The Cesàro operator on weighted $c_0$ spaces

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Joint work with A.A. Albanese and W.J. Ricker

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Aim of the lecture

**AIM**

Characterize the continuity, the compactness, the mean ergodicity and determine the spectrum of the Cesàro operator $C$ acting on the weighted Banach sequence space $c_0(w)$.

We report on joint work with Angela A. Albanese (Univ. Lecce, Italy) and Werner J. Ricker (Univ. Eichstaett, Germany).
Ernesto Cesàro (1859-1906)

José Bonet

The Cesàro operator on weighted $c_0$ spaces
Albanese and Ricker

Angela Albanese

Werner Ricker

José Bonet

The Cesàro operator on weighted $c_0$ spaces
The discrete Cesàro operator

The Cesàro operator $C$ is defined for a sequence $x = (x_n)_n \in \mathbb{C}^\mathbb{N}$ of complex numbers by

$$C(x) = \left( \frac{1}{n} \sum_{k=1}^{n} x_k \right)_n, \quad x = (x_n)_n \in \mathbb{C}^\mathbb{N}.$$ 

**Proposition.**

The operator $C : \mathbb{C}^\mathbb{N} \rightarrow \mathbb{C}^\mathbb{N}$ is a bicontinuous isomorphism of $\mathbb{C}^\mathbb{N}$ onto itself with

$$C^{-1}(y) = (ny_n - (n - 1)y_{n-1})_n, \quad y = (y_n)_n \in \mathbb{C}^\mathbb{N}, \quad (1)$$

where we set $y_{-1} := 0$.

Recall that $\mathbb{C}^\mathbb{N}$ is a Fréchet space for the topology of coordinatewise convergence.
Theorem. Hardy. 1920.

Let $1 < p < \infty$. The Cesàro operator maps the Banach space $\ell^p$ continuously into itself, which we denote by $C^{(p)}: \ell^p \rightarrow \ell^p$, and $\|C^{(p)}\| = p'$, where $\frac{1}{p} + \frac{1}{p'} = 1$, for all $n \in \mathbb{N}$.

In particular, **Hardy’s inequality** holds:

$$\|C^{(p)}\|_p \leq p' \|x\|_p, \quad x \in \ell^p.$$ 

Clearly $C$ is not continuous on $\ell_1$, since $C(e_1) = (1, 1/2, 1/3, ...).$
Proposition.

The Cesàro operators $C^{(\infty)} : \ell^\infty \rightarrow \ell^\infty$, $C^{(c)} : c \rightarrow c$ and $C^{(0)} : c_0 \rightarrow c_0$ are continuous, and $\|C^{(\infty)}\| = \|C^{(c)}\| = \|C^{(0)}\| = 1$.

Moreover, $\lim Cx = \lim x$ for each $x \in c$. 
$X$ is a Hausdorff locally convex space (lcs).

$L(X)$ (resp. $\mathcal{K}(X)$) is the space of all continuous (resp. compact) linear operators on $X$.

The resolvent set $\rho(T, X)$ of $T \in L(X)$ consists of all $\lambda \in \mathbb{C}$ such that $R(\lambda, T) := (\lambda I - T)^{-1}$ exists in $L(X)$.

The spectrum of $T$ is the set $\sigma(T, X) := \mathbb{C} \setminus \rho(T, X)$. The point spectrum is the set $\sigma_{pt}(T, X)$ of those $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not injective. The elements of $\sigma_{pt}(T, X)$ are called eigenvalues of $T$. 

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Notation:

\[ \Sigma := \{ \frac{1}{m} : m \in \mathbb{N} \} \text{ and } \Sigma_0 := \Sigma \cup \{0\}. \]

Proposition.

(i) \( \sigma(C; \mathbb{C}^N) = \sigma_{pt}(C; \mathbb{C}^N) = \Sigma. \)

(ii) Fix \( m \in \mathbb{N}. \) Let \( x^{(m)} := (x_n^{(m)})_n \in \mathbb{C}^N \) where \( x_n^{(m)} := 0 \) for \( n \in \{1, \ldots, m-1\}, \) \( x_m^{(m)} := 1 \) and \( x_n^{(m)} := \frac{(n-1)!}{(m-1)!(n-m)!} \) for \( n > m. \)

Then the eigenspace

\[ \text{Ker} \left( \frac{1}{m} I - C \right) = \text{span}\{x^{(m)}\} \subseteq \mathbb{C}^N \]

is 1-dimensional.

(i) $\sigma(C; \ell^\infty) = \sigma(C; c_0) = \{\lambda \in \mathbb{C} \mid |\lambda - \frac{1}{2}| \leq \frac{1}{2}\}$.

(ii) $\sigma_{pt}(C; \ell^\infty) = \{(1, 1, 1, \ldots)\}$.

(iii) $\sigma_{pt}(C; c_0) = \emptyset$.

Let $1 < p < \infty$ and $1/p + 1/p' = 1$.

(i) $\sigma(C; \ell^p) = \{ \lambda \in \mathbb{C} \mid |\lambda - \frac{p'}{2}| \leq \frac{p'}{2} \}$.

(ii) $\sigma_{pt}(C; \ell^p) = \emptyset$.

In particular, C is not compact in the spaces $\ell^p, 1 < p \leq \infty$, or in the space $c_0$. 
Let $w = (w(n))_{n=1}^{\infty}$ be a bounded, strictly positive sequence. Define

$$c_0(w) := \left\{ x = (x_n)_{n \in \mathbb{N}} \in \mathbb{C}^\mathbb{N} : \lim_{n \to \infty} w(n)|x_n| = 0 \right\},$$

equipped with the norm $\|x\|_{0,w} := \sup_{n \in \mathbb{N}} w(n)|x_n|$ for $x \in c_0(w)$.

$c_0(w)$ is isometrically isomorphic to $c_0$ via the linear multiplication operator $\Phi_w : c_0(w) \to c_0$ given by

$$x = (x_n)_{n \in \mathbb{N}} \to \Phi_w(x) := (w(n)x_n)_{n \in \mathbb{N}}.$$  \hfill (2)

We are interested in the case when $\inf_{n \in \mathbb{N}} w(n) = 0$. Otherwise $c_0(w) = c_0$ with equivalent norms.
Theorem.

Let $w$ be a bounded, strictly positive sequence. The Cesàro operator $C^{(0,w)} \in \mathcal{L}(c_0(w))$ if and only if

$$\left\{ \frac{w(n)}{n} \sum_{k=1}^{n} \frac{1}{w(k)} \right\}_{n \in \mathbb{N}} \in \ell_{\infty}. \quad (3)$$

Moreover, $\|C^{(0,w)}\| \geq 1$.

If $w$ is decreasing, then (3) is satisfied and $\|C^{(0,w)}\| = 1$. 
Theorem.

Let $w$ be a bounded, strictly positive sequence. The following conditions are equivalent.

(a) $C^{(0,w)}$ is weakly compact.

(b) $C^{(0,w)}$ is compact.

(c) The sequence

$$\left\{ \frac{w(n)}{n} \sum_{k=1}^{n} \frac{1}{w(k)} \right\}_{n \in \mathbb{N}} \in c_0.$$
Let \( w = (w(n))_{n=1}^{\infty} \) be two strictly positive sequences. Let 
\( T_w: \mathbb{C}^\mathbb{N} \to \mathbb{C}^\mathbb{N} \) denote the linear operator given by 

\[
T_w x := \left( \frac{w(n)}{n} \sum_{k=1}^{n} \frac{x_k}{w(k)} \right)_{n \in \mathbb{N}}, \quad x = (x_n)_{n \in \mathbb{N}} \in \mathbb{C}^\mathbb{N}. \tag{5}
\]

Then \( \Phi_w C = T_w \Phi_v \). Therefore, the Cesàro operator \( C \) maps \( c_0(w) \) continuously (resp., compactly) into \( c_0(w) \) if and only if the operator 
\( T_w \in \mathcal{L}(c_0) \) (resp., \( T_w \in \mathcal{K}(c_0) \)).
**Lemma. Banach's Book.**

Let \( A = (a_{nm})_{n,m \in \mathbb{N}} \) be a matrix with entries from \( \mathbb{C} \) and \( T : \mathbb{C}^\mathbb{N} \to \mathbb{C}^\mathbb{N} \) be the linear operator defined by

\[
Tx := \left( \sum_{m=1}^{\infty} a_{nm} x_m \right)_{n \in \mathbb{N}}, \quad x = (x_n)_{n \in \mathbb{N}},
\]

interpreted to mean that \( Tx \) exists in \( \mathbb{C}^\mathbb{N} \) for every \( x \in \mathbb{C}^\mathbb{N} \).

Then \( T \in \mathcal{L}(c_0) \) if and only if the following two conditions are satisfied:

(i) \( \lim_{n \to \infty} a_{nm} = 0 \) for each fixed \( m \in \mathbb{N} \);

(ii) \( \sup_{n \in \mathbb{N}} \sum_{m=1}^{\infty} |a_{nm}| < \infty \).

In this case, \( \| T \| = \sup_{n \in \mathbb{N}} \sum_{m=1}^{\infty} |a_{nm}| \).
Examples

Let \( w(2n + 1) = \frac{1}{n+1} \) for \( n \geq 0 \) and \( w(2n) = 2^{-n} \) for \( n \geq 1 \). Clearly \( \lim_{n \to \infty} w(n) = 0 \), but \( C \) does not act continuously in \( c_0(w) \).

Let \( \alpha > 0 \) and \( w(n) := \frac{1}{n^\alpha} \) for all \( n \in \mathbb{N} \). Since \( w \) is decreasing, \( C^{(0,w)} \in \mathcal{L}(c_0(w)) \). But \( C^{(0,w)} \) is not compact, since

\[
\frac{w(n)}{n} \sum_{k=1}^{n} \frac{1}{w(k)} = \frac{1}{n^{\alpha+1}} \sum_{k=1}^{n} k^\alpha \geq \frac{1}{n^{\alpha+1}} \sum_{k=1}^{n} \int_{k-1}^{k} x^\alpha \, dx
\]

\[
= \frac{1}{n^{\alpha+1}} \int_{0}^{n} x^\alpha \, dx = \frac{1}{\alpha + 1}.
\]
Proposition.

Let $w$ be bounded, strictly positive and satisfy

$$\limsup_{n \to \infty} \frac{w(n+1)}{w(n)} \in [0, 1),$$

then $C^{(0,w)} \in K(c_0(w))$.

Moreover, $\sigma_{pt}(C^{(0,w)}) = \Sigma$; $\sigma(C^{(0,w)}) = \Sigma_0$.

One checks that that the condition (4) is valid to prove compactness.
Examples of compact operators Cesàro operators on $C^{(0,w)}$

- $C^{(0,w)} \in \mathcal{K}(c_0(w))$ for the following sequences:

  1. $w(n) := a^{-\alpha_n}$, $n \in \mathbb{N}$, with $a > 1$, $\alpha_n \uparrow \infty$ and
     $\lim_{n \to \infty} (\alpha_n - \alpha_{n-1}) = \infty$.

  2. $w(n) := \frac{n^\alpha}{a^n}$ for $n \in \mathbb{N}$, where $a > 1$ and $\alpha \in \mathbb{R}$.

  3. $w(n) := \frac{a^n}{n!}$ for $n \in \mathbb{N}$, where $a \geq 1$.

  4. $w(n) := n^{-n}$ for $n \in \mathbb{N}$.

- Let $w(n) := e^{-\sqrt{n}}$ or $w(n) := e^{-(\log n)^\beta}$, $\beta > 1$, for $n \in \mathbb{N}$. Then
  $\lim_{n \to \infty} \frac{w(n+1)}{w(n)} = 1$, but $C \in \mathcal{K}(c_0(w))$. 

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Spectrum of $C^{(0,w)}$

Given a bounded, strictly positive sequence $w$, let

$$S_w := \{ s \in \mathbb{R} : \sum_{n=1}^{\infty} \frac{1}{n^s w(n)} < \infty \}.$$ 

In case $S_w \neq \emptyset$ we define $s_0 := \inf S_w$.

Moreover, let

$$R_w := \{ t \in \mathbb{R} : \lim_{n \to \infty} n^t w(n) = 0 \}.$$ 

In case $R_w \neq \mathbb{R}$ we define $t_0 := \sup R_w$. If $R_w = \mathbb{R}$ we set $t_0 = \infty$.

Recall $\Sigma := \{ \frac{1}{m} : m \in \mathbb{N} \}$ and $\Sigma_0 := \Sigma \cup \{0\}$. 

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Theorem.

Let $w$ be a bounded, strictly positive sequence such that $C^{(0,w)} \in \mathcal{L}(c_0(w))$.

1. The following inclusion holds:
   \[ \Sigma_0 \subseteq \sigma(C^{(0,w)}). \]

2. Let $\lambda \notin \Sigma_0$. Then $\lambda \in \rho(C^{(0,w)})$ if and only if both of the conditions
   
   (i) $\lim_{n \to \infty} \frac{w(n)}{n^{1-\alpha}} = 0$, and
   
   (ii) $\sup_{n \in \mathbb{N}} \frac{1}{n} \sum_{m=1}^{n-1} \frac{w(n)n^\alpha}{w(m)m^\alpha} < \infty$,

   are satisfied, where $\alpha := \text{Re} \left( \frac{1}{\lambda} \right)$.
Theorem continued.

(3) Suppose that $R_w \neq \mathbb{R}$, i.e., $t_0 < \infty$. Then we have the inclusions

$$\left\{ \frac{1}{m} : m \in \mathbb{N}, 1 \leq m < t_0 + 1 \right\} \subseteq \sigma_{pt}(C^{(0,w)}) \subseteq \left\{ \frac{1}{m} : m \in \mathbb{N}, 1 \leq m \leq t_0 + 1 \right\}.$$

In particular, $\sigma_{pt}(C^{(0,w)})$ is a proper subset of $\Sigma$.

If $R_w = \mathbb{R}$, then

$$\sigma_{pt}(C^{(0,w)}) = \Sigma.$$
Some ingredients of the proof of the main result

First ingredient.

The dual operator.

The dual operator \((C^{0,w})' \in \mathcal{L}(\ell_1(w^{-1}))\) satisfies \(||(C^{0,w})'|| = ||C^{0,w}||\) and it is given by

\[
(C^{0,w})'y = \left( \sum_{k=n}^{\infty} \frac{y_k}{k} \right)_{n \in \mathbb{N}}, \quad y = (y_n)_{n \in \mathbb{N}} \in \ell_1(w^{-1}).
\]

It satisfies \(0 \notin \sigma_{pt}((C^{0,w})')\) and \(\Sigma \subseteq \sigma_{pt}((C^{0,w})')\).
First ingredient.

The dual operator.

Proposition.

If $S_w \neq \emptyset$, then the dual operator $(C^{(0,w)})'$ of $C^{(0,w)}$ satisfies

$$\left\{ \lambda \in \mathbb{C} : \left| \lambda - \frac{1}{2s_0} \right| < \frac{1}{2s_0} \right\} \cup \Sigma \subseteq \sigma_{pt}((C^{(0,w)})')$$, and

$$\sigma_{pt}((C^{(0,w)})') \setminus \Sigma_0 \subseteq \left\{ \lambda \in \mathbb{C} : \left| \lambda - \frac{1}{2s_0} \right| \leq \frac{1}{2s_0} \right\}.$$
Some ingredients of the proof of the main result

Second ingredient.

A result of Reade (1985)

For $n \in \mathbb{N}$ the $n$-th row of the matrix for $(C - \lambda I)^{-1} : \mathbb{C}^N \rightarrow \mathbb{C}^N$ ($\lambda \notin \Sigma_0$) has the entries

$$\frac{-1}{n\lambda^2 \prod_{k=m}^{n} (1 - \frac{1}{\chi_k})}, \quad 1 \leq m < n,$$

$$\frac{n}{1 - n\lambda} = \frac{1}{\frac{1}{n} - \lambda}, \quad m = n,$$

and all the other entries in row $n$ are equal to 0. Therefore

$$(C - \lambda I)^{-1} = D_\lambda - \frac{1}{\chi^2} E_\lambda,$$

where the $D$ is a diagonal operator and $E_\lambda = (e_{nm})_{n,m \in \mathbb{N}}$ is a lower triangular matrix.
Some ingredients of the proof of the main result

Third ingredient.
A technical lemma improving Reade (1985)

Lemma.
Let \( \lambda \in \mathbb{C} \setminus \Sigma_0 \) and set \( \alpha := \text{Re} \left( \frac{1}{\lambda} \right) \). Then there exist constants \( d > 0 \) and \( D > 0 \) (depending on \( \alpha \)) such that

\[
\frac{d}{n^\alpha} \leq \prod_{k=1}^{n} \left| 1 - \frac{1}{k \lambda} \right| \leq \frac{D}{n^\alpha}, \quad n \in \mathbb{N}.
\]

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Proposition.

Let \( w \) be a strictly positive, decreasing sequence.

(i) \[
\sigma(C^{(0,w)}) \subseteq \left\{ \lambda \in \mathbb{C} : \left| \lambda - \frac{1}{2} \right| \leq \frac{1}{2} \right\}. \tag{7}
\]

(ii) If \( S_w \neq \emptyset \), then

\[
\left\{ \lambda \in \mathbb{C} : \left| \lambda - \frac{1}{2s_0} \right| \leq \frac{1}{2s_0} \right\} \cup \Sigma \subseteq \sigma(C^{(0,w)}). \tag{8}
\]
A sequence $u = (u_n)_{n \in \mathbb{N}} \in \mathbb{C}^\mathbb{N}$ is called \textbf{rapidly decreasing} if $(n^m u_n)_{n \in \mathbb{N}} \in c_0$ for every $m \in \mathbb{N}$. The space of all rapidly decreasing, $\mathbb{C}$-valued sequences is denoted by $s$.

**Proposition.**

Let $w$ be a bounded, strictly positive sequence. If $C^{(0,w)} \in \mathcal{K}(c_0(w))$, then

$$\sigma_{pt}(C^{(0,w)}) = \Sigma \quad \text{and} \quad \sigma(C^{(0,w)}) = \Sigma_0.$$ 

Moreover, $w \in s$ and $S_w = \emptyset$.

**There exist weights $w \in s$ such that $C^{(0,w)} \not\in \mathcal{K}(c_0(w))$:** Define $w$ via $w(1) := 1$ and $w(n) := \frac{1}{j}$ if $n \in \{2^{j-1} + 1, \ldots, 2^j\}$ for $j \in \mathbb{N}$. 
(1) \( w(n) = \frac{1}{(\log(n+1))^{\gamma}} \) for \( n \in \mathbb{N} \) with \( \gamma \geq 0 \). Then \( s_0 = 1 \) and \( t_0 = 0 \).

We have

\[
\sigma(C^{(0,w)}) = \left\{ \lambda \in \mathbb{C} : \left| \lambda - \frac{1}{2} \right| \leq \frac{1}{2} \right\}, \quad \text{and}
\]

\[
\sigma_{pt}(C^{(0,w)}) = \emptyset \text{ if } \gamma = 0; \quad \sigma_{pt}(C^{(0,w)}) = \{1\} \text{ if } \gamma > 0.
\]
Spectrum of \( C^{(0,w)} \). Relevant examples

(2) \( w(n) = \frac{1}{n^\beta} \) for \( n \in \mathbb{N} \) with \( \beta > 0 \). Then \( t_0 = \beta \) and

\[
\left\{ \lambda \in \mathbb{C} : \left| \lambda - \frac{1}{2(\beta + 1)} \right| \leq \frac{1}{2(\beta + 1)} \right\} \cup \Sigma = \sigma(C^{(0,w)}), \quad \text{and}
\]

\[
\left\{ \frac{1}{m} : m \in \mathbb{N}, \; 1 \leq m < \beta + 1 \right\} = \sigma_{pt}(C^{(0,w)}).
\]
Spectrum of $C^{(0,w)}$. Relevant examples

Checking the examples requires the following technical result:

**Lemma.**

Let $\alpha$ be a real number with $\alpha < 1$. Then

$$\sup_{n \in \mathbb{N}} \frac{1}{n^{1-\alpha}} \sum_{m=1}^{n-1} \frac{1}{m^\alpha} < \infty.$$
Mean ergodic properties. Definitions

Power bounded operators

An operator \( T \in \mathcal{L}(X) \) is said to be **power bounded** if \( \{ T^m \}_{m=1}^{\infty} \) is an equicontinuous subset of \( \mathcal{L}(X) \).

If \( X \) is a Banach space, an operator \( T \) is power bounded if and only if 
\[
\sup_n \| T^n \| < \infty.
\]

If \( X \) is a Fréchet space, an operator \( T \) is power bounded if and only if the orbits \( \{ T^m(x) \}_{m=1}^{\infty} \) of all the elements \( x \in X \) under \( T \) are bounded. This is a consequence of the uniform boundedness principle.
For $T \in \mathcal{L}(X)$, we set $T[n] := \frac{1}{n} \sum_{m=1}^{n} T^m$.

**Mean ergodic operators**

An operator $T \in \mathcal{L}(X)$ is said to be *mean ergodic* if the limits

$$P_x := \lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} T^m x, \quad x \in X,$$

exist in $X$.

If $T$ is mean ergodic, then one then has the direct decomposition

$$X = \text{Ker}(I - T) \oplus (I - T)(X).$$
Uniformly mean ergodic operators

If \( \{ T[n] \}_{n=1}^{\infty} \) happens to be convergent in \( \mathcal{L}_b(X) \) to \( P \in \mathcal{L}(X) \), then \( T \) is called \textit{uniformly mean ergodic}.


Let \( T \) a (continuous) operator on a Banach space \( X \) which satisfies

\[
\lim_{n \to \infty} \| T^n / n \| = 0.
\]

The following conditions are equivalent:

1. \( T \) is uniformly mean ergodic.
2. \( (I - T)(X) \) is closed.
Hypercyclicity. Definitions

**Hypercyclic operator**

\( T \in \mathcal{L}(X) \), with \( X \) separable, is called **hypercyclic** if there exists \( x \in X \)

such that the orbit \( \{ T^n x : n \in \mathbb{N}_0 \} \) is dense in \( X \).

**Supercyclic operator**

If, for some \( z \in X \), the projective orbit \( \{ \lambda T^n z : \lambda \in \mathbb{C}, \; n \in \mathbb{N}_0 \} \) is dense in \( X \), then \( T \) is called **supercyclic**.

Clearly, hypercyclicity always implies supercyclicity.
Proposition.

- The Cesàro operator $C : \mathbb{C}^N \to \mathbb{C}^N$ is power bounded, uniformly mean ergodic and not supercyclic.

- The Cesàro operator $C^{(p)} : \ell^p \to \ell^p$, $1 < p < \infty$, is not power bounded, not mean ergodic and not supercyclic.

- The Cesàro operator $C^{(0)} : c_0 \to c_0$ is power bounded, not mean ergodic and not supercyclic.
Proposition.

Let \( w \) be a decreasing, strictly positive sequence. Then \( C^{(0,w)} \in \mathcal{L}(c_0(w)) \) is power bounded.

Moreover, the following assertions are equivalent:

(i) \( C^{(0,w)} \) is mean ergodic.

(iii) The weight \( w \) satisfies \( \lim_{n \to \infty} w(n) = 0 \).
Uniform mean ergodicity of $C$ on $c_0(w)$

Proposition.

Let $w$ be a decreasing, strictly positive sequence. Then $C^{(0,w)} \in \mathcal{L}(c_0(w))$ is uniformly mean ergodic if and only if $w$ satisfies both of the conditions

\begin{enumerate}[(i)]
    \item \(\lim_{n \to \infty} w(n) = 0\), and
    \item \(\sup_{n \in \mathbb{N}} w(n+1) \sum_{m=1}^{n-1} \frac{1}{mw(m+1)} < \infty\).
\end{enumerate}
Proposition.
If \( w \) is a decreasing, strictly positive sequence such that \( C^{(0,w)} \in \mathcal{K}(c_0(w)) \), then \( C^{(0,w)} \) is uniformly mean ergodic.

Examples.
(i) For \( w(n) = \frac{1}{(\log(n+1))^\gamma} \) for \( n \in \mathbb{N} \) with \( \gamma \geq 1 \), the operator \( C^{(0,w)} \) is not compact, mean ergodic and not uniformly mean ergodic.

(ii) For \( w(n) = \frac{1}{n^\beta} \) for \( n \in \mathbb{N} \) with \( \beta \geq 1 \), the operator \( C^{(0,w)} \) is uniformly mean ergodic but not compact.
Non superciclycity of $C$ on $c_0(w)$

**Proposition.**

Let $w$ be a bounded, strictly positive sequence such that $C^{(0,w)} \in \mathcal{L}(c_0(w))$. Then $C^{(0,w)}$ is not supercyclic and hence, also not hypercyclic.

This is a direct consequence of a general result by Ansari and Bourdon, since $\sigma_{pt}((C^{(0,w)})')$ is infinite.
