The Cesàro operator on sequence and function spaces

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Joint work with A.A. Albanese and W.J. Ricker

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Aim of the lecture

AIM

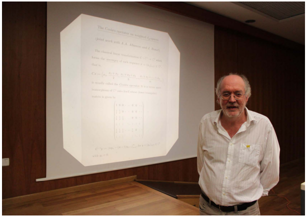
Investigate the continuity, the compactness, the mean ergodicity and determine the spectrum of the Cesàro operator C acting on weighted c_0 sequence spaces and on certain Fréchet and (LB)-spaces of analytic functions on the disc.

We report on joint work in progress with Angela A. Albanese (Univ. Lecce, Italy) and Werner J. Ricker (Univ. Eichstaett, Germany).

Ernesto Cesàro (1859-1906)

Albanese and Ricker

Angela Albanese



Werner Ricker

The discrete Cesàro operator

The *Cesàro operator* C is defined for a sequence $x = (x_n)_n \in \mathbb{C}^{\mathbb{N}}$ of complex numbers by

$$\mathrm{C}(x) = \left(\frac{1}{n} \sum_{k=1}^{n} x_k \right)_n, \quad x = (x_n)_n \in \mathbb{C}^{\mathbb{N}}.$$

Proposition.

The operator $C \colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ is a bicontinuous isomorphism of $\mathbb{C}^{\mathbb{N}}$ onto itself with

$$C^{-1}(y) = (ny_n - (n-1)y_{n-1})_n, \quad y = (y_n)_n \in \mathbb{C}^{\mathbb{N}}, \tag{1}$$

where we set $y_{-1} := 0$.

Recall that $\mathbb{C}^{\mathbb{N}}$ is a Fréchet space for the topology of coordinatewise convergence.

The Cesàro operator for analytic functions

The Cesàro operator is defined for analytic functions on the disc $\mathbb{D}$ by

$$Cf = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} a_n \right) z^n, \quad f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D}).$$

The Cesàro operator acts continuously and has the integral representation

$$Cf(z) = \frac{1}{z} \int_0^z \frac{f(\rho)}{1-\rho} \, d\rho, \quad f \in H(\mathbb{D}), \; z \in \mathbb{D}.$$

Indeed, for $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$, we have

$$\begin{aligned} Cf(z) &= \frac{1}{z}\int_0^z \frac{f(\rho)}{1-\rho}\,d\rho = \frac{1}{z}\int_0^z \left(\sum_{n=0}^{\infty} a_n \rho^n\right)\left(\sum_{m=0}^{\infty} \rho^m\right) \\ &= \sum_{n=0}^{\infty} a_n \sum_{m=0}^{\infty} \frac{1}{z}\int_0^z \rho^{n+m}\,d\rho = \sum_{n=0}^{\infty} a_n \sum_{m=0}^{\infty} \frac{z^{n+m}}{n+m+1} \\ &= \sum_{n=0}^{\infty} a_n \sum_{k=n} \frac{z^k}{k+1} = \sum_{k=0}^{\infty} \left(\frac{1}{k+1}\sum_{n=0}^{k} a_n\right) z^k. \end{aligned}$$

Theorem. Hardy. 1920.

Let $1 < p < \infty$. The Cesàro operator maps the Banach space ℓ^p continuously into itself, which we denote by $C^{(p)} \colon \ell^p \to \ell^p$, and $\|(C^{(p)})\| = p'$, where $\frac{1}{p} + \frac{1}{p'} = 1$, for all $n \in \mathbb{N}$.

In particular, **Hardy's inequality** holds:

$$\|(C^{(p)})\|_p \leq p' \|x\|_p, \quad x \in \ell^p.$$

Clearly C is not continuous on ℓ_1, since $C(e_1) = (1, 1/2, 1/3, ...)$.

Proposition.

The Cesàro operators $\mathrm{C}^{(\infty)} \colon \ell^\infty \to \ell^\infty$, $\mathrm{C}^{(c)} \colon c \to c$ and $\mathrm{C}^{(0)} \colon c_0 \to c_0$ are continuous, and $\|\mathrm{C}^{(\infty)}\| = \|\mathrm{C}^{(c)}\| = \|\mathrm{C}^{(0)}\| = 1$.

Moreover, $\lim \mathrm{C}x = \lim x$ for each $x \in c$.

Spectrum and point spectrum

X is a Hausdorff locally convex space (lcs).

$\mathcal{L}(X)$ (resp. $\mathcal{K}(X)$) is the space of all continuous (resp. compact) linear operators on X.

The **resolvent set** $\rho(T, X)$ of $T \in \mathcal{L}(X)$ consists of all $\lambda \in \mathbb{C}$ such that $R(\lambda, T) := (\lambda I - T)^{-1}$ exists in $\mathcal{L}(X)$.

The **spectrum** of T is the set $\sigma(T, X) := \mathbb{C} \setminus \rho(T, X)$. The **point spectrum** is the set $\sigma_{pt}(T, X)$ of those $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not injective. The elements of $\sigma_{pt}(T, X)$ are called eigenvalues of T.

Spectrum and point spectrum

Theorem. Leibowitz. 1972.

(i) $\sigma(\mathsf{C};\ell^\infty)=\sigma(\mathsf{C};c_0)=\{\lambda\in\mathbb{C}\mid \left|\lambda-\frac{1}{2}\right|\leq\frac{1}{2}\}$.

(ii) $\sigma_{pt}(\mathsf{C};\ell^\infty)=\{(1,1,1,\ldots)\}$.

(iii) $\sigma_{pt}(\mathsf{C};c_0)=\emptyset$.

Spectrum and point spectrum

Theorem. Leibowitz. 1972.

Let $1 < p < \infty$ and $1/p + 1/p' = 1$.

(i) $\sigma(\mathsf{C}; \ell^p) = \{\lambda \in \mathbb{C} \mid \left|\lambda - \frac{p'}{2}\right| \leq \frac{p'}{2}\}$.

(ii) $\sigma_{pt}(\mathsf{C}; \ell^p) = \emptyset$.

In particular, C is not compact in the spaces $\ell^p, 1 < p \leq \infty$, or in the space c_0.

Definitions and abstract results.

X is a Hausdorff locally convex space (lcs).

- $\rho^*(T)$ consists of all $\lambda \in \mathbb{C}$ for which there exists $\delta > 0$ such that each $\mu \in B(\lambda, \delta) := \{z \in \mathbb{C} \colon |z - \lambda| < \delta\}$ belongs to $\rho(T)$ and the set $\{R(\mu, T) \colon \ \mu \in B(\lambda, \delta)\}$ is equicontinuous in $\mathcal{L}(X)$.
- $\sigma^*(T) := \mathbb{C} \setminus \rho^*(T)$.
- $\sigma^*(T)$ is a closed set containing $\sigma(T)$. If $T \in \mathcal{L}(X)$ with X a Banach space, then $\sigma(T) = \sigma^*(T)$. There exist continuous linear operators T on a Fréchet space X such that $\overline{\sigma(T)} \subset \sigma^*(T)$ properly.

Theorem.

Let $X = \cap_{n\in\mathbb{N}} X_n$ be a Fréchet space given by the intersection of a sequence of Banach spaces $((X_n, \|\cdot\|_n))_{n\in\mathbb{N}}$ satisfying $X_{n+1} \subset X_n$ with $\|x\|_n \leq \|x\|_{n+1}$, for each $n \in \mathbb{N}$ and $x \in X_{n+1}$. Let $T \in \mathcal{L}(X)$ satisfy the following condition:

(a) For each $n \in \mathbb{N}$ there exists $T_n \in \mathcal{L}(X_n)$ such that the restriction of T_n to X (resp. of T_n to X_{n+1}) coincides with T (resp. with T_{n+1}).

Then $\sigma(T, X) \subseteq \cup_{n\in\mathbb{N}} \sigma(T_n, X_n)$ and $R(\lambda, T)$ coincides with the restriction of $R(\lambda, T_n)$ to X for each $n \in \mathbb{N}$ and each $\lambda \in \cap_{n\in\mathbb{N}} \rho(T_n, X_n)$. Moreover, if $\cup_{n\in\mathbb{N}} \sigma(T_n, X_n) \subseteq \overline{\sigma(T, X)}$, then

$$\sigma^*(T, X) = \overline{\sigma(T, X)}.$$

Abstract results.

Theorem.

Let $E = \mathrm{ind}_n(E_n, \| \ \|_n)$ be a Hausdorff inductive limit of Banach spaces. Let $T \in \mathcal{L}(E)$ satisfy the following condition:

(A) For each $n \in \mathbb{N}$ the restriction T_n of T to E_n maps E_n into itself and belongs to $\mathcal{L}(E_n)$.

Then the following properties are satisfied.

(i) $\sigma_{pt}(T, E) = \cup_{n\in\mathbb{N}} \sigma_{pt}(T_n, E_n)$.

(ii) $\sigma(T, E) \subseteq \cap_{m\in\mathbb{N}} \cup_{n=m}^{\infty} \sigma(T_n, E_n)$. Moreover, if $\lambda \in \cap_{n=m}^{\infty} \rho(T_n, E_n)$ for some $m \in \mathbb{N}$, then $R(\lambda, T_n)$ coincides with the restriction of $R(\lambda, T)$ to E_n for each $n \geq m$.

(iii) If $\cup_{n=m}^{\infty} \sigma(T_n, E_n) \subseteq \overline{\sigma(T, E)}$ for some $m \in \mathbb{N}$, then $\sigma^*(T, E) = \overline{\sigma(T, E)}$.

More about the spectrum and point spectrum

Notation:

$\Sigma := \{\frac{1}{m} \colon m \in \mathbb{N}\}$ and $\Sigma_0 := \Sigma \cup \{0\}$.

Proposition.

(i) $\sigma(\mathsf{C}; \mathbb{C}^{\mathbb{N}}) = \sigma_{pt}(\mathsf{C}; \mathbb{C}^{\mathbb{N}}) = \Sigma$.

(ii) Fix $m \in \mathbb{N}$. Let $x^{(m)} := (x_n^{(m)})_n \in \mathbb{C}^{\mathbb{N}}$ where $x_n^{(m)} := 0$ for $n \in \{1, \ldots, m-1\}$, $x_m^{(m)} := 1$ and $x_n^{(m)} := \frac{(n-1)!}{(m-1)!(n-m)!}$ for $n > m$. Then the eigenspace

$$\mathrm{Ker}\left(\frac{1}{m} I - \mathsf{C}\right) = \mathrm{span}\{x^{(m)}\} \subseteq \mathbb{C}^{\mathbb{N}}$$

is 1-dimensional.

More about the spectrum and point spectrum

Theorem

The Cesàro operator satisfies

$$(a) \quad \sigma(\mathsf{C}, H(\mathbb{D})) = \sigma_{pt}(\mathsf{C}, H(\mathbb{D})) = \{\frac{1}{m} \colon m \in \mathbb{N}\}.$$

$$(b) \quad \sigma^*(\mathsf{C}, H(\mathbb{D})) = \{\frac{1}{m} \colon m \in \mathbb{N}\} \cup \{0\}.$$

Persson showed in 2008 the following facts:

For every $m \in \mathbb{N}$ the operator $(\mathsf{C} - \frac{1}{m}I) \colon H(\mathbb{D}) \to H(\mathbb{D})$ is not injective because $\mathrm{Ker}(\mathsf{C} - \frac{1}{m}I) = \mathrm{span}\{e_m\}$, where $e_m(z) = z^{m-1}(1-z)^{-m}$, $z \in \mathbb{D}$, and it is not surjective because the function $f_m(z) := z^{m-1}$, $z \in \mathbb{D}$, does not belong to the range of $(\mathsf{C} - \frac{1}{m}I)$.

The growth spaces

For $\gamma > 0$ the growth classes $A^{-\gamma}$ and $A_0^{-\gamma}$ are the Banach spaces defined by

$$A^{-\gamma} = \{f \in H(\mathbb{D})\colon \|f\|_{-\gamma} := \sup_{z\in\mathbb{D}}(1-|z|)^{\gamma}|f(z)| < \infty\}.$$

$$A_0^{-\gamma} = \{f \in H(\mathbb{D})\colon \lim_{|z|\to 1}(1-|z|)^{\gamma}|f(z)| = 0\}.$$

$A_0^{-\gamma}$ is the closure of the polynomials on $A^{-\gamma}$.

The Cesàro operator acts continuously on $A^{-\gamma}$. Its spectrum on these (and many other spaces of analytic functions on the disc) has been studied by Aleman and Persson 2008-2010.

The result of Aleman and Persson

Theorem. Aleman, Persson.

The Cesàro operator C satisfies:

(i) $\sigma_{pt}(\mathsf{C}, A_0^{-\gamma}) = \{\frac{1}{m} \colon m \in \mathbb{N},\ m < \gamma\}$.

(ii) $\sigma(\mathsf{C}, A_0^{-\gamma}) = \sigma_{pt}(\mathsf{C}, A_0^{-\gamma}) \cup \left\{\lambda \in \mathbb{C} \colon \left|\lambda - \frac{1}{2\gamma}\right| \leq \frac{1}{2\gamma}\right\}$.

(iii) If $\left|\lambda - \frac{1}{2\gamma}\right| < \frac{1}{2\gamma}$ (or equivalently $\mathrm{Re}\left(\frac{1}{\lambda}\right) > \gamma$), then the space $\mathrm{Im}(\lambda I - \mathsf{C})$ is closed in $A_0^{-\gamma}$ and has codimension 1 in $A_0^{-\gamma}$.

(iv) $\sigma_{pt}(\mathsf{C}, A^{-\gamma}) = \{\frac{1}{m} \colon m \in \mathbb{N},\ m \leq \gamma\}$.

(v) $\sigma(\mathsf{C}, A^{-\gamma}) = \sigma(\mathsf{C}, A_0^{-\gamma})$.

The Fréchet growth spaces

Let $\gamma \geq 0$.

$$A_+^{-\gamma} := \cap_{\mu>\gamma} A^{-\mu} = \cap_{\mu>\gamma} A_0^{-\mu}.$$

The space $A_+^{-\gamma}$ is Fréchet when it is endowed with the lc-topology generated by the fundamental sequence of seminorms

$$\|f\|_k := \sup_{z\in\mathbb{D}} (1-|z|)^{\gamma+\frac{1}{k}} |f(z)|.$$

It is a Fréchet-Schwartz space because the inclusion $A^{-\mu_1} \hookrightarrow A^{-\mu_2}$ is compact for all $0 < \mu_1 < \mu_2$. In particular, every bounded subset of $A_+^{-\gamma}$ is relatively compact, i.e. the space is Montel

The (LB) growth spaces

Let $0 < \gamma \leq \infty$

$$A_-^{-\gamma} := \cup_{\mu<\gamma} A^{-\mu} = \cup_{\mu<\gamma} A_0^{-\mu},$$

and it is endowed with the finest locally convex topology such that all the inclusions $A^{-\mu} \hookrightarrow A^{-\gamma}, \ \mu < \gamma$, are continuous.
In particular, $A_-^{-\gamma}$ is the (DFS)-space

$$A_-^{-\gamma} := \operatorname{ind}_k A^{-(\gamma - \frac{1}{k})} = \operatorname{ind}_k A_0^{-(\gamma - \frac{1}{k})}.$$

As a consequence $A_-^{-\gamma}$ is a Montel space, too.

The **Korenblum space** $A_-^{-\infty}$ was introduced by Korenblum in 1975 is usually denoted by

$$A^{-\infty} = \cup_{0<\gamma<\infty} A^{-\gamma} = \cup_{n\in\mathbb{N}} A^{-n}.$$

Observe that $A_-^{-\gamma} \subseteq A_0^{-\gamma} \subseteq A^{-\gamma} \subseteq A_+^{-\gamma}$ with continuous inclusions.

All these spaces play an important role in the study of interpolation and sampling of holomorphic functions on the disc.

The Cesàro operators $\mathsf{C}\colon A_-^{-\gamma} \to A_-^{-\gamma}$ and $\mathsf{C}\colon A_+^{-\gamma} \to A_+^{-\gamma}$ are continuous because C acts continuously in every step.

The spectrum of C in the Fréchet growth spaces

Theorem

(1) Let $\gamma \in]0, \infty[$.

(a) $\sigma_{pt}(C, A_+^{-\gamma}) = \{\frac{1}{m} \colon m \in \mathbb{N}, \ m \leq \gamma\}$.

(b) $\sigma(C, A_+^{-\gamma}) = \{0\} \cup \{\frac{1}{m} \colon m, \ m \leq \gamma\} \cup \{\lambda \in \mathbb{C} \colon |\lambda - \frac{1}{2\gamma}| < \frac{1}{2\gamma}\}$.

(c) $\sigma^*(C, A_+^{-\gamma}) = \overline{\sigma(C, A_+^{-\gamma})}$.

(2) Let $\gamma = 0$.

(a) $\sigma_{pt}(C, A_+^{-0}) = \emptyset$.

(b) $\sigma(C, A_+^{-0}) = \{0\} \cup \{\lambda \in \mathbb{C} \mid \mathrm{Re}\lambda > 0\}$.

(c) $\sigma^*(C, A_+^{-0}) = \overline{\sigma(C, A_+^{-0})}$.

The spectrum of C in the (LB) growth spaces

Theorem

(1) Let $\gamma \in]0, \infty[$.

(a) $\sigma_{pt}(\mathsf{C}, A_-^{-\gamma}) = \{\frac{1}{m} \colon m \in \mathbb{N},\ m < \gamma\}$.

(b) $\sigma(\mathsf{C}, A_-^{-\gamma}) = \{\frac{1}{m} \colon m \in \mathbb{N},\ m < \gamma\} \cup \{\lambda \in \mathbb{C} \colon |\lambda - \frac{1}{2\gamma}| \leq \frac{1}{2\gamma}\}$.

(c) $\sigma^*(\mathsf{C}, A_-^{-\gamma}) = \sigma(\mathsf{C}, A_-^{-\gamma})$.

(2) For the Korenblum space $A^{-\infty}$ (i.e. $\gamma = \infty$) we have:

(a) $\sigma(\mathsf{C}, A^{-\infty}) = \sigma_{pt}(\mathsf{C}, A^{-\infty}) = \{\frac{1}{m} \colon m \in \mathbb{N}\}$.

(b) $\sigma^*(\mathsf{C}, A^{-\infty}) = \{\frac{1}{m} \colon m \in \mathbb{N}\} \cup \{0\}$.

$H(\mathbb{D})$ as a power series space

The map

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D}) \to (a_n)_{n=0}^{\infty}$$

defines an isomorphism between the Fréchet space $H(\mathbb{D})$ endowed with the topology of uniform convergence on the compact sets and the sequence space

$$\Lambda_0((n)_n) := \bigcap_{k\in\mathbb{N}} c_0(w_k),$$

$H(\mathbb{D})$ as a power series space

In the Fréchet space

$$\Lambda_0((n)_n) := \bigcap_{k\in\mathbb{N}} c_0(w_k),$$

we take $w_k(n) := (r_k)^n, k \in \mathbb{N}, n = 0, 1, 2, \ldots$ and $r_k = 1 - (1/k), k \in \mathbb{N}$, an increasing sequence tending to 1. Moreover,

$$c_0(w_k) := \left\{ x = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \colon \lim_{n\to\infty} w_k(n)|x_n| = 0 \right\},$$

equipped with the norm $\|x\|_{0,w_k} := \sup_{n\in\mathbb{N}} w_k(n)|x_n|$ for $x \in c_0(w_k)$.

$A^{-\infty}$ as a dual power series space

The map

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \to (a_n)_{n=0}^{\infty}$$

defines an isomorphism between $A^{-\infty}$ and the countable inductive limit $E_\alpha := \cup_{k\in\mathbb{N}} c_0(v_k)$ of weighted c_0 spaces defined for the weight sequence

$$v_k(n) = (n+e)^{-k}, k \in \mathbb{N}, n = 0, 1, 2, \ldots$$

The space $c_0(w)$

- Let $w = (w(n))_{n=1}^{\infty}$ be a bounded, strictly positive sequence. Define

$$c_0(w) := \left\{ x = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \colon \lim_{n\to\infty} w(n)|x_n| = 0 \right\},$$

  equipped with the norm $\|x\|_{0,w} := \sup_{n\in\mathbb{N}} w(n)|x_n|$ for $x \in c_0(w)$.

- $c_0(w)$ is isometrically isomorphic to c_0 via the linear multiplication operator $\Phi_w \colon c_0(w) \to c_0$ given by

$$x = (x_n)_{n\in\mathbb{N}} \to \Phi_w(x) := (w(n)x_n)_{n\in\mathbb{N}}. \quad (2)$$

- We are interested in the case when $\inf_{n\in\mathbb{N}} w(n) = 0$. Otherwise $c_0(w) = c_0$ with equivalent norms.

Continuity of C on $c_0(w)$

Theorem.

Let w be a bounded, strictly positive sequence.
The Cesàro operator $\mathrm{C}^{(0,w)} \in \mathcal{L}(c_0(w))$ if and only if

$$\left\{ \frac{w(n)}{n} \sum_{k=1}^{n} \frac{1}{w(k)} \right\}_{n\in\mathbb{N}} \in \ell_\infty. \tag{3}$$

Moreover, $\|\mathrm{C}^{(0,w)}\| \geq 1$.

If w is decreasing, then (3) is satisfied and $\|\mathrm{C}^{(0,w)}\| = 1$.

Compactness of C on $c_0(w)$

Theorem.

Let w be a bounded, strictly positive sequence.
The following conditions are equivalent.

(a) $\mathsf{C}^{(0,w)}$ is weakly compact.

(b) $\mathsf{C}^{(0,w)}$ is compact.

(c) The sequence

$$\left\{ \frac{w(n)}{n} \sum_{k=1}^{n} \frac{1}{w(k)} \right\}_{n\in\mathbb{N}} \in c_0. \tag{4}$$

Continuity and compactness of C on $c_0(w)$

Let $w = (w(n))_{n=1}^{\infty}$ be two strictly positive sequences. Let $T_w\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ denote the linear operator given by

$$T_w x := \left(\frac{w(n)}{n} \sum_{k=1}^{n} \frac{x_k}{w(k)} \right)_{n\in\mathbb{N}}, \quad x = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}. \tag{5}$$

Then $\Phi_w C = T_w \Phi_v$. Therefore, the Cesàro operator C maps $c_0(w)$ continuously (resp., compactly) into $c_0(w)$ if and only if the operator $T_w \in \mathcal{L}(c_0)$ (resp., $T_w \in \mathcal{K}(c_0)$).

Continuity of C on $c_0(w)$. A classical lemma

Lemma. Banach's Book.

Let $A = (a_{nm})_{n,m\in\mathbb{N}}$ be a matrix with entries from $\mathbb{C}$ and $T\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ be the linear operator defined by

$$Tx := \left(\sum_{m=1}^{\infty} a_{nm}x_m\right)_{n\in\mathbb{N}}, \quad x = (x_n)_{n\in\mathbb{N}}, \tag{6}$$

interpreted to mean that Tx exists in $\mathbb{C}^{\mathbb{N}}$ for every $x \in \mathbb{C}^{\mathbb{N}}$.
Then $T \in \mathcal{L}(c_0)$ if and only if the following two conditions are satisfied:

(i) $\lim_{n\to\infty} a_{nm} = 0$ for each fixed $m \in \mathbb{N}$;

(ii) $\sup_{n\in\mathbb{N}} \sum_{m=1}^{\infty} |a_{nm}| < \infty$.

In this case, $\|T\| = \sup_{n\in\mathbb{N}} \sum_{m=1}^{\infty} |a_{nm}|$.

Examples

- Let $w(2n+1) = \frac{1}{n+1}$ for $n \geq 0$ and $w(2n) = 2^{-n}$ for $n \geq 1$. Clearly $\lim_{n\to\infty} w(n) = 0$, but C does not act continuously in $c_0(w)$.

- Let $\alpha > 0$ and $w(n) := \frac{1}{n^\alpha}$ for all $n \in \mathbb{N}$. Since w is decreasing, $C^{(0,w)} \in \mathcal{L}(c_0(w))$. But $C^{(0,w)}$ is not compact, since

$$\begin{aligned}\frac{w(n)}{n}\sum_{k=1}^{n}\frac{1}{w(k)} &= \frac{1}{n^{\alpha+1}}\sum_{k=1}^{n}k^\alpha \geq \frac{1}{n^{\alpha+1}}\sum_{k=1}^{n}\int_{k-1}^{k}x^\alpha\,dx \\ &= \frac{1}{n^{\alpha+1}}\int_0^n x^\alpha\,dx = \frac{1}{\alpha+1}.\end{aligned}$$

Proposition.

Let w be bounded, strictly positive and satisfy

$$\limsup_{n\to\infty} \frac{w(n+1)}{w(n)} \in [0,1),$$

then $\mathrm{C}^{(0,w)} \in \mathcal{K}(c_0(w))$.

Moreover, $\sigma_{pt}(\mathrm{C}^{(0,w)}) = \Sigma; \quad \sigma(\mathrm{C}^{(0,w)}) = \Sigma_0$.

One checks that that the condition (4) is valid to prove compactness.

Examples of compact operators Cesàro operators on $C^{(0,w)}$

- $C^{(0,w)} \in \mathcal{K}(c_0(w))$ for the following sequences:

  (1) $w(n) := a^{-\alpha_n}$, $n \in \mathbb{N}$, with $a > 1$, $\alpha_n \uparrow \infty$ and $\lim_{n\to\infty}(\alpha_n - \alpha_{n-1}) = \infty$.

  (2) $w(n) := \frac{n^\alpha}{a^n}$ for $n \in \mathbb{N}$, where $a > 1$ and $\alpha \in \mathbb{R}$.

  (3) $w(n) := \frac{a^n}{n!}$ for $n \in \mathbb{N}$, where $a \geq 1$.

  (4) $w(n) := n^{-n}$ for $n \in \mathbb{N}$.

- Let $w(n) := e^{-\sqrt{n}}$ or $w(n) := e^{-(\log n)^\beta}$, $\beta > 1$, for $n \in \mathbb{N}$. Then $\lim_{n\to\infty} \frac{w(n+1)}{w(n)} = 1$, but $C \in \mathcal{K}(c_0(w))$.

Spectrum of $C^{(0,w)}$

Given a bounded, strictly positive sequence w, let

$$S_w := \{s \in \mathbb{R} \colon \sum_{n=1}^{\infty} \frac{1}{n^s w(n)} < \infty\}.$$

In case $S_w \neq \emptyset$ we define $s_0 := \inf S_w$.

Moreover, let

$$R_w := \{t \in \mathbb{R} \colon \lim_{n \to \infty} n^t w(n) = 0\}.$$

In case $R_w \neq \mathbb{R}$ we define $t_0 := \sup R_w$. If $R_w = \mathbb{R}$ we set $t_0 = \infty$.

Recall $\Sigma := \{\frac{1}{m} \colon m \in \mathbb{N}\}$ and $\Sigma_0 := \Sigma \cup \{0\}$.

Spectrum of $C^{(0,w)}$. Main result

Theorem.

Let w be a bounded, strictly positive sequence such that $C^{(0,w)} \in \mathcal{L}(c_0(w))$.

(1) The following inclusion holds:

$$\Sigma_0 \subseteq \sigma(C^{(0,w)}).$$

(2) Let $\lambda \notin \Sigma_0$. Then $\lambda \in \rho(C^{(0,w)})$ if and only if both of the conditions

(i) $\lim_{n\to\infty} \frac{w(n)}{n^{1-\alpha}} = 0$, and

(ii) $\sup_{n\in\mathbb{N}} \frac{1}{n} \sum_{m=1}^{n-1} \frac{w(n)n^{\alpha}}{w(m)m^{\alpha}} < \infty$,

are satisfied, where $\alpha := \mathrm{Re}\left(\frac{1}{\lambda}\right)$.

Spectrum of $C^{(0,w)}$. Main result continued

Theorem continued.

(3) Suppose that $R_w \neq \mathbb{R}$, i.e., $t_0 < \infty$. Then we have the inclusions

$$\left\{ \frac{1}{m} \colon m \in \mathbb{N},\ 1 \leq m < t_0 + 1 \right\} \subseteq \sigma_{pt}(C^{(0,w)}) \subseteq$$

$$\subseteq \left\{ \frac{1}{m} \colon m \in \mathbb{N},\ 1 \leq m \leq t_0 + 1 \right\}.$$

In particular, $\sigma_{pt}(C^{(0,w)})$ is a proper subset of Σ.
If $R_w = \mathbb{R}$, then

$$\sigma_{pt}(C^{(0,w)}) = \Sigma.$$

Some ingredients of the proof of the main result

First ingredient.

The dual operator.

The dual operator $(\mathsf{C}^{(0,w)})' \in \mathcal{L}(\ell_1(w^{-1}))$ satisfies $\|(\mathsf{C}^{(0,w)})'\| = \|\mathsf{C}^{(0,w)}\|$ and it is given by

$$(\mathsf{C}^{(0,w)})' y = \left(\sum_{k=n}^{\infty} \frac{y_k}{k} \right)_{n\in\mathbb{N}}, \quad y = (y_n)_{n\in\mathbb{N}} \in \ell_1(w^{-1}).$$

It satisfies $0 \notin \sigma_{pt}((\mathsf{C}^{(0,w)})')$ and $\Sigma \subseteq \sigma_{pt}((\mathsf{C}^{(0,w)})')$.

Some ingredients of the proof of the main result

First ingredient.

The dual operator.

Proposition.

If $S_w \neq \emptyset$, then the dual operator $(\mathsf{C}^{(0,w)})'$ of $\mathsf{C}^{(0,w)}$ satisfies

$$\left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{1}{2s_0}\right| < \frac{1}{2s_0}\right\} \cup \Sigma \subseteq \sigma_{pt}((\mathsf{C}^{(0,w)})'), \quad \text{and}$$

$$\sigma_{pt}((\mathsf{C}^{(0,w)})') \setminus \Sigma_0 \subseteq \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{1}{2s_0}\right| \leq \frac{1}{2s_0}\right\}.$$

Some ingredients of the proof of the main result

Second ingredient.

A result of Reade (1985)

For $n \in \mathbb{N}$ the n-th row of the matrix for $(\mathsf{C} - \lambda I)^{-1} \colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ $(\lambda \notin \Sigma_0)$ has the entries

$$\frac{-1}{n\lambda^2 \prod_{k=m}^{n} \left(1 - \frac{1}{\lambda k}\right)}, \quad 1 \leq m < n,$$

$$\frac{n}{1 - n\lambda} = \frac{1}{\frac{1}{n} - \lambda}, \quad m = n,$$

and all the other entries in row n are equal to 0. Therefore

$$(\mathsf{C} - \lambda I)^{-1} = D_\lambda - \frac{1}{\lambda^2} E_\lambda,$$

where the D is a diagonal operator and $E_\lambda = (e_{nm})_{n,m \in \mathbb{N}}$ is a lower triangular matrix.

Third ingredient.

A technical lemma improving Reade (1985)

Lemma.

Let $\lambda \in \mathbb{C} \setminus \Sigma_0$ and set $\alpha := \mathrm{Re}\left(\frac{1}{\lambda}\right)$. Then there exist constants $d > 0$ and $D > 0$ (depending on α) such that

$$\frac{d}{n^\alpha} \leq \prod_{k=1}^{n} \left|1 - \frac{1}{k\lambda}\right| \leq \frac{D}{n^\alpha}, \quad n \in \mathbb{N}.$$

Spectrum of $C^{(0,w)}$

Proposition.

Let w be a strictly positive, decreasing sequence.

(i)
$$\sigma(C^{(0,w)}) \subseteq \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\}. \qquad (7)$$

(ii) If $S_w \neq \emptyset$, then
$$\left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{1}{2s_0}\right| \leq \frac{1}{2s_0}\right\} \cup \Sigma \subseteq \sigma(C^{(0,w)}). \qquad (8)$$

Spectrum of $\mathrm{C}^{(0,w)}$

A sequence $u = (u_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ is called **rapidly decreasing** if $(n^m u_n)_{n\in\mathbb{N}} \in c_0$ for every $m \in \mathbb{N}$. The space of all rapidly decreasing, $\mathbb{C}$-valued sequences is denoted by s.

Proposition.

Let w be a bounded, strictly positive sequence. If $\mathrm{C}^{(0,w)} \in \mathcal{K}(c_0(w))$, then

$$\sigma_{pt}(\mathrm{C}^{(0,w)}) = \Sigma \quad \text{and} \quad \sigma(\mathrm{C}^{(0,w)}) = \Sigma_0.$$

Moreover, $w \in s$ and $S_w = \emptyset$.

There exist weights $w \in s$ such that $\mathrm{C}^{(0,w)} \notin \mathcal{K}(c_0(w))$: Define w via $w(1) := 1$ and $w(n) := \frac{1}{j^j}$ if $n \in \{2^{j-1}+1, \ldots, 2^j\}$ for $j \in \mathbb{N}$.

Spectrum of $C^{(0,w)}$. Relevant examples

(1) $w(n) = \frac{1}{(\log(n+1))^\gamma}$ for $n \in \mathbb{N}$ with $\gamma \geq 0$. Then $s_0 = 1$ and $t_0 = 0$. We have

$$\sigma(C^{(0,w)}) = \left\{\lambda \in \mathbb{C} \colon \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\}, \quad \text{and}$$

$$\sigma_{pt}(C^{(0,w)}) = \emptyset \text{ if } \gamma = 0; \quad \sigma_{pt}(C^{(0,w)}) = \{1\} \text{ if } \gamma > 0.$$

Spectrum of $C^{(0,w)}$. Relevant examples

(2) $w(n) = \frac{1}{n^\beta}$ for $n \in \mathbb{N}$ with $\beta > 0$. Then $t_0 = \beta$ and

$$\left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{1}{2(\beta+1)}\right| \leq \frac{1}{2(\beta+1)}\right\} \cup \Sigma = \sigma(C^{(0,w)}), \quad \text{and}$$

$$\left\{\frac{1}{m}\colon m \in \mathbb{N},\ 1 \leq m < \beta + 1\right\} = \sigma_{pt}(C^{(0,w)}).$$

Checking the examples requires the following technical result:

Lemma.

Let α be a real number with $\alpha < 1$. Then

$$\sup_{n\in\mathbb{N}} \frac{1}{n^{1-\alpha}} \sum_{m=1}^{n-1} \frac{1}{m^{\alpha}} < \infty.$$

Power bounded operators

An operator $T \in \mathcal{L}(X)$ is said to be *power bounded* if $\{T^m\}_{m=1}^{\infty}$ is an equicontinuous subset of $\mathcal{L}(X)$.

If X is a Banach space, an operator T is power bounded if and only if $\sup_n ||T^n|| < \infty$.

If X is a barrelled space, an operator T is power bounded if and only if the orbits $\{T^m(x)\}_{m=1}^{\infty}$ of all the elements $x \in X$ under T are bounded. This is a consequence of the uniform boundedness principle.

Mean ergodic properties. Definitions

For $T \in \mathcal{L}(X)$**, we set** $T_{[n]} := \frac{1}{n}\sum_{m=1}^{n} T^m$.

Mean ergodic operators

An operator $T \in \mathcal{L}(X)$ is said to be *mean ergodic* if the limits

$$Px := \lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} T^m x, \quad x \in X, \tag{9}$$

exist in X.

If T is mean ergodic, then one then has the direct decomposition

$$X = \mathrm{Ker}(I - T) \oplus \overline{(I - T)(X)}.$$

Mean ergodic properties. Definitions

Uniformly mean ergodic operators

If $\{T_{[n]}\}_{n=1}^{\infty}$ happens to be convergent in $\mathcal{L}_b(X)$ to $P \in \mathcal{L}(X)$, then T is called *uniformly mean ergodic*.

Theorem. Lin. 1974.

Let T a (continuous) operator on a Banach space X which satisfies $\lim_{n\to\infty} ||T^n/n|| = 0$. The following conditions are equivalent:

(1) T is uniformly mean ergodic.

(4) $(I - T)(X)$ is closed.

Ergodic properties of C on classical spaces

Proposition.

- The Cesàro operator $C\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ is power bounded and uniformly mean ergodic.
- The Cesàro operator $\mathrm{C}^{(p)}\colon \ell^p \to \ell^p$, $1 < p < \infty$, is not power bounded and not mean ergodic.
- The Cesàro operator $\mathrm{C}^{(0)}\colon c_0 \to c_0$ is power bounded, not mean ergodic.

Hypercyclic operator

$T \in \mathcal{L}(X)$, with X separable, is called **hypercyclic** if there exists $x \in X$ such that the orbit $\{T^n x \colon n \in \mathbb{N}_0\}$ is dense in X.

Supercyclic operator

If, for some $z \in X$, the projective orbit $\{\lambda T^n z \colon \lambda \in \mathbb{C},\ n \in \mathbb{N}_0\}$ is dense in X, then T is called *supercyclic*.

Clearly, hypercyclicity always implies supercyclicity.

Proposition.

- The Cesàro operator $C\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ is power bounded, uniformly mean ergodic and not supercyclic.
- The Cesàro operator $\mathrm{C}^{(p)}\colon \ell^p \to \ell^p$, $1 < p < \infty$, is not power bounded, not mean ergodic and not supercyclic.
- The Cesàro operator $\mathrm{C}^{(0)}\colon c_0 \to c_0$ is power bounded, not mean ergodic and not supercyclic.

Mean ergodicity of C on $c_0(w)$

Proposition.

Let w be a decreasing, strictly positive sequence. Then $C^{(0,w)} \in \mathcal{L}(c_0(w))$ is power bounded.

Moreover, the following assertions are equivalent:

(i) $C^{(0,w)}$ is mean ergodic.

(iii) The weight w satisfies $\lim_{n\to\infty} w(n) = 0$.

Proposition.

Let w be a decreasing, strictly positive sequence. Then $\mathsf{C}^{(0,w)} \in \mathcal{L}(c_0(w))$ is uniformly mean ergodic if and only if w satisfies both of the conditions

(i) $\lim_{n\to\infty} w(n) = 0$, and

(ii) $\sup_{n\in\mathbb{N}} w(n+1) \sum_{m=1}^{n-1} \frac{1}{mw(m+1)} < \infty$.

Uniform mean ergodicity of C on $c_0(w)$

Proposition.

If w is a decreasing, strictly positive sequence such that $C^{(0,w)} \in \mathcal{K}(c_0(w))$, then $C^{(0,w)}$ is uniformly mean ergodic.

Examples.

(i) For $w(n) = \frac{1}{(\log(n+1))^\gamma}$ for $n \in \mathbb{N}$ with $\gamma \geq 1$, the operator $C^{(0,w)}$ is not compact, mean ergodic and not uniformly mean ergodic.

(ii) For $w(n) = \frac{1}{n^\beta}$ for $n \in \mathbb{N}$ with $\beta \geq 1$, the operator $C^{(0,w)}$ is uniformly mean ergodic but not compact

Non superciclycity of C on $c_0(w)$

Proposition

Let w be a bounded, strictly positive sequence such that $C^{(0,w)} \in \mathcal{L}(c_0(w))$. Then $C^{(0,w)}$ is not supercyclic and hence, also not hypercyclic.

This is a direct consequence of a general result by Ansari and Bourdon, since $\sigma_{pt}((C^{(0,w)})')$ is infinite.

Ergodicity of C on $H(\mathbb{D})$

Theorem

The Cesàro operator C acting on $H(\mathbb{D})$ is power bounded, uniformly mean ergodic and not supercyclic, hence not hypercyclic.

As a consequence, C is not supercyclic on the spaces $A_+^{-\gamma}$, $\gamma \geq 0$, and $A_-^{-\gamma}, 0 < \gamma \leq \infty$.

Ergodicity of C on the Fréchet growth spaces

Proposition

Let $\gamma \in [0, \infty[$.

The following conditions are equivalent:

(a) C is power bounded on $A_+^{-\gamma}$.

(b) C is (uniformly) mean ergodic on $A_+^{-\gamma}$.

(c) $1 \leq \gamma < \infty$.

Ergodicity of C on the (LB) growth spaces

Proposition

Let $\gamma \in]0, \infty]$.

The following conditions are equivalent:

(a) C is power bounded on $A_-^{-\gamma}$.

(b) C is (uniformly) mean ergodic on $A_-^{-\gamma}$.

(c) $1 < \gamma \leq \infty$.

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