The Cesàro operator on weighted c_0 spaces

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Aim of the lecture

AIM

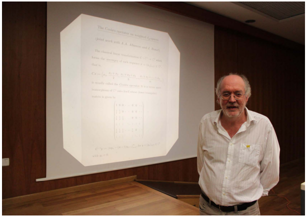
Characterize the continuity, the compactness, the mean ergodicity and determine the spectrum of the Cesàro operator C acting on the weighted Banach sequence space $c_0(w)$.

We report on joint work with Angela A. Albanese (Univ. Lecce, Italy) and Werner J. Ricker (Univ. Eichstaett, Germany).

Ernesto Cesàro (1859-1906)

Albanese and Ricker

Angela Albanese



Werner Ricker

The discrete Cesàro operator

The *Cesàro operator* C is defined for a sequence $x = (x_n)_n \in \mathbb{C}^{\mathbb{N}}$ of complex numbers by

$$\mathrm{C}(x) = \left(\frac{1}{n} \sum_{k=1}^{n} x_k \right)_n, \quad x = (x_n)_n \in \mathbb{C}^{\mathbb{N}}.$$

Proposition.

The operator $C \colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ is a bicontinuous isomorphism of $\mathbb{C}^{\mathbb{N}}$ onto itself with

$$C^{-1}(y) = (ny_n - (n-1)y_{n-1})_n, \quad y = (y_n)_n \in \mathbb{C}^{\mathbb{N}}, \tag{1}$$

where we set $y_{-1} := 0$.

Recall that $\mathbb{C}^{\mathbb{N}}$ is a Fréchet space for the topology of coordinatewise convergence.

Theorem. Hardy. 1920.

Let $1 < p < \infty$. The Cesàro operator maps the Banach space ℓ^p continuously into itself, which we denote by $\mathsf{C}^{(p)} \colon \ell^p \to \ell^p$, and $\|(\mathsf{C}^{(p)})\| = p'$, where $\frac{1}{p} + \frac{1}{p'} = 1$, for all $n \in \mathbb{N}$.

In particular, **Hardy's inequality** holds:

$$\|(\mathsf{C}^{(p)})\|_p \leq p' \|x\|_p, \quad x \in \ell^p.$$

Clearly C is not continuous on ℓ_1, since $\mathsf{C}(e_1) = (1, 1/2, 1/3, ...)$.

Proposition.

The Cesàro operators $\mathsf{C}^{(\infty)}\colon \ell^\infty \to \ell^\infty$, $\mathsf{C}^{(c)}\colon c \to c$ and $\mathsf{C}^{(0)}\colon c_0 \to c_0$ are continuous, and $\|\mathsf{C}^{(\infty)}\| = \|\mathsf{C}^{(c)}\| = \|\mathsf{C}^{(0)}\| = 1$.

Moreover, $\lim \mathsf{C}x = \lim x$ for each $x \in c$.

Spectrum and point spectrum

X is a Hausdorff locally convex space (lcs).

$\mathcal{L}(X)$ (resp. $\mathcal{K}(X)$) is the space of all continuous (resp. compact) linear operators on X.

The **resolvent set** $\rho(T, X)$ of $T \in \mathcal{L}(X)$ consists of all $\lambda \in \mathbb{C}$ such that $R(\lambda, T) := (\lambda I - T)^{-1}$ exists in $\mathcal{L}(X)$.

The **spectrum** of T is the set $\sigma(T, X) := \mathbb{C} \setminus \rho(T, X)$. The **point spectrum** is the set $\sigma_{pt}(T, X)$ of those $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not injective. The elements of $\sigma_{pt}(T, X)$ are called eigenvalues of T.

Spectrum and point spectrum

Notation:

$\Sigma := \{\frac{1}{m} \colon m \in \mathbb{N}\}$ and $\Sigma_0 := \Sigma \cup \{0\}$.

Proposition.

(i) $\sigma(\mathsf{C}; \mathbb{C}^{\mathbb{N}}) = \sigma_{pt}(\mathsf{C}; \mathbb{C}^{\mathbb{N}}) = \Sigma$.

(ii) Fix $m \in \mathbb{N}$. Let $x^{(m)} := (x_n^{(m)})_n \in \mathbb{C}^{\mathbb{N}}$ where $x_n^{(m)} := 0$ for $n \in \{1, \ldots, m-1\}$, $x_m^{(m)} := 1$ and $x_n^{(m)} := \frac{(n-1)!}{(m-1)!(n-m)!}$ for $n > m$. Then the eigenspace

$$\mathrm{Ker}\left(\frac{1}{m}I - \mathsf{C}\right) = \mathrm{span}\{x^{(m)}\} \subseteq \mathbb{C}^{\mathbb{N}}$$

is 1-dimensional.

Spectrum and point spectrum

Theorem. Leibowitz. 1972.

(i) $\sigma(\mathsf{C}; \ell^\infty) = \sigma(\mathsf{C}; c_0) = \{\lambda \in \mathbb{C} \mid \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\}$.

(ii) $\sigma_{pt}(\mathsf{C}; \ell^\infty) = \{(1, 1, 1, ...)\}$.

(iii) $\sigma_{pt}(\mathsf{C}; c_0) = \emptyset$.

Spectrum and point spectrum

Theorem. Leibowitz. 1972.

Let $1 < p < \infty$ and $1/p + 1/p' = 1$.

(i) $\sigma(\mathsf{C}; \ell^p) = \{\lambda \in \mathbb{C} \mid \left|\lambda - \frac{p'}{2}\right| \leq \frac{p'}{2}\}$.

(ii) $\sigma_{pt}(\mathsf{C}; \ell^p) = \emptyset$.

In particular, C is not compact in the spaces $\ell^p, 1 < p \leq \infty$, or in the space c_0.

The space $c_0(w)$

- Let $w = (w(n))_{n=1}^{\infty}$ be a bounded, strictly positive sequence. Define

$$c_0(w) := \left\{ x = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \colon \lim_{n\to\infty} w(n)|x_n| = 0 \right\},$$

equipped with the norm $\|x\|_{0,w} := \sup_{n\in\mathbb{N}} w(n)|x_n|$ for $x \in c_0(w)$.

- $c_0(w)$ is isometrically isomorphic to c_0 via the linear multiplication operator $\Phi_w \colon c_0(w) \to c_0$ given by

$$x = (x_n)_{n\in\mathbb{N}} \to \Phi_w(x) := (w(n)x_n)_{n\in\mathbb{N}}. \qquad (2)$$

- We are interested in the case when $\inf_{n\in\mathbb{N}} w(n) = 0$. Otherwise $c_0(w) = c_0$ with equivalent norms.

Continuity of C on $c_0(w)$

Theorem.

Let w be a bounded, strictly positive sequence.
The Cesàro operator $C^{(0,w)} \in \mathcal{L}(c_0(w))$ if and only if

$$\left\{ \frac{w(n)}{n} \sum_{k=1}^{n} \frac{1}{w(k)} \right\}_{n\in\mathbb{N}} \in \ell_\infty. \tag{3}$$

Moreover, $\|C^{(0,w)}\| \geq 1$.

If w is decreasing, then (3) is satisfied and $\|C^{(0,w)}\| = 1$.

Compactness of C on $c_0(w)$

Theorem.

Let w be a bounded, strictly positive sequence.
The following conditions are equivalent.

(a) $C^{(0,w)}$ is weakly compact.

(b) $C^{(0,w)}$ is compact.

(c) The sequence

$$\left\{ \frac{w(n)}{n} \sum_{k=1}^{n} \frac{1}{w(k)} \right\}_{n \in \mathbb{N}} \in c_0. \tag{4}$$

Continuity and compactness of C on $c_0(w)$

Let $w = (w(n))_{n=1}^{\infty}$ be two strictly positive sequences. Let $T_w \colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ denote the linear operator given by

$$T_w x := \left(\frac{w(n)}{n} \sum_{k=1}^{n} \frac{x_k}{w(k)} \right)_{n\in\mathbb{N}}, \quad x = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}. \tag{5}$$

Then $\Phi_w C = T_w \Phi_v$. Therefore, the Cesàro operator C maps $c_0(w)$ continuously (resp., compactly) into $c_0(w)$ if and only if the operator $T_w \in \mathcal{L}(c_0)$ (resp., $T_w \in \mathcal{K}(c_0)$).

Continuity of C on $c_0(w)$. A classical lemma

Lemma. Banach's Book.

Let $A = (a_{nm})_{n,m\in\mathbb{N}}$ be a matrix with entries from $\mathbb{C}$ and $T\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ be the linear operator defined by

$$Tx := \left(\sum_{m=1}^{\infty} a_{nm}x_m\right)_{n\in\mathbb{N}}, \quad x = (x_n)_{n\in\mathbb{N}}, \tag{6}$$

interpreted to mean that Tx exists in $\mathbb{C}^{\mathbb{N}}$ for every $x \in \mathbb{C}^{\mathbb{N}}$.
Then $T \in \mathcal{L}(c_0)$ if and only if the following two conditions are satisfied:

(i) $\lim_{n\to\infty} a_{nm} = 0$ for each fixed $m \in \mathbb{N}$;

(ii) $\sup_{n\in\mathbb{N}} \sum_{m=1}^{\infty} |a_{nm}| < \infty$.

In this case, $\|T\| = \sup_{n\in\mathbb{N}} \sum_{m=1}^{\infty} |a_{nm}|$.

Examples

- Let $w(2n+1) = \frac{1}{n+1}$ for $n \geq 0$ and $w(2n) = 2^{-n}$ for $n \geq 1$. Clearly $\lim_{n\to\infty} w(n) = 0$, but C does not act continuously in $c_0(w)$.
- Let $\alpha > 0$ and $w(n) := \frac{1}{n^\alpha}$ for all $n \in \mathbb{N}$. Since w is decreasing, $\mathsf{C}^{(0,w)} \in \mathcal{L}(c_0(w))$. But $\mathsf{C}^{(0,w)}$ is not compact, since

$$\begin{aligned}\frac{w(n)}{n}\sum_{k=1}^{n}\frac{1}{w(k)} &= \frac{1}{n^{\alpha+1}}\sum_{k=1}^{n}k^\alpha \geq \frac{1}{n^{\alpha+1}}\sum_{k=1}^{n}\int_{k-1}^{k}x^\alpha\,dx \\ &= \frac{1}{n^{\alpha+1}}\int_0^n x^\alpha\,dx = \frac{1}{\alpha+1}.\end{aligned}$$

Proposition.

Let w be bounded, strictly positive and satisfy

$$\limsup_{n\to\infty} \frac{w(n+1)}{w(n)} \in [0,1),$$

then $C^{(0,w)} \in \mathcal{K}(c_0(w))$.

Moreover, $\sigma_{pt}(C^{(0,w)}) = \Sigma; \quad \sigma(C^{(0,w)}) = \Sigma_0$.

One checks that that the condition (4) is valid to prove compactness.

Examples of compact operators Cesàro operators on $C^{(0,w)}$

- $C^{(0,w)} \in \mathcal{K}(c_0(w))$ for the following sequences:

  (1) $w(n) := a^{-\alpha_n}$, $n \in \mathbb{N}$, with $a > 1$, $\alpha_n \uparrow \infty$ and $\lim_{n\to\infty}(\alpha_n - \alpha_{n-1}) = \infty$.

  (2) $w(n) := \frac{n^\alpha}{a^n}$ for $n \in \mathbb{N}$, where $a > 1$ and $\alpha \in \mathbb{R}$.

  (3) $w(n) := \frac{a^n}{n!}$ for $n \in \mathbb{N}$, where $a \geq 1$.

  (4) $w(n) := n^{-n}$ for $n \in \mathbb{N}$.

- Let $w(n) := e^{-\sqrt{n}}$ or $w(n) := e^{-(\log n)^\beta}$, $\beta > 1$, for $n \in \mathbb{N}$. Then $\lim_{n\to\infty} \frac{w(n+1)}{w(n)} = 1$, but $C \in \mathcal{K}(c_0(w))$.

Spectrum of $C^{(0,w)}$

Given a bounded, strictly positive sequence w, let

$$S_w := \{s \in \mathbb{R} \colon \sum_{n=1}^{\infty} \frac{1}{n^s w(n)} < \infty\}.$$

In case $S_w \neq \emptyset$ we define $s_0 := \inf S_w$.

Moreover, let

$$R_w := \{t \in \mathbb{R} \colon \lim_{n\to\infty} n^t w(n) = 0\}.$$

In case $R_w \neq \mathbb{R}$ we define $t_0 := \sup R_w$. If $R_w = \mathbb{R}$ we set $t_0 = \infty$.

Recall $\Sigma := \{\frac{1}{m} \colon m \in \mathbb{N}\}$ and $\Sigma_0 := \Sigma \cup \{0\}$.

Spectrum of $C^{(0,w)}$. Main result

Theorem.

Let w be a bounded, strictly positive sequence such that $C^{(0,w)} \in \mathcal{L}(c_0(w))$.

(1) The following inclusion holds:

$$\Sigma_0 \subseteq \sigma(C^{(0,w)}).$$

(2) Let $\lambda \notin \Sigma_0$. Then $\lambda \in \rho(C^{(0,w)})$ if and only if both of the conditions

(i) $\lim_{n\to\infty} \frac{w(n)}{n^{1-\alpha}} = 0$, and

(ii) $\sup_{n\in\mathbb{N}} \frac{1}{n} \sum_{m=1}^{n-1} \frac{w(n)n^{\alpha}}{w(m)m^{\alpha}} < \infty$,

are satisfied, where $\alpha := \mathrm{Re}\left(\frac{1}{\lambda}\right)$.

Spectrum of $C^{(0,w)}$. Main result continued

Theorem continued.

(3) Suppose that $R_w \neq \mathbb{R}$, i.e., $t_0 < \infty$. Then we have the inclusions

$$\left\{ \frac{1}{m} \colon m \in \mathbb{N},\ 1 \leq m < t_0 + 1 \right\} \subseteq \sigma_{pt}(C^{(0,w)}) \subseteq$$

$$\subseteq \left\{ \frac{1}{m} \colon m \in \mathbb{N},\ 1 \leq m \leq t_0 + 1 \right\}.$$

In particular, $\sigma_{pt}(C^{(0,w)})$ is a proper subset of Σ.
If $R_w = \mathbb{R}$, then

$$\sigma_{pt}(C^{(0,w)}) = \Sigma.$$

First ingredient.

The dual operator.

The dual operator $(\mathsf{C}^{(0,w)})' \in \mathcal{L}(\ell_1(w^{-1}))$ satisfies $\|(\mathsf{C}^{(0,w)})'\| = \|\mathsf{C}^{(0,w)}\|$ and it is given by

$$(\mathsf{C}^{(0,w)})' y = \left(\sum_{k=n}^{\infty} \frac{y_k}{k} \right)_{n \in \mathbb{N}}, \quad y = (y_n)_{n \in \mathbb{N}} \in \ell_1(w^{-1}).$$

It satisfies $0 \notin \sigma_{pt}((\mathsf{C}^{(0,w)})')$ and $\Sigma \subseteq \sigma_{pt}((\mathsf{C}^{(0,w)})')$.

Some ingredients of the proof of the main result

First ingredient.

The dual operator.

Proposition.

If $S_w \neq \emptyset$, then the dual operator $(\mathsf{C}^{(0,w)})'$ of $\mathsf{C}^{(0,w)}$ satisfies

$$\left\{\lambda \in \mathbb{C} \colon \left|\lambda - \frac{1}{2s_0}\right| < \frac{1}{2s_0}\right\} \cup \Sigma \subseteq \sigma_{pt}((\mathsf{C}^{(0,w)})'), \quad \text{and}$$

$$\sigma_{pt}((\mathsf{C}^{(0,w)})') \setminus \Sigma_0 \subseteq \left\{\lambda \in \mathbb{C} \colon \left|\lambda - \frac{1}{2s_0}\right| \leq \frac{1}{2s_0}\right\}.$$

Some ingredients of the proof of the main result

Second ingredient.

A result of Reade (1985)

For $n \in \mathbb{N}$ the n-th row of the matrix for $(\mathsf{C} - \lambda I)^{-1} \colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ ($\lambda \notin \Sigma_0$) has the entries

$$\frac{-1}{n\lambda^2 \prod_{k=m}^{n} \left(1 - \frac{1}{\lambda k}\right)}, \quad 1 \leq m < n,$$

$$\frac{n}{1 - n\lambda} = \frac{1}{\frac{1}{n} - \lambda}, \quad m = n,$$

and all the other entries in row n are equal to 0. Therefore

$$(\mathsf{C} - \lambda I)^{-1} = D_\lambda - \frac{1}{\lambda^2} E_\lambda,$$

where the D is a diagonal operator and $E_\lambda = (e_{nm})_{n,m \in \mathbb{N}}$ is a lower triangular matrix.

Some ingredients of the proof of the main result

Third ingredient.

A technical lemma improving Reade (1985)

Lemma.

Let $\lambda \in \mathbb{C} \setminus \Sigma_0$ and set $\alpha := \mathrm{Re}\left(\frac{1}{\lambda}\right)$. Then there exist constants $d > 0$ and $D > 0$ (depending on α) such that

$$\frac{d}{n^\alpha} \leq \prod_{k=1}^{n} \left|1 - \frac{1}{k\lambda}\right| \leq \frac{D}{n^\alpha}, \quad n \in \mathbb{N}.$$

Proposition.

Let w be a strictly positive, decreasing sequence.

(i)

$$\sigma(C^{(0,w)}) \subseteq \left\{ \lambda \in \mathbb{C} \colon \left| \lambda - \frac{1}{2} \right| \leq \frac{1}{2} \right\}. \tag{7}$$

(ii) If $S_w \neq \emptyset$, then

$$\left\{ \lambda \in \mathbb{C} \colon \left| \lambda - \frac{1}{2s_0} \right| \leq \frac{1}{2s_0} \right\} \cup \Sigma \subseteq \sigma(C^{(0,w)}). \tag{8}$$

Spectrum of $C^{(0,w)}$

A sequence $u = (u_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ is called **rapidly decreasing** if $(n^m u_n)_{n\in\mathbb{N}} \in c_0$ for every $m \in \mathbb{N}$. The space of all rapidly decreasing, $\mathbb{C}$-valued sequences is denoted by s.

Proposition.

Let w be a bounded, strictly positive sequence. If $C^{(0,w)} \in \mathcal{K}(c_0(w))$, then

$$\sigma_{pt}(C^{(0,w)}) = \Sigma \quad \text{and} \quad \sigma(C^{(0,w)}) = \Sigma_0.$$

Moreover, $w \in s$ and $S_w = \emptyset$.

There exist weights $w \in s$ such that $C^{(0,w)} \notin \mathcal{K}(c_0(w))$: Define w via $w(1) := 1$ and $w(n) := \frac{1}{j^j}$ if $n \in \{2^{j-1}+1, \ldots, 2^j\}$ for $j \in \mathbb{N}$.

Spectrum of $C^{(0,w)}$. Relevant examples

(1) $w(n) = \frac{1}{(\log(n+1))^{\gamma}}$ for $n \in \mathbb{N}$ with $\gamma \geq 0$. Then $s_0 = 1$ and $t_0 = 0$. We have

$$\sigma(C^{(0,w)}) = \left\{\lambda \in \mathbb{C} \colon \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\}, \quad \text{and}$$

$$\sigma_{pt}(C^{(0,w)}) = \emptyset \text{ if } \gamma = 0; \quad \sigma_{pt}(C^{(0,w)}) = \{1\} \text{ if } \gamma > 0.$$

(2) $w(n) = \frac{1}{n^\beta}$ for $n \in \mathbb{N}$ with $\beta > 0$. Then $t_0 = \beta$ and

$$\left\{\lambda \in \mathbb{C} \colon \left|\lambda - \frac{1}{2(\beta+1)}\right| \leq \frac{1}{2(\beta+1)}\right\} \cup \Sigma = \sigma(C^{(0,w)}), \quad \text{and}$$

$$\left\{\frac{1}{m} \colon m \in \mathbb{N},\ 1 \leq m < \beta + 1\right\} = \sigma_{pt}(C^{(0,w)}).$$

Checking the examples requires the following technical result:

Lemma.

Let α be a real number with $\alpha < 1$. Then

$$\sup_{n\in\mathbb{N}} \frac{1}{n^{1-\alpha}} \sum_{m=1}^{n-1} \frac{1}{m^{\alpha}} < \infty.$$

Mean ergodic properties. Definitions

Power bounded operators

An operator $T \in \mathcal{L}(X)$ is said to be *power bounded* if $\{T^m\}_{m=1}^{\infty}$ is an equicontinuous subset of $\mathcal{L}(X)$.

If X is a Banach space, an operator T is power bounded if and only if $\sup_n ||T^n|| < \infty$.

If X is a Fréchet space, an operator T is power bounded if and only if the orbits $\{T^m(x)\}_{m=1}^{\infty}$ of all the elements $x \in X$ under T are bounded. This is a consequence of the uniform boundedness principle.

Mean ergodic properties. Definitions

For $T \in \mathcal{L}(X)$**, we set** $T_{[n]} := \frac{1}{n}\sum_{m=1}^{n} T^m$.

Mean ergodic operators

An operator $T \in \mathcal{L}(X)$ is said to be *mean ergodic* if the limits

$$Px := \lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} T^m x, \quad x \in X, \tag{9}$$

exist in X.

If T is mean ergodic, then one then has the direct decomposition

$$X = \mathrm{Ker}(I - T) \oplus \overline{(I - T)(X)}.$$

Mean ergodic properties. Definitions

Uniformly mean ergodic operators

If $\{T_{[n]}\}_{n=1}^{\infty}$ happens to be convergent in $\mathcal{L}_b(X)$ to $P \in \mathcal{L}(X)$, then T is called *uniformly mean ergodic*.

Theorem. Lin. 1974.

Let T a (continuous) operator on a Banach space X which satisfies $\lim_{n\to\infty} ||T^n/n|| = 0$. The following conditions are equivalent:

(1) T is uniformly mean ergodic.

(4) $(I - T)(X)$ is closed.

Hypercyclicity. Definitions

Hypercyclic operator

$T \in \mathcal{L}(X)$, with X separable, is called **hypercyclic** if there exists $x \in X$ such that the orbit $\{T^n x \colon n \in \mathbb{N}_0\}$ is dense in X.

Supercyclic operator

If, for some $z \in X$, the projective orbit $\{\lambda T^n z \colon \lambda \in \mathbb{C},\ n \in \mathbb{N}_0\}$ is dense in X, then T is called *supercyclic*.

Clearly, hypercyclicity always implies supercyclicity.

Dynamics of C on classical spaces

Proposition.

- The Cesàro operator $C\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ is power bounded, uniformly mean ergodic and not supercyclic.
- The Cesàro operator $\mathrm{C}^{(p)}\colon \ell^p \to \ell^p$, $1 < p < \infty$, is not power bounded, not mean ergodic and not supercyclic.
- The Cesàro operator $\mathrm{C}^{(0)}\colon c_0 \to c_0$ is power bounded, not mean ergodic and not supercyclic.

Mean ergodicity of C on $c_0(w)$

Proposition.

Let w be a decreasing, strictly positive sequence. Then $C^{(0,w)} \in \mathcal{L}(c_0(w))$ is power bounded.

Moreover, the following assertions are equivalent:

(i) $C^{(0,w)}$ is mean ergodic.

(iii) The weight w satisfies $\lim_{n\to\infty} w(n) = 0$.

Uniform mean ergodicity of C on $c_0(w)$

Proposition.

Let w be a decreasing, strictly positive sequence. Then $C^{(0,w)} \in \mathcal{L}(c_0(w))$ is uniformly mean ergodic if and only if w satisfies both of the conditions

(i) $\lim_{n\to\infty} w(n) = 0$, and

(ii) $\sup_{n\in\mathbb{N}} w(n+1) \sum_{m=1}^{n-1} \frac{1}{mw(m+1)} < \infty$.

Uniform mean ergodicity of C on $c_0(w)$

Proposition.

If w is a decreasing, strictly positive sequence such that $\mathsf{C}^{(0,w)} \in \mathcal{K}(c_0(w))$, then $\mathsf{C}^{(0,w)}$ is uniformly mean ergodic.

Examples.

(i) For $w(n) = \frac{1}{(\log(n+1))^\gamma}$ for $n \in \mathbb{N}$ with $\gamma \geq 1$, the operator $\mathsf{C}^{(0,w)}$ is not compact, mean ergodic and not uniformly mean ergodic.

(ii) For $w(n) = \frac{1}{n^\beta}$ for $n \in \mathbb{N}$ with $\beta \geq 1$, the operator $\mathsf{C}^{(0,w)}$ is uniformly mean ergodic but not compact

Non superciclycity of C on $c_0(w)$

Proposition.

Let w be a bounded, strictly positive sequence such that $\mathrm{C}^{(0,w)} \in \mathcal{L}(c_0(w))$. Then $\mathrm{C}^{(0,w)}$ is not supercyclic and hence, also not hypercyclic.

This is a direct consequence of a general result by Ansari and Bourdon, since $\sigma_{pt}((\mathrm{C}^{(0,w)})')$ is infinite.

References

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3. **A. A. Albanese, J. Bonet, W. J. Ricker,** Mean ergodicity and spectrum of the Cesàro operator on weighted c_0 spaces, Preprint (2015).