Continuity and spectrum of Volterra operators on weighted Banach spaces of entire functions

José Bonet (IUMPA, UPV)

Eichstaett (Germany), July, 2015

On joint work with Jari Taskinen

Project Prometeo II/2013/013

UNIVERSIDAD POLITECNICA DE VALENCIA

GENERALITAT VALENCIANA
CONSELLERIA D'EDUCACIÓ, CULTURA I ESPORT

Preliminaries

- $H(\mathbb{C})$ and $\mathcal{P}$ denote the space of entire functions and the space of polynomials, respectively. The space $H(\mathbb{C})$ will be endowed with the compact open topology τ_{co}.
- The differentiation operator $Df(z) = f'(z)$ and the integration operator $Jf(z) = \int_0^z f(\zeta)d\zeta$ are continuous on $H(\mathbb{C})$.
- Given an entire function $g \in H(\mathbb{C})$, the **Volterra operator** V_g with symbol g is defined on $H(\mathbb{C})$ by

$$V_g(f)(z) := \int_0^z f(\zeta)g'(\zeta)d\zeta \quad (z \in \mathbb{C}).$$

For $g(z) = z$ this reduces to the integration operator. Clearly V_g defines a continuous operator on $H(\mathbb{C})$.

Aim of the talk

Our purpose is to characterize boundedness, compactness and weak compactness of Volterra operators V_g acting between different weighted Banach spaces $H_v^\infty(\mathbb{C})$ of entire functions with sup-norms in terms of the symbol g. We also investigate the spectrum of V_g acting on $H_v^\infty(\mathbb{C})$.

We complement recent work by **Bassallote, Contreras, Hernández-Mancera, Martín and Paul** in 2012 for spaces of holomorphic functions on the disc and by **Constantin and Peláez** in 2013 for reflexive weighted Fock spaces.

The case of functions on the disc was considered by **Pommerenke, Aleman, Siskakis, Pau, Peláez and Rättyä**, among others.

New results about weak compactness, the spectrum and about operators on the smaller spaces $H_v^0(\mathbb{C})$ are presented.

Definitions

A **weight** v is a continuous function $v : [0, \infty[\to]0, \infty[$, which is non-increasing on $[0, \infty[$ and satisfies $\lim_{r \to \infty} r^m v(r) = 0$ for each $m \in \mathbb{N}$. If necessary, we extend v to $\mathbb{C}$ by $v(z) := v(|z|)$.

The **weighted Banach spaces of entire functions** are defined by

$$H_v^\infty(\mathbb{C}) := \{f \in H(\mathbb{C}) \mid \|f\|_v := \sup_{z \in \mathbb{C}} v(|z|)|f(z)| < \infty\},$$
$$H_v^0(\mathbb{C}) := \{f \in H(\mathbb{C}) \mid \lim_{|z| \to \infty} v(|z|)|f(z)| = 0\},$$

and they are endowed with the weighted sup norm $\| \cdot \|_v$.

Definitions

- $H_v^\infty(\mathbb{C})$ coincides with the **weighted Fock space** $\mathcal{F}_\infty^\phi$ of order infinity when $v(z) = \exp(-\phi(|z|))$, and $\phi : [0, \infty[\to]0, \infty[$ is a twice continuously differentiable increasing function.
- $H_v^0(\mathbb{C})$ is a closed subspace of $H_v^\infty(\mathbb{C})$.
- The polynomials are contained and dense in $H_v^0(\mathbb{C})$ but the monomials do not in general form a Schauder basis (**Lusky**). The Cesàro means of the Taylor polynomials satisfy $\|C_n f\|_v \leq \|f\|_v$ for each $f \in H_v^\infty(\mathbb{C})$ and the sequence $(C_n f)_n$ is $\|\cdot\|_v$-convergent to f when $f \in H_v^0(\mathbb{C})$

Definitions

For a weight v, the associated weight $\tilde{v}$ is defined by

$$\tilde{v}(z) := \left(\sup \left\{ |f(z)| \mid f \in H_v^\infty(\mathbb{C}), \|f\|_v \leq 1 \right\} \right)^{-1} = \left(\|\delta_z\|_v \right)^{-1}, \; z \in \mathbb{C},$$

where δ_z denotes the point evaluation of z.

- $\tilde{v}$ is continuous, radial, $\tilde{v} \geq v > 0$, and for each $z \in \mathbb{D}$ we can find $f_z \in H_v^\infty$, $\|f_z\|_v = 1$ with $|f_z(z)|\tilde{v}(z) = 1$.
- $H_{\tilde{v}}^\infty(\mathbb{C})$ coincides isometrically with $H_v^\infty(\mathbb{C})$, and $H_{\tilde{v}}^0(\mathbb{C})$ with $H_v^0(\mathbb{C})$.

The Volterra operator

The **Volterra operator** V_g with symbol $g \in H(\mathbb{C})$ is defined on $H(\mathbb{C})$ by

$$V_g(f)(z) := \int_0^z f(\zeta)g'(\zeta)d\zeta \quad (z \in \mathbb{C}).$$

Question

How it acts on $H_v^\infty(\mathbb{C})$ or $H_v^\infty(\mathbb{C})$?

A result

Theorem

Let v be a weight and let $w(r) := \exp(-\alpha r^p)$, where $\alpha > 0, p > 0$ are constants. The following conditions are equivalent for $g \in H(\mathbb{C})$:

(1) $V_g : H_v^\infty(\mathbb{C}) \to H_w^\infty(\mathbb{C})$ is continuous.

(2) $V_g : H_v^0(\mathbb{C}) \to H_w^0(\mathbb{C})$ is continuous.

(3) There exists a constant $C > 0$ such that $|g'(z)| \leq C|z|^{p-1}\exp(\alpha|z|^p)\tilde{v}(|z|)$ for all $z \in \mathbb{C}$, $|z| \geq 1$.

The following conditions are also equivalent for $g \in H(\mathbb{C})$:

(1) $V_g : H_v^\infty(\mathbb{C}) \to H_w^\infty(\mathbb{C})$ is compact.

(2) $V_g : H_v^0(\mathbb{C}) \to H_w^0(\mathbb{C})$ is compact.

(3) $|g'(z)| = o\big(|z|^{p-1}\exp(\alpha|z|^p)\tilde{v}(|z|)\big)$ as $|z| \to \infty$.

Multiplication operators

Proposition

Let v and w be weights. The following conditions are equivalent for an entire function $h \in H(\mathbb{C})$:

(1) $M_h : H_v^\infty(\mathbb{C}) \to H_w^\infty(\mathbb{C})$ is continuous.

(2) $M_h : H_v^0(\mathbb{C}) \to H_w^0(\mathbb{C})$ is continuous.

(3) $\sup \frac{w(z)|h(z)|}{\tilde{v}(z)} < \infty$.

(4) $\sup \frac{\tilde{w}(z)|h(z)|}{\tilde{v}(z)} < \infty$.

Multiplication operators

Proposition

Let v and w be weights. The following conditions are equivalent for an entire function $g \in H(\mathbb{C})$:

(1) $M_h : H_v^\infty(\mathbb{C}) \to H_w^\infty(\mathbb{C})$ is compact.

(2) $M_h : H_v^0(\mathbb{C}) \to H_w^0(\mathbb{C})$ is compact.

(3) $\lim_{|z|\to\infty} \frac{w(z)|h(z)|}{\tilde{v}(z)} = 0$.

(4) $\lim_{|z|\to\infty} \frac{\tilde{w}(z)|h(z)|}{\tilde{v}(z)} = 0$.

Multiplication operators

A sequence $(z_j)_j$ in $\mathbb{C}$ is called **interpolating for** $H_v^\infty(\mathbb{C})$ if for every sequence $(\alpha_j)_j$ with $\sup_{j\in\mathbb{N}} v(z_j)|\alpha_j| < \infty$, there is $g \in H_v^\infty(\mathbb{C})$ such that $f(z_j) = \alpha_j$ for each $j \in \mathbb{N}$. This property holds for example for $v(z) = e^{-\alpha|z|^p}, \alpha > 0, p > 0$ (**Marco, Massaneda, Ortega-Cerdà**).

Proposition

Let v and w be weights. If every discrete sequence in $\mathbb{C}$ has a subsequence that is interpolating for $H_{\tilde{v}}^\infty(\mathbb{C})$, the following conditions are equivalent for $g \in H(\mathbb{C})$:

(1) $M_h : H_v^\infty(\mathbb{C}) \to H_w^\infty(\mathbb{C})$ is compact.

(2) $M_h : H_v^\infty(\mathbb{C}) \to H_w^\infty(\mathbb{C})$ is weakly compact.

(3) $M_h : H_v^0(\mathbb{C}) \to H_w^0(\mathbb{C})$ is weakly compact.

Volterra operators. Reduction arguments. Definitions

Let $\varphi : [0, \infty[\to]0, \infty[$ be a continuous non-decreasing function, C^1 on $[r_\varphi, \infty[$ for some $r_\varphi \geq 0$. Suppose that φ' is non-decreasing in $[r_\varphi, \infty[$, $\varphi'(r_\varphi) > 0$ and $r^n = O(\varphi'(r))$ as $r \to \infty$ for each $n \in \mathbb{N}$. This implies $r^n = O(\varphi(r))$ as $r \to \infty$ for each $n \in \mathbb{N}$, and that

$$w_\varphi(z) := 1/\varphi(|z|), z \in \mathbb{C},$$

and

$$u_\varphi(z) := 1/\max\{\varphi'(r_\varphi), \varphi'(|z|)\} = \begin{cases} 1/\varphi'(r_\varphi) \quad , & |z| \leq r_\varphi \\ 1/\varphi'(|z|) \quad , & |z| \geq r_\varphi \end{cases}$$

are weights.

If $\varphi(r) = \exp(\alpha r^p), r \geq 0, \alpha > 0, p > 0$, then $w_\varphi(z) = \exp(-\alpha|z|^p), z \in \mathbb{C}$, and $u_\varphi(z) = \alpha^{-1}p^{-1}|z|^{1-p}\exp(-\alpha|z|^p)$ for $|z|$ large enough.

Volterra operators. Reduction arguments

Proposition

The integration operators $J : H^\infty_{u_\varphi}(\mathbb{C}) \to H^\infty_{w_\varphi}(\mathbb{C})$ and $J : H^0_{u_\varphi}(\mathbb{C}) \to H^0_{w_\varphi}(\mathbb{C})$ are continuous.

Proposition

If the function φ is of smoothness C^2 on $[r_\varphi, \infty[$ for some $r_\varphi > 0$ and it satisfies $\sup_{r \geq r_\varphi} \frac{\varphi''(r)\varphi(r)}{(\varphi'(r))^2} < \infty$ in addition to the general assumptions, then the differentiation operators $D : H^\infty_{w_\varphi}(\mathbb{C}) \to H^\infty_{u_\varphi}(\mathbb{C})$ and $D : H^0_{w_\varphi}(\mathbb{C}) \to H^0_{u_\varphi}(\mathbb{C})$, $Df := f'$, are continuous.

Volterra operators. Reduction arguments

Under the assumptions on φ in the last Proposition, $M(f,r) = O(\varphi(r))$, when $r \to \infty$ if and only if $M(f',r) = O(\varphi'(r))$ for $r \to \infty$. The argument can be traced back, at least, to **Pavlovic** (1999).

Examples of functions φ that satisfy the assumptions of Proposition can be found in the work of Hardy. For example one can take

$$\varphi(r) := r^a (\log)^b \exp(cr^d + k(\log r)^m),$$

for large r, where $c > 0, d > 0$ or $c = 0, k > 0, m > 1$.

A Littlewood-Paley-type formula

A Littlewood-Paley-type formula for entire functions and growth estimates of infinite order:

Proposition

Let φ be of smoothness C^2 on $[r_\varphi, \infty[$ for some $r_\varphi > 0$ and let it satisfy $\sup_{r \geq r_\varphi} \frac{\varphi''(r)\varphi(r)}{(\varphi'(r))^2} < \infty$ in addition to the general assumptions.
An entire function f satisfies $f \in H^\infty_{w_\varphi}(\mathbb{C})$ (resp. $f \in H^0_{w_\varphi}(\mathbb{C})$) if and only if $f' \in H^\infty_{u_\varphi}(\mathbb{C})$ (resp. $f' \in H^0_{u_\varphi}(\mathbb{C})$).
Moreover, there are constants $C, C', C'' > 0$ such that, for each $f \in H^\infty_{w_\varphi}(\mathbb{C})$,

$$\|f'\|_{u_\varphi} \leq C\|f\|_{w_\varphi}$$

and

$$\|f\|_{w_\varphi} \leq C'|f(0)| + C''\|f'\|_{u_\varphi}.$$

Theorem

Let φ be of smoothness C^2 on $[r_\varphi, \infty[$ for some $r_\varphi > 0$ and let it satisfy $\sup_{r \geq r_\varphi} \frac{\varphi''(r)\varphi(r)}{(\varphi'(r))^2} < \infty$ in addition to the general assumptions.

The following conditions are equivalent for an entire function $g \in H(\mathbb{C})$:

(1) $V_g : H_v^\infty(\mathbb{C}) \to H_{w_\varphi}^\infty(\mathbb{C})$ is continuous.

(2) $V_g : H_v^0(\mathbb{C}) \to H_{w_\varphi}^0(\mathbb{C})$ is continuous.

(3) $\sup_{|z| \geq r_\varphi} \frac{|g'(z)|}{\varphi'(|z|)\tilde{v}(z)} < \infty$.

Volterra operators

Proof.

Assume that condition (1) holds. The differentiation operator $D : H^\infty_{w_\varphi}(\mathbb{C}) \to H^\infty_{u_\varphi}(\mathbb{C})$ is continuous. We can apply (1) and the identity $DV_g = M_{g'}$ to conclude that $M_{g'} : H^\infty_v(\mathbb{C}) \to H^\infty_{u_\varphi}(\mathbb{C})$ is continuous. Now condition (3) follows from the characterization of bounded multiplication operators, since $u_\varphi(z) = 1/\varphi'(|z|), |z| \geq r_\varphi$.

Conversely, if condition (3) holds, the operator $M_{g'} : H^\infty_v(\mathbb{C}) \to H^\infty_{u_\varphi}(\mathbb{C})$ is continuous by the characterization of bounded multiplication operators. We apply the boundedness of $J : H^\infty_{u_\varphi}(\mathbb{C}) \to H^\infty_{w_\varphi}(\mathbb{C})$ to get that $V_g = J \circ M_{g'} : H^\infty_v(\mathbb{C}) \to H^\infty_{w_\varphi}(\mathbb{C})$ is continuous.

Volterra operators

Theorem

Let φ be of smoothness C^2 on $[r_\varphi, \infty[$ for some $r_\varphi > 0$ and let it satisfy $\sup_{r \geq r_\varphi} \frac{\varphi''(r)\varphi(r)}{(\varphi'(r))^2} < \infty$ in addition to the general assumptions.

The following conditions are equivalent for an entire function $g \in H(\mathbb{C})$:

(1) $V_g : H_v^\infty(\mathbb{C}) \to H_{w_\varphi}^\infty(\mathbb{C})$ is compact.

(2) $V_g : H_v^0(\mathbb{C}) \to H_{w_\varphi}^0(\mathbb{C})$ is compact.

(3) $\lim_{|z| \to \infty} \frac{|g'(z)|}{\varphi'(|z|)\tilde{v}(z)} = 0$.

Theorem

Let φ be of smoothness C^2 on $[r_\varphi, \infty[$ for some $r_\varphi > 0$ and let it satisfy $\sup_{r \geq r_\varphi} \frac{\varphi''(r)\varphi(r)}{(\varphi'(r))^2} < \infty$ in addition to the general assumptions. If every discrete sequence in $\mathbb{C}$ has a subsequence that is interpolating for $H^\infty_{\tilde{v}}(\mathbb{C})$, then the following conditions are equivalent for an entire function $g \in H(\mathbb{C})$:

(1) $V_g : H_v^\infty(\mathbb{C}) \to H_{w_\varphi}^\infty(\mathbb{C})$ is weakly compact.

(2) $V_g : H_v^0(\mathbb{C}) \to H_{w_\varphi}^0(\mathbb{C})$ is weakly compact.

(3) $\lim_{|z|\to\infty} \frac{|g'(z)|}{\varphi'(|z|)\tilde{v}(z)} = 0$.

We have already proved

Theorem

Let v be a weight and let $w(r) := \exp(-\alpha r^p)$, where $\alpha > 0, p > 0$ are constants. The following conditions are equivalent for $g \in H(\mathbb{C})$:

(1) $V_g : H_v^\infty(\mathbb{C}) \to H_w^\infty(\mathbb{C})$ is continuous.

(2) $V_g : H_v^0(\mathbb{C}) \to H_w^0(\mathbb{C})$ is continuous.

(3) There exists a constant $C > 0$ such that $|g'(z)| \leq C|z|^{p-1}\exp(\alpha|z|^p)\tilde{v}(|z|)$ for all $z \in \mathbb{C}$, $|z| \geq 1$.

The following conditions are also equivalent for $g \in H(\mathbb{C})$:

(1) $V_g : H_v^\infty(\mathbb{C}) \to H_w^\infty(\mathbb{C})$ is compact.

(2) $V_g : H_v^0(\mathbb{C}) \to H_w^0(\mathbb{C})$ is compact.

(3) $|g'(z)| = o\big(|z|^{p-1}\exp(\alpha|z|^p)\tilde{v}(|z|)\big)$ as $|z| \to \infty$.

Further examples and results

Consider $\phi : [0,\infty[\to [0,\infty[$ of smoothness C^2, $\phi' > 0$ on $[0,\infty[$, and

$$r\phi'(r) \to \infty \text{ as } r \to \infty\,, \text{ and } \phi''(r) \le (1-\delta)\phi'(r)^2 \ \ \forall r \in [R_0,\infty[, \quad (1)$$

for some constants $R_0 \ge 0$ and $0 < \delta < 1$. Assume further that there exists constants $r_0, c > 0$ such that the inequality

$$\phi'(r) + r\phi''(r) \ge \frac{c}{r} \quad (2)$$

holds for all $r \ge r_0$.

Examples

Under the assumptions (1) and (2) on ϕ, it follows that $v(r) = e^{-\phi(r)}$ is a weight that is equivalent to $\tilde{v}$. This depends on a result due to **Borichev** (1998).

Examples:

$$(1) \quad v(r) = \exp\left(-\alpha r^p + \beta(\log r)^q\right) \quad , \quad r \geq 2,$$

with $\alpha, p, q > 0$, $\beta \in \mathbb{R}$, assuming the function is extended to $[0, 2]$ properly.

$$(2) \quad v(r) = \exp(-(\log r)^p), \quad p \geq 2, r \geq 2.$$

Two corollaries

Corollary

Let $v(r) = \exp(-\phi(r))$, where ϕ satisfies (1) and (2). Then, $V_g : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ is continuous if and only if there exists a constant $C > 0$ such that

$$|g'(z)| \leq C\phi'(|z|) \ \ \forall z \in \mathbb{C}.$$

In particular, if $v(r) = \exp(-\alpha r^p)$, $\alpha > 0, p \geq 1$, then $V_g : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ is continuous if and only if g is a polynomial of degree less than or equal to the integer part of p.

Two corollaries

Corollary

If $v(r) = \exp(-\alpha r^p)$, $\alpha > 0, p > 0$, then $V_g : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ is compact if and only if it is weakly compact if and only if g is a polynomial of degree less than or equal to the integer part of $p - 1$.

Spectrum of Volterra operators

Now we investigate the **spectrum of the Volterra operator** when it acts continuously on a weighted Banach space of entire functions $H_v^\infty(\mathbb{C})$.

Aleman and Constantin in 2009 and **Aleman and Peláez** in 2012 investigated the spectra of Volterra operators on several spaces of of holomorphic functions on the disc. **Constantin** started in 2012 the study of the spectrum of Volterra operator on spaces of entire functions, more precisely on the classical Fock spaces.

We assume that $g \in H(\mathbb{C})$ be a non-constant entire function such that $g(0) = 0$ and V_g is the Volterra operator.

Definitions

X is a Hausdorff locally convex space (lcs).

$\mathcal{L}(X)$ is the space of all continuous linear operators on X.

The **resolvent set** $\rho(T, X)$ of T consists of all $\lambda \in \mathbb{C}$ such that $R(\lambda, T) := (\lambda I - T)^{-1}$ exists in $\mathcal{L}(X)$.

The **spectrum** of T is the set $\sigma(T, X) := \mathbb{C} \setminus \rho(T, X)$. The **point spectrum** is the set $\sigma_{pt}(T, X)$ of those $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not injective.

Preliminary results

Proposition

The operator $V_g - \lambda I : H(\mathbb{C}) \to H(\mathbb{C})$ is injective for each $\lambda \in \mathbb{C}$. In particular $\sigma_{pt}(V_g, H(\mathbb{C})) = \emptyset$. Moreover, $0 \in \sigma(V_g, H(\mathbb{C}))$.

Lemma

Given $\lambda \in \mathbb{C}$, $\lambda \neq 0$, and $h \in H(\mathbb{C})$, the equation $f - (1/\lambda)V_g f = h$ has a unique solution given by

$$f(z) = R_{\lambda,g}h(z) = h(0)e^{\frac{g(z)}{\lambda}} + e^{\frac{g(z)}{\lambda}} \int_0^z e^{-\frac{g(\zeta)}{\lambda}} h'(\zeta)d\zeta, \qquad z \in \mathbb{C}.$$

Preliminary results

Proposition

Let $g \in H(\mathbb{C})$ be a non-constant entire function such that $g(0) = 0$. The Volterra operator V_g satisfies $\sigma(V_g, H(\mathbb{C})) = \{0\}$ and $\sigma_{pt}(V_g, H(\mathbb{C})) = \emptyset$.

Proposition

Let $X \subset H(\mathbb{C})$ be a locally convex space that contains the constants and such that the inclusion $X \subset H(\mathbb{C})$ is continuous. Assume that $V_g : X \to X$ is continuous for some non-constant entire function g such that $g(0) = 0$. Then

$$\{0\} \cup \{\lambda \in \mathbb{C} \setminus \{0\} \mid e^{\frac{g}{\lambda}} \notin X\} \subset \sigma(V_g, X).$$

If X is a Banach space, then

$$\{0\} \cup \overline{\{\lambda \in \mathbb{C} \setminus \{0\} \mid e^{\frac{g}{\lambda}} \notin X\}} \subset \sigma(V_g, X).$$

Preliminary results

Lemma

Let $X \subset H(\mathbb{C})$ be a locally convex space that contains the constants and such that the inclusion $X \to H(\mathbb{C})$ is continuous. Assume that $V_g : X \to X$ is continuous for some non-constant entire function g such that $g(0) = 0$. The following conditions are equivalent:

(i) $\lambda \in \rho(V_g, X)$.

(ii) $R_{\lambda,g} : X \to X$ is continuous.

(iii) (a) $e^{\frac{g}{\lambda}} \in X$, and

(b) $S_{\lambda,g} : X_0 \to X_0,\ S_{\lambda,g}h(z) := e^{\frac{g(z)}{\lambda}} \int_0^z h'(\zeta)e^{-\frac{g(\zeta)}{\lambda}} d\zeta,\ z \in \mathbb{C}$, is continuous on the subspace X_0 of X of all the functions $h \in X$ with $h(0) = 0$.

Preliminary results

Lemma

Let $X \subset H(\mathbb{C})$ be a locally convex space that contains the constants and such that the inclusion $X \to H(\mathbb{C})$ is continuous. Let X_0 be the subspace of X of all the functions $h \in X$ with $h(0) = 0$. The following conditions are equivalent for $\lambda \in \mathbb{C} \setminus \{0\}$.

(i) $S_{\lambda,g} : X_0 \to X_0,\ S_{\lambda,g}h(z) := e^{\frac{g(z)}{\lambda}} \int_0^z h'(\zeta) e^{-\frac{g(\zeta)}{\lambda}} d\zeta,\ z \in \mathbb{C}$, is continuous.

(ii) $T : X_0 \to X_0$, $Th(z) := e^{\frac{g(z)}{\lambda}} \int_0^z h(\zeta) g'(\zeta) e^{-\frac{g(\zeta)}{\lambda}} d\zeta,\ z \in \mathbb{C}$, is continuous.

The proof is obtained integrating by parts.

We first deal with the Volterra operator acting on the Banach space $H_v^\infty(\mathbb{C})$, with $v(r) = \exp(-\alpha r^p)$, where $\alpha, p > 0$. Recall:

Proposition

Assume that $v(r) = \exp(-\alpha r^p)$, $\alpha > 0$, $p > 0$.

(i) $V_g : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ is continuous if and only if g is a polynomial of degree less than or equal to the integer part of p.

(ii) $V_g : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ is compact if and only if g is a polynomial of degree less than or equal to the integer part of $p - 1$.

Lemma

Let v be a weight such that $v(r)e^{\alpha r^n}$ is non-increasing on $[r_0, \infty[$ for some $r_0 > 0$, $\alpha > 0$ and $n \in \mathbb{N}$. The operator $T_\gamma : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ defined by

$$T_\gamma h(z) := e^{\gamma z^n} \int_0^z \zeta^{n-1} h(\zeta) e^{-\gamma \zeta^n} d\zeta, \quad z \in \mathbb{C},$$

is continuous if $|\gamma| < \alpha$.

Theorem

Assume that $v(r) = \exp(-\alpha r^p)$, $\alpha > 0$, $p > 0$. Let g be a polynomial of degree n less than or equal to the integer part of p with $g(0) = 0$.

(i) If the degree n of g satisfies $n < p$, then $\sigma(V_g, H_v^\infty(\mathbb{C})) = \{0\}$.

(ii) If $p = n \in \mathbb{N}$ and $g(z) = \beta z^n + k(z)$, k a polynomial of degree strictly less than n, then $\sigma(V_g, H_v^\infty(\mathbb{C})) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq |\beta|/\alpha\}$.

Moreover, we have
$\sigma(V_g, H_v^\infty(\mathbb{C})) = \{0\} \cup \overline{\{\lambda \in \mathbb{C} \setminus \{0\} \mid e^{\frac{g}{\lambda}} \notin H_v^\infty(\mathbb{C})\}}$.

In this case we also have $\sigma(V_g, H_v^0(\mathbb{C})) = \sigma(V_g, H_v^\infty(\mathbb{C}))$.

Idea of the proof.

(i) **If n is less than or equal to the integer part of $p-1$,** then $V_g : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ is compact.

Assume now that $p-1 < n < p$. For each $\lambda \neq 0$, $e^{\frac{g}{\lambda}} \in H_v^\infty(\mathbb{C})$.

Suppose first that $g(z) = \beta z^n$ for some $\beta \neq 0$. For $\lambda \neq 0$, take $\gamma > |\beta|/|\lambda|$. Clearly $v(r)e^{\gamma r^n}$ is non-increasing on $[r_0, \infty[$ for some $r_0 > 0$. Our Lemmas above imply $\lambda \in \rho(V_g, H_v^\infty(\mathbb{C}))$.

Suppose now that $g(z) = \beta z^n + k(z)$ for some $\beta \neq 0$ and some polynomial k of degree strictly less than n. Setting $g_1(z) := \beta z^n$, we have $V_g = V_{g_1} + V_k$, and V_k is a compact injective operator on $H_v^\infty(\mathbb{C})$. If $\lambda \neq 0$, we have $V_g - \lambda I = (V_{g_1} - \lambda I) + V_k$. A classical result on operator theory yields $\sigma(V_g, H_v^\infty(\mathbb{C})) = \sigma(V_{g_1}, H_v^\infty(\mathbb{C})) = \{0\}$.

Idea of the proof continued.

(ii) **We suppose now that** $v(r) = \exp(-\alpha r^n)$**,** $\alpha > 0$, **and that** g **is a polynomial of degree exactly** n**.**

Consider first the case $g(z) = \beta z^n$. For $\lambda \in \mathbb{C} \setminus \{0\}$, we have $e^{\frac{g}{\lambda}} \in H_v^\infty(\mathbb{C})$ if and only if $|\beta|/|\lambda| \leq \alpha$. Therefore, $\{\lambda \mid |\lambda| \leq |\beta|/\alpha\} \subset \sigma(V_g, H_v^\infty(\mathbb{C}))$.

Now take $\lambda \in \mathbb{C}$ with $|\lambda| > |\beta|/\alpha$. Since $v(r)\exp(\alpha r^n) = 1$, our Lemmas above imply $\sigma(V_g, H_v^\infty(\mathbb{C})) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq |\beta|/\alpha\}$ in the present case.

In the general case $g(z) = \beta z^n + k(z), \beta \neq 0$ and some polynomial k of degree strictly less than n, we proceed as in the proof of part (i).

Spectra of Volterra operators on radial Hörmander algebras

Growth functions.

A function $p : \mathbb{C} \to]0, \infty[$ is called a **growth function** if it satisfies:

- (w1) p is continuous and subharmonic.
- (w2) p is radial, that is, $p(z) = p(|z|), z \in \mathbb{C}$.
- (w3) $\log(1 + |z|^2) = o(p(z))$ as $|z| \to \infty$.
- (w4) p is doubling, i.e. $p(2z) = O(p(z))$ as $|z| \to \infty$.

Spectra of Volterra operators on radial Hörmander algebras

Given p, we define the following weighted (LB)-space of entire functions.

$$A_p(\mathbb{C}) := \{f \in \mathcal{H}(\mathbb{C}) : \text{ there is } A > 0 : \sup_{z\in\mathbb{C}} |f(z)| \exp(-Ap(z)) < \infty\},$$

endowed with the inductive limit topology, for which it is a (DFN)-algebra.

Spectra of Volterra operators on radial Hörmander algebras

Given p, we define the following weighted Fréchet space of entire functions.

$$A_p^0(\mathbb{C}) := \{f \in \mathcal{H}(\mathbb{C}) : \text{ for all } \varepsilon > 0 : \sup_{z \in \mathbb{C}} |f(z)| \exp(-\varepsilon p(z)) < \infty\},$$

endowed with the projective topology, for which it is a nuclear Fréchet algebra.

Spectra of Volterra operators on radial Hörmander algebras

- $A_p^0(\mathbb{C}) \subset A_p(\mathbb{C})$.
- Condition (w3) implies that $A_p^0(\mathbb{C})$ contains the polynomials.
- Condition (w4) implies that the spaces are stable under differentiation.
- The differentiation operator D and the integration operator J are continuous on A_p and on A_p^0. The spectrum of these two operators on A_p and on A_p^0 was investigated recently by Beltrán, Bonet and Fernández.

Spectra of Volterra operators on radial Hörmander algebras

Examples:

- When $p(z) = |z|^s$, then $A_p(\mathbb{C})$ consists of all entire functions of order s and finite type or order less than s.
- When $p(z) = |z|^s$, then $A_p^0(\mathbb{C})$ is the space of all entire functions of order at most s and type 0.
- For $s = 1$, $p(z) = |z|$, $A_p(\mathbb{C})$ is the space of all entire functions of exponential type, and $A_p^0(\mathbb{C})$ is the space of entire functions of infraexponential type.

Proposition

Let g be an entire function.

(i) $V_g : A_p \to A_p$ is continuous if and only if $g \in A_p$.

(ii) $V_g : A_p^0 \to A_p^0$ is continuous if and only if $g \in A_p^0$.

Spectra of Volterra operators on radial Hörmander algebras

Lemma

Let $p : \mathbb{C} \to [0, \infty[$ be a growth condition and let h be an entire function.

(i) The function e^h belongs to A_p if and only if $M(h, r) = O(p(r))$ as $r \to \infty$. If this is the case, then h is a polynomial.

(ii) The function e^h belongs to A_p^0 if and only if $M(h, r) = o(p(r))$ as $r \to \infty$. If this is the case, then h is a polynomial.

This is a consequence of an inequality of Caratheodory about the behaviour of the real part of entire functions.

Spectra of Volterra operators on radial Hörmander algebras

Theorem

Let $p : \mathbb{C} \to [0, \infty[$ be a growth condition and let $g \in A_p$ be non-constant.

(i) If $M(g, r) = O(p(r)), r \to \infty$, is not satisfied (which happens in particular when $p(r) = o(r), r \to \infty$), then $\sigma(V_g, A_p) = \mathbb{C}$.

(ii) If $M(g, r) = O(p(r)), r \to \infty$, then $\sigma(V_g, A_p) = \{0\}$. In this case g is a polynomial and $r = O(p(r)), r \to \infty$.

Moreover, in both cases we have
$\sigma(V_g, A_p) = \{0\} \cup \overline{\{\lambda \in \mathbb{C} \setminus \{0\} \mid e^{\frac{g}{\lambda}} \notin A_p\}}$.

Spectra of Volterra operators on radial Hörmander algebras

Idea of the proof.

First observe that $M(g/\lambda, r) = (1/|\lambda|)M(g, r)$ for each $\lambda \in \mathbb{C} \setminus \{0\}$ and $r > 0$. Therefore $e^{\frac{g}{\lambda}} \in A_p$ for some (all) $\lambda \neq 0$, if and only if $e^g \in A_p$, that is equivalent to $M(g, r) = O(p(r))$ as $r \to \infty$.

(i) If $M(g, r) = O(p(r))$ as $r \to \infty$ is not satisfied, then $e^{\frac{g}{\lambda}} \notin A_p$ for each $\lambda \neq 0$. We conclude $\sigma(V_g, A_p) = \mathbb{C}$.

Observe that in case $p(r) = o(r), r \to \infty$, then $M(g, r) = O(p(r))$ as $r \to \infty$ is not satisfied, since otherwise we would have $M(g, r) = O(p(r)) = o(r)$ as $r \to \infty$, that implies that g is constant; a contradiction.

Spectra of Volterra operators on radial Hörmander algebras

Idea of the proof.

(ii) If $M(g,r) = O(p(r)), r \to \infty$, then $e^{\frac{g}{\lambda}} \in A_p$ for each $\lambda \neq 0$. Given $\lambda \in \mathbb{C} \setminus \{0\}$, setting $G = e^{\frac{g}{\lambda}},\ 1/G = e^{-\frac{g}{\lambda}}$, the associated operator $S_{\lambda,g}$ satisfies $S_{\lambda,g} = M_G \circ J \circ M_{1/G} \circ D$. These four operators are continuous on the algebra A_p. Therefore $\lambda \in \rho(V_g, A_p)$, and $\sigma(V_g, A_p) = \{0\}$.

In this case, since g must be a non constant polynomial, the assumption in (ii) implies $r = O(p(r)), r \to \infty$.

Spectra of Volterra operators on radial Hörmander algebras

Theorem

Let $p : \mathbb{C} \to [0, \infty[$ be a growth condition and let $g \in A_p^0$ be non-constant.

(i) If $M(g, r) = o(p(r)), r \to \infty$, is not satisfied (which happens in case $p(r) = O(r), r \to \infty$), then $\sigma(V_g, A_p^0) = \mathbb{C}$.

(ii) If $M(g, r) = o(p(r)), r \to \infty$, then $\sigma(V_g, A_p^0) = \{0\}$. In this case g is a polynomial and $r = o(p(r)), r \to \infty$.

Moreover, in both cases, we have
$\sigma(V_g, A_p^0) = \{0\} \cup \overline{\{\lambda \in \mathbb{C} \setminus \{0\} \mid e^{\frac{g}{\lambda}} \notin A_p^0\}}$.

References

- **M. Basallote, M.D. Contreras, C. Hernández-Mancera, M.J. Martín, P.J. Paúl,** Volterra operators and semigroups in weighted Banach spaces of analytic functions, Collect. Math. 65 (2014), 233-249.

- **J. Bonet,** The spectrum of Volterra operators on weighted spaces of entire functions, Quart. J. Math. (to appear). DOI: 10.1093/qmath/hav019.

- **J. Bonet, J. Taskinen,** A note about Volterra operators on weighted Banach spaces of entire functions, Math. Nachr. (to appear). DOI: 10.1002/mana.201400099.

- **O. Constantin, J.A. Peláez,** Integral operators, embedding theorems and a Littlewood-Paley formula on weighted Fock spaces, Journal of Geometric Analysis (2015), DOI: 10.1007/s12220-015-9585-7.