The Cesàro operator in growth Banach spaces of analytic functions

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Aim of the lecture

AIM

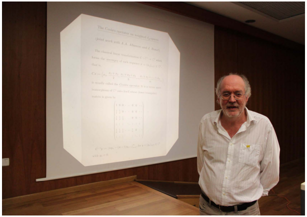
Investigate the norm and the mean ergodicity of the Cesàro operator C acting on certain Banach spaces of analytic functions on the disc.

We report on joint work in progress with Angela A. Albanese (Univ. Lecce, Italy) and Werner J. Ricker (Univ. Eichstaett, Germany).

Ernesto Cesàro (1859-1906)

Albanese and Ricker

Angela Albanese



Werner Ricker

The discrete Cesàro operator

The *Cesàro operator* C is defined for a sequence $x = (x_n)_n \in \mathbb{C}^{\mathbb{N}}$ of complex numbers by

$$\mathrm{C}(x) = \left(\frac{1}{n} \sum_{k=1}^{n} x_k \right)_n, \quad x = (x_n)_n \in \mathbb{C}^{\mathbb{N}}.$$

Proposition.

The operator $C \colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ is a bicontinuous isomorphism of $\mathbb{C}^{\mathbb{N}}$ onto itself with

$$C^{-1}(y) = (ny_n - (n-1)y_{n-1})_n, \quad y = (y_n)_n \in \mathbb{C}^{\mathbb{N}}, \tag{1}$$

where we set $y_{-1} := 0$.

Recall that $\mathbb{C}^{\mathbb{N}}$ is a Fréchet space for the topology of coordinatewise convergence.

The Cesàro operator for analytic functions

The Cesàro operator is defined for analytic functions on the disc $\mathbb{D}$ by

$$Cf = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} a_n \right) z^n, \quad f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D}).$$

The Cesàro operator acts continuously and has the integral representation

$$Cf(z) = \frac{1}{z} \int_0^z \frac{f(\rho)}{1-\rho} \, d\rho, \quad f \in H(\mathbb{D}), \; z \in \mathbb{D}.$$

The Cesàro operator for analytic functions

Indeed, for $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$, we have

$$\begin{aligned}
Cf(z) &= \frac{1}{z}\int_0^z \frac{f(\rho)}{1-\rho}\, d\rho = \frac{1}{z}\int_0^z \left(\sum_{n=0}^{\infty} a_n \rho^n\right)\left(\sum_{m=0}^{\infty} \rho^m\right) \\
&= \sum_{n=0}^{\infty} a_n \sum_{m=0}^{\infty} \frac{1}{z}\int_0^z \rho^{n+m}\, d\rho = \sum_{n=0}^{\infty} a_n \sum_{m=0}^{\infty} \frac{z^{n+m}}{n+m+1} \\
&= \sum_{n=0}^{\infty} a_n \sum_{k=n} \frac{z^k}{k+1} = \sum_{k=0}^{\infty} \left(\frac{1}{k+1}\sum_{n=0}^{k} a_n\right) z^k.
\end{aligned}$$

Spectrum and point spectrum

X is a Hausdorff locally convex space (lcs).

$\mathcal{L}(X)$ (resp. $\mathcal{K}(X)$) is the space of all continuous (resp. compact) linear operators on X.

The **resolvent set** $\rho(T, X)$ of $T \in \mathcal{L}(X)$ consists of all $\lambda \in \mathbb{C}$ such that $R(\lambda, T) := (\lambda I - T)^{-1}$ exists in $\mathcal{L}(X)$.

The **spectrum** of T is the set $\sigma(T, X) := \mathbb{C} \setminus \rho(T, X)$. The **point spectrum** is the set $\sigma_{pt}(T, X)$ of those $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not injective. The elements of $\sigma_{pt}(T, X)$ are called eigenvalues of T.

More about the spectrum and point spectrum

Theorem

The Cesàro operator satisfies

$$(a) \quad \sigma(\mathsf{C}, H(\mathbb{D})) = \sigma_{pt}(\mathsf{C}, H(\mathbb{D})) = \{\frac{1}{m} \colon m \in \mathbb{N}\}.$$

$$(b) \quad \sigma^*(\mathsf{C}, H(\mathbb{D})) = \{\frac{1}{m} \colon m \in \mathbb{N}\} \cup \{0\}.$$

Persson showed in 2008 the following facts:

For every $m \in \mathbb{N}$ the operator $(\mathsf{C} - \frac{1}{m}I) \colon H(\mathbb{D}) \to H(\mathbb{D})$ is not injective because $\mathrm{Ker}(\mathsf{C} - \frac{1}{m}I) = \mathrm{span}\{e_m\}$, where $e_m(z) = z^{m-1}(1-z)^{-m}$, $z \in \mathbb{D}$, and it is not surjective because the function $f_m(z) := z^{m-1}$, $z \in \mathbb{D}$, does not belong to the range of $(\mathsf{C} - \frac{1}{m}I)$.

The growth spaces

For $\gamma > 0$ the growth classes $A^{-\gamma}$ and $A_0^{-\gamma}$ are the Banach spaces defined by

$$A^{-\gamma} = \{f \in H(\mathbb{D})\colon \|f\|_{-\gamma} := \sup_{z\in\mathbb{D}}(1-|z|)^{\gamma}|f(z)| < \infty\}.$$

$$A_0^{-\gamma} = \{f \in H(\mathbb{D})\colon \lim_{|z|\to 1}(1-|z|)^{\gamma}|f(z)| = 0\}.$$

$A_0^{-\gamma}$ is the closure of the polynomials on $A^{-\gamma}$.

The Cesàro operator acts continuously on $A^{-\gamma}$. Its spectrum on these (and many other spaces of analytic functions on the disc) has been studied by Aleman and Persson 2008-2010.

The result of Aleman and Persson

Theorem. Aleman, Persson.

Let $\gamma > 0$. The Cesàro operator $\mathrm{C}_{\gamma,0} \colon A_0^{-\gamma} \to A_0^{-\gamma}$ has the following properties.

(i) $\sigma_{pt}(\mathrm{C}_{\gamma,0}) = \{\frac{1}{m} \colon m \in \mathbb{N},\ m < \gamma\}$.

(ii) $\sigma(\mathrm{C}_{\gamma,0}) = \sigma_{pt}(\mathrm{C}_{\gamma,0}) \cup \left\{\lambda \in \mathbb{C} \colon \left|\lambda - \frac{1}{2\gamma}\right| \leq \frac{1}{2\gamma}\right\}$.

(iii) If $\left|\lambda - \frac{1}{2\gamma}\right| < \frac{1}{2\gamma}$ (equivalently $\mathrm{Re}\left(\frac{1}{\lambda}\right) > \gamma$), then $\mathrm{Im}(\lambda I - \mathrm{C}_{\gamma,0})$ is a closed subspace of $A_0^{-\gamma}$ and has codimension 1.

Moreover, the Cesàro operator $\mathrm{C}_\gamma : A^{-\gamma} \to A^{-\gamma}$ satisfies

(iv) $\sigma_{pt}(\mathrm{C}_\gamma) = \{\frac{1}{m} \colon m \in \mathbb{N},\ m \leq \gamma\}$, and

(v) $\sigma(\mathrm{C}_\gamma) = \sigma(\mathrm{C}_{\gamma,0})$.

The norm

The continuity of C_γ and $C_{\gamma,0}$ as established by Aleman and Persson gives no quantitative estimate for their operator norm.

Theorem.

(i) Let $\gamma \geq 1$. Then $\|C_\gamma^n\| = \|C_{\gamma,0}^n\| = 1$ for all $n \in \mathbb{N}$.

(ii) Let $0 < \gamma < 1$. Then $\|C_\gamma^n\| = \|C_{\gamma,0}^n\| = 1/\gamma^n$ for all $n \in \mathbb{N}$.

Mean ergodic properties. Definitions

Power bounded operators

An operator $T \in \mathcal{L}(X)$ is said to be *power bounded* if $\{T^m\}_{m=1}^{\infty}$ is an equicontinuous subset of $\mathcal{L}(X)$.

If X is a Banach space, an operator T is power bounded if and only if $\sup_n ||T^n|| < \infty$.

If X is a barrelled space, an operator T is power bounded if and only if the orbits $\{T^m(x)\}_{m=1}^{\infty}$ of all the elements $x \in X$ under T are bounded. This is a consequence of the uniform boundedness principle.

Mean ergodic properties. Definitions

For $T \in \mathcal{L}(X)$**, we set** $T_{[n]} := \frac{1}{n}\sum_{m=1}^{n} T^m$.

Mean ergodic operators

An operator $T \in \mathcal{L}(X)$ is said to be *mean ergodic* if the limits

$$Px := \lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} T^m x, \quad x \in X, \tag{2}$$

exist in X.

If T is mean ergodic, then one then has the direct decomposition

$$X = \mathrm{Ker}(I - T) \oplus \overline{(I - T)(X)}.$$

Mean ergodic properties. Definitions

Uniformly mean ergodic operators

If $\{T_{[n]}\}_{n=1}^{\infty}$ happens to be convergent in $\mathcal{L}_b(X)$ to $P \in \mathcal{L}(X)$, then T is called *uniformly mean ergodic*.

Theorem. Lin. 1974.

Let T a (continuous) operator on a Banach space X which satisfies $\lim_{n\to\infty} ||T^n/n|| = 0$. The following conditions are equivalent:

(1) T is uniformly mean ergodic.

(4) $(I - T)(X)$ is closed.

Theorem

(i) Let $0 < \gamma < 1$. Both of the operators C_γ and $C_{\gamma,0}$ fail to be power bounded and are not mean ergodic. Moreover,

$$\mathrm{Ker}(I - C_\gamma) = \mathrm{Ker}(I - C_{\gamma,0}) = \{0\},$$

and $\mathrm{Im}(I - C_\gamma)$ (resp. $\mathrm{Im}(I - C_{\gamma,0})$) is a proper closed subspace of $A^{-\gamma}$ (resp. of $A_0^{-\gamma}$).

(ii) Both of the operators C_1 and $C_{1,0}$ are power bounded but not mean ergodic. Moreover, $\mathrm{Im}(I - C_1)$ (resp. $\mathrm{Im}(I - C_{1,0})$) is not a closed subspace of $A^{-\gamma}$ (resp. of $A_0^{-\gamma}$).

Ergodicity of C on growth spaces

Theorem continued

(iii) Let $\gamma > 1$. Both of the operators C_γ and $\mathsf{C}_{\gamma,0}$ are power bounded and uniformly mean ergodic. Moreover, $\mathrm{Im}(I - \mathsf{C}_\gamma)$ (resp. $\mathrm{Im}(I - \mathsf{C}_{\gamma,0})$) is a proper closed subspace of $A^{-\gamma}$ (resp. of $A_0^{-\gamma}$). In addition,

$$\mathrm{Im}(I - \mathsf{C}_\gamma) = \{h \in A^{-\gamma} : h(0) = 0\}. \tag{3}$$

Moreover, with $\varphi(z) := 1/(1-z)$, for $z \in \mathbb{D}$, the linear projection operator $P_\gamma : A^{-\gamma} \to A^{-\gamma}$ given by

$$P_\gamma(f) := f(0)\varphi, \qquad f \in A^{-\gamma},$$

is continuous and satisfies $\lim_{n\to\infty} (\mathsf{C}_\gamma)_{[n]} = P_\gamma$ in the operator norm.

Hypercyclicity. Definitions

Hypercyclic operator

$T \in \mathcal{L}(X)$, with X separable, is called **hypercyclic** if there exists $x \in X$ such that the orbit $\{T^n x \colon n \in \mathbb{N}_0\}$ is dense in X.

Supercyclic operator

If, for some $z \in X$, the projective orbit $\{\lambda T^n z \colon \lambda \in \mathbb{C},\ n \in \mathbb{N}_0\}$ is dense in X, then T is called *supercyclic*.

Clearly, hypercyclicity always implies supercyclicity.

Ergodicity of C on $H(\mathbb{D})$

Theorem

The Cesàro operator C acting on $H(\mathbb{D})$ is power bounded, uniformly mean ergodic and not supercyclic, hence not hypercyclic.

As a consequence, C is not supercyclic on the spaces $A^{-\gamma}$, $\gamma \geq 0$, and $A_0^{-\gamma}, 0 < \gamma \leq \infty$.

The optimal domain

The *optimal domain* $[\mathrm{C}, X]$ of the Cesàro operator C when it acts in any Banach space of analytic functions X on $\mathbb{D}$ is defined by

$$[\mathrm{C}, X] := \{f \in H(\mathbb{D}) \ : \ \mathrm{C}(f) \in X\},$$

$$\|f\|_{[\mathrm{C},X]} := \|\mathrm{C}(f)\|_X, \qquad f \in [\mathrm{C}, X].$$

It is a Banach space as of a consequence of $\mathrm{C} : H(\mathbb{D}) \to H(\mathbb{D})$ being a topological Fréchet space isomorphism.

If C acts in X, then $X \subseteq [\mathrm{C}, X]$ and the natural inclusion map is continuous. Moreover, $[\mathrm{C}, X]$ is the *largest* of all Banach spaces of analytic functions Y on $\mathbb{D}$ that C maps continuously into X

The optimal domain

Theorem

Let $\gamma > 0$ and $\varphi(z) := 1/(1-z)$ for $z \in \mathbb{D}$.
The optimal domain $[\mathrm{C}, A^{-\gamma}]$ of $\mathrm{C}_\gamma : A^{-\gamma} \to A^{-\gamma}$ is isometrically isomorphic to $A^{-\gamma}$ and is given by

$$[\mathrm{C}, A^{-\gamma}] = \{f \in H(\mathbb{D}) \ : \ f\varphi \in A^{-(\gamma+1)}\}. \tag{4}$$

Moreover, the norm $\|.\|_{[\mathrm{C}, A^{-\gamma}]}$ is equivalent to the norm $f \to \|f\varphi\|_{-(\gamma+1)}$ and the containment $A^{-\gamma} \subseteq [\mathrm{C}, A^{-\gamma}]$ is proper.

A similar result holds for the optimal domain $[\mathrm{C}, A_0^{-\gamma}]$ of $\mathrm{C}_{\gamma,0} : A_0^{-\gamma} \to A_0^{-\gamma}$.

References

1. **A. A. Albanese, J. Bonet, W. J. Ricker,** The Cesàro operator in growth Banach spaces of analytic functions, Integr. Equ. Oper. Theory DOI 10.1007/s00020-016-2316-z