambda}\right) \in S_w(p)$ and so $\text{Re} \left(\frac{p'}{\lambda}\right) \geq s_p$.

Equivalently, $\text{Re} \left(\frac{1}{\lambda}\right) \geq \frac{s_p}{p}$, i.e., $\lambda \in \left\{\mu \in \mathbb{C} : \left|\mu - \frac{p'}{2sp}\right| \leq \frac{p'}{2sp}\right\}$.

It is clear that Steps 1-4 establish the two containments in (3.6) and (3.7).

For every $T \in \mathcal{L}(X)$ with $X$ a Banach space, it is known that $\sigma_{\text{pl}}(T') \subseteq \sigma(T)$, [9, p.581], with $\sigma(T)$ a closed subset of $\mathbb{C}$. Accordingly, (3.8) follows from (3.6).

**Step 5.** $\sigma(C(p,w)) \subseteq \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{p'}{2}\right| \leq \frac{p'}{2}\right\}$.

It suffices to show that every $\lambda \in \mathbb{C}$ with $\left|\lambda - \frac{p'}{2}\right| > \frac{p'}{2}$ belongs to $\rho(C(p,w))$.

To do this we argue as in [7]. We recall the formula for $(C - \lambda I)^{-1} : \mathbb{C}^\mathbb{N} \to \mathbb{C}^\mathbb{N}$ whenever $\lambda \notin \Sigma_0$, [18, p.266]. For $n \in \mathbb{N}$ the $n$-th row of the matrix for $(C - \lambda I)^{-1}$...
has the entries
\[
\frac{-1}{n\lambda^2 \prod_{k=m}^n (1 - \frac{1}{\lambda k})}, \quad 1 \leq m < n,
\]
\[
\frac{n}{1 - n\lambda} = \frac{1}{n - \lambda}, \quad m = n,
\]
and all the other entries in row \( n \) are equal to 0. So, we can write
\[
(C - \lambda I)^{-1} = D_\lambda - \frac{1}{\lambda^2} E_\lambda, \quad (3.15)
\]
where the diagonal operator \( D_\lambda = (d_{nm})_{n,m\in\mathbb{N}} \) is given by \( d_{nn} := \frac{1}{n - \lambda} \) and \( d_{nm} := 0 \) if \( n \neq m \). The operator \( E_\lambda = (e_{nm})_{n,m\in\mathbb{N}} \) is then the lower triangular matrix with \( e_{1m} = 0 \) for all \( m \in \mathbb{N} \), and for every \( n \geq 2 \) with \( e_{nm} := \frac{1}{n \prod_{k=m}^n (1 - \frac{1}{\lambda k})} \) if \( 1 \leq m < n \) and \( e_{nm} := 0 \) if \( m \geq n \).

If \( \lambda \notin \Sigma_0 \), then \( d(\lambda) := \text{dist}(\lambda, \Sigma_0) > 0 \) and \( |d_{mn}| \leq \frac{1}{d(\lambda)} \) for \( n \in \mathbb{N} \). Hence, for every \( x \in \ell_p(w) \), we have
\[
\|D_\lambda(x)\|_{p,w} = \left( \sum_{n=1}^{\infty} |d_{nn} x_n|^p w(n) \right)^{1/p} \leq \frac{1}{d(\lambda)} \left( \sum_{n=1}^{\infty} |x_n|^p w(n) \right)^{1/p} = \|x\|_{p,w}.
\]
This means that \( D_\lambda \in \mathcal{L}(\ell_p(w)) \). So, by \( (3.15) \) it remains to show that \( E_\lambda \in \mathcal{L}(\ell_p(w)) \) whenever \( \lambda \in \mathcal{C} \) satisfies \( |\lambda - \frac{p'}{2}| > \frac{p'}{2} \). To this end, we note that if \( \lambda \in \mathcal{C} \setminus \Sigma_0 \) then, with \( \alpha := \text{Re} \left( \frac{1}{\lambda} \right) \), it follows from \( (3.2) \) that
\[
|e_{1n}| \leq \frac{d^{-1}}{n^{1-\alpha}}, \quad n \geq 2,
\]
\[
|e_{nm}| \leq \frac{d^{-1} D'}{n^{1-\alpha} m^n}, \quad 2 \leq m < n,
\]
for some constants \( d > 0 \) and \( D' > 0 \) depending on \( \lambda \). So, for every \( \lambda \in \mathcal{C} \setminus \Sigma_0 \) there exists \( c = c(\lambda) > 0 \) such that
\[
|\langle E_\lambda(x) \rangle_n| \leq c(\mathcal{G}_\lambda(|x|))_n, \quad x \in \mathbb{C}^\mathbb{N}, \ n \in \mathbb{N}, \quad (3.16)
\]
where \( (\mathcal{G}_\lambda(x))_n := \sum_{k=1}^n \frac{x_k^{1/p}}{n^{1-\alpha} k^\alpha} \) with \( \alpha := \text{Re} \left( \frac{1}{\lambda} \right) \) and for all \( x \in \mathbb{C}^\mathbb{N} \) and \( n \in \mathbb{N} \). Then \( (3.16) \) implies that \( E_\lambda \in \mathcal{L}(\ell_p(w)) \) whenever \( \lambda \in \mathcal{C} \setminus \Sigma_0 \) satisfies \( |\lambda - \frac{p'}{2}| > \frac{p'}{2} \).

Claim: \( \mathcal{G}_\lambda \in \mathcal{L}(\ell_p(w)) \) whenever \( \lambda \in \mathcal{C} \) satisfies \( |\lambda - \frac{p'}{2}| > \frac{p'}{2} \).

To establish this claim fix \( \lambda \in \mathcal{C} \) with \( |\lambda - \frac{p'}{2}| > \frac{p'}{2} \). Then necessarily \( \lambda \notin \Sigma_0 \) with \( \alpha := \text{Re} \left( \frac{1}{\lambda} \right) < \frac{1}{2p} \) and so \( (1 - \alpha)p > 1 \). This implies that \( \alpha < 1 \). Observe that \( \mathcal{G}_\lambda \in \mathcal{L}(\ell_p(w)) \) if, and only if, the operator \( \tilde{\mathcal{G}}_\lambda : \mathbb{C}^\mathbb{N} \to \mathbb{C}^\mathbb{N} \) given by
\[
(\tilde{\mathcal{G}}_\lambda(x))_n = w(n)^{1/p} \sum_{k=1}^n \frac{w(k)^{-1/p}}{n^{1-\alpha} k^\alpha} x_k, \quad x \in \mathbb{C}^\mathbb{N}, \ n \in \mathbb{N},
\]
defines a continuous linear operator on \( \ell_p \) (the proof of this is along the lines of that of Lemma 2.1). To prove that indeed \( \tilde{\mathcal{G}}_\lambda \in \mathcal{L}(\ell_p) \) we need to distinguish the three cases: a) \( \alpha = 0 \); b) \( \alpha < 0 \) and c) \( 0 < \alpha < 1 \) and establish relevant inequalities in each case.
Case a). Since \(w\) is decreasing, we have, for every \(n \in \mathbb{N}\), that

\[
\sum_{k=1}^{n} \frac{1}{w(k)^{1/(p-1)k^{\alpha p/(p-1)}}} = \sum_{k=1}^{n} \frac{1}{w(k)^{1/(p-1)}} \leq \frac{n}{w(n)^{1/(p-1)}}
\]

and hence, for every \(m \in \mathbb{N}\), that

\[
\sum_{n=1}^{m} \left( \frac{w(n)^{1/p}}{n} \sum_{k=1}^{n} \frac{1}{w(k)^{1/(p-1)}} \right)^p \leq \sum_{n=1}^{m} \frac{1}{w(n)^{1/(p-1)}}. \quad (3.17)
\]

Case b). Observe, for every \(n \in \mathbb{N}\), that

\[
\sum_{k=1}^{n} \frac{1}{w(k)^{1/(p-1)k^{\alpha p/(p-1)}}} \leq \frac{1}{w(n)^{1/(p-1)}} \int_{1}^{n+1} x^{-\alpha p/(p-1)} \, dx = \frac{1}{w(n)^{1/(p-1)}} \left( (n + 1)^{-\frac{\alpha p}{p-1} + 1} - 1 \right) \leq \frac{p(1 - \alpha)}{(p - 1)} \frac{(n + 1)^{\frac{p(1 - \alpha)}{p - 1}} - 1}{w(n)^{1/(p-1)}}.
\]

Setting \(c := \frac{p-1}{p(1-\alpha) - 1} > 0\) it follows, for every \(m \in \mathbb{N}\), that

\[
\sum_{n=1}^{m} \frac{1}{w(n)^{1/(p-1)\frac{p(1-\alpha)}{p-1}}} \leq \frac{c}{w(n)^{1/(p-1)\frac{p(1-\alpha)}{p-1}}} \sum_{n=1}^{m} \frac{1}{w(n)^{1/(p-1)n^{\frac{p(1-\alpha)}{p-1}}}} \leq 2 \frac{p(1-\alpha)}{p-1} c^p \sum_{n=1}^{m} \frac{1}{w(n)^{1/(p-1)n^{\frac{p(1-\alpha)}{p-1}}}}. \quad (3.18)
\]

Case c). We have, for every \(n \in \mathbb{N}\), still with \(c = \frac{p-1}{p(1-\alpha) - 1}\), that

\[
\sum_{k=2}^{n} \frac{1}{w(k)^{1/(p-1)k^{\alpha p/(p-1)}}} \leq \frac{1}{w(n)^{1/(p-1)}} \int_{1}^{n} x^{-\alpha p/(p-1)} \, dx = \frac{c}{w(n)^{1/(p-1)}} \left( n^{\frac{p(1-\alpha)}{p-1}} - 1 \right).
\]
Since \((1 - \alpha)p > 1\) (i.e., \((1 - \alpha)p - 1 > 0\)) and \(\alpha p > 0\) with \(\frac{1}{w(1)} \leq \frac{1}{w(n)}\), this implies, for every \(n \in \mathbb{N}\), that

\[
\left( \frac{w(n)^{1/p}}{n^{1-\alpha}} \sum_{k=1}^{n} \frac{1}{w(k)^{1/(p-1)} k^{\alpha p/(p-1)}} \right)^p \leq \left[ \frac{w(n)^{1/p}}{n^{1-\alpha}w(1)^{1/(p-1)}} + \frac{w(n)^{1/p}c}{n^{1-\alpha}w(n)^{1/(p-1)}} \right]^p \leq \left[ \frac{w(n)^{1/p}}{n^{1-\alpha}w(n)^{1/(p-1)}} + \frac{w(n)^{1/p}c}{n^{1-\alpha}w(n)^{1/(p-1)}} \right]^p = \left( 1 - c \right) \frac{w(n)^{1/p}}{n^{1-\alpha}w(n)^{1/(p-1)}} + \frac{w(n)^{1/p}c}{n^{1-\alpha}w(n)^{1/(p-1)}} \leq \left( \frac{w(n)^{1/(p-1)}c}{n^{1-\alpha}} \right)^p \leq c^p w(n)^{-\alpha p/(p-1)}.
\]

Hence, for every \(m \in \mathbb{N}\), we have that

\[
\sum_{n=1}^{m} \left( \frac{w(n)^{1/p}}{n^{1-\alpha}} \sum_{k=1}^{n} \frac{1}{w(k)^{1/(p-1)} k^{\alpha p/(p-1)}} \right)^p \leq c^p \sum_{n=1}^{m} \frac{1}{w(n)^{1/(p-1)} n^{\alpha p/(p-1)}}.
\]

The inequalities (3.17), (3.18) and (3.19) imply that \(G_\lambda \in \mathcal{C}(\ell_p)\); indeed, in each case, suitable choices of \(a_n\) and \(b_k\) (with \(p = q\)) allow us to apply Theorem 2(ii) of [5]. This establishes the claim and hence, also Step 5.

**Step 6.** \(\sigma(C(p,w)) \subseteq \{ \lambda \in \mathbb{C} : |\lambda| \leq \|C(p,w)\| \} \).

This is well known, [9, Ch.VII Lemma 3.4].

Steps 5 and 6 clearly yield (3.9). The proof of part (i) is thereby complete.

(ii) Suppose first that \(R_w \neq \mathbb{R}\). Fix any \(1 < p < \infty\).

**Step 7. Both of the inclusions in (3.10) are valid.**

The Cesàro operator \(C(p,w)\) is clearly injective. So, \(0 \notin \sigma_{pl}(C(p,w))\). Let \(\lambda \in \mathbb{C} \setminus \{0\}\). Consider the equation \((\lambda I - C)x = 0\) with \(x = (x_n)_{n \in \mathbb{N}} \in \ell_p^\mathbb{N}\). Then \(x_1 = \lambda x_1\) and \((2\lambda - 1)x_2 = x_1\) and \(x_n = \lambda(n-1)x_{n-1}\) for all \(n \geq 3\).

If \(m \in \mathbb{N}\) denotes the smallest positive integer such that \(x_m \neq 0\), then it follows that \(\lambda = \frac{1}{m}\) and so \(x_n = \frac{n-1}{n-m} x_{n-1}\) for all \(n > m\). Thus, we deduce that

\[
x_n = x_{m+(n-m)} = \frac{(n-1)!}{(m-1)!(n-m)!} x_m, \quad n > m.
\]

According to (3.3) we have \(\frac{(n-1)!}{(m-1)!(n-m)!} \approx \frac{1}{(m-1)!} \cdot n^{m-1}\), for each \(m \in \mathbb{N}\). So, \(x \in \ell_p(w)\) if, and only if, the series \(\sum_{n=m+1}^{\infty} n^{(m-1)p} w(n)\) converges. But, the series \(\sum_{n=m+1}^{\infty} n^{(m-1)p} w(n)\) converges precisely when \((m-1)p \in R_w\). In this case, \((m-1)p \leq t_0\), i.e., \(m \leq \frac{t_0}{p} + 1\). So, \(\sigma_{pl}(C(p,w)) \subseteq \{ \frac{1}{m} : m \in \mathbb{N}, 1 \leq m \leq \frac{t_0}{p} + 1 \} \).
Conversely, if \( m < \frac{t_0}{p} + 1 \) for some \( m \in \mathbb{N} \), i.e., \((m-1)p < t_0\), then \((m-1)p \in R_w\) as \( t_0 = \sup R_w \). Then the vector \( x \in \mathbb{C}^n \) defined according to (3.20), with \( x_1 = \ldots = x_{m-1} = 0 \) and for any arbitrary \( x_m \neq 0 \), belongs to \( \ell_p(w) \). Therefore, \( \frac{1}{m} \in \sigma_{pt}(C(p,w)) \).

**Step 8.** Assume now that \( R_w = \mathbb{R} \). Then (3.11) is valid.

Fix \( 1 < p < \infty \). As argued in Step 7, the point \( \frac{1}{m} \in \sigma_{pt}(C(p,w)) \) if and only if \((m-1)p \in R_w\). But, for \( R_w = \mathbb{R} \), this is satisfied for every \( m \in \mathbb{N} \) and so \( \Sigma \subseteq \sigma_{pt}(C(p,w)) \). On the other hand, it is also shown in the proof of Step 7 that every eigenvalue \( \lambda \) of \( C : \mathbb{C}^N \to \mathbb{C}^N \) must have the form \( \lambda = \frac{1}{m} \) for some \( m \in \mathbb{N} \). Since every eigenvalue of \( C(p,w) \) is also an eigenvalue of \( C \) (as \( \ell_p(w) \subseteq \mathbb{C}^N \)), it follows that \( \sigma_{pt}(C(p,w)) \subseteq \Sigma \).

**Remark 3.4.** (i) If \( s_p \not\in S_w(p) \), for some \( 1 < p < \infty \), then the argument of Step 4 in the proof of Theorem 3.3 implies that (3.6) reduces to the equality

\[
\sigma_{pt}(C(p,w)) = \left\{ \lambda \in \mathbb{C} : \left| \lambda - \frac{p'}{2sp} \right| < \frac{p'}{2sp} \right\} \cup \Sigma.
\]

Also, if \( t_0 \not\in R_w \), then (3.10) reduces to the equality

\[
\sigma_{pt}(C(p,w)) = \left\{ \frac{1}{m} : m \in \mathbb{N}, 1 \leq m < \frac{t_0}{p} + 1 \right\}, 1 < p < \infty.
\]

(ii) For \( w(n) = 1 \) for all \( n \in \mathbb{N} \), in which case \( \ell_p(w) = \ell_p \) and \( s_p = 1 \), we have that \( C(p,w) = C(p) \) for all \( 1 < p < \infty \) with \( \|C(p,w)\| = \|C(p)\| = p' \). Then (3.8) and (3.9) imply the known fact that

\[
\sigma(C(p)) = \left\{ \lambda \in \mathbb{C} : \left| \lambda - \frac{p'}{2} \right| \leq \frac{p'}{2} \right\}.
\]

Since \( t_0 = -1 \), we also recover from (3.10) the known fact that \( \sigma_{pt}(C(p)) = \emptyset \).

(iii) According to (3.8), for \( w \) positive, decreasing and with \( S_w(p) \neq \emptyset \) we have

\[
\max\left\{ 1, \frac{p'}{sp} \right\} \leq \|C(p,w)\| \leq p'.
\]

In particular, whenever \( s_p = 1 \) (see e.g., Example 3.5(i) below), the inequalities in (3.22) imply that necessarily \( \|C(p,w)\| = p' \) is as large as possible.

For the special case when \( w(n) = \frac{1}{n} \), \( n \in \mathbb{N} \), for some \( \alpha > 0 \), direct calculation yields that \( s_p = 1 + \frac{\alpha p'}{p} \) and so \( S_w(p) \neq \emptyset \) for all \( 1 < p < \infty \). It follows that

\[
\frac{p'}{sp} = \frac{p}{p' + \alpha + p - 1} = m_1,
\]

where \( m_1 \) occurs in the lower bound for \( \|C(p,w)\| \) as given in (2.4); see Proposition 2.3. Hence, (3.22) yields that \( m_1 \leq \|C(p,w)\| \). Combined with Example 2.4(iii) we can conclude that

\[
\max\{m_1, m_2\} \leq \|C(p,w)\|.
\]

This provides an alternate proof, to that in [11], of the same estimate in (2.4).

(iv) An examination of the argument for Step 2 in the proof of Theorem 3.3(i) shows that the assumption \( S_w(p) \neq \emptyset \) is not used there, i.e., we always have

\[
\Sigma \subseteq \sigma_{pt}(C(p,w))
\]
for every $1 < p < \infty$ and every positive, decreasing weight $w$.

We now present some relevant examples.

**Examples 3.5.** (i) Suppose that $w(n) = \frac{1}{(\log(n+1))^p}$ for $n \in \mathbb{N}$ with $\gamma \geq 0$. Then $\sum_{n=1}^{\infty} \frac{1}{n^\gamma w(n)^{p'}} < \infty$ if and only if $s > 1$ and hence, $s_p = 1$ for every $1 < p < \infty$. In view of Remark 3.4(iii) we have that $\|C(p,w)\| = p'$. Moreover, $\sum_{n=1}^{\infty} n^\gamma w(n) < \infty$ if and only if $t < -1$ or $t \leq -1$ in case $\gamma > 1$. Hence, $t_0 = -1$. According to Theorem 3.3 we have, for each $1 < p < \infty$, that

$$\sigma(C(p,w)) = \left\{ \lambda \in \mathbb{C}: \left| \lambda - \frac{p'}{2(p' + 1)} \right| \leq \frac{p'}{2(p' + 1)} \right\} \cup \Sigma \subseteq \sigma(C(p,w))$$

and

$$\sigma_{pl}(C(p,w)) = \left\{ \frac{1}{m}: m \in \mathbb{N}, 1 \leq m < \frac{\beta}{p} + 1 \right\}.$$ 

In particular, $\sigma_{pl}(C(p,w)) = \emptyset$ whenever $\beta \in [0,1]$. We claim that actually

$$\left\{ \lambda \in \mathbb{C}: \left| \lambda - \frac{p'}{2(p' + 1)} \right| \leq \frac{p'}{2(p' + 1)} \right\} \cup \Sigma = \sigma(C(p,w)),$$

which shows that equality may occur in (3.8).

Keeping in mind the argument for Step 5 in the proof of Theorem 3.3, to verify this identity it suffices to prove that every $\lambda \in \mathbb{C} \setminus \{0\}$ satisfying $\left| \lambda - \frac{p'}{2(p' + 1)} \right| > \frac{p'}{2(p' + 1)}$ belongs to $\rho(C(p,w))$, i.e., that the operator $\tilde{G}_\lambda \in \mathcal{L}(\ell_p)$. So, fix such a $\lambda$ and note that $\alpha := \text{Re} \left( \frac{1}{\lambda} \right) < \left( \frac{\beta}{p} + 1 \right) / p' = \frac{\beta}{p} + \frac{1}{p'}$. We also observe, for our particular $w$, that the operator $\tilde{G}_\lambda$ is given by

$$\tilde{G}_\lambda(x)_n = \frac{1}{n^{1-\alpha+(\beta/p)} \log^{\gamma/p}(n+1)} \sum_{k=1}^{n} \frac{x_k}{k^{\alpha-(\beta/p) \log^{-\gamma/p}(k+1)}}, x = (x_n)_{n \in \mathbb{N}} \in \mathbb{C}^\mathbb{N}.$$ 

So, $\tilde{G}_\lambda$ is given by the factorable matrix with $a_n := n^{-1-\alpha+(\beta/p)} \log^{-\gamma/p}(n+1)$ and $b_k := k^{-\alpha-(\beta/p) \log^{-\gamma/p}(k+1)}$, where $\alpha < \frac{\beta}{p} + \frac{1}{p'} = \frac{\beta}{p} + 1 - \frac{1}{p}$. implies that $1 - \alpha + \frac{\beta}{p} > \frac{1}{p}$ and we have that 

$$\left( 1 - \alpha + \frac{\beta}{p} \right) + \left( \alpha - \frac{\beta}{p} \right) = 1 = \frac{1}{p} + \frac{1}{p}$$

and also that

$$\left( \frac{2}{p} \right) + \left( -\frac{2}{p} \right) = 0.$$ 

According to Corollary 9(ii) of [5] it follows that $\tilde{G}_\lambda \in \mathcal{L}(\ell_p)$ and the claim is proved.
Proof. (i) (1)⇔(2) follows from the definition of $R_w$.

(2)⇔(3). That (2)⇒(3) is immediate from $\ell_1 \subseteq c_0$.

Assume (3). Fix $t \in \mathbb{N}$ and set $m = t + 2$. Then $(n^m w(n))_{n \in \mathbb{N}} \in c_0$ implies that $\sup_{n \in \mathbb{N}} n^m w(n) < \infty$. Accordingly,

$$\sum_{n=1}^{\infty} n^t w(n) = \sum_{n=1}^{\infty} \frac{1}{n^{2}} n^m w(n) \leq \frac{\pi^2}{6} \sup_{n \in \mathbb{N}} n^m w(n) < \infty.$$ 

Since $t$ is arbitrary, we can conclude that (2) holds.

(2)⇔(4). Clear from the definition of the space $s$.

(ii) Since $C^{(p,w)}$ is injective, $0 \notin \sigma_{pt}(C^{(p,w)})$. By (3.3) and (3.20), $\lambda \in \mathbb{C} \setminus \{0\}$ is an eigenvalue of $C^{(p,w)}$ if and only if $\lambda = \frac{1}{m}$ for some $m \in \mathbb{N}$ with the corresponding 1-dimensional eigenspace generated by a vector $x^{[m]} = (x^{[m]}_n)_{n \in \mathbb{N}} \in \mathbb{C}^\mathbb{N}$ satisfying $x^{[m]}_n \sim n^{m-1}$. So, $\Sigma \subseteq \sigma_{pt}(C^{(p,w)})$ if and only if $(n^{m-1})_{n \in \mathbb{N}} \in \ell_p(w)$ for all $m \in \mathbb{N}$, that is, if and only if $(n^{m-1} w(n)^{1/p})_{n \in \mathbb{N}} \in \ell_p$ for all $m \in \mathbb{N}$, which is equivalent to (6) via Lemma 3.2(iii).

(iii) Follows immediately from parts (i) and (ii) and the fact that (2)⇒(6) since $\ell_1 \subseteq \ell_p$ for every $1 < p < \infty$.

(iv) Immediate from $\ell_p \subseteq c_0$ for every $1 < p < \infty$. \hfill \Box

Given a decreasing sequence $w = (w(n))_{n \in \mathbb{N}}$ of positive real numbers, set $\alpha_n := -\log w(n)$, for $n \in \mathbb{N}$. Then $w(n) = e^{-\alpha_n}$, for $n \in \mathbb{N}$. Moreover, $\alpha_n \to \infty$ for $n \to \infty$ if and only if $w(n) \to 0$ for $n \to \infty$.

Corollary 3.7. Let $w = (w(n))_{n \in \mathbb{N}}$ be a decreasing, positive sequence.

(i) If $w \in s$, then $\lim_{n \to \infty} \frac{\log n}{\alpha_n} = 0$.

(ii) If $\lim_{n \to \infty} \frac{\log n}{\alpha_n} = 0$ and $w(N) < 1$ for some $N$, then $w \in s$.\hfill \Box
Proof. (i) Since \( w \in s \), condition (3) in Proposition 3.6 implies that
\[
\forall m \in \mathbb{N} \exists n_m \in \mathbb{N} \forall n \geq n_m : n^m w(n) = \frac{n^m}{e^{\alpha n}} < 1,
\]
i.e., that
\[
\forall m \in \mathbb{N} \exists n_m \in \mathbb{N} \forall n \geq n_m : n^m < e^{\alpha n}.
\]
It follows that
\[
\forall m \in \mathbb{N} \exists n_m \in \mathbb{N} \forall n \geq n_m : m \log n < \alpha_n.
\]
This implies that necessarily \( \alpha_n > 0 \) for all \( n \geq n_m \) and so
\[
\forall m \in \mathbb{N} \exists n_m \in \mathbb{N} \forall n \geq n_m : \frac{\log n}{\alpha_n} < \frac{1}{m}.
\]
This means precisely that \( \lim_{n \to \infty} \frac{\log n}{\alpha_n} = 0 \).

(ii) Fix \( m \in \mathbb{N} \). Then there is \( n_0 \in \mathbb{N} \) with \( n_0 \geq N \) such that
\[
\frac{\log n}{\alpha_n} < \frac{1}{m+1}
\]
for all \( n \geq n_0 \). Since \( w(N) < 1 \) implies that \( \alpha_n = -\log w(n) > 0 \) for all \( n \geq n_0 \),
we can conclude that \( (m + 1) \log n < \alpha_n \), i.e., \( n^{m+1} w(n) < 1 \) for all \( n \geq n_0 \).
So,
\[
\sup_{n \in \mathbb{N}} n^{m+1} w(n) < \infty.
\]
It follows that
\[
n^m w(n) \leq \frac{1}{n} \sup_{r \in \mathbb{N}} r^{m+1} w(r), \quad n \in \mathbb{N},
\]
with \( \frac{1}{n} \sup_{r \in \mathbb{N}} r^{m+1} w(r) \to 0 \) as \( n \to \infty \). By (3) \( \Leftrightarrow \) (4) in Proposition 3.6(i)
it follows that \( w \in s \). \( \square \)

Remark 3.8. (i) Concerning condition (5) in Proposition 3.6 (for any given \( 1 < p < \infty \)), we claim that the entire set \( \Sigma \subseteq \sigma_{pt}(C^{(p,w)}) \) whenever \( \sigma_{pt}(C^{(p,w)}) \)
is an infinite set. To see this, suppose that \( \frac{1}{m} \in \sigma_{pt}(C^{(p,w)}) \) for some \( m \in \mathbb{N} \).
According to the argument in Step 7 of the proof of Theorem 3.3, we can conclude that \( (n^{m-1})_{n \in \mathbb{N}} \in \ell_p(w) \).
So, for all \( 1 \leq k < m \), it follows that
\[
\sum_{n=1}^{\infty} (n^{k})^p w(n) \leq \sum_{n=1}^{\infty} (n^{m-1})^p w(n) < \infty
\]
and hence, via (3.3), that the vector \((x_n)_{n \in \mathbb{N}} \in \mathbb{C}^N\) given by (3.20), with \( k \)
in place of \( m \), also belongs to \( \ell_p(w) \), i.e., it is an eigenvector of \( C^{(p,w)} \) corresponding to \( \lambda = \frac{1}{k} \).
This shows that \( \{ \frac{1}{k} \}_{k=1}^{m} \subseteq \sigma_{pt}(C^{(p,w)}) \) whenever \( \frac{1}{m} \in \sigma_{pt}(C^{(p,w)}) \), which
clearly implies the stated claim.

(ii) Let \( 1 < p_0 < \infty \). The constant vector \( \mathbf{1} := (1, 1, \ldots) \in \mathbb{C}^N \) satisfies \( \mathbf{C} \mathbf{1} = \mathbf{1} \)
and so \( 1 \in \sigma_{pt}(C^{(0,\infty,w)}) \) if and only if, \( \mathbf{1} \in \ell_{p_0}(w) \), i.e., if, and only if, \( w \in \ell_1 \). In
this case, \( 1 \in \sigma_{pt}(C^{(p,w)}) \) for every \( 1 < p < \infty \). Then Theorem 3.3(ii) implies that necessarily \( t_0 \in (0, \infty) \).

(iii) Let \( w(n) = \frac{1}{n^\alpha} \), for all \( n \in \mathbb{N} \) and some \( \alpha > 0 \). Then \( \sum_{n=1}^{\infty} n^\alpha w(n) < \infty \)
if, and only if, \( t < (\alpha - 1) \) and so \( t_0 = (\alpha - 1) \). In particular, \( R_w \neq \mathbb{R} \). Moreover,
for any \( 1 < p < \infty \), we have
\[
\left\{ \frac{1}{m} : m \in \mathbb{N}, \ 1 \leq m < \frac{t_0}{p} + 1 \right\} = \left\{ \frac{1}{m} : m \in \mathbb{N}, \ 1 \leq m < \frac{\alpha - 1}{p} + 1 \right\}.
\]
So, given any \( 1 < p < \infty \), it is possible to choose an appropriate \( \alpha > 0 \) such that \( \sigma_{pt}(C^{(p,w)}) \) is a finite set with any pre-assigned cardinality; see (3.10).
(iv) Condition (1) of Proposition 3.6, i.e., \( R_w = \mathbb{R} \), implies that necessarily \( S_w(p) = \emptyset \) for every \( 1 < p < \infty \); see Proposition 3.1(i).

Let \( w = (w(n))_{n \in \mathbb{N}} \) be any decreasing, (strictly) positive sequence and let \( 1 < p < \infty \). The Cesàro operator \( C^{(p,w)} \) is similar (via an isometry) to a continuous linear operator \( T_w \) acting on \( \ell_p \) which is defined by the factorable matrix \( A(w) = (a_{nk})_{n,k \in \mathbb{N}} \) with entries \( a_{nk} = a_n b_k = \frac{w(n)^{1/p}}{n} \cdot w(k)^{-1/p} \) for \( 1 \leq k \leq n \) and \( a_{nk} = 0 \) for \( k > n \) (see the proof of Lemma 2.1). In particular, \( \sigma(C^{(p,w)}) = \sigma(T_w) \).

Moreover, the matrix \( A(w) \) satisfies the following two conditions:

(i) \( \sup_{n \in \mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}| = \sup_{n \in \mathbb{N}} \frac{w(n)^{1/p}}{n} \sum_{k=1}^{n} w(k)^{-1/p} \leq 1, \)

because \( w \) decreasing implies that \( \sum_{k=1}^{n} w(k)^{-1/p} \leq nw(n)^{-1/p}, n \in \mathbb{N}, \)

and

(ii) \( f_k := \lim_{n \to \infty} a_{nk} = w(k)^{-1/p} \lim_{n \to \infty} \frac{w(n)^{1/p}}{n} = 0, k \in \mathbb{N}, \)

because \( w \in \ell_\infty. \)

If, in addition, the matrix \( A(w) \) also satisfies the condition

(iii) \( \alpha := \lim_{n \to \infty} \sum_{k=1}^{\infty} a_{nk} = \lim_{n \to \infty} \frac{w(n)^{1/p}}{n} \sum_{k=1}^{n} w(k)^{-1/p} \) exists,

then the linear operator corresponding to \( A(w) \) is a selfmap of \( c \), the space of all convergent sequences, that is, \( A(w) \) is conservative, [19, p.112].

Suppose now that the matrix \( A(w) \) satisfies condition (iii) with \( \alpha = 1 \). Then \( A(w) \) is regular and the linear operator corresponding to \( A(w) \) is limit preserving over \( c \), [19, p.114]. Define \( \eta := \lim \sup_{n \to \infty} a_n b_n \). For the operator \( T_w \) (which is similar to the Cesàro operator \( C^{(p,w)} \)) it turns out that \( \eta = 0 \) and so a result of Rhoades and Yildirim [19, Theorem 3] yields that

\[
\left\{ \lambda \in \mathbb{C} : \left| \lambda - \frac{1}{2} \right| \leq \frac{1}{2} \right\} \subseteq \sigma(C^{(p,w)}),
\]

after noting that \( S := \{a_n b_n : n \in \mathbb{N}\} = \Sigma_0 \subseteq \{ \lambda \in \mathbb{C} : \left| \lambda - \frac{1}{2} \right| \leq \frac{1}{2} \}. \)

It is worthwhile to compare (3.8) with (3.23). So, let \( 1 < p < \infty \) and \( w \) be a positive, decreasing sequence such that \( S_w(p) \neq \emptyset \). Then

\[
\left\{ \lambda \in \mathbb{C} : \left| \lambda - \frac{1}{2} \right| \leq \frac{1}{2} \right\} \subseteq \left\{ \lambda \in \mathbb{C} : \left| \lambda - \frac{p}{2s_p} \right| \leq \frac{p}{2s_p} \right\} \subseteq \sigma(C^{(p,w)}),
\]

with the first inclusion holding if and only if \( s_p \leq p' \). Observe that if \( \left( \frac{w(n)^{-1/p}}{n} \right)_{n \in \mathbb{N}} \in \ell_{p'} \), then \( s_p \leq p' \) is valid and conversely, if \( s_p < p' \), then \( \left( \frac{w(n)^{-1/p}}{n} \right)_{n \in \mathbb{N}} \notin \ell_{p'} \). In this case, (3.8) is a better inclusion than (3.23). For instance, if \( w(n) := \frac{1}{n} \) for all \( n \in \mathbb{N} \) and some \( r > 0 \), then \( \left( \frac{w(n)^{-1/p}}{n} \right)_{n \in \mathbb{N}} \in \ell_{p'} \) if, and only if, \( r < 1 \). On the other hand, the reverse inclusion

\[
\left\{ \lambda \in \mathbb{C} : \left| \lambda - \frac{p'}{2s_p} \right| \leq \frac{p}{2s_p} \right\} \subseteq \left\{ \lambda \in \mathbb{C} : \left| \lambda - \frac{1}{2} \right| \leq \frac{1}{2} \right\}
\]

holds if and only if \( p' \leq s_p \). Observe that if \( \left( \frac{w(n)^{-1/p}}{n} \right)_{n \in \mathbb{N}} \notin \ell_{p'} \), then \( p' \leq s_p \) is valid and conversely, if \( p' < s_p \), then \( \left( \frac{w(n)^{-1/p}}{n} \right)_{n \in \mathbb{N}} \notin \ell_{p'} \). In this case, modulo
the additional requirement that $\alpha = 1$ (see condition (iii)), in which case (3.23) is actually valid, we see that (3.23) is a better inclusion than (3.8).

The following example shows that condition (iii) above and the property $S_w(p) \neq \emptyset$ can be compatible.

**Example 3.9.** Fix $1 < p < \infty$. For each $n \in \mathbb{N}$ set $w(n) = \frac{1}{(\log(n+1))^{p'}}$, in which case $w(n) \downarrow 0$. Then $S_w(p) = (1, \infty)$ and

$$\left\{ \lambda \in \mathbb{C} : \left| \lambda - \frac{p'}{2} \right| \leq \frac{p'}{2} \right\} = \sigma(C^{(p,w)}) \text{ with } \sigma_{pt}(C^{(p,w)}) = \emptyset;$$

see Example 3.5(i) with $\gamma = p$. Moreover, concerning condition (iii) observe that

$$\frac{w(n)^{1/p}}{n} \sum_{k=1}^{n} w(k)^{-1/p} = \frac{1}{n \log(n+1)} \sum_{k=1}^{n} \log(k+1), \quad n \in \mathbb{N}.$$

Then the inequalities

$$[(n+1) \log(n+1) - n] \leq \sum_{k=1}^{n} \log(k+1) \leq [(n+2) \log(n+2) - n - 2 \log 2], \quad n \in \mathbb{N},$$

imply that

$$\alpha = \lim_{n \to \infty} \frac{w(n)^{1/p}}{n} \sum_{k=1}^{n} w(k)^{-1/p} = 1.$$

We also note that $\left( \frac{w(n)^{-1/p}}{n} \right)_{n \in \mathbb{N}} = \left( \frac{(\log(n+1))^{-1}}{n} \right)_{n \in \mathbb{N}} \in \ell_{p'}$.

We conclude this section with some comments about the mean ergodicity and the linear dynamics of $C^{(p,w)}$. For $X$ a Banach space, recall that $T \in \mathcal{L}(X)$ is **mean ergodic** if its sequence of Cesàro averages $T_{[n]} := \frac{1}{n} \sum_{m=1}^{n} T^m$, for $n \in \mathbb{N}$, converges to some operator $P \in \mathcal{L}(X)$ for the strong operator topology, i.e.,

$$\lim_{n \to \infty} T_{[n]} x = Px \text{ for each } x \in X,$$

in which case $T_{[n]} \to P$ in the strong operator topology. In particular, the sequence $\{ \frac{1}{n}(C^{(p,w)})^nx : n \in \mathbb{N} \}$ cannot converge to 0 for the strong operator topology in $\mathcal{L}(\ell_p(w))$. Accordingly, $C^{(p,w)}$ **fails to be mean ergodic whenever** $s_p < p'$. This is the case when $w(n) = 1$, for all $n \in \mathbb{N}$, in which case $s_p = 1$, and we recover the known fact that the classical Cesàro operator $C^{(p)}$ fails to be mean ergodic for every $1 < p < \infty$; see [3, Section 4], where it is also shown that the Cesàro operator fails to be mean ergodic in the classical Banach sequence spaces $c_0, c, \ell_p$ $(1 < p \leq \infty)$, $bv_0$ and $bv$ but, that it is mean ergodic in $bv_p$ $(1 < p < \infty)$.
Concerning the dynamics of a continuous linear operator $T$ defined on a separable Banach space $X$, recall that $T$ is hypercyclic if there exists $x \in X$ such that the orbit $\{T^n x : n \in \mathbb{N}_0\}$ is dense in $X$. If, for some $x \in X$, the projective orbit $\{\lambda T^nx : \lambda \in \mathbb{C}, \ n \in \mathbb{N}_0\}$ is dense in $X$, then $X$ is called supercyclic. Clearly, hypercyclicity always implies supercyclicity.

Let now $w$ be a positive, decreasing sequence and $1 < p < \infty$. According to Remark 3.4(iv) the infinite set $\Sigma \subseteq \sigma_{pt}(C(p,w))$. Then, by a result of Ansari and Bourdon [4, Theorem 3.2], $C(p,w)$ is not supercyclic and hence, also not hypercyclic.

4. Compactness of $C(p,w)$

According to (3.21), for each $1 < p < \infty$ the classical Cesàro operator $C(p) \in \mathcal{L}(\ell_p)$ is surely not compact. However, in the presence of a positive weight $w \downarrow 0$, this may no longer be the case for $C(p,w)$ acting on $\ell_p(w)$. We begin with the following fact.

**Proposition 4.1.** Let $w$ be a positive, decreasing weight.

(i) For every $1 < p < \infty$ we have $\Sigma \subseteq \sigma(C(p,w))$.

(ii) Suppose that $C(p,w)$ is a compact operator, for some $1 < p < \infty$. Then

$$\sigma(C(p,w)) = \Sigma_0 \text{ and } \sigma_{pt}(C(p,w)) = \Sigma.$$  \hfill (4.1)

Moreover, $w \in s$ and $r(C(p,w)) < \|C(p,w)\|$.

**Proof.** (i) According to Remark 3.4(iv) we have $\Sigma \subseteq \sigma_{pt}(C(p,w))$. But, always $\sigma_{pt}(C(p,w)) \subseteq \sigma(C(p,w))$, [9, p. 581], and so $\Sigma \subseteq \sigma(C(p,w))$.

(ii) Since $C(p,w)$ is injective, $0 \not\in \sigma_{pt}(C(p,w))$. The compactness of $C(p,w)$ then implies that $\sigma_{pt}(C(p,w)) = \sigma(C(p,w)) \setminus \{0\}$, [15, Theorem 3.4.23]. According to the proof of Step 8 for Theorem 3.3 we also have that $\sigma_{pt}(C(p,w)) \subseteq \Sigma$. In view of part (i), the equalities in (4.1) follow.

By Theorem 3.3(ii) we must have $R_w = \mathbb{R}$ (if not, then $t_0$ is finite and so (3.10) would imply that $\sigma_{pt}(C(p,w))$ is finite which is a contradiction to (4.1)). Then, via Proposition 3.6(i), we can conclude that $w \in s$.

It follows from (2.3) and the equality $r(C(p,w)) = 1$ (see (4.1)) that $r(C(p,w)) < \|C(p,w)\|$.

To decide when $C(p,w)$ is compact, first observe that $C(p,w) = \Phi_w^{-1}T_w\Phi_w$ (see Lemma 2.1 and its proof), where $T_w \in \mathcal{L}(\ell_p)$ is given by (2.2). Given any $x \in B_p := \{x \in \ell_p : \|x\| \leq 1\}$ and $n \in \mathbb{N}$, it follows from Hölder’s inequality that

$$\sum_{n=1}^{\infty} |(T_w x)_n|^p = \sum_{n=1}^{\infty} \frac{w(n)}{n^p} \left| \sum_{k=1}^{n} \frac{1}{w(k)^{\frac{1}{p'}}} \cdot x_k \right|^p \leq \sum_{n=1}^{\infty} \frac{w(n)}{n^p} \left( \sum_{k=1}^{n} \frac{1}{w(k)^{\frac{1}{p'}}} \right)^{p/p'}.$$

So, $T_w$ (hence, also $C(p,w)$) will be compact whenever $w$ satisfies the following

$$\text{Compactness criterion: } \sum_{n=1}^{\infty} \frac{w(n)}{n^p} \left( \sum_{k=1}^{n} \frac{1}{w(k)^{\frac{1}{p'}}} \right)^{p/p'} < \infty. \quad (4.2)$$
Indeed, (4.2) implies that \( \lim_{n \to \infty} \sum_{i=1}^{\infty} |(T_w x)_i|^p = 0 \) uniformly with respect to \( x \in B_p \), from which the relative compactness in \( \ell_p \) of the bounded set \( T_w(B_p) \subseteq \ell_p \) follows, [9, pp.338-339].

We introduce some notation. Let \( w \) be a positive, decreasing sequence. Define

\[
A_n(p, w) := w(n)^{p'/p} \sum_{k=1}^{n} \frac{1}{w(k)^{p'/p}}, \quad n \in \mathbb{N}, \quad 1 < p < \infty.
\]

The compactness criterion (4.2) then states that \( C(p, w) \) is a compact operator if \( \sum_{n=1}^{\infty} (A_n(p, w))^{p'/n} < \infty \).

**Theorem 4.2.** Suppose, for some \( 1 < p < \infty \), that there exist constants \( M > 0 \) and \( 0 \leq \alpha < 1 \) such that

\[
A_n(p, w) \leq M n^{\alpha}, \quad n \in \mathbb{N}.
\]

Then \( C(q, w) \) is a compact operator for every \( 1 < q \leq p \). In particular, \( w \in s \).

**Proof.** Observe, for fixed \( 1 < q \leq p \), that

\[
\gamma := \frac{q'}{q} - \frac{p'}{p} = \frac{1}{q-1} - \frac{1}{p-1} = \frac{p-q}{(q-1)(p-1)} \geq 0.
\]

For each \( n \in \mathbb{N} \) we have

\[
\sum_{k=1}^{n} \frac{1}{w(k)^{q'/q}} = \sum_{k=1}^{n} \frac{1}{w(k)^{p'/p} \cdot w(k)^{\gamma}}.
\]

Accordingly, for each \( n \in \mathbb{N} \),

\[
A_n(q, w) = \frac{w(n)^{q'/q}}{w(n)^{p'/p} \cdot w(n)^{p'/p}} \cdot \sum_{k=1}^{n} \frac{1}{w(k)^{p'/p} \cdot w(k)^{\gamma}} = w(n)^{p'/p} \sum_{k=1}^{n} \frac{1}{w(k)^{p'/p}} \cdot \left( \frac{w(n)}{w(k)} \right)^{\gamma}.
\]

Since \( w \) is decreasing, \( \frac{w(n)}{w(k)} \leq 1 \) for all \( 1 \leq k \leq n \) and so

\[
A_n(q, w) \leq w(n)^{p'/p} \sum_{k=1}^{n} \frac{1}{w(k)^{p'/p}} = A_n(p, w) \leq M n^{\alpha}.
\]

Accordingly,

\[
\sum_{n=1}^{\infty} \frac{(A_n(q, w))^{q'/q}}{n^q} \leq M^{q'/q} \sum_{n=1}^{\infty} \frac{n^{\alpha q'/q}}{n^q} = M^{q'/q} \sum_{n=1}^{\infty} \frac{1}{n^{q-(\alpha q'/q)}}.
\]

But, \( q - \frac{\alpha q}{q} = q - \alpha(q-1) = q(1-\alpha) + \alpha > (1-\alpha) + \alpha = 1 \) and so

\[
\sum_{n=1}^{\infty} \frac{(A_n(q, w))^{q'/q}}{n^q} < \infty.
\]

Then the compactness criterion yields that \( C(q, w) \) is a compact operator.

That \( w \in s \) is a consequence of Proposition 4.1(ii).

The following consequence of Theorem 4.2 leads to a rich supply of weights \( w \) for which \( C(p, w) \) is compact.
**Corollary 4.3.** Let \( w \) be a positive weight with \( w \downarrow 0 \). If the limit

\[
l := \lim_{n \to \infty} \frac{w(n)}{w(n-1)}
\]

exists in \( \mathbb{R} \setminus \{1\} \), then \( C^{(p,w)} \) is compact for every \( 1 < p < \infty \).

**Proof.** Fix \( 1 < p < \infty \). According to Theorem 4.2 (with \( \alpha = 0 \)) it suffices to prove that \( \sup_{n \in \mathbb{N}} A_n(p,w) < \infty \). Set \( a_n := \sum_{k=1}^{n} w(k)^{-p'/p} \) and \( b_n := w(n)^{-p'/p} \) for \( n \in \mathbb{N} \). Since \( w \downarrow 0 \), we have \( b_n \uparrow \infty \). Moreover, the limit

\[
\lim_{n \to \infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = \lim_{n \to \infty} \frac{w(n)^{-p'/p}}{w(n) - w(n-1)} = \frac{1}{1 - (w(n)/w(n-1))^{p'/p}}
\]

exists in \( \mathbb{R} \) as \( l \neq 1 \). According to the Stolz-Cesàro criterion, [16, Theorem 1.22], it follows that also \( \lim_{n \to \infty} \frac{a_n}{b_n} = 1/(1 - l^{p'/p}) \in \mathbb{R} \), i.e., \( \lim_{n \to \infty} A_n(p,w) = 1/(1 - l^{p'/p}) \in \mathbb{R} \). In particular, \( \sup_{n \in \mathbb{N}} A_n(p,w) < \infty \) is indeed satisfied. \( \square \)

**Remark 4.4.** (i) Let \( w \) be a positive, decreasing weight.

(a) According to (3.8), if \( C^{(p,w)} \) is a compact operator for some \( 1 < p < \infty \), then \( S_w(p) = \emptyset \).

(b) The condition \( w \downarrow 0 \) by itself need not imply that \( S_w(p) = \emptyset \) (see Examples 3.5, for instance).

(ii) Suppose \( S_w(p) \neq \emptyset \) for some \( 1 < p < \infty \). Then \( C^{(q,w)} \) fails to be compact for every \( q \in [p,\infty) \). This follows from part (i)(a) and Proposition 3.1(iii).

(iii) The following examples (a)-(c) all fall within the scope of Corollary 4.3.

So, in each case \( w \in s \) and the identities in (4.1) hold; see Proposition 4.1.

(a) For any fixed \( a > 1 \) and \( r \geq 0 \) set \( w(n) := n^r/a^n \) for \( n \in \mathbb{N} \). Then

\[
\lim_{n \to \infty} \frac{w(n)}{w(n-1)} = a^{-1} \neq 1.
\]

(b) For any fixed \( a \geq 1 \), the weight \( w(n) := a^n/n! \) for \( n \in \mathbb{N} \) satisfies

\[
\lim_{n \to \infty} \frac{w(n)}{w(n-1)} = 0 \neq 1.
\]

(c) The weight \( w(n) := 1/n^n \) for \( n \in \mathbb{N} \) also satisfies (4.5).

We point out, since \( w \) is decreasing, that \( \frac{w(n)}{w(n-1)} \in (0,1] \) for all \( n \in \mathbb{N} \). Hence, whenever the limit (4.4) exists, then necessarily \( l \in [0,1] \).

As an application, suppose that the positive, decreasing weight \( w \) has the property that \( l := \lim_{n \to \infty} \frac{w(n)}{w(n-1)} \) exists in \( [0,1) \). Then, for each \( r > 0 \), the positive, decreasing weight \( w^r : n \mapsto w(n)^r \), for \( n \in \mathbb{N} \), satisfies \( \lim_{n \to \infty} \frac{w(n)^r}{w(n-1)^r} = l^r \in [0,1) \). Hence, \( C^{(p,w^r)} \) is a compact operator for every \( 1 < p < \infty \).

(iv) The following criterion is sufficient to ensure that the limit (4.4) exists in \( \mathbb{R} \setminus \{1\} \). Hence, both Proposition 4.1 and Corollary 4.3 are applicable to such a weight \( w \). In particular, \( w \in s \).
Let $\beta = (\beta_n)_{n \in \mathbb{N}}$ be a positive, increasing sequence with $\beta \uparrow \infty$ such that \(\lim_{n \to \infty} (\beta_n - \beta_{n-1}) = \infty\). Then the weight \(w(n) := e^{-\beta_n}\), for \(n \in \mathbb{N}\), satisfies
\[l := \lim_{n \to \infty} \frac{w(n)}{w(n+1)} = 0 \neq 1.\]

It is routine to verify that \(\lim_{n \to \infty} \frac{w(n)}{w(n+1)} = 0\).

For the weight \(w(n) := a^{-n}\) for \(n \in \mathbb{N}\) (with \(a > 1\)) we have that \(\beta_n := -\log w(n) = n \log(a) \uparrow \infty\) but, \((\beta_n - \beta_{n-1}) \log(a) \not\to \infty\) for \(n \to \infty\). So, the above criterion is not applicable to this weight. However, according to part (iii)(a) of this remark (with \(r = 0\)) the weight \(w\) is admissible for Corollary 4.3.

The following examples illustrate that Theorem 4.2 is more general than Corollary 4.3.

**Examples 4.5.** (i) Fix \(0 < \beta < 1\) and set \(w_\beta(n) := e^{-n^\beta}\) for \(n \in \mathbb{N}\), in which case \(w \downarrow 0\), but
\[
\lim_{n \to \infty} \frac{w_\beta(n)}{w_\beta(n-1)} = \lim_{n \to \infty} e^{(n-1)^\beta - n^\beta} = \lim_{n \to \infty} e^{-\beta/n(1 - \beta)} = 1,
\]
as \((n-1)^\beta - n^\beta = n^\beta \left(1 - \frac{1}{n}\right)^\beta - 1\) \(= n^\beta \left[1 - \frac{\beta}{n} + o\left(\frac{1}{n}\right)\right] \approx -\frac{\beta}{n-1}\) for \(n \to \infty\). So, Corollary 4.3 is not applicable.

We show that Theorem 4.2 does apply.

Fix \(1 < p < \infty\) and set \(\gamma := \frac{\beta}{p}\). Then, for each \(n \in \mathbb{N}\), we have that
\[
A_n(p, w) = \sum_{k=1}^n e^{\gamma k^\beta} \leq e^{-\gamma n^\beta} \int_1^{n+1} e^{\gamma x^\beta} dx,
\]
where \(m \in \mathbb{N}\) is chosen minimal such that \((m - 1) < \frac{1}{\beta} - 1 \leq m\). An integration by parts \((m + 1)\)-times yields that
\[
\int_1^{n+1} e^{\gamma t^m} dt \leq a_0 + a_1(n+1)^{\beta} e^{\gamma(n+1)^\beta} + a_2(n+1)^{2\beta} e^{\gamma(n+1)^\beta} + \ldots + a_m(n+1)^{m\beta} e^{\gamma(n+1)^\beta}
\]
for positive constants \(a_0, a_1, \ldots, a_m\). It follows that
\[
\int_1^{n+1} e^{\gamma t^m} dt \leq M(1 + n)^{m\beta} e^{\gamma(1+n)^\beta}, \quad n \in \mathbb{N},
\]
for some constant \(M > 0\). Accordingly,
\[
A_n(p, w) \leq M \frac{(1 + n)^{m\beta} e^{\gamma(1+n)^\beta}}{\beta}, \quad n \in \mathbb{N}.
\]
Since \((n + 1)^\beta - n^\beta \approx \frac{\beta}{n^\beta}\) and \((1 + n)^{m\beta} \approx n^{m\beta}\) for \(n \to \infty\), there exists \(K > 0\) (independent of \(n\)) such that
\[
A_n(p, w) \leq Kn^{m\beta}, \quad i \in \mathbb{N}.
\]
Since \((m - 1) < \frac{1}{\beta} - 1\) implies that \(\alpha := m\beta \in (0,1)\), Theorem 4.2 yields that \(C_{p,w_{\beta}}\) is compact.
Proposition 4.6. Let \( p, w \) be a positive, decreasing sequence. Suppose, for some \( 1 < p < \infty \), that there exists 0 \( \leq \alpha < 1 \) such that

\[
A_n(p, w) \leq Mn^\alpha, \quad n \in \mathbb{N},
\]

for some constant \( M > 0 \).
Let $v$ be any positive, decreasing sequence such that $\frac{v(n)}{w(n)} \in \ell_\infty$ and satisfying

$$w(n) \leq Kn^\beta v(n), \quad n \in \mathbb{N}, \tag{4.9}$$

for some $0 \leq \beta < (p-1)(1-\alpha)$ and some constant $K > 0$. Then $C^{(q,v)}$ is a compact operator for every $1 < q \leq p$.

**Proof.** Let $L := \sup_{n \in \mathbb{N}} \frac{v(n)}{w(n)}$. Then, for each $n \in \mathbb{N}$, we have via (4.8) and (4.9) that

$$A_n(p,v) = v(n)^{p'/p} \sum_{k=1}^{n} \frac{1}{v(k)^{p'/p}} w(n)^{p'/p} \sum_{k=1}^{n} \frac{1}{w(k)^{p'/p}} \cdot \left( \frac{w(k)}{v(k)} \right)^{p'/p}$$

$$\leq L^{p'/p} w(n)^{p'/p} \sum_{k=1}^{n} \frac{1}{w(k)^{p'/p}} (Kk^\beta)^{p'/p} \leq (LK)^{p'/p} w(n)^{p'/p} \sum_{k=1}^{n} \frac{1}{w(k)^{p'/p}} n^{\beta p'/p}$$

$$= (LK)^{p'/p} n^{\beta p'/p} A_n(p,w) \leq M(LK)^{p'/p} n^{\alpha + (\beta p'/p)}.$$  

Moreover, $\alpha + \beta p'/p = \alpha + \frac{\beta}{p-1} < 1$ because $0 \leq \beta < (p-1)(1-\alpha)$ implies $\frac{\beta}{p-1} < (1-\alpha)$ which implies $\alpha + \frac{\beta}{p-1} < 1$. So, Theorem 4.2 applied to $v$ (with $\alpha + \beta$ in place of $\alpha$) implies $C^{(q,v)}$ is compact for all $1 < q \leq p$. \hfill \Box

**Example 4.7.** Let $v(n) := e^{n\gamma \log(n+1)}$ for $n \in \mathbb{N}$, where $0 < \beta < 1$ and $\gamma > 0$. Then $C^{(p,v)}$ is compact for every $1 < p < \infty$. Observe that $\lim_{n \to \infty} \frac{v(n)}{v(n-1)} = 1$ and so Corollary 4.3 is not applicable.

So, fix $1 < p < \infty$. Define $w(n) := e^{-n^\beta}$ for $n \in \mathbb{N}$. According to Example 4.5(i), there exist constants $M > 0$ and $0 < \alpha < 1$ such that

$$A_n(p,w) \leq Mn^\alpha, \quad n \in \mathbb{N}.$$  

Since $v(n) \leq w(n)$ for $n \in \mathbb{N}$, it is clear that $\left\{ \frac{v(n)}{w(n)} \right\} \in \ell_\infty$. Choose any $r \in (0, (p-1)(1-\alpha))$. Then

$$\frac{w(n)}{v(n)} = \log^\gamma (n+1) = \frac{\log^\gamma (n+1)}{n^r} \cdot n^r \leq Kn^r, \quad n \in \mathbb{N},$$

for some $K > 0$ (as $\lim_{n \to \infty} \frac{\log^\gamma (n+1)}{n^r} = 0$). According to Proposition 4.6, we can conclude that $C^{(p,v)}$ is compact.

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**References**


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