# DYNAMICS AND SPECTRUM OF THE CESÀRO OPERATOR ON $C^\infty(\mathbb{R}_+)$

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Abstract. The spectrum and point spectrum of the Cesàro averaging operator $\mathsf{C}$ acting on the Fréchet space $C^\infty(\mathbb{R}_+)$ of all $C^\infty$ functions on the interval $[0,\infty)$ are determined. We employ an approach via $C_0$-semigroup theory for linear operators. A spectral mapping theorem for the resolvent of a closed operator acting on a locally convex space is established; it constitutes a useful tool needed to establish the main result. The dynamical behaviour of $\mathsf{C}$ is also investigated.

## 1. Introduction and main results

One approach, perhaps somewhat curious, to compute the spectrum $\sigma(T)$ of a *continuous* linear operator $T$ acting on a Banach space, is to first compute the spectrum of a related *closed operator* $A$ and then to apply a suitable *spectral mapping theorem* to $A$ (if available) to compute $\sigma(T)$ in terms of $\sigma(A)$. For example, this method was used by D.W. Boyd to compute the spectrum of the Cesàro operator $\mathsf{C}_{(p)}$ acting in the Banach space $L^p(\mathbb{R}_+)$, $1<p<\infty$, with $\mathbb{R}_+ := [0,\infty)$, [10]. Closed operators often arise as the infinitesimal generator of a strongly continuous $C_0$-semigroup of operators. This point was taken up by A.G. Siskakis who realized that the Cesàro operator $\mathsf{C}^{(p)}$ acting in the Hardy space $H^p(\mathbb{D})$, $1<p<\infty$, occurs as a resolvent operator of the infinitesimal generator of a related $C_0$-semigroup of operators, which then allowed him to determine $\sigma(\mathsf{C}^{(p)})$, [19]. Via this "method of semigroups" W. Arendt was able to determine the spectrum of $\mathsf{C}_{(p)}$ acting on $L^p(\mathbb{R}_+)$, [8], thereby providing an alternate argument to that of Boyd.

Our aim is to extend the $C_0$-semigroup approach mentioned above in order to determine the spectrum and point spectrum of the Cesàro operator $\mathsf{C}$ acting on the *Fréchet space* $C^\infty(\mathbb{R}_+)$ of infinitely differentiable

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functions on the interval $\mathbb{R}_+$; see Theorem 1.2. This complements analogous results in [6], [7]. An important tool that is needed is a spectral mapping theorem for the resolvent of a closed operator (not necessarily the generator of a $C_0$-semigroup) acting on a locally convex space (cf. Theorem 1.1), which is also of independent interest. Since the spectral theory of linear operators in locally convex spaces is more diverse than for operators in Banach spaces, some care is required for this result.

Various properties of the dynamical behaviour of the Cesàro operator $\mathsf{C}$ on $C^\infty(\mathbb{R}_+)$ are also presented; see Proposition 3.5.

Let $C^\infty(\mathbb{R}_+)$ be the space of all $\mathbb{C}$-valued, infinitely differentiable functions on $\mathbb{R}_+$. It is routine to verify that $C^\infty(\mathbb{R}_+)$ is a Fréchet space with respect to the lc-topology generated by the increasing sequence of seminorms

$$p_n(f) := \max_{0\leq j\leq n} \max_{x\in[0,n]} |f^{(j)}(x)|, \quad f \in C^\infty(\mathbb{R}_+),\ n \in \mathbb{N}.$$

For $f \in C^\infty(\mathbb{R}_+)$ the Cesàro operator $\mathsf{C}$ is defined pointwise by

$$\mathsf{C}f(x) := \frac{1}{x}\int_0^x f(t)\,dt, \quad x \in (0,\infty), \tag{1.1}$$

with $Cf(0) := f(0)$, in which case it acts continuously in $C^\infty(\mathbb{R}_+)$; see Lemma 3.1.

In order to formulate the main results we require some notation and relevant concepts. Let $A\colon D(A) \subseteq X \to X$ be a linear operator defined on a subspace $D(A)$ of a locally convex Hausdorff space $X$ (lcHs for brevity). Whenever $\lambda \in \mathbb{C}$ is such that $(\lambda - A)\colon D(A) \to X$ is injective, the linear operator $(\lambda - A)^{-1}$ is always understood to have domain $\mathrm{Im}(\lambda - A) := \{(\lambda - A)x : x \in D(A)\}$. Of course, $\mathrm{Im}(\lambda - A)^{-1} = D(A)$. The *resolvent set* of $A$ is defined by

$$\rho(A) := \{\lambda \in \mathbb{C} :\ (\lambda - A)\colon D(A) \to X \text{ is bijective and } (\lambda - A)^{-1} \in \mathcal{L}(X)\}$$

and the *spectrum* of $A$ is defined by $\sigma(A) := \mathbb{C} \setminus \rho(A)$, where $\mathcal{L}(X)$ denotes the space of all continuous linear operators of $X$ into itself. For $\lambda \in \rho(A)$ we also write $R(\lambda, A) := (\lambda - A)^{-1}$ and refer to $R(\lambda, A)$ as the *resolvent operator* of $A$ at $\lambda$. Recall that $A$ is called *closed* if the conditions $(x_\alpha)_\alpha \subseteq D(A)$ converges to $x$ in $X$ and $(Ax_\alpha)_\alpha$ converges to $y$ in $X$ imply that $x \in D(A)$ and $y = Ax$. We point out for $A$ closed, that also $\lambda - A$ is closed for all $\lambda \in \mathbb{C}$ and that $(\lambda - A)^{-1}$ is closed whenever $\lambda - A$ is injective.

The *point spectrum* $\sigma_{pt}(A)$ of $A$ consists of all $\lambda \in \mathbb{C}$ such that $(\lambda I - A)$ is not injective. Whenever $\lambda, \mu \in \rho(A)$ we have the *resolvent identity* $R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A)$. Even if $A \in \mathcal{L}(X)$, for non-Banach spaces $X$ it may happen that $\rho(A) = \emptyset$ or that $\rho(A)$

is not open in $\mathbb{C}$. This is why some authors prefer the subset $\rho^*(A)$ of $\rho(A)$ consisting of all $\lambda \in \mathbb{C}$ for which there exists $\delta > 0$ such that the open ball $B(\lambda, \delta) := \{z \in \mathbb{C}\colon |z - \lambda| < \delta\}$ lies within $\rho(A)$ and the set $\{R(\mu, A)\colon\ \mu \in B(\lambda, \delta)\}$ is equicontinuous in $\mathcal{L}(X)$. If $X$ is a Fréchet space, then it suffices that this set is bounded in $\mathcal{L}_s(X)$, with $\mathcal{L}_s(X)$ denoting $\mathcal{L}(X)$ equipped with the strong operator topology (i.e., the topology of uniform convergence on the finite subsets of $X$, [17, p.275]). The advantage of $\rho^*(A)$, whenever it is non–empty, is that it is open and the resolvent map $R\colon \lambda \mapsto R(\lambda, A)$ is holomorphic from $\rho^*(A)$ into $\mathcal{L}_b(X)$, with $\mathcal{L}_b(X)$ denoting $\mathcal{L}(X)$ equipped with the topology of uniform convergence on the bounded subsets of $X$, [17, p.275]; see [3, Proposition 3.4]. Define $\sigma^*(A) := \mathbb{C} \setminus \rho^*(A)$, which is a closed set containing $\sigma(A)$. If $A \in \mathcal{L}(X)$ with $X$ a Banach space, then $\sigma(A) = \sigma^*(A)$. In [3, Remark 3.5(vi), p.265] an example of an operator $A \in \mathcal{L}(X)$, with $X$ a Fréchet space, is presented such that $\overline{\sigma(A)} \subset \sigma^*(A)$ properly.

The following spectral mapping theorem will be established in Section 2.

**Theorem 1.1.** *Let $X$ be a lcHs and $A\colon D(A) \subseteq X \to X$ be a closed linear operator. Then the following assertions hold.*

(i) $\sigma(R(\lambda, A)) \setminus \{0\} = \{\frac{1}{\lambda-\mu}\colon \mu \in \sigma(A)\}$ *for every* $\lambda \in \rho(A)$.
(ii) $\sigma_{pt}(R(\lambda, A)) \setminus \{0\} = \{\frac{1}{\lambda-\mu}\colon \mu \in \sigma_{pt}(A)\}$ *for every* $\lambda \in \rho(A)$.
(iii) $\sigma^*(R(\lambda, A)) \setminus \{0\} = \{\frac{1}{\lambda-\mu}\colon \mu \in \sigma^*(A)\}$ *for every* $\lambda \in \rho^*(A)$.

As alluded to above, Theorem 1.1 provides the tool (together with an appropriate $C_0$-semigroup) needed to establish the main result of this paper, namely

**Theorem 1.2.** *The spectra of the Cesàro operator* $\mathsf{C}$ *acting on the Fréchet space* $C^\infty(\mathbb{R}_+)$ *are given by*

$$\sigma_{pt}(\mathsf{C}) = \left\{\frac{1}{n}\colon n \in \mathbb{N}\right\} \tag{1.2}$$

*and*

$$\sigma(\mathsf{C}) = \sigma^*(\mathsf{C}) = \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\}. \tag{1.3}$$

We point out that the spectra of the Cesàro operator $\mathsf{C}$, when considered to be acting on the Fréchet space $C(\mathbb{R}_+)$ of all $\mathbb{C}$-valued, continuous functions on $\mathbb{R}_+$ endowed with the topology of uniform convergence on the compact intervals $[0, n]$, for $n \in \mathbb{N}$, are also given by (1.3) but, its point spectrum (namely, $\{\lambda \in \mathbb{C}\colon\ |\lambda - \frac{1}{2}| \leq \frac{1}{2}\} \setminus \{0\}$) is *vastly different* to that given by (1.2), [6, Proposition 2.1].

## 2. A spectral mapping theorem for the resolvent

This section is devoted entirely to establishing Theorem 1.1. So, let us begin the proof immediately.

Let $\lambda \in \rho(A)$ and $\mu \in \mathbb{C} \setminus \{0\}$. From the identity $AR(\lambda, A) = \lambda R(\lambda, A) - I$ it follows that

$$(\mu I - R(\lambda, A))x = \mu \left[\left(\lambda - \frac{1}{\mu}\right) I - A\right] R(\lambda, A)x, \quad x \in X, \tag{2.1}$$

and that

$$(\mu I - R(\lambda, A))z = R(\lambda, A)\mu \left[\left(\lambda - \frac{1}{\mu}\right) I - A\right] z, \quad z \in D(A). \tag{2.2}$$

The identity (2.1) implies that

$$\mathrm{Im}(\mu I - R(\lambda, A)) = \mathrm{Im}\left[\left(\lambda - \frac{1}{\mu}\right) I - A\right], \tag{2.3}$$

whereas the identity (2.2) implies that

$$\mathrm{Ker}(\mu I - R(\lambda, A)) = \mathrm{Ker}\left[\left(\lambda - \frac{1}{\mu}\right) I - A\right]. \tag{2.4}$$

(i) We first prove, for fixed $\lambda \in \rho(A)$, that

$$\rho(R(\lambda, A)) \setminus \{0\} = \left\{\frac{1}{\lambda - \eta} : \eta \in \rho(A) \setminus \{\lambda\}\right\}. \tag{2.5}$$

Let $\mu \in \rho(R(\lambda, A)) \setminus \{0\}$. Then (2.3) implies that $\mathrm{Im}\left[\left(\lambda - \frac{1}{\mu}\right) I - A\right] = X$ and (2.4) implies that $\mathrm{Ker}\left[\left(\lambda - \frac{1}{\mu}\right) I - A\right] = \{0\}$. Therefore the operator $\left[\left(\lambda - \frac{1}{\mu}\right) I - A\right] : D(A) \to X$ is bijective. By (2.1) the inverse operator $\left[\left(\lambda - \frac{1}{\mu}\right) I - A\right]^{-1} : X \to D(A)$ satisfies the identity

$$(\mu I - R(\lambda, A))^{-1} = \frac{1}{\mu}(\lambda I - A)\left[\left(\lambda - \frac{1}{\mu}\right) I - A\right]^{-1}. \tag{2.6}$$

It follows that

$$\left[\left(\lambda - \frac{1}{\mu}\right) I - A\right]^{-1} = \mu R(\lambda, A)(\mu I - R(\lambda, A))^{-1}. \tag{2.7}$$

Accordingly, $\left[\left(\lambda - \frac{1}{\mu}\right) I - A\right]^{-1} \in \mathcal{L}(X)$ and so $\eta := \left(\lambda - \frac{1}{\mu}\right) \in \rho(A) \setminus \{\lambda\}$ (as $\frac{1}{\mu} \neq 0$), that is, $\mu = \frac{1}{\lambda - \eta}$ with $\eta \in \rho(A) \setminus \{\lambda\}$.

To establish the other inclusion in (2.5), let $\eta \in \rho(A) \setminus \{\lambda\}$ and set $\mu := \frac{1}{\lambda - \eta} \neq 0$; hence, $\frac{1}{\mu} = \lambda - \eta$, i.e., $\eta = \lambda - \frac{1}{\mu}$. We apply (2.3) and (2.4) to conclude that the operator $(\mu I - R(\lambda, A))\colon X \to X$ is bijective. By (2.1) the operator $(\mu I - R(\lambda, A))^{-1}\colon X \to X$ satisfies (2.6). Substitute $\lambda I = \left(\lambda - \frac{1}{\mu}\right) I + \frac{1}{\mu} I$ into the right-side of (2.6) yields

$$(\mu I - R(\lambda, A))^{-1} = \frac{1}{\mu} I + \frac{1}{\mu^2}\left[\left(\lambda - \frac{1}{\mu}\right) I - A\right]^{-1}. \tag{2.8}$$

Accordingly, $(\mu I - R(\lambda, A))^{-1} \in \mathcal{L}(X)$ and so $\frac{1}{\lambda - \eta} = \mu \in \rho(R(\lambda, A)) \setminus \{0\}$. Thereby (2.5) is established.

We claim that (2.5) is equivalent to the identity

$$\sigma(R(\lambda, A)) \setminus \{0\} = \left\{\frac{1}{\lambda - \eta} : \eta \in \sigma(A)\right\}. \tag{2.9}$$

Observe, since $\lambda \in \rho(A)$ implies $(\lambda - \eta) \neq 0$ for all $\eta \in \sigma(A)$, that the right-side of (2.9) is well defined. Given $\mu \in \left\{\frac{1}{\lambda-\eta} : \eta \in \sigma(A)\right\}$, we have $\mu = \frac{1}{\lambda - \eta}$ for some $\eta \in \sigma(A)$ (with $\eta \neq \lambda$) and $\mu \neq 0$. Hence, $\eta \notin \rho(A)$ and $\eta \notin \rho(A) \setminus \{\lambda\}$. From (2.5) we can conclude that $\mu \notin \rho(R(\lambda, A)) \setminus \{0\}$ and hence, that $\mu \in \sigma(R(\lambda, A)) \cup \{0\}$. But, $\mu \neq 0$ and so $\mu \in \sigma(R(\lambda, A)) \setminus \{0\}$.

Now choose $\mu \in \sigma(R(\lambda, A)) \setminus \{0\}$. Then, $\mu = \frac{1}{1/\mu} = \frac{1}{\lambda - (\lambda - (1/\mu))}$, where by (2.5) we see that $(\lambda - (1/\mu)) \notin \rho(A) \setminus \{\lambda\}$. It follows that $(\lambda - (1/\mu)) \in \sigma(A)$ which implies that $\mu \in \left\{\frac{1}{\lambda-\eta} : \eta \in \sigma(A)\right\}$. So, the identity (2.9) follows from (2.5). In a similar way, one shows that (2.5) follows from (2.9).

(ii) This is a direct consequence of (2.4).

(iii) On account of the arguments given at the end of the proof of part (i) above, it is clear that (iii) follows if we can verify that

$$\rho^*(R(\lambda, A)) \setminus \{0\} = \left\{\frac{1}{\lambda - \eta} : \eta \in \rho^*(A) \setminus \{\lambda\}\right\}, \quad \lambda \in \rho^*(A). \tag{2.10}$$

Fix $\mu \in \rho^*(R(\lambda, A)) \setminus \{0\}$ and set $T := R(\lambda, A)$. Then there exists $\delta > 0$ such that $\overline{B(\mu, \delta)} \subseteq \rho(R(\lambda, A)) \setminus \{0\}$ (as $\mu \neq 0$ and so $0 \notin \overline{B(\mu, \delta)}$ for $\delta$ small enough) and the set $\{R(\zeta, T) \colon \zeta \in \overline{B(\mu, \delta)}\}$ is equicontinuous in $\mathcal{L}(X)$, i.e., for each $p \in \Gamma_X$ (the family of continuous seminorms on $X$) there exist $q \in \Gamma_X$ and a constant $M_p > 0$ such that

$$p(R(\zeta, T)x) \leq M_p q(x), \quad \forall \zeta \in \overline{B(\mu, \delta)},\ x \in X. \tag{2.11}$$

Since $\rho^*((R(\lambda, A)) \subseteq \rho((R(\lambda, A))$, it follows from (2.5) that $\mu = \frac{1}{\lambda - \eta}$ for some $\eta \in \rho(A) \setminus \{\lambda\}$. In particular, the function $\varphi\colon \mathbb{C} \setminus \{\lambda\} \to$

$\mathbb{C} \setminus \{0\}$ given by $\varphi(\zeta) := \frac{1}{\lambda-\zeta}$, for $\zeta \neq \lambda$, is continuous and bijective. So, there exists $\varepsilon > 0$ such that $\overline{B(\eta,\varepsilon)} \subseteq \varphi^{-1}(\overline{B(\mu,\delta)})$, i.e., $\left\{\frac{1}{\lambda-\zeta} \colon \zeta \in \overline{B(\eta,\varepsilon)}\right\} \subseteq \overline{B(\mu,\delta)}$. Again applying (2.5) it follows that $\overline{B(\eta,\varepsilon)} \subseteq \rho(A) \setminus \{\lambda\}$.

Next, we claim that (2.7) and (2.11) together with the fact that $R(\lambda, A) \in \mathcal{L}(X)$, imply that the set $\{R(w,A) \colon w \in \overline{B(\eta,\varepsilon)}\}$ is equicontinuous in $\mathcal{L}(X)$. Indeed, given $r \in \Gamma_X$ there exist $N_r > 0$ and $p \in \Gamma_X$ such that $r(R(\lambda,A)x) \leq N_r p(x)$ for all $x \in X$. On the other hand, by (2.11) there exist $M_p > 0$ and $q \in \Gamma_X$ such that $p(R(\zeta,T)x) \leq M_p q(x)$ for all $\zeta \in \overline{B(\mu,\delta)}$ and $x \in X$. So, via (2.7) it follows, for every $w \in \overline{B(\eta,\varepsilon)}$ (i.e., $\zeta = \frac{1}{\lambda-w} \in \overline{B(\mu,\delta)}$ and so $w = \lambda - \frac{1}{\zeta}$) and $x \in X$, that

$$r(R(w,A)x) = |\zeta| r(R(\lambda,A)R(\zeta,T)x) \leq DN_r p(R(\zeta,T)x) \leq DN_r M_p q(x),$$

where $D := \max\{|\zeta| \colon \zeta \in \overline{B(\mu,\delta)}\}$. Since $r \in \Gamma_X$ is arbitrary, the equicontinuity of $\{R(w,A) \colon w \in \overline{B(\eta,\varepsilon)}\}$ in $\mathcal{L}(X)$ follows and hence, $\eta \in \rho^*(A) \setminus \{\lambda\}$. Thus, $\rho^*(R(\lambda,A)) \setminus \{0\} \subseteq \left\{\frac{1}{\lambda-\eta} \colon \eta \in \rho^*(A) \setminus \{\lambda\}\right\}$.

To establish the reverse inclusion in (2.10), let $\eta \in \rho^*(A) \setminus \{\lambda\}$. Then there exists $\delta > 0$ such that $\overline{B(\eta,\delta)} \subseteq \rho(A) \setminus \{\lambda\}$ (as $\eta \neq \lambda$ and so $\lambda \notin \overline{B(\eta,\delta)}$ for $\delta$ small enough) and the set $\left\{R(\zeta,A) \colon \zeta \in \overline{B(\eta,\delta)})\right\}$ is equicontinuous in $\mathcal{L}(X)$, i.e., for each $p \in \Gamma_X$ there exist $q \in \Gamma_X$ and a constant $M_p > 0$ such that

$$p(R(\zeta,A)x) \leq M_p q(x), \quad \forall \zeta \in \overline{B(\eta,\delta)},\ x \in X. \tag{2.12}$$

By (2.5) we have that $\mu := \frac{1}{\lambda-\eta} \in \rho(R(\lambda,A)) \setminus \{0\}$. Moreover, by continuity of the function $\psi \colon \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{\lambda\}$ with $\psi := \varphi^{-1}$, there exists $\varepsilon > 0$ such that $\overline{B(\mu,\varepsilon)} \subseteq \psi^{-1}(\overline{B(\eta,\delta)})$, i.e., $\left\{\lambda - \frac{1}{z} \colon z \in \overline{B(\mu,\varepsilon)}\right\} \subseteq \overline{B(\eta,\delta)}$ and $0 \notin \overline{B(\mu,\varepsilon)}$. Hence, by (2.5) it follows that $\overline{B(\mu,\varepsilon)} \subseteq \rho(R(\lambda,A)) \setminus \{0\}$. Set $d^{-1} := \min\{|z| \colon z \in \overline{B(\mu,\varepsilon)}\} > 0$.

We claim (2.8) and (2.12) imply that the set $\{R(z,T) \colon z \in \overline{B(\mu,\varepsilon)}\}$ is equicontinuous in $\mathcal{L}(X)$. To see this, let $p \in \Gamma_X$. By (2.12) there exist $M_p > 0$ and $q \in \Gamma_X$ (with $q \geq p$) such that $p(R(\zeta,A)x) \leq M_p q(x)$ for all $\zeta \in \overline{B(\eta,\delta)}$ and $x \in X$. So, via (2.8) it follows, for every $x \in X$ and $z \in \overline{B(\mu,\varepsilon)}$ (i.e., $\zeta := \lambda - \frac{1}{z} \in \overline{B(\eta,\delta)}$), that

$$p(R(z,T)x) \leq \frac{1}{|z|} p(x) + \frac{1}{|z|^2} p\left(R\left(\zeta,A\right)x\right) \leq (d + d^2 M_p) q(x).$$

Since $p \in \Gamma_X$ is arbitrary, the equicontinuity of $\{R(z,T)\colon z \in \overline{B(\mu,\varepsilon)}\}$ in $\mathcal{L}(X)$ has been established, i.e., $\mu \in \rho^*(R(\lambda,A)) \setminus \{0\}$. Thus, $\left\{\frac{1}{\lambda-\eta}\colon \eta \in \rho^*(A) \setminus \{\lambda\}\right\} \subseteq \rho^*(R(\lambda,A)) \setminus \{0\}$.

## 3. The Cesàro operator on $C^\infty(\mathbb{R}_+)$

The main aim of this section is to establish Theorem 1.2. In the latter part of the section we describe various features concerning the dynamical behaviour of $\mathsf{C}$ acting on $C^\infty(\mathbb{R}_+)$; see Proposition 3.5. We begin with some basic facts.

**Lemma 3.1.** (i) *The Fréchet space $C^\infty(\mathbb{R}_+)$ is nuclear.*

(ii) *The operator $\mathsf{C} \in \mathcal{L}(C^\infty(\mathbb{R}_+))$. Moreover,*

$$(Cf)^{(n)}(0) = \frac{f^{(n)}(0)}{n+1}, \quad f \in C^\infty(\mathbb{R}_+),\ n \in \mathbb{N}_0 := \{0,1,2,\ldots\}.$$

*Proof.* (i) Although this is essentially known, we include a proof for the convenience of the reader.

For each $n \in \mathbb{N}$ let $C^\infty([0,n])$ denote the space of all $\mathbb{C}$-valued, infinitely differentiable functions on $[0,n]$. Recall that $C^\infty([0,n])$ is a Fréchet space with respect to the lc-topology generated by the increasing sequence of seminorms

$$q_m^{(n)}(f) := \max_{0\le j\le m} \max_{x\in[0,1]} |f^{(j)}(x)|, \quad f \in C^\infty([0,n]),\ m \in \mathbb{N}.$$

In particular, $C^\infty([0,n])$ is topologically isomorphic to the Fréchet space $s$ of all rapidly decreasing, $\mathbb{C}$-valued sequences, [17, Example 29.5(4)], and hence, it is nuclear.

The claim is that $C^\infty(\mathbb{R}_+)$ is isomorphic to a closed subspace of the countable product $Y := \prod_{n=1}^\infty C^\infty([0,n])$ endowed with the product lc-topology. To see this, let $T\colon C^\infty(\mathbb{R}_+) \to Y$ be defined by restriction in each coordinate, i.e., $Tf := (f|_{[0,n]})_{n\in\mathbb{N}}$, for $f \in C^\infty(\mathbb{R}_+)$. Then $T$ is a well defined, injective and continuous linear operator. So, to conclude the proof of the claim it remains to prove that the range $\mathrm{Im}\,(T)$ is a closed subspace of $Y$.

Let $(f_j)_{j\in\mathbb{N}}$ be a sequence in $C^\infty(\mathbb{R}_+)$ such that $(Tf_j)_{j\in\mathbb{N}}$ converges to $h = (h_n)_{n\in\mathbb{N}}$ in $Y$ as $j \to \infty$. Then, for each $n \in \mathbb{N}$ the sequence $(f_j|_{[0,n]})_{j\in\mathbb{N}}$ converges to $h_n$ in $C^\infty([0,n])$ as $j \to \infty$. Accordingly, $h_{n+1}|_{[0,n]} = h_n$ for each $n \in \mathbb{N}$. Setting $\tilde{h}(x) := h_n(x)$ if $x \in [0,n]$ for some $n \in \mathbb{N}$, defines a function $\tilde{h} \in C^\infty(\mathbb{R}_+)$. In particular, the sequence $(f_j)_{j\in\mathbb{N}}$ converges to $\tilde{h}$ in $C^\infty(\mathbb{R}_+)$ as $j \to \infty$. So, by continuity of $T$, it follows that $T\tilde{h} = h$. Therefore, $\mathrm{Im}\,(T)$ is a closed subspace of $Y$.

Since $Y$ is a nuclear Fréchet space, [13, Proposition 21.1.7], and $C^\infty(\mathbb{R}_+)$ is isomorphic to a closed subspace of $Y$, it follows that $C^\infty(\mathbb{R}_+)$ is also a nuclear Fréchet space, [13, Proposition 21.1.7].

(ii) It is known that $\mathsf{C}$ acts continuously on the Fréchet space $C(\mathbb{R}_+)$, [6]. Accordingly, by the closed graph theorem for Fréchet spaces, it suffices to prove that $\mathsf{C}f \in X$ for each $f \in X := C^\infty(\mathbb{R}_+)$. So, fix $f \in X$. By the Seeley extension theorem (for $n = 0$), [18, Theorem], there exists $\tilde{f} \in C^\infty(\mathbb{R})$ such that $\tilde{f}(x) = f(x)$ for each $x \in [0, \infty)$. Define $g(x) := \int_0^x \tilde{f}(s)\, ds$, $x \in \mathbb{R} \setminus \{0\}$, and $g(0) := 0$. Clearly $g \in C^\infty(\mathbb{R})$. Since $g(0) = 0$, we can apply a result due to L. Schwartz, [16, Proposition 1.1], to ensure the existence of $h \in C^\infty(\mathbb{R})$ such that $g(x) = xh(x)$ for $x \in \mathbb{R}$. In particular, $h(x) = Cf(x)$ for each $x \in \mathbb{R}_+$ (for $x = 0$ use the continuity of $h$ and $\mathsf{C}f$) and so $\mathsf{C}f \in C^\infty(\mathbb{R}_+)$. Hence, $\mathsf{C} \in \mathcal{L}(X)$.

For each $n \in \mathbb{N}$, we have $g^{(n)}(x) = xh^{(n)}(x) + nh^{(n-1)}(x)$, for $x \in \mathbb{R}$. In particular, $g^{(n)}(0) = nh^{(n-1)}(0)$ for $n \in \mathbb{N}$. Thus $h(0) = g'(0) = \tilde{f}(0) = f(0) = \mathsf{C}f(0)$, and

$$(\mathsf{C}f)^{(n)}(0) = h^{(n)}(0) = \frac{g^{(n+1)}(0)}{n+1} = \frac{\tilde{f}^{(n)}(0)}{n+1} = \frac{f^{(n)}(0)}{n+1}, \quad n \in \mathbb{N}.$$

□

We now turn our attention to various aspects of a particular $C_0$-semigroup acting in the Fréchet space $C^\infty(\mathbb{R}_+)$ which is closely related to $\mathsf{C}$.

For each $t \in \mathbb{R}$ and $f \in X := C^\infty(\mathbb{R}_+)$, define $T(t)f \in X$ by

$$T(t)f(x) := f(e^{-t}x), \quad x \in \mathbb{R}_+. \tag{3.1}$$

The 1-parameter family of linear operators $T(t)\colon X \to X$, for $t \in \mathbb{R}$, as determined by (3.1), clearly satisfies $T(0) = I$ and $T(s+t) = T(s)T(t)$ for all $s, t \in \mathbb{R}$. Moreover, $(T(t))_{t\in\mathbb{R}} \subseteq \mathcal{L}(X)$. Indeed, for every $t \in \mathbb{R}$, $f \in X$ and $j \in \mathbb{N}_0$, we have $(T(t)f)^{(j)}(x) = e^{-jt}f^{(j)}(e^{-t}x)$, for $x \in \mathbb{R}_+$. For every $n \in \mathbb{N}$ and $f \in X$, it then follows for all $t \geq 0$ that

$$p_n(T(t)f) = \max_{0\leq j\leq n}\max_{x\in[0,n]} |e^{-jt}f^{(j)}(e^{-t}x)| \leq \max_{0\leq j\leq n}\max_{y\in[0,n]} |f^{(j)}(y)| = p_n(f). \tag{3.2}$$

On the other hand, for every $t < 0$ and $n \in \mathbb{N}$ the positive integer $m := [e^{-t}n] + 1$ (which depends on $t$ and $n$) satisfies $m \geq n$ and

$$\begin{aligned} p_n(T(t)f) &= \max_{0\leq j\leq n}\max_{x\in[0,n]} |e^{-jt}f^{(j)}(e^{-t}x)| \\ &\leq e^{-nt}\max_{0\leq j\leq n}\max_{y\in[0,m]} |f^{(j)}(y)| \leq e^{-nt}p_m(f), \end{aligned} \tag{3.3}$$

for every $f \in X$.

The terminology in the following result is explained within the proof; see also [3, Section 2], [14], for instance.

**Proposition 3.2.** *The family $(T(t))_{t\in\mathbb{R}}$ is a uniformly continuous, locally equicontinuous $C_0$-group on $C^\infty(\mathbb{R}_+)$. The infinitesimal generator $A$ of $(T(t))_{t\in\mathbb{R}}$ is the continuous, everywhere defined linear operator*

$$(Af)(x) := -xf'(x), \quad x \in \mathbb{R}_+,\ f \in C^\infty(\mathbb{R}_+). \tag{3.4}$$

*Moreover, $(T(t))_{t\geq 0}$ is an equicontinuous $C_0$-semigroup on $C^\infty(\mathbb{R}_+)$.*

*Concerning the spectra of $A$, it is the case that*

$$\sigma_{pt}(A) = \{-n\colon n \in \mathbb{N}_0\} \tag{3.5}$$

*and that*

$$\sigma(A) = \sigma^*(A) = \{\lambda \in \mathbb{C}\colon \mathrm{Re}\,(\lambda) \leq 0\}. \tag{3.6}$$

*Proof.* To show that $(T(t))_{t\in\mathbb{R}}$ is a $C_0$-*group* on $X := C^\infty(\mathbb{R}_+)$ we need to verify that $\lim_{t\to 0} T(t) = I$ in $\mathcal{L}_s(X)$. So, fix $f \in X$, $n \in \mathbb{N}$ and $\varepsilon > 0$. Since each function $f^{(j)}$ is uniformly continuous on $[0, n]$ for $j = 0, \ldots, n$, there exists $\delta > 0$ satisfying

$$\forall x,\ x' \in [0, n] \text{ with } |x-x'| < \delta\colon |f^{(j)}(x)-f^{(j)}(x')| < \frac{\varepsilon}{4e^n},\ j = 0, \ldots, n. \tag{3.7}$$

Select $\{x_i\}_{i=1}^k \subseteq [0, n]$ with the property that for every $x \in [0, n]$ there exists $i \in \{1, \ldots, k\}$ satisfying $|x - x_i| < \frac{\delta}{e}$. Note, for every $i = 1, \ldots, k$ and $j = 0, \ldots, n$, that $\lim_{t\to 0} e^{-jt} f^{(j)}(e^{-t}x_i) = f^{(j)}(x_i)$, since each $f^{(j)}$ is continuous on $\mathbb{R}_+$. Hence, there exists $\tau \in (0, 1)$ such that

$$|e^{-jt} f^{(j)}(e^{-t}x_i) - f^{(j)}(x_i)| < \frac{\varepsilon}{4},\ i = 1, \ldots, k,\ j = 0, \ldots, n,\ |t| < \tau. \tag{3.8}$$

Now, fix $x \in [0, n]$. Select $i \in \{1, \ldots, k\}$ satisfying $|x - x_i| < \delta/e$, in which case $|e^{-t}x - e^{-t}x_i| = e^{-t}|x - x_i| \leq e|x - x_i| < \delta$ for all $t \in [-1, 1]$. It follows via (3.7) and (3.8), for every $|t| < \tau(< 1)$ and $j = 0, \ldots, n$, that

$$\begin{aligned} |e^{-jt} f^{(j)}(e^{-t}x) - f^{(j)}(x)| &\leq |e^{-jt} f^{(j)}(e^{-t}x) - e^{-jt} f^{(j)}(e^{-t}x_i)| \\ &\quad + |e^{-jt} f^{(j)}(e^{-t}x_i) - f^{(j)}(x_i)| \\ &\quad + |f^{(j)}(x_i) - f^{(j)}(x)| \\ &\leq e^n |f^{(j)}(e^{-t}x) - f^{(j)}(e^{-t}x_i)| + \frac{\varepsilon}{4} \\ &\quad + |f^{(j)}(x_i) - f^{(j)}(x)| \\ &< e^n \frac{\varepsilon}{4e^n} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4e^n} < \frac{3\varepsilon}{4}. \end{aligned}$$

Since $x \in [0,n]$ is arbitrary, for all $|t| < \tau$ we conclude that

$$p_n((T(t)-I)f) = \max_{0\le j\le n} \max_{x\in[0,n]} |e^{-jt} f^{(j)}(e^{-t}x) - f^{(j)}(x)| \le \frac{3\varepsilon}{4} < \varepsilon,$$

that is, $\lim_{t\to 0} T(t)f = f$ in $X$ for each $f \in X$. Hence, $\lim_{t\to 0} T(t) = I$ in $\mathcal{L}_s(X)$. According to (3.2) and (3.3) the $C_0$-group given by (3.1) has the property that $\{T(t)\colon |t| \le k\}$ is an equicontinuous subset of $\mathcal{L}(X)$ for each $k > 0$, i.e., $(T(t))_{t\in\mathbb{R}}$ is *locally equicontinuous*. It then follows that $\lim_{t\to s} T(t) = T(s)$ in $\mathcal{L}_s(X)$ for each $s \in \mathbb{R}$, [3, p.256], i.e., the $C_0$-group $(T(t))_{t\in\mathbb{R}}$ is *strongly continuous.*

Since $X$ is Montel (cf. Lemma 3.1(i) and [17, Remark 24.24, Corollary 28.5]), it follows that $(T(t))_{t\in\mathbb{R}}$ is even *uniformly continuous*, i.e., $\lim_{t\to s} T(t) = T(s)$ in $\mathcal{L}_b(X)$ for every $s \in \mathbb{R}$ (apply Theorem 7 in [2], for instance, after recalling that every Montel space is a GDP-space, [2, p.312]). Moreover, by (3.2) the $C_0$-semigroup $(T(t))_{t\ge 0}$ is equicontinuous.

We now prove that the infinitesimal generator of $(T(t))_{t\in\mathbb{R}}$ is given by (3.4). To see this fix $f \in X$. Then, for every $j \in \mathbb{N}_0$ we have

$$(Af)^{(j)}(x) = -jf^{(j)}(x) - xf^{(j+1)}(x), \quad x \in \mathbb{R}_+, \tag{3.9}$$

and so, for every fixed $t \in \mathbb{R}\setminus\{0\}$ and $x \in \mathbb{R}_+$, we have

$$\begin{aligned}
&\frac{(T(t)f-f)^{(j)}(x)}{t} - (Af)^{(j)}(x) = \\
&= \frac{e^{-jt}f^{(j)}(e^{-t}x) - f^{(j)}(x)}{t} + jf^{(j)}(x) + xf^{(j+1)}(x) \\
&= \frac{e^{-jt}(f^{(j)}(e^{-t}x) - f^{(j)}(x))}{t} + \frac{e^{-jt}-1}{t} f^{(j)}(x) + jf^{(j)}(x) + xf^{(j+1)}(x) \\
&= -e^{-jt}x \int_0^1 (e^{-st} f^{(j+1)}(e^{-st}x) - e^{jt} f^{(j+1)}(x))\, ds \\
&+ \left(\frac{e^{-jt}-1}{t} + j\right) f^{(j)}(x),
\end{aligned} \tag{3.10}$$

where, for the last equality, we have used the identity $\frac{d}{ds}(f^{(j)}(e^{-st}x)) = -txe^{-st}f^{(j+1)}(e^{-st}x)$ on $[0,1]$ in order to be able to invoke the stated integral expression. Fix $n \in \mathbb{N}$ and $\varepsilon > 0$. Since $f^{(j+1)}$ is uniformly continuous on $[0,n]$ for each $j = 0,\dots,n$, there exists $\delta > 0$ satisfying

$$\forall x,\ x' \in [0,n] \text{ with } |x-x'| < \delta\colon |f^{(j+1)}(x) - f^{(j+1)}(x')| < \frac{\varepsilon}{4ne^{2n}}, \tag{3.11}$$

for every $j = 0,\dots,n$. Choose $\{x_i\}_{i=1}^k \subseteq [0,n]$ with the property that for every $x \in [0,n]$ there exists $i \in \{1,\dots,k\}$ satisfying $|x-x_i| < \frac{\delta}{e}$. On the other hand, an application of the dominated convergence theorem shows that $\int_0^1 (e^{-st} f^{(j+1)}(e^{-st}x_i) - e^{jt} f^{(j+1)}(x_i))\, ds \to 0$ as $t \to 0$ for all $i = 1, \dots, k$ and $j = 0, \dots, n$ (as each $f^{(j+1)}$ is continuous at $x_i \in \mathbb{R}_+$). Also $\left(\frac{e^{-jt}-1}{t} + j\right) \to 0$ as $t \to 0$ for each $j = 0, \dots, n$. Hence, there exists $\tau \in (0,1)$ such that, for every $i = 1, \dots, k$ and $j = 0, \dots, n$ and every $|t| < \tau$, we have

$$\left| \int_0^1 (e^{-st} f^{(j+1)}(e^{-st}x_i) - e^{jt} f^{(j+1)}(x_i))\, ds \right| < \frac{\varepsilon}{8ne^n}, \tag{3.12}$$

and that

$$\left| \frac{e^{-jt}-1}{t} + j \right| < \frac{\varepsilon}{2(p_n(f)+1)}. \tag{3.13}$$

Now, fix $x \in [0,n]$. Then there exists $i \in \{1, \dots, k\}$ satisfying $|x - x_i| < \delta/e$ and so $|e^{-st}x - e^{-st}x_i| = e^{-st}|x - x_i| < \delta$ for all $t \in [-1,1]$ and $s \in [0,1]$. So, it follows via (3.11) and (3.12) that, for every $|t| < \tau (< 1)$ and $j = 0, \dots, n$, we have

$$\begin{aligned}
&\left| e^{-jt} x \int_0^1 \left(e^{-st} f^{(j+1)}(e^{-st}x) - e^{jt} f^{(j+1)}(x)\right) ds \right| \\
&\le e^n n \left| \int_0^1 e^{-st} \left(f^{(j+1)}(e^{-st}x) - f^{(j+1)}(e^{-st}x_i)\right) ds \right| \\
&+ e^n n \left| \int_0^1 \left(e^{-st} f^{(j+1)}(e^{-st}x_i) - e^{jt} f^{(j+1)}(x_i)\right) ds \right| \\
&+ n \left| f^{(j+1)}(x_i) - f^{(j+1)}(x) \right| \\
&< e^n n \int_0^1 e^{-st} |f^{(j+1)}(e^{-st}x) - f^{(j+1)}(e^{-st}x_i)|\, ds + \frac{\varepsilon}{8} + \frac{\varepsilon}{4e^{2n}} \\
&< e^n n \frac{e\varepsilon}{4ne^{2n}} + \frac{\varepsilon}{8} + \frac{\varepsilon}{4e^{2n}} < \frac{e\varepsilon}{4e^n} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} < \frac{\varepsilon}{2}.
\end{aligned} \tag{3.14}$$

On the other hand, by (3.13) we have, for all $j = 0, \dots, n$ and $|t| < \tau$, that

$$\left| \left( \frac{e^{-jt}-1}{t} + j \right) f^{(j)}(x) \right| < \frac{\varepsilon}{2(p_n(f)+1)} p_n(f) \le \frac{\varepsilon}{2}. \tag{3.15}$$

Moreover, from (3.10) combined with (3.14) and (3.15) it follows, for every $j = 0, \dots, n$ and $x \in [0,n]$, that

$$\left| \frac{(T(t)f - f)^{(j)}(x)}{t} - (Af)^{(j)}(x) \right| < \varepsilon, \quad \forall\ |t| < \tau.$$

Accordingly,

$$p_n\left(\frac{(T(t)f-f)}{t}-Af\right)\leq \varepsilon, \quad \forall\ |t|<\tau.$$

Since $n\in\mathbb{N}$ is arbitrary, we can conclude that $\lim_{t\to 0}\frac{T(t)f-f}{t}=Af$ in $X$. Hence, $A$ as given by (3.4), is the infinitesimal generator of $(T(t))_{t\in\mathbb{R}}$.

Finally, (3.9) implies that the operator $A$ satisfies

$$p_n(Af)\leq 2np_{n+1}(f), \quad f\in X,\ n\in\mathbb{N}.$$

Consequently, $A\in\mathcal{L}(X)$.

Since $(T(t))_{t\geq 0}$ is an equicontinuous $C_0$-semigroup on $X$, we can apply [20, Corollary 1, p.241] to conclude that

$$\mathbb{C}_{0^+}:=\{\lambda\in\mathbb{C}\colon \operatorname{Re}\lambda>0\}\subseteq\rho(A). \tag{3.16}$$

The containment (3.16) is crucial to establish (3.5) and (3.6). A further consequence of the equicontinuity of $(T(t))_{t\geq 0}$ is that also $\mathbb{C}_{0^+}\subseteq\rho^*(A)\,(\subseteq\rho(A))$. Indeed, fix $\lambda\in\mathbb{C}_{0^+}$ and set $\delta:=\frac{\operatorname{Re}\lambda}{2}>0$. Then $B(\lambda,\delta)\subseteq\mathbb{C}_{0^+}\subseteq\rho(A)$; see (3.16). By [20, Corollary 1, p.241], for fixed $f\in X$ and $\mu\in B(\lambda,\delta)\subseteq\rho(A)$ we have (as an $X$-valued Riemann integral) that

$$R(\mu,A)f=\int_0^\infty e^{-\mu t}T(t)f\,dt.$$

Applying (3.2) it follows, for every $n\in\mathbb{N}$, that

$$p_n(R(\mu,A)f)\leq p_n(f)\int_0^\infty e^{-(\operatorname{Re}\mu)t}\,dt=\frac{1}{\operatorname{Re}\mu}p_n(f)\leq\frac{2}{\operatorname{Re}\lambda}p_n(f),\ f\in X.$$

This shows that $\{R(\mu,A)\colon \mu\in B(\lambda,\delta)\}$ is equicontinuous in $\mathcal{L}(X)$, i.e., $\lambda\in\rho^*(A)$.

The next step is to show that, actually, $\rho^*(A)=\rho(A)=\mathbb{C}_{0^+}$, which then immediately yields (3.6). For this, it suffices to prove that the closure $\overline{\mathbb{C}_{0^-}}=\{\lambda\in\mathbb{C}\colon \operatorname{Re}\lambda\leq 0\}\subseteq\sigma(A)$. So, fix $\lambda\in\overline{\mathbb{C}_{0^-}}$ and consider the homogeneous differential equation

$$\lambda f(x)-Af(x)=0, \quad x\in\mathbb{R}_+,$$

or, equivalently,

$$xf'(x)+\lambda f(x)=0, \quad x\in\mathbb{R}_+.$$

This is a first order, ordinary differential equation; each solution has the form $f_\lambda(x)=\eta x^{-\lambda}$, $x\in\mathbb{R}_+$, for some $\eta\in\mathbb{C}$. But, $f_\lambda\in X$ if and only if $(-\lambda)\in\mathbb{N}_0$. Accordingly, $\sigma_{pt}(A)=\{-n\colon n\in\mathbb{N}_0\}$ which is the assertion (3.5).

For fixed $g \in X$, the solutions of the non-homogeneous differential equation

$$\lambda f(x) - Af(x) = g(x), \quad x \in \mathbb{R}_+,$$

or, equivalently, of

$$xf'(x) + \lambda f(x) = g(x), \quad x \in \mathbb{R}_+,$$

are then given by $f(x) = x^{-\lambda}(\int_0^x g(t)t^{\lambda-1}\,dt + c)$, for $c \in \mathbb{C}$. Since $\mathrm{Re}\,(\lambda - 1) \leq -1$ for such a $\lambda \in \overline{\mathbb{C}_{0^-}}$, the integral $\int_0^x g(t)t^{\lambda-1}\,dt$ necessarily diverges for the constant function $g(x) = 1$, for $x \in \mathbb{R}_+$. So, $\mathrm{Im}\,(\lambda I - A)$ is a proper subspace of $X$, i.e., $\lambda \in \sigma(A)$. Accordingly, $\overline{\mathbb{C}_{0^-}} \subseteq \sigma(A)$ as claimed.

Therefore, $\sigma(A) = \sigma^*(A) = \overline{\mathbb{C}_{0^-}}$ and hence, $\rho(A) = \rho^*(A) = \mathbb{C}_{0^+}$. □

*Proof of Theorem 1.2.* Since $1 \in \rho(A)$ (cf. (3.6)), the resolvent operator $R(1, A) \in \mathcal{L}(X)$ with

$$R(1, A)f(x) = \int_0^\infty e^{-t}T(t)f(x)\,dt = \int_0^\infty e^{-t}f(e^{-t}x)\,ds = \frac{1}{x}\int_0^x f(s)\,ds, \tag{3.17}$$

for all $f \in X$, $x \in \mathbb{R}_+$, [20, Corollary 1, p.241]. Thus the Cesáro operator satisfies $\mathsf{C} = R(1, A)$. This identity provides the possibility to apply Theorem 1.1. Indeed, since $A \in \mathcal{L}(X)$ is surely a *closed operator*, via Theorem 1.1 we can deduce (1.2) and (1.3) immediately from (3.5) and (3.6), respectively. □

In the final part of this section we analyze the dynamical behaviour of the operators $A$, $\mathsf{C} \in \mathcal{L}(C^\infty(\mathbb{R}_+))$. Recall that an operator $T \in \mathcal{L}(X)$, with $X$ a lcHs, is *power bounded* if $\{T^n : n \in \mathbb{N}\}$ is an equicontinuous subset of $\mathcal{L}(X)$. Given $T \in \mathcal{L}(X)$, define its Cesáro means by

$$T_{[n]} := \frac{1}{n}\sum_{m=1}^{n} T^m, \quad n \in \mathbb{N}.$$

Then $T$ is called *mean ergodic* (resp. *uniformly mean ergodic*) if $\{T_{[n]}\}_{n=1}^\infty$ is a convergent sequence in $\mathcal{L}_s(X)$ (resp. in $\mathcal{L}_b(X)$). In certain Fréchet spaces every power bounded operator is necessarily mean ergodic, [1, Corollary 2.7]. The converse is not true in general. E. Hille exhibited a classical kernel operator $T$ in the Banach space $L^1([0, 1])$ which fails to be power bounded (actually, $\|T^n\| = O(n^{1/4})$) but, nevertheless, *is* mean ergodic, [12, §6]. For the notions of an operator being hypercyclic, supercyclic and weakly supercyclic we refer to the monographs [9], [11].

**Proposition 3.3.** *For the infinitesimal generator $A \in \mathcal{L}(C^\infty(\mathbb{R}_+))$ given by* (3.4) *the following assertions hold.*

(i) *$A$ is not mean ergodic.*

(ii) *The range of $A$ is the proper, closed subspace of $C^\infty(\mathbb{R}_+)$ given by*

$$\mathrm{Im}\,(A) = \{g \in C^\infty(\mathbb{R}_+) \colon g(0) = 0\}. \tag{3.18}$$

(iii) *$A$ is not weakly supercyclic and hence, neither supercyclic nor hypercyclic.*

*Proof.* (i) The function $h(x) = x^2$, for $x \in \mathbb{R}_+$, belongs to $X := C^\infty(\mathbb{R}_+)$ and satisfies

$$A^n h = (-1)^n 2^n h, \quad n \in \mathbb{N}. \tag{3.19}$$

Accordingly, the Cesáro means $A$ are given by

$$A_{[n]}h = \left( \frac{1}{n} \sum_{k=1}^{n} (-1)^k 2^k \right) h, \quad n \in \mathbb{N}.$$

Setting $n = 2m$ for $m \in \mathbb{N}$ it follows that

$$A_{[2m]}h = \left( \frac{1}{2m} \sum_{k=1}^{2m} (-1)^k 2^k \right) h = \frac{(4^m - 1)}{3m} h, \quad m \in \mathbb{N}.$$

Clearly, the set $\{A_{[n]}h \colon n \in \mathbb{N}\}$ is then unbounded in $X$ and so the sequence $\{A_{[n]}\}_{n=1}^\infty$ cannot be convergent in $\mathcal{L}_s(X)$. Hence, $A$ is not mean ergodic.

(ii) We first establish (3.18). Given $f \in X$, the function $h := Af$ clearly belongs to $X$. Moreover, $h(0) = (Af)(0) = 0$. So, $h \in \{g \in X \colon g(0) = 0\}$. To show the reverse containment, fix $h \in \{g \in X \colon g(0) = 0\}$. By the Seeley extension theorem, [18, Theorem], there exists $\tilde{h} \in C^\infty(\mathbb{R})$ such that $\tilde{h}(x) = h(x)$ for each $x \in \mathbb{R}_+$. Since $h(0) = 0$, also $\tilde{h}(0) = 0$. So, we can apply a result due to L. Schwartz, [16, Proposition 1.1], to ensure the existence of $k \in C^\infty(\mathbb{R})$ such that $\tilde{h}(x) = xk(x)$ for $x \in \mathbb{R}$. In particular, $h(x) = xk(x)$ for $x \in \mathbb{R}_+$. Setting $f(x) := -\int_0^x k(s)\,ds$ for each $x \in \mathbb{R}_+$, it follows that $f \in X$ and $Af = h$ in $\mathbb{R}_+$. Thus, (3.18) is established.

Since $\{g \in X \colon g(0) = 0\}$ is clearly a closed subspace of $X$, it follows from (3.18) that also $\mathrm{Im}\,(A)$ is a closed subspace of $X$.

Now, it is also evident from (3.18) that $\mathbf{1} \notin \mathrm{Im}\,(A)$ and so $\mathrm{Im}\,(A)$ is a proper closed subspace of $X$.

(iii) Since each projective orbit $\{\lambda A^n f \colon \lambda \in \mathbb{C},\ n \in \mathbb{N}_0\} \subseteq \mathrm{Im}\,(A)$, for $f \in X$, with $\mathrm{Im}\,(A)$ a closed *proper* subspace of $X$, it is clear that $A$ cannot be weakly supercyclic and hence, neither supercyclic nor hypercyclic. □

*Remark* 3.4. It is clear from (3.19) that $A$ is *not* power bounded. However, the discussion immediately prior to Proposition 3.3 shows that this by itself is insufficient to deduce that $A$ is not mean ergodic in $C^\infty(\mathbb{R}_+)$.

Our final result shows that the Cesàro operator $\mathsf{C}$ on $C^\infty(\mathbb{R}_+)$ exhibits features more desirable than those of $A$.

**Proposition 3.5.** *(i) The Cesàro operator* $\mathsf{C} \in \mathcal{L}(C^\infty(\mathbb{R}_+))$ *is power bounded and uniformly mean ergodic. Moreover,* $\mathrm{Im}\,(I - \mathsf{C})$ *is a proper closed subspace of* $C^\infty(\mathbb{R}_+)$ *and we have the direct decomposition*

$$C^\infty(\mathbb{R}_+) = \mathrm{Ker}(I - \mathsf{C}) \oplus \mathrm{Im}\,(I - \mathsf{C}). \tag{3.20}$$

*(ii) The Cesàro operator* $\mathsf{C}$ *is not weakly supercyclic on* $C^\infty(\mathbb{R}_+)$*.*

*Proof.* (i) It follows from (3.17) that $\mathsf{C} = R(1, A)$ and from [20, p.242] that

$$\mathsf{C}^h f(x) = R(1, A)^h f(x) = \frac{1}{(h-1)!} \int_0^\infty e^{-t} t^{h-1} T(t) f(x)\, dt, \quad x \in \mathbb{R}_+,$$

for all $f \in X := C^\infty(\mathbb{R}_+)$ and $h \in \mathbb{N}$. It then follows from (3.2), for every $n,\ h \in \mathbb{N}$, that

$$p_n(\mathsf{C}^h f) \le \frac{1}{(h-1)!} \int_0^\infty e^{-t} t^{h-1} p_n(T(t)f)\, dt \le p_n(f), \quad f \in X.$$

Hence, $\mathsf{C}$ is *power bounded.* Since $X$ is Montel, it follows that $\mathsf{C}$ is necessarily *uniformly mean ergodic* on $X$, [1, Proposition 2.8].

According to Theorem 2.2 of [1] the direct decomposition

$$C^\infty(\mathbb{R}_+) = \mathrm{Ker}(I - \mathsf{C}) \oplus \overline{\mathrm{Im}\,(I - \mathsf{C})} \tag{3.21}$$

is then valid. Since $\mathsf{C} = R(1, A)$, it follows from [5, Lemma 2.9] that $\mathrm{Im}\,(I - \mathsf{C}) = \mathrm{Im}\,(A)$ and hence, via Proposition 3.3(ii), that $\mathrm{Im}\,(I - \mathsf{C})$ is a *closed* subspace of $C^\infty(\mathbb{R}_+)$. So, (3.21) reduces to (3.20). Moreover, the identities $\mathsf{C} = R(1, A)$ and $D(A) = \mathbb{C}^\infty(\mathbb{R}_+)$ together with Lemma 3.6(i) of [3] imply that $\mathrm{Ker}(A) = \mathrm{Ker}(I - \mathsf{C})$. From (3.4) it is clear that $\mathrm{Ker}(A) = \mathrm{span}\{\mathbf{1}\}$ and so also $\mathrm{Ker}(I - \mathsf{C}) = \mathrm{span}\{\mathbf{1}\}$. In particular, $\mathrm{Im}\,(I - \mathsf{C})$ is a *proper* closed subspace of $C^\infty(\mathbb{R}_+)$; see (3.20).

(ii) It remains to show that $\mathsf{C}$ is not weakly supercyclic. First observe that the Cesàro operator $\mathsf{C}$ acting on the Banach space $C([0, 1])$ is not weakly supercyclic. Indeed, proceeding by contradiction, suppose that there exists a weakly supercyclic vector $g \in C([0, 1])$ for $\mathsf{C}$, i.e., the set $\{\lambda \mathsf{C}^n g \colon \lambda \in \mathbb{C},\ n \in \mathbb{N}_0\}$ is weakly dense in $C([0, 1])$. Then there exist a sequence $\{\lambda_k\}_{k=1}^\infty \subseteq \mathbb{C}$ and an increasing sequence $\{n_k\}_{k=1}^\infty \subseteq \mathbb{N}$ such that $\delta_0(\lambda_k \mathsf{C}^{n_k} g) \to \delta_0(\mathbf{1})$ for $k \to \infty$, i.e., $\lambda_k \mathsf{C}^{n_k} g(0) = \lambda_k g(0) \to 1$ for $k \to \infty$ and so $g(0) \neq 0$; here $\delta_t \in C([0,1])'$ is the Dirac point measure at any $t \in [0,1]$. On the other hand, for the particular function $f(x) = x$ for $x \in [0,1]$, there exist a sequence $\{\mu_r\}_{r=1}^\infty \subseteq \mathbb{C}$ and an increasing sequence $\{m_r\}_{r=1}^\infty \subseteq \mathbb{N}$ such that $\delta_0(\mu_r \mathsf{C}^{m_r} g) \to \delta_0(f)$ and $\delta_1(\mu_r \mathsf{C}^{m_r} g) \to \delta_1(f)$ for $r \to \infty$. Hence, $\mu_r \mathsf{C}^{m_r} g(0) = \mu_r g(0) \to f(0) = 0$ and $\delta_1(\mu_r \mathsf{C}^{m_r} g) \to f(1) = 1$ for $r \to \infty$. So, $\mu_r \to 0$ for $r \to \infty$ (as $g(0) \neq 0$). Since $\|\mu_r \mathsf{C}^{m_r} g\|_\infty \leq |\mu_r| \cdot \|g\|_\infty$ for all $r \in \mathbb{N}$ (as $\|\mathsf{C}\| \leq 1$), it then follows that $\mu_r \mathsf{C}^{m_r} g \to 0$ in $C([0,1])$ for $r \to \infty$ and hence, $\delta_1(\mu_r \mathsf{C}^{m_r} g) \to \delta_1(0) = 0$. This is a contradiction.

Since $\mathsf{C}$ acting on $C([0,1])$ is not weakly supercyclic, the Cesàro operator $\mathsf{C}$ acting on the Fréchet space $C(\mathbb{R}_+)$ is also not weakly supercyclic; just "repeat" the argument given in the proof of Theorem 3.1 in [6] where it was shown that $\mathsf{C}$ is not supercyclic on $C(\mathbb{R}_+)$. Finally, observe that the space $X$ is continuously included in $C(\mathbb{R}_+)$ with dense range and that $X$ is an invariant subspace of $\mathsf{C}\colon C(\mathbb{R}_+) \to C(\mathbb{R}_+)$. So, if $\mathsf{C}\colon X \to X$ were weakly supercyclic, then also $\mathsf{C}\colon C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ would be weakly supercyclic, a contradiction. □

*Remark* 3.6. A result of M. Lin asserts for an operator $T \in \mathcal{L}(X)$ satisfying $\lim_{n\to\infty} \frac{\|T^n\|}{n} = 0$, with $X$ a Banach space, that $T$ is uniformly mean ergodic if and only if the range $\operatorname{Im}(I - T)$ is closed in $X$, [15]. For a *prequojection* Fréchet space $X$ (i.e., $X$ has no infinite-dimensional Köthe nuclear quotient which admits a continuous norm) and $T \in \mathcal{L}(X)$ satisfying $\lim_{n\to\infty} \frac{T^n}{n} = 0$ in $\mathcal{L}_b(X)$ it is also known that $T$ is uniformly mean ergodic if and only if the range $\operatorname{Im}(I-T)$ is closed in $X$, [4, Theorem 3.5]. Examples of Fréchet spaces $X$ and uniformly mean ergodic operators $T \in \mathcal{L}(X)$ which are power bounded (in particular, $\lim_{n\to\infty} \frac{T^n}{n} = 0$ in $\mathcal{L}_b(X)$) but $\operatorname{Im}(I-T)$ is *not* closed in $X$ can be found in [1, Example 2.17] and [4, Propositions 3.1 and 3.3], for instance. Since the nuclear Fréchet space $C^\infty(\mathbb{R}_+)$ is *not* a prequojection, it was necessary in the proof of Proposition 3.5 to check separately each of the properties that the Cesàro operator $\mathsf{C} \in \mathcal{L}(C^\infty(\mathbb{R}_+))$ is uniformly mean ergodic and that $\operatorname{Im}(I - \mathsf{C})$ is closed in $C^\infty(\mathbb{R}_+)$.

It is known that $C^\infty(\mathbb{R}_+)$ is not a prequojection; one sees this as follows. The Fréchet space $C^\infty([0,1])$ was introduced in the proof of Lemma 3.1(i), where it was noted to be topologically isomorphic to the Fréchet space $s$. Hence, it is an infinite-dimensional Köthe nuclear Fréchet space with a continuous norm.

Next, let $R\colon C^\infty(\mathbb{R}_+) \to C^\infty([0,1])$ be defined by $Rf = f|_{[0,1]}$, for $f \in C^\infty(\mathbb{R}_+)$, i.e., $R$ is the restriction map to the closed interval $[0,1]$.

Clearly, $R$ is continuous as a linear operator between Fréchet spaces. Moreover, $R$ is also surjective. To see this, fix $g \in C^\infty([0,1])$. By applying [21, (9), p.373] with A=[0,1] and C=[0,2], we can conclude that there exists $h \in C^\infty([0,2])$ satisfying $h|_{[0,1]} = g$. Now choose $k \in C^\infty(\mathbb{R}_+)$ satisfying $k(x) = 1$ for $x \in [0,1]$ and $h(x) = 0$ for $x \in [3/2, \infty)$. Setting $f := h.k$, it then follows that $f \in C^\infty(\mathbb{R}_+)$ and $Rf = (h.k)|_{[0,1]} = g$. Since $g$ is arbitrary, the operator $R$ is surjective.

We can now apply the open mapping theorem for Fréchet spaces to conclude that $R$ is also open, [17, Theorem 8.5]. Therefore, $C^\infty([0,1])$ is isomorphic to a quotient space of $C^\infty(\mathbb{R}_+)$. But, $C^\infty([0,1]) \simeq s$ and so $C^\infty(\mathbb{R}_+)$ cannot be a prequojection.

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