THE CESÁRO OPERATOR ON DUALS OF POWER SERIES
SPACES OF INFINITE TYPE

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ABSTRACT. A detailed investigation is made of the continuity, spectrum and
mean ergodic properties of the Cesàro operator $C$ when acting on the strong
duals of power series spaces of infinite type. There is a dramatic difference in
the nature of the spectrum of $C$ depending on whether or not the strong dual
space (which is always Schwartz) is nuclear.

1. INTRODUCTION AND NOTATION.

The discrete Cesàro operator $C$ is defined on the linear space $\mathbb{C}^N$ (consisting of
all scalar sequences) by

$$C x := \left( x_1, \frac{x_1 + x_2}{2}, \ldots, \frac{x_1 + \ldots + x_n}{n}, \ldots \right), \quad x = (x_n)_{n \in \mathbb{N}} \in \mathbb{C}^N. \quad (1.1)$$

The linear operator $C$ is said to act in a vector subspace $X \subseteq \mathbb{C}^N$ if it maps $X$
into itself. Of particular interest is the situation when $X$ is a Fréchet space or an
(LF)-space. Two fundamental questions in this case are: Is $C : X \to X$ continuous
and, if so, what is its spectrum? For a large collection of classical Banach spaces
$X \subseteq \mathbb{C}^N$ where precise answers are known we refer to the Introductions in [4],
[6], for example. The discrete Cesàro operator $C$ acting on the Fréchet sequence
space $\mathbb{C}^N$, on $\ell^p := \cap_{p>q} \ell^q$, and on the power series spaces $\Lambda_0(\alpha) := \Lambda_0^*(\alpha)$ of
finite type was investigated in [3], [5], [6], respectively. The aim of this paper is to
investigate the behaviour of $C$ when it acts on the strong duals $(\Lambda_0^*(\alpha))'$ of power
series spaces $\Lambda_0^*(\alpha)$ of infinite type. Power series spaces of infinite type play an
important role in the isomorphic classification of Fréchet spaces, [17], [21], [22].
The reason for concentrating on the infinite type dual spaces $(\Lambda_0^*(\alpha))'$ is that the
Cesàro operator $C$ fails to be continuous on “most” of the finite type dual spaces
$(\Lambda_0^*(\alpha))'$. This is explained more precisely in an Appendix (Section 5) at the end
of the paper.

In order to describe the main results we require some notation and definitions.

Let $X$ be a locally convex Hausdorff space (briefly, lHs) and $\Gamma_X$ a system of
continuous seminorms determining the topology of $X$. Let $X'$ denote the space of
all continuous linear functionals on $X$. The family of all bounded subsets of $X$
is denoted by $\mathcal{B}(X)$. Denote the identity operator on $X$ by $I$. Let $\mathcal{L}(X)$ denote
the space of all continuous linear operators from $X$ into itself. For $T \in \mathcal{L}(X)$, the

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resolvent set \( \rho(T) \) of \( T \) consists of all \( \lambda \in \mathbb{C} \) such that \( R(\lambda, T) := (\lambda I - T)^{-1} \) exists in \( \mathcal{L}(X) \). The set \( \sigma(T) := \mathbb{C} \setminus \rho(T) \) is called the spectrum of \( T \). The point spectrum \( \sigma_p(T) \) of \( T \) consists of all \( \lambda \in \mathbb{C} \) such that \( (\lambda I - T) \) is not injective. If we need to stress the space \( X \), then we also write \( \sigma(T; X) \), \( \sigma_p(T; X) \) and \( \rho(T; X) \). Given \( \lambda, \mu \in \rho(T) \) the resolvent identity \( R(\lambda, T) - R(\mu, T) = (\mu - \lambda)R(\lambda, T)R(\mu, T) \) holds. Unlike for Banach spaces, it may happen that \( \rho(T) = \emptyset \) (cf. Remark 2.6(ii)) or that \( \rho(T) \) is not open in \( \mathbb{C} \); see Proposition 2.9(i) for example. That is why some authors prefer the subset \( \rho^*(T) \) of \( \rho(T) \) consisting of all \( \lambda \in \mathbb{C} \) for which there exists \( \delta > 0 \) such that the open disc \( B(\lambda, \delta) := \{ z \in \mathbb{C} : |z - \lambda| < \delta \} \subseteq \rho(T) \) and \( \{ R(\mu, T) : \mu \in B(\lambda, \delta) \} \) is equicontinuous in \( \mathcal{L}(X) \). If \( X \) is a Fréchet space or even an (LF)-space, then it suffices that such sets are bounded in \( \mathcal{L}_s(X) \), where \( \mathcal{L}_s(X) \) denotes \( \mathcal{L}(X) \) endowed with the strong operator topology \( \tau_\sigma \) which is determined by the seminorms \( T \mapsto q_B(T) := q(Tx), \) for all \( x \in X \) and \( q \in \Gamma_X \). The advantage of \( \rho^*(T) \), whenever it is non-empty, is that it is open and the resolvent map \( R : \lambda \mapsto R(\lambda, T) \) is holomorphic from \( \rho^*(T) \) into \( \mathcal{L}_b(X) \), [2, Proposition 3.4]. Here \( \mathcal{L}_b(X) \) denotes \( \mathcal{L}(X) \) endowed with the lcH-topology \( \tau_\beta \) of uniform convergence on members of \( \mathcal{B}(X) \); it is determined by the seminorms \( T \mapsto q_B(T) := \sup_{x \in B} q(Tx), \) for \( T \in \mathcal{L}(X) \), for all \( B \in \mathcal{B}(X) \) and \( q \in \Gamma_X \). Define \( \sigma^*(T) := \mathbb{C} \setminus \rho^*(T) \), which is a closed set containing \( \sigma(T) \). If \( T \in \mathcal{L}(X) \) with \( X \) a Banach space, then \( \sigma(T) = \sigma^*(T) \). In [2, Remark 3.5(vi), p.265] an example of a continuous linear operator \( T \) on a Fréchet space \( X \) is presented such that \( \sigma(T) \subset \sigma^*(T) \) properly. For undefined concepts concerning lcHs see [12, 17].

Each positive, strictly increasing sequence \( \alpha = (\alpha_n) \) which tends to infinity generates a power series space \( \Lambda_1^{\alpha}/(\alpha) \) of infinite type; see Section 2. The strong dual \( E_\alpha \subseteq \mathbb{C}^N \) of \( \Lambda_1^{\alpha}/(\alpha) \) is then a co-echelon space, i.e., a particular kind of inductive limit of Banach spaces (of sequences), which is necessarily a Schwartz space in our setting. It turns out (cf. Proposition 2.1) that always \( \mathbb{C} \subseteq \mathcal{L}(E_\alpha) \). Furthermore, it is known that the nuclearity of the space \( E_\alpha \) is characterized by the condition \( \sup_{n \in \mathbb{N}} \log(n)/\alpha_n < \infty \). Remarkably, this is equivalent to the operator \( \mathbb{C} \subseteq \mathcal{L}(E_\alpha) \) being invertible, i.e., \( 0 \in \rho(\mathbb{C}; E_\alpha) \); see Proposition 2.4. Actually, the main results of this section (namely, Proposition 2.9 and Corollary 2.10) establish the equivalence of the following assertions:

(i) \( E_\alpha \) is nuclear.

(ii) \( \sigma(\mathbb{C}; E_\alpha) = \sigma_p(\mathbb{C}; E_\alpha) \).

(iii) \( \sigma(\mathbb{C}; E_\alpha) = \{ 1/n : n \in \mathbb{N} \} \).

Moreover, in this case we have \( \sigma^*(\mathbb{C}; E_\alpha) = \{ 0 \} \cup \sigma(\mathbb{C}; E_\alpha) \). So, whenever \( E_\alpha \) is nuclear, the spectra \( \sigma_p(\mathbb{C}; E_\alpha), \sigma(\mathbb{C}; E_\alpha) \) and \( \sigma^*(\mathbb{C}; E_\alpha) \) are completely identified. In particular, these spectra of \( \mathbb{C} \) are independent of \( \alpha \).

The operator \( D \in \mathcal{L}(\mathbb{C}^N) \) of differentiation (defined in the obvious way) is closely connected to the Cesàro operator \( \mathbb{C} \subseteq \mathcal{L}(\mathbb{C}^N) \) via the identity (valid in \( \mathcal{L}(\mathbb{C}^N) \))

\[ C^{-1} = (I - S_t)DS_t, \]

where \( S_t \in \mathcal{L}(\mathbb{C}^N) \) is the right-shift operator. It is always the case that \( S_t \in \mathcal{L}(E_\alpha) \) whenever \( \alpha_n \uparrow \infty \). Moreover, it follows from (i)-(iii) above that \( C^{-1} \in \mathcal{L}(E_\alpha) \) precisely when \( E_\alpha \) is nuclear. So, the above identity for \( C^{-1} \) suggests that there should be a connection between the continuity of \( D \) on \( E_\alpha \) and the nuclearity of \( E_\alpha \). This is clarified by Proposition 2.5. Namely, \( D \) is continuous on \( E_\alpha \) if and
only if $E_\alpha$ is both nuclear and $\sup_{n \in \mathbb{N}} \frac{\alpha_{n+1}}{\alpha_n} < \infty$. Remark 2.6(i) shows that these two conditions are independent of one another.

Section 3 identifies the spectra of $C \in \mathcal{L}(E_\alpha)$ in the case when $E_\alpha$ is not nuclear. We have seen if $E_\alpha$ is nuclear, then $\sigma(C; E_\alpha)$ is a bounded, infinite and countable set with no accumulation points. For $E_\alpha$ non-nuclear the spectrum of $C$ is very different. Indeed, in this case

$$\sigma(C; E_\alpha) = \{0, 1\} \cup \{\lambda \in \mathbb{C} : |\lambda - \frac{1}{2}| < \frac{1}{2}\}$$

and $\sigma^*(C; E_\alpha) = \{\lambda \in \mathbb{C} : |\lambda - \frac{1}{2}| \leq \frac{1}{2}\}$ whenever $\sup_{n \in \mathbb{N}} \frac{\log(\log(n))}{\alpha_n} < \infty$, whereas

$$\sigma(C; E_\alpha) = \sigma^*(C; E_\alpha) = \{\lambda \in \mathbb{C} : |\lambda - \frac{1}{2}| \leq \frac{1}{2}\}$$

otherwise; see Proposition 3.4. Again the spectra of $C$ are independent of $\alpha$.

J. von Neumann (1931) proved that unitary operators $T$ in Hilbert space are mean ergodic, i.e., the sequence of its averages $\frac{1}{n} \sum_{m=1}^{n} T^m$, for $n \in \mathbb{N}$, converges for the strong operator topology (to a projection). Ever since, intensive research has been undertaken to identify the mean ergodicity of individual (and classes) of operators both in Banach spaces and non-normable lchS's; see [1], [15] for example, and the references therein. In Section 4 it is shown, for every sequence $\alpha$ with $\alpha_n \uparrow \infty$, that the Cesàro operator $C \in \mathcal{L}(E_\alpha)$ is always power bounded, (uniformly) mean ergodic and $E_\alpha = \text{Ker}(I - C) \oplus (I - C)(E_\alpha)$; see Proposition 4.1. Actually, even the sequence $\{C^n\}_{n=1}^{\infty}$ of the iterates of $C$ (not just its averages) turns out to be convergent, not only in $\mathcal{L}_1(E_\alpha)$ but also in $\mathcal{L}_b(E_\alpha)$; see Proposition 4.2. Furthermore, if $E_\alpha$ is nuclear, then the range $(I - C)^m(E_\alpha)$ of the operator $(I - C)^m$ is a closed subspace of $E_\alpha$ for each $m \in \mathbb{N}$ (cf. Proposition 4.3). For $m = 1$ this is an analogue, for the operator $C \in \mathcal{L}(E_\alpha)$, of a result of M. Lin for arbitrary uniformly mean ergodic Banach space operators $T$ which satisfy $\lim_{n \to \infty} \frac{\|T^n\|}{n} = 0$, [16].

2. The Spectrum of $C$ in the Nuclear Case

Let $\alpha := (\alpha_n)$ be a positive, strictly increasing sequence tending to infinity, briefly, $\alpha_n \uparrow \infty$. Let $(s_k) \subseteq (1, \infty)$ be another strictly increasing sequence satisfying $s_k \uparrow \infty$. For each $k \in \mathbb{N}$, define $v_k : \mathbb{N} \to (0, \infty)$ by $v_k(n) := s_k^{-\alpha_n}$ for $n \in \mathbb{N}$. Then $v_k(n) \geq v_k(n+1)$, for $n \in \mathbb{N}$, i.e., $v_k$ is a decreasing sequence, and $v_k \geq v_{k+1}$ pointwise on $\mathbb{N}$ for all $k \in \mathbb{N}$. Set $\mathcal{V} := (v_k)$ and note that $v_k \in c_0$ for all $k \in \mathbb{N}$.

Define the co-echelon spaces $E_\alpha := \text{ind}_k c_0(v_k)$, that is, $E_\alpha$ is the (increasing) union of the weighted Banach spaces $c_0(v_k)$, $k \in \mathbb{N}$, endowed with the finest lch-topology such that each natural inclusion map $c_0(v_k) \to E_\alpha$ is continuous. Since $\lim_{n \to \infty} \frac{v_{k+1}(n)}{v_k(n)} = 0$, for $k \in \mathbb{N}$, implies that $\ell_\infty(v_k) \subseteq c_0(v_{k+1})$ continuously, for $k \in \mathbb{N}$, it follows that also $E_\alpha := \text{ind}_k \ell_\infty(v_k)$. Observing that the power series space $A_{\alpha_0}(\alpha) := \text{proj}_k \ell_1(v_k^{-1})$ of infinite type is Fréchet-Schwartz (hence, distinguished), [17, p. 357], it follows that $E_\alpha := \text{ind}_k c_0(v_k) = \text{ind}_k \ell_\infty(v_k) = (A_{\alpha_0}(\alpha))^\prime$ is the strong dual of $A_{\alpha_0}(\alpha)$, [17, Remark 25.13]. The condition $\frac{v_{k+1}}{v_k} \in c_0$ for $k \in \mathbb{N}$ implies that $E_\alpha$ is always a (DFS)-space, [17, p. 304], and in particular, a Montel space, [17, Remark 24.24]. Note that power series spaces in [17, Chapter 24] are defined using $\ell_2$-norms. It follows from [17, Proposition 29.6] that $A_{\alpha_0}(\alpha)$ is a nuclear Fréchet space (equivalently, $E_\alpha$ is a (DFN)-space) if and only if
$\sup_{n \in \mathbb{N}} \frac{\log n}{\alpha_n} < \infty$. This criterion plays a relevant role throughout this section. As the space $E_\alpha$ does not change if $(s_k)$ is replaced by any other strictly increasing sequence in $(1, \infty)$ tending to infinity, we sometimes choose $s_k := e^k$, $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, define the norm

$$q_k(x) := \sup_{n \in \mathbb{N}} v_k(n)|x_n|, \quad x = (x_n) \in \ell_\infty(v_k),$$

whose restriction to $c_0(v_k)$ is the norm of $c_0(v_k)$. Observe, for each $k \in \mathbb{N}$, that $c_0(v_k) \subseteq c_0(v_l)$ for every $l \in \mathbb{N}$ with $l \geq k$, and

$$q_l(x) \leq q_k(x), \quad x \in c_0(v_k). \quad (2.1)$$

As general references for co-echelon spaces we refer to [8], [9], [14], [17], for example.

**Proposition 2.1.** For each $\alpha_n \uparrow \infty$ the Cesàro operator satisfies $C \in \mathcal{L}(E_\alpha)$.

**Proof.** Since each sequence $v_k$, for $k \in \mathbb{N}$, is decreasing, Corollary 2.3(i) of [4] implies that the Cesàro operator at each step, namely $C: c_0(v_k) \to c_0(v_k)$, for $k \in \mathbb{N}$, is continuous. The result then follows from the general theory of (LB)-spaces as $E_\alpha = \text{ind}_k c_0(v_k)$. □

**Lemma 2.2.** Let $\alpha_n \uparrow \infty$. The following conditions are equivalent.

(i) $\sup_{n \in \mathbb{N}} \frac{\log n}{\alpha_n} < \infty$.

(ii) For each $\gamma > 0$ there exists $M(\gamma) \in \mathbb{N}$ such that $\sup_{n \in \mathbb{N}} n^{\gamma}e^{-M(\gamma)\alpha_n} < \infty$.

(iii) For some $\gamma > 0$ and $M(\gamma) \in \mathbb{N}$ we have $\sup_{n \in \mathbb{N}} n^{\gamma}e^{-M(\gamma)\alpha_n} < \infty$.

**Proof.** (i)⇒(ii). Fix any $\gamma > 0$. By assumption there exists $D > 0$ such that

$$\log n \leq Dao_n \quad \text{for all} \ n \in \mathbb{N}.$$ 

Let $M(\gamma) \in \mathbb{N}$ satisfy $M(\gamma) \geq \gamma D$. Then

$$\gamma \log n \leq \gamma Dao_n \leq M(\gamma)\alpha_n \quad \text{for all} \ n \in \mathbb{N} \ 	ext{and hence}, \ n^{\gamma} \leq e^{M(\gamma)\alpha_n} \quad \text{for all} \ n \in \mathbb{N}.$$ 

(ii)⇒(iii) is clear.

(iii)⇒(i). By assumption $\sup_{n \in \mathbb{N}} n^{\gamma}e^{-M(\gamma)\alpha_n} < \infty$. So, there exists $D > 1$ such that

$$n^{\gamma} \leq De^{M(\gamma)\alpha_n} \quad \text{for all} \ n \in \mathbb{N}.$$ 

It follows for each $n \in \mathbb{N}$ that

$$\frac{\log n}{\alpha_n} \leq \frac{\log D}{\gamma\alpha_n} + \frac{M}{\gamma}.$$ 

Since $\alpha_n \to \infty$, we can conclude that $\sup_{n \in \mathbb{N}} \frac{\log n}{\alpha_n} < \infty$. □

We now turn our attention to the spectrum of $C \in \mathcal{L}(E_\alpha)$, for which we introduce the notation $\Sigma := \{1/n : n \in \mathbb{N}\}$ and $\Sigma_0 := \{0\} \cup \Sigma$. The Cesàro matrix $C$, when acting in $C^N$, is similar to the diagonal matrix diag($\frac{1}{n^2}$). Indeed, $C = \Delta \text{diag}((\frac{1}{n^2})) \Delta$ with $\Delta = \Delta^{-1} = (\Delta_{nk})_{n,k \in \mathbb{N}} \in \mathcal{L}(C^N)$ the lower triangular matrix where, for each $n \in \mathbb{N}$, $\Delta_{nk} = (-1)^{k-1}(n-1)/2k$ for $1 \leq k < n$ and $\Delta_{nk} = 0$ if $k > n$, [13, pp. 247-249]. Thus $\sigma_{pl}(C; C^N) = \Sigma$ and each eigenvalue $\frac{1}{n}$ has multiplicity 1 with eigenvector $\Delta e_n$, where $e_n := (\delta_{nk})_{k \in \mathbb{N}}$, for $n \in \mathbb{N}$, are the canonical basis vectors in $C^N$. Moreover, $\lambda I - C$ is invertible for each $\lambda \in \mathbb{C} \setminus \Sigma$. If $X$ is a lcHs continuously contained in $C^N$ and $C(X) \subseteq X$, then

$$\sigma_{pl}(C; X) = \{1/n : n \in \mathbb{N}, \ \Delta e_n \in X\} \subseteq \Sigma. \quad (2.2)$$

In case the space $\varphi$ (of all finitely supported vectors in $C^N$) is densely contained in $X$, then $\varphi \subseteq X'$ and $\Sigma \subseteq \sigma_{pl}(C; X') \subseteq \sigma(C; X')$, where $C'$ is the dual operator of $C$. Observe that always $\Delta e_1 = 1 := (1)_{n \in \mathbb{N}} \in c_0(v_1) \subseteq E_\alpha$ whenever $\alpha_n \uparrow \infty$. Since $\varphi$ is dense in $E_\alpha$ for every $\alpha$ with $\alpha_n \uparrow \infty$, we conclude that always

$$1 \in \sigma_{pl}(C; E_\alpha) \subseteq \sigma(C; E_\alpha). \quad (2.3)$$
We point out that C does not act in the vector space \( \varphi := \text{ind}_k \mathbb{C}^k \subseteq \mathbb{C}^N \) because \( \epsilon_1 \notin \varphi \) but \( C_{\epsilon_1} = \left( \frac{1}{n} \right) \notin \varphi \).

**Proposition 2.3.** For \( \alpha \) with \( \alpha_n \uparrow \infty \) the following assertions are equivalent.

(i) \( E_\alpha \) is nuclear.

(ii) \( \sup_{n \in \mathbb{N}} \log n \frac{\log n}{\alpha_n} < \infty. \)

(iii) \( \sigma_{pt}(C; E_\alpha) = \Sigma. \)

(iv) \( \sigma_{pt}(C; E_\alpha) \setminus \{1\} \neq \emptyset. \)

**Proof.** (i) \( \iff \) (ii). See the introduction to this section.

(ii) \( \Rightarrow \) (iii). Observe that \( \Delta e_m \), for fixed \( m \in \mathbb{N} \), behaves asymptotically like \( (n^{m-1})_{n \in \mathbb{N}}, i.e., \lfloor (\Delta e_m) \rfloor \approx n^{m-1} \) for \( n \to \infty. \) By Lemma 2.2 each \( \Delta e_m \in E_\alpha \) for \( m \in \mathbb{N}. \) Hence, (2.2) yields that \( \sigma_{pt}(C; E_\alpha) = \Sigma. \)

(iii) \( \Rightarrow \) (iv). Obvious.

(iv) \( \Rightarrow \) (ii). For this proof select \( v_k(n) := e^{-k\alpha_n}, n \in \mathbb{N}, \) for each \( k \in \mathbb{N}. \)

By (2.3) and the assumption (iv) there exists \( m \in \mathbb{N} \) with \( m > 1 \) such that \( \frac{k}{\alpha_n} \in \sigma_{pt}(C; E_\alpha), i.e., \Delta e_m \in E_\alpha. \) As seen in the proof of (ii) \( \Rightarrow \) (iii) we then have \( (n^{m-1})_{n \in \mathbb{N}} \in E_\alpha. \) Hence, for some \( k \in \mathbb{N}, (n^{m-1})_{n \in \mathbb{N}} \in c_0(v_k) \) and so there exists \( M > 1 \) such that \( n^{m-1}v_k(n) = n^{m-1}e^{-k\alpha_n} \leq M \) for all \( n \in \mathbb{N}. \) It follows from Lemma 2.2 that (ii) holds. \( \square \)

**Proposition 2.4.** Let \( \alpha_n \uparrow \infty. \) The following conditions are equivalent.

(i) \( \sup_{n \in \mathbb{N}} \log n \frac{\log n}{\alpha_n} < \infty, i.e., E_\alpha \) is nuclear.

(ii) \( C \in \mathcal{L}(E_\alpha) \) is invertible, i.e., \( 0 \notin \rho(C; E_\alpha). \)

**Proof.** Note that \( C: \mathbb{C}^N \to \mathbb{C}^N \) is bijective with inverse \( C^{-1}: \mathbb{C}^N \to \mathbb{C}^N \) given by

\[
C^{-1}y = (n y_n - (n-1) y_{n-1}), \quad y = (y_n) \in \mathbb{C}^N,
\]

with \( y_0 := 0. \) Accordingly, \( 0 \notin \sigma(C; E_\alpha) \) if and only if \( C^{-1}: E_\alpha \to E_\alpha \) is continuous if and only if for each \( k \in \mathbb{N} \) there exists \( l \geq k \) such that \( C^{-1}: c_0(v_k) \to c_0(v_l) \) is continuous.

For the rest of the proof we select \( v_k(n) := e^{-k\alpha_n} \) for \( k, n \in \mathbb{N}, i.e., s_k := e^k. \)

(i) \( \Rightarrow \) (ii). By Lemma 2.2 there exists \( m \in \mathbb{N} \) with \( D := \sup_{n \in \mathbb{N}} ne^{-m\alpha_n} < \infty. \)

Fix \( k \in \mathbb{N} \) and set \( l := m + k. \) Let \( y = (y_n) \in c_0(v_k). \) For each \( n \in \mathbb{N} \), we have

\[
|v_l(n)|(C^{-1}y) = e^{-l\alpha_n}|n y_n - (n-1) y_{n-1}| \leq e^{-l\alpha_n}n|y_n| + e^{-l\alpha_n}(n-1)|y_{n-1}| \leq D(e^{-k\alpha_n}|y_n| + e^{-k\alpha_n-1}|y_{n-1}|) \leq 2Dq_k(y).
\]

Forming the supremum relative to \( n \in \mathbb{N} \) yields \( q(C^{-1}y) \leq 2Dq_k(y) \) for all \( y \in c_0(v_k). \) Accordingly, \( C^{-1}: c_0(v_k) \to c_0(v_l) \) is continuous. Since \( k \in \mathbb{N} \) is arbitrary, it follows that \( C^{-1}: E_\alpha \to E_\alpha \) is continuous and so \( 0 \notin \rho(C; E_\alpha). \)

(ii) \( \Rightarrow \) (i). By assumption \( C^{-1}: E_\alpha \to E_\alpha \) is continuous. So, there exists \( l \in \mathbb{N} \) such that \( C^{-1}: c_0(v_l) \to c_0(v_l) \) is continuous, that is, there exists \( D > 1 \) such that \( q_l(C^{-1}y) \leq Dq_l(y) \) for all \( y \in c_0(v_l). \) Since \( C^{-1}e_n = ne_n - ne_{n+1} \) and \( q_l(C^{-1}e_n) = \max\{n y_l(n), n y_l(n+1)\} = n y_l(n) = e^{-l\alpha_n}, \) with \( q_l(e_n) = y_l(n) = e^{-\alpha_n}, \) for all \( n \in \mathbb{N}, \) it follows that \( ne^{-l\alpha_n} \leq De^{-\alpha_n}, \) for \( n \in \mathbb{N}. \) Hence, \( ne^{(1-l)\alpha_n} \leq D, \) for \( n \in \mathbb{N}, \) which implies that \( \sup_{n \in \mathbb{N}} \log n \frac{\log n}{\alpha_n} < \infty. \) \( \square \)

The operator of differentiation \( D \) acts on \( \mathbb{C}^N \) via

\[
D(x_1, x_2, x_3, \ldots) := (x_2, 2x_3, 3x_4, \ldots), \quad x = (x_n) \in \mathbb{C}^N.
\]
Clearly $D \in \mathcal{L}(\mathbb{C}^N)$. According to (2.4) and a routine calculation the inverse operator $C^{-1} \in \mathcal{L}(\mathbb{C}^N)$ is given by

$$C^{-1} = (I - S_r)DS_r,$$  \hspace{1cm} (2.5)

where $S_r \in \mathcal{L}(\mathbb{C}^N)$ is the right-shift operator, i.e., $S_rx := (0, x_1, x_2, \ldots)$ for $x \in \mathbb{C}^N$. Fix $k \in \mathbb{N}$. Since $v_k$ is decreasing on $\mathbb{N}$, it follows that

$$q_k(S_rx) := \sup_{n \in \mathbb{N}} v_k(n + 1)|x_n| \leq \sup_{n \in \mathbb{N}} v_k(n)|x_n| = q_k(x), \quad x \in c_0(v_k).$$

Hence, $S_r : c_0(v_k) \to c_0(v_k)$ is continuous for each $k \in \mathbb{N}$ which implies (for every $\alpha_n \uparrow \infty$) that $S_r \in \mathcal{L}(E_{\alpha})$. Moreover, Proposition 2.4 shows that $C^{-1} \in \mathcal{L}(E_{\alpha})$ if and only if $E_{\alpha}$ is nuclear. The identity (2.5) suggests there should be a connection between the nuclearity of $E_{\alpha}$ and the continuity of $D$ on $E_{\alpha}$. The following result addresses this point. Recall that $E_{\alpha}$ is shift stable if $\lim_{n \to \infty} \frac{\alpha_{n+1}}{\alpha_n} < \infty$, [23].

**Proposition 2.5.** For $\alpha$ with $\alpha_n \uparrow \infty$ the following assertions are equivalent.

(i) $D(E_{\alpha}) \subseteq E_{\alpha}$, i.e., $D$ acts in $E_{\alpha}$.

(ii) The differentiation operator $D \in \mathcal{L}(E_{\alpha})$.

(iii) For every $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ with $l > k$ such that $D : c_0(v_k) \to c_0(v_l)$ is continuous.

(iv) For every $k \in \mathbb{N}$ there exist $l \in \mathbb{N}$ with $l > k$ and $M > 0$ such that

$$nv_l(n) \leq Mv_k(n + 1), \quad n \in \mathbb{N}.$$  

(v) The space $E_{\alpha}$ is both nuclear and shift stable.

**Proof.** (i)$\Leftrightarrow$(ii) is immediate from the closed graph theorem for $(\text{LB})$-spaces, [17, Theorem 24.31 and Remark 24.36].

(ii)$\Leftrightarrow$(iii) is a general fact about continuous linear operators between $(\text{LB})$-spaces.

(iii)$\Rightarrow$(iv). Fix $k \in \mathbb{N}$. By (iii) there exists $l \in \mathbb{N}$ with $l > k$ such that $D : c_0(v_k) \to c_0(v_l)$ is continuous. Hence, there is $M > 0$ satisfying

$$q_l(Dx) = \sup_{n \in \mathbb{N}} v_l(n)|(Dx)| \leq Mq_k(x) = M \sup_{n \in \mathbb{N}} v_k(n)|x_n|, \quad x \in c_0(v_k).$$

For each $j \in \mathbb{N}$ with $j \geq 2$ substitute $x := e_j$ in the previous inequality (noting that $Dx = D(e_j) = (j - 1)e_{j-1}$) yields $(j - 1)v_l(j - 1) \leq Mv_k(j)$. Since $j \geq 2$ is arbitrary, this is precisely (iv).

(iv)$\Rightarrow$(iii). Given any $k \in \mathbb{N}$ select $l > k$ and $M > 0$ which satisfy (iv). Fix $x \in c_0(v_k)$. Then, for each $n \in \mathbb{N}$, we have via (iv) that

$$v_l(n)|(Dx)| = nv_l(n)|x_{n+1}| \leq Mv_k(n + 1).$$

Forming the supremum relative to $n \in \mathbb{N}$ of both sides of this inequality yields

$$q_l(Dx) \leq Mq_k(x), \quad x \in c_0(v_k),$$

which is precisely (iii).

(iv)$\Rightarrow$(v). For $k = 1$, condition (iv) ensures the existence of $l > 1$ and $M > 1$ such that

$$nv_1(n) \leq Mv_1(n + 1) \leq Mv_1(n), \quad n \in \mathbb{N}. \hspace{1cm} (2.6)$$

For the remainder of the proof of this proposition, choose $s_k := e_k$ for $k \in \mathbb{N}$. It follows from (2.6) that $ne^{-\alpha_n} \leq Me^{-\alpha_n}$ for all $n \in \mathbb{N}$. By Lemma 2.2 one can conclude that $E_{\alpha}$ is nuclear.
To prove that $E_α$ is shift stable observe that the left-inequality in (2.6) is $ne^{-λα_n} ≤ Me^{-α_{n+1}}$ for $n ∈ N$. Taking logarithms and rearranging yields

$$\frac{α_{n+1}}{α_n} ≤ l + \frac{\log(M)}{α_n} - \frac{\log(n)}{α_n}, \quad n ∈ N.$$ 

Since $\sup_{n \in N} \frac{\log(n)}{α_n} < ∞$ (as $E_α$ is nuclear) and $\sup_{n \in N} \frac{\log(M)}{α_n} < ∞$ it follows that $\sup_{n \in N} \frac{α_{n+1}}{α_n} < ∞$, i.e., $E_α$ is shift-stable.

(v)⇒(iv). Fix $k ∈ N$. Since $E_α$ is shift stable, there exists $h ∈ N$ such that $α_{n+1} ≤ hα_n$ for $n ∈ N$. Because of the nuclearity of $E_α$, Lemma 2.2 implies the existence of $M ∈ N$ which satisfies $L := \sup_{n \in N} ne^{-Μα_n} < ∞$. Set $l := M + hk$. Then $l ∈ N$ and, for each $n ∈ N$, it follows that

$$nv_l(n) = ne^{-λα_n} = ne^{-Mα_n} e^{-hkα_n} ≤ Le^{-k(hα_n)} ≤ Le^{-kα_{n+1}} = Lv_k(n+1).$$

This is precisely condition (iv). □

**Remark 2.6.** (i) There exist nuclear spaces $E_α$ for which $D$ is not continuous on $E_α$. Let $α_n := n^α$ for $n ∈ N$. Then $E_α$ is nuclear but, not shift stable. Proposition 2.5 implies that $D ∉ L(E_α)$. On the other hand, for $α_n := \log(\log(n))$ for $n ≥ 3$, the space $E_α$ is shift stable but, not nuclear; again $D ∉ L(E_α)$.

(ii) Because $v_1 \downarrow 0$, it is clear that $\ell_∞ ≤ \ell_∞(v_1) \subseteq E_α := \text{ind}_k \ell_∞(v_k)$ for every $α$ with $α_n ↑ ∞$. Accordingly, if $x_α := (λ^{n-1})_{n∈N}$ for $λ ∈ C$, then clearly $\{x_λ : λ ∈ C\} ⊆ E_α$ and so $\{x_λ : λ ∈ C\} ⊆ E_α$. Since $Dx_λ = λx_λ$ for each $λ ∈ C$, we have established (via Proposition 2.5) the following fact.

Let $α$ with $α_∞ ↑ ∞$ be a sequence such that $E_α$ is both nuclear and shift stable. Then $D ∈ L(E_α)$ and

$$σ_{pt}(D; E_α) = σ(D; E_α) = σ^+(D; E_α) = C.$$ 

In order to determine $σ(C; E_α)$ we require some further preliminaries. Define the continuous function $α : C \setminus \{0\} → R$ by $a(z) := \text{Re}(\frac{1}{z})$ for $z ∈ C \setminus \{0\}$. The following result is a refinement of [19, Lemma 7].

**Lemma 2.7.** Let $λ ∈ C \setminus Σ_0$. Then there exists $δ = δ_λ > 0$ and positive constants $d_δ, D_δ$ such that $B(λ, δ) ∩ Σ_0 = 0$ and

$$\frac{d_δ}{N^{α(μ)}} ≤ \prod_{n=1}^{N} \left|1 - \frac{1}{μ_n}\right| ≤ D_δ \frac{N^{α(μ)}}{N^{α(μ)}}, \quad ∀N ∈ N, \quad μ ∈ B(λ, δ).$$

(2.7)

**Proof.** Fix $λ ∈ C \setminus Σ_0$ and write $\frac{1}{λ} = α + iβ$ with $α, β ∈ R$, i.e., $α = a(λ)$. Observe that

$$1 - \frac{2α}{n} + \frac{(α^2 + β^2)}{n^2} = 1 - \frac{α}{n}^2 + \frac{β^2}{n^2} > 0, \quad n ∈ N.$$ 

Using the inequality $(1 + x) ≤ e^x$ for $x ∈ R$ we conclude that $(1 + x)^{1/2} ≤ e^{x/2}$ for all $x ≥ -1$. In particular, for $x := -\frac{2α}{n} + \frac{(α^2 + β^2)}{n^2}$ it follows that

$$\left(1 - \frac{2α}{n} + \frac{(α^2 + β^2)}{n^2}\right)^{1/2} ≤ \exp\left(-\frac{α}{n} + \frac{(α^2 + β^2)}{2n^2}\right), \quad n ∈ N.$$
Fix $N \in \mathbb{N}$. Since $\sum_{n=1}^{N} \frac{1}{n^2} < 2$, we conclude that
\[
\prod_{n=1}^{N} \left| 1 - \frac{1}{n^2} \lambda \right| = \prod_{n=1}^{N} \left( 1 - \frac{2\alpha}{n} + \frac{(\alpha^2 + \beta^2)}{2n^2} \right)^{1/2}
\leq \exp \left( \sum_{n=1}^{N} -\frac{\alpha}{n} + \frac{(\alpha^2 + \beta^2)}{2n^2} \right) \leq \exp(\alpha^2 + \beta^2) \exp \left( -\alpha \sum_{n=1}^{N} \frac{1}{n} \right)
= \exp \left( \frac{1}{|\lambda|^2} \right) \exp \left( -\alpha \sum_{n=1}^{N} \frac{1}{n} \right).
\]

By considering separately the cases when $\alpha \leq 0$ and $\alpha > 0$ and employing the inequalities
\[
\log(k + 1) \leq \sum_{n=1}^{k} \frac{1}{n} \leq 1 + \log(k), \quad k \in \mathbb{N}, \quad (2.8)
\]
it turns out that
\[
\exp \left( -\alpha \sum_{n=1}^{N} \frac{1}{n} \right) = \frac{e^{[a(\lambda)]}}{N^{a(\lambda)}} \leq \frac{e^{1/|\lambda|}}{N^{a(\lambda)}}.
\]

Accordingly, we have that
\[
\prod_{n=1}^{N} \left| 1 - \frac{1}{n^2} \lambda \right| \leq \exp \left( \frac{1}{|\lambda|^2} + \frac{1}{|\lambda|^2} \right) \frac{e^{1/|\lambda|}}{N^{a(\lambda)}}, \quad N \in \mathbb{N}. \quad (2.9)
\]

From above, for each $n \in \mathbb{N}$, we have $|1 - \frac{1}{n^2}|^{-1} = (1 + x_n)^{-1/2}$, where $x_n := -\frac{2\alpha}{n} + \frac{(\alpha^2 + \beta^2)}{n^2}$ satisfies $x_n > -1$. Applying Taylor’s formula to the function $f(x) = (1 + x)^{-1/2}$ for $x > -1$ yields, for each $n \in \mathbb{N}$, that
\[
(1 + x_n)^{-1/2} = f(0) + f'(0)x_n + \frac{f''(\theta_n x_n)}{2!} x_n^2
\]
\[
= 1 - \frac{1}{2} x_n + \frac{3}{4}(1 + \theta_n x_n)^{-5/2} x_n^2
\]
for some $\theta_n \in (0, 1)$. Substituting for $x_n$ its definition and rearranging we get
\[
(1 + x_n)^{-1/2} = 1 + \frac{\alpha}{n} - \frac{(\alpha^2 + \beta^2)}{2n^2} + \frac{3}{4}(1 - \theta_n + \theta_n |1 - \frac{1}{n^2}|)^{-5/2} \left( -\frac{2\alpha}{n} + \frac{(\alpha^2 + \beta^2)}{n^2} \right)^2,
\]
for each $n \in \mathbb{N}$. Defining $d(\lambda) := \text{dist}(\lambda, \Sigma_0) \leq |\lambda|$ we have
\[
|1 - \frac{1}{n^2}| = |\lambda - \frac{1}{n}| \geq \frac{d(\lambda)}{|\lambda|}, \quad n \in \mathbb{N}.
\]
Hence, for each $n \in \mathbb{N}$, it follows that
\[
1 - \theta_n + \theta_n |1 - \frac{1}{n^2}| \geq 1 - \theta_n + \theta_n \frac{d(\lambda)}{|\lambda|} \geq \min \left\{ 1, \frac{d(\lambda)}{|\lambda|} \right\} = \frac{d(\lambda)}{|\lambda|},
\]
where we have used the inequality
\[
1 - x + \gamma x \geq \min \{1, \gamma\}, \quad \forall \gamma \in \mathbb{R}, \ x \in [0, 1].
\]
Accordingly, \((1 - \theta_n + \theta_n \left| 1 - \frac{1}{\lambda_n} \right|)^{-5/2} \leq \left( \frac{\lambda}{d(\lambda)} \right)^{5/2} \), for \(n \in \mathbb{N} \), which implies (see above), for each \(n \in \mathbb{N} \), that

\[
\left| \frac{1}{\lambda_n} \right|^{-1} \leq 1 + \frac{\alpha}{n} + \frac{1}{n^2} \left( -\frac{\alpha^2 + \beta^2}{2} + \frac{3}{4} \left( \frac{|\lambda|}{d(\lambda)} \right)^{5/2} \left( -2\alpha + \frac{(\alpha^2 + \beta^2)}{n} \right) \right)
\]

\[
\leq 1 + \frac{\alpha}{n} + \frac{3}{4n^2} \left( \frac{|\lambda|}{d(\lambda)} \right)^{5/2} (2|\alpha| + \alpha^2 + \beta^2)^2.
\]

But, \((2|\alpha| + \alpha^2 + \beta^2)^2 \leq \left( \frac{2}{|\lambda|} + \frac{1}{|\lambda|} \right)^2 \leq 4 \left( \frac{1}{|\lambda|} + \frac{1}{|\lambda|^2} \right)^2\) and so

\[
\left| \frac{1}{\lambda_n} \right|^{-1} \leq 1 + \frac{\alpha}{n} + \frac{D(\lambda)}{n^2}, \quad n \in \mathbb{N},
\]

with \(D(\lambda) := \frac{3(1+|\lambda|)^2}{|\lambda|^{3/2}(d(\lambda))^{3/2}}\). Accordingly, for fixed \(N \in \mathbb{N}\), we have

\[
\prod_{n=1}^{N} \left| \frac{1}{\lambda_n} \right|^{-1} \leq \prod_{n=1}^{N} \left( 1 + \frac{\alpha}{n} + \frac{D(\lambda)}{n^2} \right) \leq \exp \left( \alpha \sum_{n=1}^{N} \frac{1}{n} \right) \exp \left( D(\lambda) \sum_{n=1}^{N} \frac{1}{n^2} \right)
\]

\[
\leq e^{2D(\lambda)} \exp \left( \alpha \sum_{n=1}^{N} \frac{1}{n} \right).
\]

By considering separately the cases when \(\alpha < 0\) and \(\alpha \geq 0\) and applying (2.8) yields

\[
\exp \left( \alpha \sum_{n=1}^{N} \frac{1}{n} \right) \leq e^{|\alpha|N} \leq e^{|\alpha|} N^{|\alpha|}.
\]

Accordingly, \(\prod_{n=1}^{N} \left| 1 - \frac{1}{\lambda_n} \right|^{-1} \leq N^{|\alpha|} \exp(2D(\lambda) + \frac{1}{|\lambda|})\) and hence,

\[
\frac{\exp(-\frac{1}{|\lambda|} - 2D(\lambda))}{N^{|\alpha|}} \leq \prod_{n=1}^{N} \left| 1 - \frac{1}{\lambda_n} \right|, \quad N \in \mathbb{N}.
\]

(2.10)

It follows from (2.9) and (2.10), for any given \(\lambda \in \mathbb{C} \setminus \Sigma_0\), that

\[
\frac{u(\lambda)}{N^{\alpha(\lambda)}} \leq \prod_{n=1}^{N} \left| 1 - \frac{1}{\lambda_n} \right| \leq \frac{v(\lambda)}{N^{\alpha(\lambda)}}, \quad N \in \mathbb{N},
\]

(2.11)

where \(v(\lambda) := \exp \left( \frac{1}{|\lambda|} + \frac{1}{|\lambda|^2} \right)\) and \(u(\lambda) := \exp \left( -\frac{1}{|\lambda|} - \frac{6(1+|\lambda|)^2}{|\lambda|^{3/2}(d(\lambda))^{3/2}} \right)\).

Fix now a point \(\lambda \in \mathbb{C} \setminus \Sigma_0\) and choose any \(\delta > 0\) satisfying \(B(\lambda, \delta) \cap \Sigma_0 = \emptyset\). According to (2.11) we have

\[
\frac{u(\mu)}{N^{\alpha(\mu)}} \leq \prod_{n=1}^{N} \left| 1 - \frac{1}{\mu_{n}} \right| \leq \frac{v(\mu)}{N^{\alpha(\mu)}}, \quad \forall N \in \mathbb{N}, \mu \in B(\lambda, \delta).
\]

(2.12)

By the continuity (and form) of the functions \(u\) and \(v\) on \(\mathbb{C} \setminus \Sigma_0\) and the compactness of the set \(B(\lambda, \delta) \subseteq (\mathbb{C} \setminus \Sigma_0)\) it follows that \(D_\delta := \sup \{ v(\mu) : \mu \in B(\lambda, \delta) \} < \infty\) and \(d_\delta := \inf \{ u(\mu) : \mu \in B(\lambda, \delta) \} > 0\). It is then clear that (2.4) follows from (2.12). \(\square\)
Lemma 2.8. Let \( w = (w_n) \) be any strictly positive, decreasing sequence. Then
\[
\sigma(C; c_0(w)) \subseteq \{ \lambda \in \mathbb{C} : |\lambda - \frac{1}{2}| \leq \frac{1}{2} \}. \tag{2.13}
\]
Moreover, for each \( \lambda \in \mathbb{C} \) satisfying \( |\lambda - \frac{1}{2}| > \frac{1}{2} \) there exist constants \( \delta_\lambda > 0 \) and \( M_\lambda > 0 \) such that
\[
\|(\mu I - C)^{-1}\|_\text{op} \leq \frac{M_\lambda}{1 - a(\mu)}, \quad \mu \in B(\lambda, \delta_\lambda),
\]
where \( \| : \|_\text{op} \) denotes the operator norm in \( \mathcal{L}(c_0(w)) \).

Proof. According to [4, Corollary 2.3(i)] the Cesàro operator \( C : c_0(w) \to c_0(w) \) is continuous. Then Corollary 3.6 of [4] implies that (2.13) is satisfied.

Set \( A := \{ \lambda \in \mathbb{C} : |\lambda - \frac{1}{2}| \leq \frac{1}{2} \} \) and fix \( \lambda \in \mathbb{C} \setminus A \). Define \( \delta_\lambda := \frac{1}{2} \text{dist}(\lambda, A) > 0 \) and \( C_\lambda := B(\lambda, \delta) \), in which case (2.13) implies that \( \text{dist}(C_\lambda, \sigma(C; c_0(w))) \geq \text{dist}(C_\lambda, A) = \delta_\lambda \). According to Lemma 6.11 of [10, p. 590] there is a constant \( K > 0 \) such that (setting \( \varepsilon := \delta_\lambda \) in that lemma)
\[
\|(\mu I - C)^{-1}\|_\text{op} \leq \frac{K}{\delta_\lambda}, \quad \mu \in C_\lambda.
\] (2.14)

Now, each \( \mu \in B(\lambda, \delta_\lambda) \) satisfies \( a(\mu) < 1, [4, \text{Remark 3.5}] \), and so
\[
\frac{K}{\delta_\lambda} = \frac{K\delta_\lambda^{-1}(1 - a(\mu))}{1 - a(\mu)} \leq \frac{K\delta_\lambda^{-1}(1 + \frac{1}{|\mu|})}{1 - a(\mu)} \leq \frac{M_\lambda}{1 - a(\mu)}, \tag{2.15}
\]
where \( M_\lambda := \sup \{ \frac{K}{\delta_\lambda}(1 + \frac{1}{|z|}) : z \in C_\lambda \} < \infty \) as the set \( C_\lambda \subseteq (\mathbb{C} \setminus \{0\}) \) is compact and the function \( z \mapsto \frac{K}{\delta_\lambda}(1 + \frac{1}{|z|}) \) is continuous on \( \mathbb{C} \setminus \{0\} \). The desired inequality follows from (2.14) and (2.15).

Recall that a Hausdorff inductive limit \( E = \text{ind}_k E_k \) of Banach spaces is called \textit{regular} if every \( B \in \mathbb{B}(E) \) is contained and bounded in some step \( E_k \). In particular, for every \( \alpha \) with \( \alpha_n \uparrow \infty \) the space \( E_\alpha = \text{ind}_k c_0(v_k) \) is regular, [17, Proposition 25.19].

Proposition 2.9. Let \( \alpha \) satisfy \( \alpha_n \uparrow \infty \) with \( E_\alpha \) nuclear. Then

(i) \( \sigma(C; E_\alpha) = \sigma_{pd}(C; E_\alpha) = \Sigma \), and

(ii) \( \sigma^*(C; E_\alpha) = \sigma(C; E_\alpha) \cup \{0\} = \Sigma_0 \).

Proof. By Proposition 2.3 we have \( \Sigma = \sigma_{pd}(C; E_\alpha) \subseteq \sigma(C; E_\alpha) \) and hence,
\[
\Sigma_0 = \sigma(C; E_\alpha) \subseteq \sigma^*(C; E_\alpha) \subseteq \sigma^*(C; E_\alpha).
\]
Moreover, Proposition 2.4 yields \( 0 \notin \sigma(C; E_\alpha) \). So, it remains to show that \( (\mathbb{C} \setminus \Sigma_0) \subseteq \rho^*(C; E_\alpha) \). To this end, we need to show, for each \( \lambda \in \mathbb{C} \setminus \Sigma_0 \), that there exists \( \delta > 0 \) with the property that \( (C - \mu I)^{-1} : E_\alpha \to E_\alpha \) is continuous for each \( \mu \in B(\lambda, \delta) \) and the set \( \{ (C - \mu I)^{-1} : \mu \in B(\lambda, \delta) \} \) is equicontinuous in \( \mathcal{L}(E_\alpha) \). We recall that \( (C - \mu I)^{-1} : \mathbb{C}^N \to \mathbb{C}^N \) exists in \( \mathcal{L}(\mathbb{C}^N) \) for each \( \mu \in \mathbb{C} \setminus \Sigma \).

For this proof we select the weights \( v_k(n) = e^{-\kappa_n}, n \in \mathbb{N}, \) for each \( k \in \mathbb{N} \). Fix \( \lambda \in \mathbb{C} \setminus \Sigma_0 \). First, choose \( \delta_1 > 0 \) such that \( B(\lambda, \delta_1) \cap \Sigma_0 = \emptyset \). Later \( \delta > 0 \) will be selected in such a way that \( 0 < \delta < \delta_1 \).
According to Lemma 5.4 in the Appendix it suffices to find a $\delta > 0$ satisfying the following condition: for each $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ with $l \geq k$ and $D_k > 0$ such that

$$q_l((C - \mu I)^{-1} x) \leq D_k q_k(x), \quad \forall \mu \in B(\lambda, \delta), \ x \in c_0(v_k). \quad (2.16)$$

**Case (i).** Suppose that $|\lambda - \frac{1}{2}| > \frac{1}{2}$ (equivalently, $a(\lambda) < 1$, [4, Remark 3.5]). To establish the condition (2.16) we proceed as follows. Fix $k \in \mathbb{N}$. Since $a(\lambda) < 1$, we can select $\varepsilon > 0$ such that $a(\lambda) < 1 - \varepsilon$. By continuity of the function $a: \mathbb{C} \setminus \{0\} \to \mathbb{R}$ there exists $\delta_2 \in (0, \delta_1)$ such that $a(\mu) < 1 - \varepsilon$ for all $\mu \in B(\lambda, \delta_2)$.

Applying Lemma 2.8 (with $v_k$ in place of $w$), it follows that there exist $\delta \in (0, \delta_2)$ and $M_{k, \lambda} > 0$ satisfying

$$q_k((C - \mu I)^{-1} x) \leq \frac{M_{k, \lambda}}{1 - a(\mu)} q_k(x) \leq \frac{M_{k, \lambda}}{1 - \varepsilon} q_k(x)$$

for all $\mu \in B(\lambda, \delta)$ and $x \in c_0(v_k)$. So, inequality (2.16) is then satisfied with $l := k$ and $D_k := \frac{M_{k, \lambda}}{\varepsilon}$. Since $k \in \mathbb{N}$ is arbitrary, condition (2.16) holds.

**Case (ii).** Suppose now that $|\lambda - \frac{1}{2}| \leq \frac{1}{2}$ (equivalently, $a(\lambda) \geq 1$, [4, Remark 3.5]). We recall the formula for the inverse operator $(C - \mu I)^{-1}: \mathbb{C}^\mathbb{N} \to \mathbb{C}^\mathbb{N}$ whenever $\mu \not\in \Sigma_0, [19, \text{p. 266}].$ For $n \in \mathbb{N}$ the $n$-th row of the matrix for $(C - \mu I)^{-1}$ has the entries

$$-\frac{1}{n \mu^2} \prod_{k=m}^{n} \left(1 - \frac{1}{m_k}\right), \quad 1 \leq m < n,$$

$$\frac{n}{1 - n \mu} = \frac{1}{n - \mu}, \quad m = n,$$

and all the other entries in row $n$ are equal to $0$. So, we can write

$$(C - \mu I)^{-1} = D_{\mu} - \frac{1}{n^2} E_{\mu}, \quad \mu \in \mathbb{C} \setminus \Sigma_0, \quad (2.17)$$

where the diagonal operator $D_{\mu} = (d_{nm}(\mu))_{n, m \in \mathbb{N}}$ is given by $d_{nm}(\mu) := \frac{1}{n - \mu}$ and $d_{mn}(\mu) := 0$ if $n \neq m$. The operator $E_{\mu} = (e_{nm}(\mu))_{n, m \in \mathbb{N}}$ is then the lower triangular matrix with $e_{1m}(\mu) = 0$ for all $m \in \mathbb{N}$, and for every $n \geq 2$ with $e_{nm}(\mu) := \frac{1}{n \prod_{k=m}^{n - 1} (1 - \frac{1}{m_k})}$ if $1 \leq m < n$ and $e_{nm}(\mu) := 0$ if $m \geq n$.

Since $d_0(\lambda) := \text{dist}(B(\lambda, \delta_1), \Sigma_0) > 0$, we have $|d_{mn}(\mu)| \leq \frac{1}{d_0(\lambda)}$ for all $\mu \in B(\lambda, \delta_1)$ and $n \in \mathbb{N}$. Fix $k \in \mathbb{N}$. Then, for every $x \in c_0(v_k)$ and $\mu \in B(\lambda, \delta_1)$, we have

$$q_k(D_{\mu}(x)) = \sup_{n \in \mathbb{N}} |d_{mn}(\mu)x_n|v_k(n) \leq \frac{1}{d_0(\lambda)} \sup_{n \in \mathbb{N}} |x_n|v_k(n) = \frac{1}{d_0(\lambda)} q_k(x).$$

So, $\{D_{\mu} : \mu \in B(\lambda, \delta_1)\} \subseteq L(c_0(v_k))$. Moreover, for every $l \in \mathbb{N}$ with $l \geq k$ it follows that

$$q_l(D_{\mu}(x)) \leq q_k(D_{\mu}(x)) \leq \frac{1}{d_0(\lambda)} q_k(x), \quad \forall x \in c_0(v_k), \ \mu \in B(\lambda, \delta_1). \quad (2.18)$$

Via (2.17) it remains to investigate the operator $E_{\mu}: E_{\alpha} \to E_{\alpha}$ in order to show the validity of condition (2.16) for $(C - \mu I)^{-1}$. To this end we first observe, for each $k \in \mathbb{N}$, that $c_0(v_k)$ is isometrically isomorphic to $c_0$ via the linear multiplication operator $\Phi_k: c_0(v_k) \to c_0$ given by $\Phi_k(x) := (v_k(n)x_n)$, for $x = (x_n) \in c_0(v_k)$. Of
course, each $\Phi_k$ is also a bicontinuous isomorphism of $\mathbb{C}^N$ onto $\mathbb{C}^N$. So, it suffices to show, for every $k \in \mathbb{N}$, that there exist $l \in \mathbb{N}$ with $l \geq k$ and $D_k > 0$ such that $\|\Phi_l E_\mu \Phi_k^{-1} x\|_0 \leq D_k \|x\|_0$ for all $x \in c_0$ and $\mu \in \overline{B}(\lambda, \delta_1)$; here $\|\cdot\|_0$ denotes the usual norm of $c_0$. For each $k, l \in \mathbb{N}$ with $l \geq k$, define $\tilde{E}_{\mu, k, l} := \Phi_l E_\mu \Phi_k^{-1} \in \mathcal{L}(\mathbb{C}^N)$, for $\mu \in \mathbb{C} \setminus \Sigma_0$.

Fix $k \in \mathbb{N}$. For each $l \geq k$ the operator $\tilde{E}_{\mu, k, l}$, for $\mu \in B(\lambda, \delta_1)$, is the restriction to $c_0$ of

$$
\tilde{E}_{\mu, k, l}(x) = ((\tilde{E}_{\mu, k, l}(x))) = \left(v_l(n) \sum_{m=1}^{n-1} e_{nm}(\mu) x_m, \ x = (x_n) \in \mathbb{C}^N,
$$

with $(\tilde{E}_{\mu, k, l}(x))_1 := 0$. Moreover, observe that $\tilde{E}_{\mu, k, l} = (\tilde{e}_{nm}^{k,l}(\mu))_{n, m \in \mathbb{N}}$ is the lower triangular matrix given by $\tilde{e}_{nn}^{k,l}(\mu) = 0$ for $m \in \mathbb{N}$ and $\tilde{e}_{nm}^{k,l}(\mu) = \frac{v_l(n)}{v_k(m)} e_{nm}(\mu)$ for $n \geq 2$ and $1 \leq m < n$.

So, it suffices to verify, for some $l \geq k$ and $\delta > 0$, that $\tilde{E}_{\mu, k, l} \in \mathcal{L}(c_0)$ for $\mu \in B(\lambda, \delta)$ and $\tilde{E}_{\mu, k, l}$ is equicontinuous in $\mathcal{L}(c_0)$. To prove this first observe from the definition of $e_{nm}(\mu)$ that Lemma 2.7 implies, for every $l \geq k$, every $m, n \in \mathbb{N}$ and all $\mu \in B(\lambda, \delta_2)$ that

$$
|\tilde{e}_{nm}^{k,l}(\mu)| = \frac{v_l(n)}{v_k(m)} |e_{nm}(\mu)| \leq D'_\lambda n^{a(\mu) - 1} v_l(n) \sum_{m=1}^{n-1} \frac{1}{m^{a(\mu)} v_k(m)}, \quad (2.19)
$$

for some constant $D'_\lambda > 0$ and $\delta_2 \in (0, \delta_1)$. Because the function $a : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{R}$ is continuous, there exists $\delta \in (0, \delta_2)$ such that $a(\lambda) - \frac{1}{2} < a(\mu) < a(\lambda) + \frac{1}{2}$, for all $\mu \in \overline{B}(\lambda, \delta)$. This implies, for each $\mu \in B(\lambda, \delta)$ that $a(\mu) > a(\lambda) - \frac{1}{2} > \frac{1}{2}$; recall that $a(\lambda) \geq 1$. Let $c := \max\{2, a(\lambda) + \frac{1}{2}\}$. According to Lemma 2.2 there exists $t \in \mathbb{N}$ such that $S_\lambda := \sup_{n \in \mathbb{N}} n^c e^{-\lambda n} < \infty$. Set $l := k + t$. By (2.19) and the fact that $\tilde{e}_{nm}^{k,l}(\mu) = 0$ for $1 \leq m < n$, it follows for every $n \in \mathbb{N}$ and $\mu \in B(\lambda, \delta)$ that

$$
\sum_{m=1}^{\infty} |\tilde{e}_{nm}^{k,l}(\mu)| = \sum_{m=1}^{n-1} |\tilde{e}_{nm}^{k,l}(\mu)| \leq D'_\lambda n^{a(\mu) - 1} v_l(n) \sum_{m=1}^{n-1} \frac{1}{m^{a(\mu)} v_k(m)}
$$

$$
= D'_\lambda n^{a(\mu) - 1} e^{-\lambda n} \sum_{m=1}^{n-1} \frac{e^{\lambda m}}{m^{a(\mu)}} \leq D'_\lambda n^{a(\mu) - 1} e^{-\lambda n} \sum_{m=1}^{n-1} e^{\lambda m}
$$

$$
\leq D'_\lambda n^{a(\mu) - 1} e^{-\lambda n} (n - 1) e^{\lambda n} \leq D'_\lambda n^{a(\mu)} e^{(k-t)\alpha n}
$$

$$
= D'_\lambda n^{a(\mu)} e^{-\lambda n} \leq D'_\lambda S_\lambda.
$$

Hence, for every $\mu \in \overline{B(\lambda, \delta)}$, we have the inequality

$$
\sup_{n \in \mathbb{N}} \sum_{m=1}^{\infty} |\tilde{e}_{nm}^{k,l}(\mu)| \leq D'_\lambda S_\lambda,
$$

that is, condition (ii) of Lemma 2.1 in [4] is satisfied for all $\mu \in \overline{B(\lambda, \delta)}$. Moreover, since $n^{a(\mu) - 1} v_l(n) = n^{a(\mu) - 1} e^{-\lambda n} = n^{a(\mu) - 1 - c} e^{-\lambda n} e^{-k \alpha n} \rightarrow 0$ for $n \rightarrow \infty$ (because $S_\lambda := \sup_{n \in \mathbb{N}} n^c e^{-\lambda n} < \infty$, $e^{-k \alpha n} \leq 1$, and $a(\mu) < a(\lambda) + \frac{1}{2} \leq c + 1$), the inequality (2.19) implies for each fixed $\mu \in \overline{B(\lambda, \delta)}$ and $m \in \mathbb{N}$ that

$$
\lim_{n \rightarrow \infty} \tilde{e}_{nm}^{k,l}(\mu) = 0.
$$
Also the condition (i) of Lemma 2.1 in [4] is satisfied, for all \( \mu \in B(\lambda, \delta) \). Accordingly, [4, Lemma 2.1] implies, for every \( \mu \in B(\lambda, \delta) \), that \( \tilde{E}_{\mu,k,l} \in L(c_0) \) with \( \| \tilde{E}_{\mu,k,l} \|_{op} \leq D'(S) \), that is, \( \{ \tilde{E}_{\mu,k,l} : \mu \in B(\lambda, \delta) \} \) is equiconstant in \( L(c_0) \). Finally, in view of (2.18), we have shown that condition (2.16) is indeed satisfied.

**Corollary 2.10.** For \( \alpha \) with \( \alpha_n \uparrow \infty \) the following assertions are equivalent.

(i) \( E_\alpha \) is nuclear.

(ii) \( \sigma(C; E_\alpha) = \sigma_{pt}(C; E_\alpha) \).

(iii) \( \sigma(C; E_\alpha) = \Sigma \).

**Proof.** (i)\(\Rightarrow\)(ii) and (i)\(\Rightarrow\)(iii) are clear from Proposition 2.9(i).

(ii)\(\Rightarrow\)(i). The equality in (ii) together with the fact that \( \sigma_{pt}(C; E_\alpha) \subseteq \Sigma \) (see the discussion prior to Proposition 2.3) implies \( 0 \in \rho(C; E_\alpha) \). Hence, \( E_\alpha \) is nuclear; see Proposition 2.4.

(iii)\(\Rightarrow\)(i). The equality in (iii) implies \( 0 \in \rho(C; E_\alpha) \) and so \( E_\alpha \) is nuclear (cf. Proposition 2.4).

Recall that an operator \( T \in L(X) \), with \( X \) a lChs, is compact (resp. weakly compact) if there exists a neighbourhood \( U \) of 0 such that \( T(U) \) is a relatively compact (resp. relatively weakly compact) subset of \( X \).

**Corollary 2.11.** Let \( \alpha \) satisfy \( \alpha_n \uparrow \infty \) with \( E_\alpha \) nuclear. Then the Cesàro operator \( C \in L(E_\alpha) \) is neither compact nor weakly compact.

**Proof.** Since \( E_\alpha \) is Montel, there is no distinction between \( C \) being compact or weakly compact. So, suppose that \( C \) is compact. Then \( \sigma(C; E_\alpha) \) is necessarily a compact set in \( C \), [11, Theorem 9.10.2], which contradicts Proposition 2.9(i).

The identity \( C = \Delta \text{diag}((\frac{1}{n})) \Delta \) holds in \( L(\mathbb{C}^N) \) and all the three operators \( C, \Delta \) and \( \text{diag}((\frac{1}{n})) \) are continuous; see the discussion prior to Proposition 2.3. For every positive sequence \( \alpha_n \uparrow \infty \) we also have that \( C \in L(E_\alpha) \) (cf. Proposition 2.1) and \( \text{diag}((\frac{1}{n})) \in L(c_0) \) (because \( \text{diag}((\frac{1}{n})) \in L(c_0(v_k)) \) for every \( k \in \mathbb{N} \)). If \( \Delta \) acts in \( E_\alpha \), then \( \Delta \alpha_n \in E_\alpha \) for all \( n \in \mathbb{N} \) and so \( \sigma_{pt}(C; E_\alpha) = \Sigma \); see (2.2). Accordingly, \( E_\alpha \) is necessarily nuclear via Proposition 2.3. However, this condition alone is not sufficient for the continuity of \( \Delta \).

**Proposition 2.12.** For \( \alpha \) with \( \alpha_n \uparrow \infty \) the following assertions are equivalent.

(i) The operator \( \Delta \in L(E_\alpha) \).

(ii) \( \sup_{n \in \mathbb{N}} \frac{n}{\alpha_n} < \infty \).

**Proof.** For each \( k \in \mathbb{N} \), the surjective isometric isomorphism \( \Phi_k : c_0(v_k) \rightarrow c_0 \) was defined in the proof of Proposition 2.9. Because \( E_\alpha = \text{ind}_k c_0(v_k) \) it follows that \( \Delta \in L(E_\alpha) \) if and only if for each \( k \in \mathbb{N} \) there exists \( l \in \mathbb{N} \) with \( l > k \) such that \( \Delta : c_0(v_k) \rightarrow c_0(v_l) \) is continuous. Moreover, the continuity of \( \Delta : c_0(v_k) \rightarrow c_0(v_l) \) is equivalent to continuity of the operator \( D^{k,l} : c_0 \rightarrow c_0 \), where \( D^{k,l} := \Phi_l \Delta \Phi_k^{-1} \).

Note that \( \Phi_l = \text{diag}((v_l(n))) \) and \( \Phi^{-1}_l = \text{diag}((\frac{1}{v_l(n)})) \) are diagonal matrices and \( \Delta = (\Delta_{nm})_{n,m \in \mathbb{N}} \) is a lower triangular matrix, a direct calculation shows that \( D^{k,l} = (d_{nm}^{k,l})_{n,m \in \mathbb{N}} \) is the lower triangular matrix where, for each \( n \in \mathbb{N} \),

\[
d_{nm}^{k,l} = (-1)^{m-1} \frac{v_k(n)}{v_k(m)} (m-1), \quad 1 \leq m < n \quad \text{and} \quad d_{nm}^{k,l} = 0 \quad \text{if} \quad m > n.
\]

It follows
from [20, Theorem 4.51-C] that a matrix $A = (a_{nm})_{n,m \in \mathbb{N}}$ acts continuously on $c_0$ if and only if the matrix $(|a_{nm}|)_{n,m \in \mathbb{N}}$ does so and hence, by the same result in [20], that $\Delta \in \mathcal{L}(E_{\alpha})$ if and only if for each $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ with $l > k$ such that the lower triangular matrix $(|d_{nm}|)_{n,m \in \mathbb{N}}$ satisfies both

$$\lim_{n \to \infty} |d_{nm}| = \lim_{n \to \infty} \frac{v_l(n)}{v_k(m)} \left( \frac{n - 1}{m - 1} \right) = 0, \quad \forall m \in \mathbb{N},$$

and

$$\sup_{n \in \mathbb{N}} \sum_{m=1}^{\infty} |d_{nm}| = \sup_{n \in \mathbb{N}} \sum_{m=1}^{n} \frac{v_l(n)}{v_k(m)} \left( \frac{n - 1}{m - 1} \right) < \infty. \quad (2.21)$$

Actually, (2.21) implies (2.20). Indeed, if (2.21) holds, then there exists $L > 0$ satisfying $v_l(n) \sum_{m=1}^{n} \frac{1}{v_k(m)} \left( \frac{n - 1}{m - 1} \right) \leq L$ for all $n \in \mathbb{N}$ and hence, as $\frac{1}{v_k(m)} = e^{\alpha_m}$, $\alpha_m > 1$ for all $m \in \mathbb{N}$, also $2^{n-1}v_l(n) = v_l(n) \sum_{m=1}^{n} \left( \frac{n - 1}{m - 1} \right) \leq L$ for all $n \in \mathbb{N}$. Then, for fixed $m \in \mathbb{N}$, it follows that

$$n^{m-1}v_l(n) = \frac{n^{m-1}}{2^{m-1}} \cdot 2^{n-1}v_l(n) \leq \frac{L \cdot n^{m-1}}{2^{m-1}}, \quad n \in \mathbb{N}.$$ 

Since $(\frac{n^{m-1}}{2^{m-1}})_{n \in \mathbb{N}}$ is a null sequence and $(\frac{n - 1}{m - 1}) \simeq n^{m-1}$ for $n \to \infty$ the condition (2.20) follows. So, we have established that the continuity of $\Delta : E_{\alpha} \to E_{\alpha}$ is equivalent to the following

**Condition** (δ): For every $k \in \mathbb{N}$ there exists $l > k$ such that (2.21) is satisfied.

(i)⇒(ii). Since Condition (δ) holds, for the choice $k = 1$ there exist $l \in \mathbb{N}$ with $l > 1$ and $M > 1$ such that

$$2^{n-1}v_l(n) = v_l(n) \sum_{m=1}^{n} \left( \frac{n - 1}{m - 1} \right) \leq \sum_{m=1}^{n} \frac{v_l(n)}{v_1(m)} \left( \frac{n - 1}{m - 1} \right) \leq M, \quad n \in \mathbb{N}.$$ 

Hence, $2^{n}v_l(n) \leq 2M$ from which it follows that

$$\exp(n \log(2) - la_n) \leq 2M = \exp(\log(2M)), \quad n \in \mathbb{N}.$$ 

Rearranging this inequality yields

$$\frac{n}{\alpha_n} \leq \frac{l}{\log(2)} + \frac{\log(2M)}{\alpha_n \log(2)}, \quad n \in \mathbb{N}.$$ 

Since $\alpha_n \uparrow \infty$, it follows that $\sup_{n \in \mathbb{N}} \frac{n}{\alpha_n} < \infty$.

(ii)⇒(i). Choose $M \in \mathbb{N}$ such that $n \leq M\alpha_n$ for $n \in \mathbb{N}$. In order to verify Condition (δ) fix $k \in \mathbb{N}$. Then $l := (k + M) \in \mathbb{N}$ and $l > k$. Since $v_k$ is decreasing on $\mathbb{N}$ we have

$$\sum_{m=1}^{n} \frac{v_l(n)}{v_k(m)} \left( \frac{n - 1}{m - 1} \right) \leq \frac{v_l(n)}{v_k(n)} \sum_{m=1}^{n} \left( \frac{n - 1}{m - 1} \right) \leq 2^n \frac{v_l(n)}{v_k(n)}, \quad n \in \mathbb{N}.$$ 

Furthermore, for each $n \in \mathbb{N}$, it is also the case that

$$2^n \frac{v_l(n)}{v_k(n)} = 2^n e^{-\alpha_n(l-k)} = e^{n\log(2)} e^{-M\alpha_n} \leq e^n e^{-M\alpha_n} \leq 1.$$ 

The previous two sets of inequalities imply (2.21) and hence, Condition (δ) is satisfied, i.e., $\Delta \in \mathcal{L}(E_{\alpha})$. \qed
Remark 2.13. (i) Clearly \( \sup_{n \in \mathbb{N}} \frac{\alpha_n}{\alpha_n} < \infty \) implies \( E_\alpha \) is a nuclear space (cf. Proposition 2.4). On the other hand, the sequence \( \alpha_n := \log(n) \), \( n \in \mathbb{N} \), has the property that \( E_\alpha \) is nuclear but, \( \Delta \notin \mathcal{L}(E_\alpha) \) by Proposition 2.12.

(ii) The continuity of the operators \( \Delta \) and \( D \) on \( E_\alpha \) is unrelated. Indeed, consider \( \alpha_n := \sqrt{n} \), for \( n \in \mathbb{N} \). Then \( D \) is continuous because \( E_\alpha \) is both nuclear and shift stable (cf. Proposition 2.5) whereas \( \Delta \) is not continuous (cf. Proposition 2.12). On the other hand, \( \Delta \) is continuous on \( E_\alpha \) for \( \alpha_n := n^n \), \( n \in \mathbb{N} \) (via Proposition 2.12), but \( D \) fails to be continuous on this space; see Remark 2.6.

We end this section with an application. Consider the space of germs of holomorphic functions at 0, namely the regular (LB)-space defined by \( H_0 := \text{ind}_k A\overline{(B(0, \frac{1}{k}))} \). Here, for each \( k \in \mathbb{N} \), \( A\overline{(B(0, \frac{1}{k}))} \) is the disc algebra consisting of all holomorphic functions on the open disc \( B(0, \frac{1}{k}) \subseteq \mathbb{C} \) which have a continuous extension to its closure \( \overline{B(0, \frac{1}{k})} \): it is a Banach algebra for the norm

\[
\|f\|_k := \sup_{|z| \leq \frac{1}{k}} |f(z)| = \sup_{|z| = \frac{1}{k}} |f(z)|, \quad f \in A\overline{(B(0, \frac{1}{k}))}.
\]

It is known that the linking maps \( A\overline{(B(0, \frac{1}{k}))} \to A\overline{(B(0, \frac{1}{k+1}))} \) for \( k \in \mathbb{N} \), which are given by restriction, are injective and absolutely summing. By Köthe duality theory, \( H_0 \) is isomorphic to the strong dual of the nuclear Fréchet space \( H(\mathbb{C}) \). In particular, \( H_0 \) is a (DFN)-space. We refer to [9, Section 2, Example 5] and [14, Ch. 5.27, Sections 3.4] for further information concerning spaces of holomorphic germs and their strong duals. Define \( \alpha = (\alpha_n) \) for \( n \in \mathbb{N} \) in which case \( \lim_{n \to \infty} \frac{\log(n)}{\alpha_n} = 0 \). Then \( H(\mathbb{C}) \) is isomorphic to the power series space \( \Lambda_\infty^1(\alpha) \) of infinite type, [17, Example 29.4(2)], and its strong dual \( E_\alpha \) is isomorphic to \( H_0 \). Indeed, a topological isomorphism of \( H_0 \) onto \( E_\alpha \) is given by the linear map which sends \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) (an element of \( A\overline{(B(0, \frac{1}{k}))} \) for some \( k \in \mathbb{N} \)) to \((a_{n-1})_{n \in \mathbb{N}} \in E_\alpha \). The proof of this (known) fact relies on the following estimates.

(i) If \( f \in A\overline{(B(0, \varepsilon))} \) for some \( 0 < \varepsilon < 1 \) (with \( f(z) = \sum_{n=0}^{\infty} a_n z^n \)), then the Cauchy estimates for \( f \) imply \( |a_n| \leq \frac{1}{\varepsilon^k} \max_{|z| = \varepsilon} |f(z)| \) for \( n \in \mathbb{N}_0 := \{0\} \cup \mathbb{N} \).

Hence, if \( f \in A\overline{(B(0, \frac{1}{k}))} \) for some \( k \in \mathbb{N} \), then

\[
|a_n| \leq k^n \max_{|z| = \frac{1}{k}} |f(z)| = k^n \|f\|_k, \quad n \in \mathbb{N}_0.
\]

(ii) Let \( a := (a_n)_{n \in \mathbb{N}_0} \in \ell_\infty(v_k) \) for some \( k \in \mathbb{N} \), where \( v_k(n) := \frac{1}{(1+k)^n} \) for \( n \in \mathbb{N}_0 \), \( k \in \mathbb{N} \); we have taken here \( s_k := \log(k+1) \). Then \( |a_n| \leq q_k(a) k^n \) for \( n \in \mathbb{N}_0 \) and each fixed \( k \in \mathbb{N} \). Hence, if \( |z| \leq \frac{1}{k} \), then \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) satisfies

\[
|f(z)| \leq \sum_{n=0}^{\infty} |a_n| \cdot |z|^n \leq q_k(a) \sum_{n=0}^{\infty} k^n \frac{1}{(2k)^n} = 2q_k(a).
\]

Accordingly, \( f \in A\overline{(B(0, \frac{1}{2k}))} \).

The above facts, combined with Proposition 2.9 and Corollary 2.11, yield the following result.

Proposition 2.14. The Cesàro operator \( C : H_0 \to H_0 \) is continuous with spectra

\[
\sigma(C, H_0) = \sigma_{pt}(C, H_0) = \Sigma \quad \text{and} \quad \sigma^*(C, H_0) = \Sigma_0.
\]
In particular, $\mathcal{C}$ is not (weakly) compact.

3. The spectrum of $\mathcal{C}$ in the non-nuclear case

The aim of this section is to give a complete description of the spectrum of $\mathcal{C} \in \mathcal{L}(E_\alpha)$ for the case when $E_\alpha$ is not nuclear. It turns out that $\sigma(\mathcal{C}; E_\alpha)$ and $\sigma^*(\mathcal{C}; E_\alpha)$ are dramatically different to that when $E_\alpha$ is nuclear. The following fact, which we record for the sake of explicit reference, is immediate from (2.3) and Propositions 2.3 and 2.4.

**Proposition 3.1.** For $\alpha$ with $\alpha_n \uparrow \infty$ the following assertions are equivalent.

(i) $E_\alpha$ is not nuclear.
(ii) $\sigma_{pl}(\mathcal{C}; E_\alpha) = \{1\}$.
(iii) $0 \in \sigma(\mathcal{C}; E_\alpha)$.

The following general result will be useful in the sequel. For each $r > 0$ we adopt the notation $D(r) := \{\lambda \in \mathbb{C} : |\lambda - \frac{1}{2r}| < \frac{1}{2r}\}$.

**Proposition 3.2.** Let $\alpha$ satisfy $\alpha_n \uparrow \infty$. Then

$$\Sigma \subseteq \sigma(\mathcal{C}; E_\alpha) \subseteq \overline{D(1)}.$$  

**Proof.** Since $\mathcal{C} \in \mathcal{L}(E_\alpha)$, its dual operator $\mathcal{C}'$ is defined, continuous on the strong dual $E'_\alpha = \bigcap_{k \in \mathbb{N}} \ell_1(\frac{1}{n_k}) = \text{proj}_k \ell_1(\frac{1}{n_k})$ of $E_\alpha$ and is given by the formula

$$\mathcal{C}'y := \left( \sum_{j=n}^{\infty} \frac{y_j}{j} \right)_{n \in \mathbb{N}}, \quad y = (y_n) \in E'_\alpha;$$

see (3.7) in [4, p. 774], for example, after noting that $E'_\alpha \subseteq \ell_1(\frac{1}{n_k})$. Given $\lambda \in \Sigma$ there is $m \in \mathbb{N}$ with $\lambda = \frac{1}{m}$. Define $u^{(m)}$ by $u^{(m)}_n := \prod_{k=1}^{n-1} (1 - \frac{1}{m_k})$ for $1 < n \leq m$

(with $u^{(m)}_1 := 1$) and $u^{(m)}_n := 0$ for $n > m$. It is routine to verify that $u^{(m)} \in E'_\alpha$ (as $u^{(m)} \in \varphi$) and $\mathcal{C}'u^{(m)} = \frac{1}{m} u^{(m)}$, i.e., $\lambda \in \sigma_{pl}(\mathcal{C}; E'_\alpha)$.

It follows that $\lambda \in \sigma(\mathcal{C}; E_\alpha)$. Indeed, if not, then $\lambda \in \rho(\mathcal{C}; E_\alpha)$ and so $(\mathcal{C} - \lambda I)(E_\alpha) = E_\alpha$. This implies, for each $z \in E_\alpha$, that there exists $x \in E_\alpha$ satisfying $(\mathcal{C} - \lambda I)x = z$. Hence,

$$\langle z, u^{(m)} \rangle = \langle (\mathcal{C} - \lambda I)x, u^{(m)} \rangle = \langle x, (\mathcal{C}' - \lambda I)u^{(m)} \rangle = 0,$$

that is, $\langle z, u^{(m)} \rangle = 0$ for all $z \in E_\alpha$. Since $u^{(m)} \neq 0$, this is a contradiction. So, $\lambda \in \sigma(\mathcal{C}; E_\alpha)$. This establishes that $\Sigma \subseteq \sigma(\mathcal{C}; E_\alpha)$.

According to Lemma 2.8 we see that $\sigma(\mathcal{C}_k; c_0(v_k)) \subseteq \overline{D(1)}$ for all $k \in \mathbb{N}$, where $\mathcal{C}_k : c_0(v_k) \to c_0(v_k)$ is the restriction of $\mathcal{C} \in \mathcal{L}(\mathbb{C}^m)$. Hence,

$$\bigcap_{m \in \mathbb{N}} \left( \bigcup_{k=m}^{\infty} \sigma(\mathcal{C}_k; c_0(v_k)) \right) \subseteq \overline{D(1)}$$

and so $\sigma(\mathcal{C}; E_\alpha) \subseteq \overline{D(1)}$; see Lemma 5.5 in the Appendix.

The following result identifies a large part of $\sigma(\mathcal{C}; E_\alpha)$.

**Proposition 3.3.** Let $\alpha$ satisfy $\alpha_n \uparrow \infty$ and such that $E_\alpha$ is not nuclear. Then

$$\{0,1\} \cup D(1) \subseteq \sigma(\mathcal{C}; E_\alpha) \subseteq \overline{D(1)}.$$
The Cesàro Operator

Proof. It follows from Propositions 3.1 and 3.2 that $\Sigma_0 \subseteq \sigma(C; E_\alpha) \subseteq \overline{D(1)}$. So, it remains to verify that $(D(1) \setminus \Sigma) \subseteq \sigma(C; E_\alpha)$. This is achieved via a contradiction argument.

Let $\lambda \in D(1) \setminus \Sigma$ and suppose that $\lambda \in \rho(C; E_\alpha)$. Note that $\beta := \text{Re}(\frac{1}{\lambda}) > 1$.

Since $(C - \lambda I)^{-1} : E_\alpha \to E_\alpha$ is continuous, for $k = 1$ there exists $l \in \mathbb{N}$ with $l > 1$ such that $(C - \lambda I)^{-1} : c_0(v_l) \to c_0(v_l)$ is continuous. In the notation of the proof of Proposition 2.9 it follows that the linear map $\widetilde{E}_{\lambda,1,l} : c_0 \to c_0$ is continuous, where $\widetilde{E}_{\lambda,1,l} = (\tilde{e}_{nm}(\lambda))_{n,m \in \mathbb{N}}$ is the lower triangular matrix given by

$$\tilde{e}_{nm}^{1,l}(\lambda) = \frac{v_l(n)}{v_1(m)} e_{nm}(\lambda), \quad \forall n \geq 2, \ 1 \leq m < n, \quad (3.1)$$

and $\tilde{e}_{nm}^{1,l}(\lambda) = 0$ otherwise. Here $e_{n,m}(\lambda) = \frac{1}{n! \prod_{k=m}^{n-1} (1 - \frac{k}{n})}$ if $1 \leq m < n$ and $e_{nm}(\lambda) = 0$ if $m \geq n$. According to the inequality (3.10) in [4, p. 776], there exist positive constants $c, d$ such that

$$\frac{c}{n^{1-\beta}} \leq |e_{n1}(\lambda)| \leq \frac{d}{n^{1-\beta}}, \quad n \geq 2. \quad (3.2)$$

Since $\widetilde{E}_{\lambda,1,l} \in \mathcal{L}(c_0)$, a well known criterion, [4, Lemma 2.1], [20, Theorem 4.51-C], implies that necessarily

$$\lim_{n \to \infty} \tilde{e}_{nm}^{1,l}(\lambda) = 0, \quad m \in \mathbb{N}. \quad (3.3)$$

It now follows from (3.1), the left-inequality in (3.2), and (3.3) with $m = 1$, that

$$\lim_{n \to \infty} n^{\beta-1} e^{-\lambda n} = \lim_{n \to \infty} n^{\beta-1} v_l(n) = 0.$$

Since $\beta > 1$, it follows from Lemma 2.2 that $\sup_{n \in \mathbb{N}} \frac{\log(n)}{\alpha_n} < \infty$ which contradicts the non-nuclearity of $E_\alpha$ (cf. Proposition 2.3). Hence, no $\lambda \in D(1) \setminus \Sigma$ exists with $\lambda \in \rho(C; E_\alpha)$.

We now come to the main result of this section.

Proposition 3.4. Let $\alpha$ satisfy $\alpha_n \uparrow \infty$ and such that $E_\alpha$ is not nuclear.

(i) If $\sup_{n \in \mathbb{N}} \frac{\log(\log(n))}{\alpha_n} < \infty$, then

$$\sigma(C; E_\alpha) = \{0, 1\} \cup D(1) \quad \text{and} \quad \sigma^*(C; E_\alpha) = \overline{D(1)}.$$ 

(ii) If $\sup_{n \in \mathbb{N}} \frac{\log(\log(n))}{\alpha_n} = \infty$, then

$$\sigma(C; E_\alpha) = \overline{D(1)} = \sigma^*(C; E_\alpha).$$

Proof. In the notation of the proof of Proposition 2.9, for each $\lambda \in \mathbb{C} \setminus \Sigma_0$ the inverse operator $(C - \lambda I)^{-1} \in \mathcal{L}(\mathbb{C}^N)$ satisfies

$$(C - \lambda I)^{-1} = D_\lambda - \frac{1}{\lambda^2} E_\lambda;$$

see (2.17). It is also argued there (as a consequence of the fact that the diagonal in $D_\lambda$ is a bounded sequence) that $(C - \lambda I)^{-1} : E_\alpha \to E_\alpha$ is continuous if and only if $E_\lambda \in \mathcal{L}(E_\alpha)$; the nuclearity of $E_\alpha$ is not used for this part of the argument. Moreover, since $E_\alpha$ is an inductive limit, general theory yields that $E_\lambda \in \mathcal{L}(E_\alpha)$ if and only if for each $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ with $l > k$ such that $E_\lambda : c_0(v_l) \to c_0(v_l)$ is continuous. With $\widetilde{E}_{\lambda,k,l} = (\tilde{e}_{nm}(\lambda))_{n,m \in \mathbb{N}}$, where
\( e_{nm}(\lambda) := \frac{v_l(n)}{v_k(m)} e_{nm}(\lambda) \) for \( n, m \in \mathbb{N} \), it follows via the argument used in Case (ii) of the proof of Proposition 2.9 (see also the proof of Proposition 3.3, where \( k = 1 \) can be replaced by an arbitrary \( k \in \mathbb{N} \)) that \( E_\lambda : c_0(v_k) \to c_0(v_l) \) is continuous if and only if \( \tilde{E}_{\lambda,k,l} : c_0 \to c_0 \) is continuous. Via [20, Theorem 4.51-C] this is equivalent to both of the following conditions being satisfied:

\[
\lim_{n \to \infty} |\frac{v_l(n)}{v_k(m)} e_{nm}(\lambda)| = 0, \quad \forall m \in \mathbb{N}, \quad (3.4)
\]

and

\[
\sup_{n \in \mathbb{N}} \sum_{m=1}^{\infty} |\frac{v_l(n)}{v_k(m)} e_{nm}(\lambda)| = \sup_{n \in \mathbb{N}} \sum_{m=1}^{n-1} |\frac{v_l(n)}{v_k(m)} e_{nm}(\lambda)| < \infty. \quad (3.5)
\]

Next, if \( \lambda \notin \{0, 1\} \) belongs to the boundary \( \partial D(1) \) of \( D(1) \), then \( \beta := \text{Re} \left( \frac{1}{\lambda} \right) = 1 \) and \( \lambda \notin \Sigma_0 \). Accordingly, Lemma 3.3 of [4] ensures the existence of positive constants \( c, d \) such that \( c \leq |e_{n1}(\lambda)| \leq d \) for all \( n \in \mathbb{N} \) and

\[
\frac{c}{m} \leq |e_{nm}(\lambda)| \leq \frac{d}{m}, \quad \forall n \in \mathbb{N}, \quad 2 \leq m < n. \quad (3.6)
\]

In order to deduce (3.6) from [4, Lemma 3.3] we have used the formula

\[
|e_{nm}(\lambda)| = \frac{1}{(m-1)!} \left( \alpha \frac{m-1}{m} \right)^{n-1} \prod_{k=1}^{n-1} \frac{1}{1 - \frac{k}{1+m}}, \quad \forall n \in \mathbb{N}, \quad 2 \leq m < n.
\]

Henceforth we use \( v_l(n) := e^{-\alpha n} \) for all \( r, n \in \mathbb{N} \). Note that (3.4) is satisfied for every \( \lambda \in \partial D(1) \setminus \{0, 1\} \). Indeed, for fixed \( m \in \mathbb{N} \), we have via (3.6) that

\[
\frac{v_l(n)}{v_k(m)} |e_{nm}(\lambda)| \leq \frac{de^{\alpha m}}{me^{\alpha n}} \leq \frac{d}{e^{\alpha n}}, \quad n \in \mathbb{N},
\]

from which (3.4) is clear.

(i) Since \( \sup_{n \in \mathbb{N}} \frac{\log(\log(n))}{\alpha n} < \infty \), there exists \( M \in \mathbb{N} \) such that \( \log(\log(n)) \leq M \alpha n \), equivalently \( \log(n) \leq e^{M \alpha n} \) for \( n \in \mathbb{N} \). Fix \( \lambda \in \partial D(1) \setminus \{0, 1\} \); in particular, \( \lambda \notin \Sigma_0 \). Given \( k \in \mathbb{N} \) define \( l := k + M \). Then, for every \( n \geq 2 \), it follows from (2.8), (3.6) and (l - k) = \( M \) that

\[
\sum_{m=1}^{n-1} |\frac{v_l(n)}{v_k(m)} e_{nm}(\lambda)| \leq \frac{d}{e^{\alpha n}} \sum_{m=1}^{n-1} e^{\alpha m} \leq \frac{de^{\alpha n}}{e^{\alpha n}} \sum_{m=1}^{n-1} \frac{1}{m} \leq 1 + \log(n) = e^{M \alpha n} + \log(n) \leq 2.
\]

Accordingly, (3.5) is satisfied. Since (3.4) holds, we conclude that \( \tilde{E}_{\lambda,k,l} : c_0 \to c_0 \) is continuous, equivalently that \( (C - \lambda I)^{-1} \in \mathcal{L}(E_\alpha) \). It follows that \( \partial D(1) \setminus \{0, 1\} \subseteq \rho(C; E_\alpha) \) and so \( \sigma(C; E_\alpha) = \{0, 1\} \cup D(1) \); see Proposition 3.3.

It was shown in the proof of Proposition 3.2 that \( \bigcup_{k=1}^{\infty} \sigma(C_k; c_0(v_k)) \subseteq \overline{D(1)} \). Since \( \sigma(C; E_\alpha) = \{0, 1\} \cup D(1) \), we have \( \overline{\sigma(C; E_\alpha)} = \overline{D(1)} \) and so \( \bigcup_{k=1}^{\infty} \sigma(C_k; c_0(v_k)) \subseteq \sigma(C; E_\alpha) \). It follows from Lemma 5.5(iii) in the Appendix that \( \sigma^c(C; E_\alpha) = \overline{D(1)} \).
(ii) Fix \( \lambda \in \partial D(1) \setminus \{0, 1\} \). Observe first, for \( k = 1 \) and \( l \in \mathbb{N} \) arbitrary, that it follows from (2.8) and (3.6) that
\[
\sum_{m=1}^{n-1} \frac{v_j(n)}{v_k(m)} e_{nm}(\lambda) \geq \frac{c}{\varepsilon^{\alpha_m}} \sum_{m=1}^{n-1} e^{\alpha_m} \sum_{m=1}^{n-1} \frac{1}{m} \geq \frac{c \log(n)}{\varepsilon^{\alpha_m}},
\]
for all \( n \geq 2 \). Suppose now that \( \lambda \in \rho(C; E_\alpha) \). Then for \( k = 1 \) there exists \( l \in \mathbb{N} \) with \( l > 1 \) such that (3.5) is satisfied. It then follows from (3.7) that
\[
\sup_{n \in \mathbb{N}} \frac{\log(n)}{\varepsilon^{\alpha_m}} < \infty.
\]
So, there exists \( K > 1 \) such that \( \log(n) \leq K e^{\alpha_m} \), equivalently that
\[
\log(n) \leq l\alpha_m + \log(K), \quad n \geq 3.
\]
A rearrangement yields \( \frac{\log(n)}{\varepsilon^{\alpha_m}} \leq l + \frac{\log(K)}{\varepsilon^{\alpha_m}} \) for \( n \geq 3 \), and so \( \sup_{n \in \mathbb{N}} \frac{\log(n)}{\varepsilon^{\alpha_m}} < \infty \); contradiction! So, no \( \lambda \in \partial D(1) \setminus \{0, 1\} \) exists which satisfies \( \lambda \in \rho(C; E_\alpha) \), i.e., \( \partial D(1) \setminus \{0, 1\} \subseteq \sigma(C; E_\alpha) \). It now follows from Proposition 3.3 that \( \sigma(C; E_\alpha) = \overline{D(1)} \).

It was observed in the proof of part (i) that \( \bigcup_{n=1}^{\infty} \sigma(C_k; e_0(v_k)) \subseteq \overline{D(1)} \). Since \( \overline{D(1)} = \sigma(C; E_\alpha) = \sigma(C; \overline{E_\alpha}) \), it again follows from Lemma 5.5(iii) in the Appendix that \( \sigma(C; E_\alpha) = \sigma(C; E_\alpha) \).

**Remark 3.5.** (i) Let \( \alpha \) satisfy \( \alpha_n \uparrow \infty \). Then \( \sigma(C; E_\alpha) \) is a compact subset of \( C \) if and only if \( \sup_{n \in \mathbb{N}} \frac{\log(n)}{\varepsilon^{\alpha_n}} \) is finite. This follows from Corollary 2.10, Proposition 3.4 and the fact that the condition \( \sup_{n \in \mathbb{N}} \frac{\log(n)}{\varepsilon^{\alpha_n}} \) is finite implies \( \sup_{n \in \mathbb{N}} \frac{\log(n)}{\varepsilon^{\alpha_n}} = \infty \), i.e., \( E_\alpha \) is automatically non-nuclear.

(ii) The sequence \( \alpha_n := \log(n) \) for \( n \geq 3^3 > e^e \) (with \( 1 < \alpha_1 < \ldots < \alpha_{26} < \log(3^3) \) arbitrary) satisfies \( 1 < \alpha_n \uparrow \infty \) with \( E_\alpha \) not nuclear and \( \sup_{n \in \mathbb{N}} \frac{\log(n)}{\varepsilon^{\alpha_n}} = \infty \). Proposition 3.4(i) shows that \( \sigma(C; E_\alpha) = \{0, 1\} \cup \overline{D(1)} \).

On the other hand, the sequence \( \alpha_n := \log(\log(n)) \) for \( n \geq 3^{27} > e^{e^e} \) (with \( 1 < \alpha_1 < \ldots < \alpha_{3^7-1} < \log(\log(3^{27})) \) arbitrary) satisfies \( 1 < \alpha_n \uparrow \infty \) with \( E_\alpha \) not nuclear and \( \sup_{n \in \mathbb{N}} \frac{\log(n)}{\varepsilon^{\alpha_n}} = \infty \). In this case Proposition 3.4(ii) reveals that \( \sigma(C; E_\alpha) = \overline{D(1)} \).

4. Mean ergodicity of the Cesàro operator.

An operator \( T \in L(X) \), with \( X \) a lcHs, is power bounded if \( \{T^n\}_{n=1}^{\infty} \) is an equicontinuous subset of \( L(X) \). Given \( T \in L(X) \), the averages
\[
T_{[n]} := \frac{1}{n} \sum_{m=1}^{n} T^m, \quad n \in \mathbb{N},
\]
are called the Cesàro means of \( T \). The operator \( T \) is said to be mean ergodic (resp. uniformly mean ergodic) if \( \{T_{[n]}\}_{n=1}^{\infty} \) is a convergent sequence in \( L(X) \) (resp., in \( L_b(X) \)). A relevant text for mean ergodic operators is [15].

**Proposition 4.1.** Let \( \alpha_n \uparrow \infty \). The Cesàro operator \( C \in L(E_\alpha) \) is power bounded and uniformly mean ergodic. In particular,
\[
E_\alpha = \text{Ker}(I - C) \oplus (I - C)(E_\alpha)
\] (4.1)
with
\[ \text{Ker}(I - C) = \{1\} \text{ and } (I - C)(E_0) = \{x \in E_0 : x_1 = 0\} = \text{span}\{e_n\}_{n \geq 2}. \]

**Proof.** Since each weight \( v_k \) for \( k \in \mathbb{N} \) is decreasing, it is known that \( C \in \mathcal{L}(c_0(v_k)) \) and \( q_k(Cx) \leq q_k(x) \) for all \( x \in c_0(v_k) \), [4, Corollary 2.3(i)]. It follows, via (2.1), for every \( k \in \mathbb{N} \) that
\[
q_k(C^m x) \leq q_k(x), \quad \forall x \in c_0(v_k), \ m \in \mathbb{N}.
\]
Accordingly, for each \( k \in \mathbb{N} \), (5.5) is satisfied with \( l := k \) and \( D = 1 \). Then Lemma 5.4 in the Appendix implies that \( \mathcal{H} := \{C^m : m \in \mathbb{N}\} \subseteq \mathcal{L}(E_0) \) is equicontinuous, i.e., the Cesàro operator \( C \) is power bounded in \( E_0 \). Since \( E_0 \) is Montel, it follows via [1, Proposition 2.8] that the Cesàro operator \( C \) is uniformly mean ergodic in \( E_0 \) and hence, (4.1) is also satisfied, [1, Theorem 2.4]. The facts that each \( x \in E_0 \) belongs to \( c_0(v_k) \) for some \( k \in \mathbb{N} \), that the inclusion \( c_0(v_k) \subseteq E_0 \) is continuous and that the canonical vectors \( e_n := (\delta_{nk})_{k \in \mathbb{N}} \) for \( n \in \mathbb{N} \), form a Schauder basis in \( c_0(v_k) \) implies \( \{e_n : n \in \mathbb{N}\} \) is a Schauder basis for \( E_0 \). The proof of the identities in (4.2) now follow by applying the same (algebraic) arguments as used in the proof of [3, Proposition 4.1].

**Proposition 4.2.** Let \( \alpha_n \uparrow \infty \). The sequence \( \{C^m\}_{m \in \mathbb{N}} \) converges in \( L_b(E_0) \) to the projection onto \( \text{span}\{1\} \) along \( (I - C)(E_0) \).

**Proof.** Using Proposition 4.1 we proceed as in the proof of the analogous result when \( C \) acts in the Fréchet space \( \Lambda_0(\alpha) \), [6, Proposition 3.2]. Indeed, for each \( x \in E_0 \), we have that \( x = y + z \) with \( y \in \text{Ker}(I - C) = \text{span}\{1\} \) and \( z \in (I - C)(E_0) = \text{span}\{e_n\}_{n \geq 2} \). So, for each \( m \in \mathbb{N} \) we have \( C^m x = C^m y + C^m z \), with \( C^m y = y \rightarrow y \) in \( E_0 \) as \( m \rightarrow \infty \). The claim is that the sequence \( \{C^m z\}_{m \in \mathbb{N}} \) is also convergent in \( E_0 \). Indeed, proceeding as in the proof of Proposition 3.2 of [6] one shows, for each \( r \geq 2 \) and \( m \in \mathbb{N} \), that \( |(C^m e_r)(n)| \leq \frac{1}{r - 1} a_m \), where \( (a_m)_{m \in \mathbb{N}} \) is a sequence of positive numbers satisfying \( \lim_{m \rightarrow \infty} a_m = 0 \). Since \( v_1(n) |(C^m e_r)(n)| \leq v_1(1) \frac{1}{r - 1} a_m \), for each \( r \geq 2 \) and \( n \in \mathbb{N} \), with \( 1 \geq v_1(1) \geq v_1(n) \) for all \( n \in \mathbb{N} \) it follows that \( q_1(C^m e_r) \leq \frac{1}{r - 1} a_m \). We deduce, for each \( r \geq 2 \), that \( C^m e_r \rightarrow 0 \) in \( c_0(v_1) \) and hence, also in \( E_0 \) as \( m \rightarrow \infty \). Since \( \{C^m\}_{m \in \mathbb{N}} \subseteq \mathcal{L}(E_0) \) is equicontinuous and (by (4.2)) the linear span of \( \{e_n\}_{n \geq 2} \) is dense in \( (I - C)(E_0) \), it follows that \( C^m z \rightarrow 0 \) in \( E_0 \) as \( m \rightarrow \infty \) for each \( z \in (I - C)(E_0) \). So, it has been shown that \( C^m x = C^m y + C^m z \rightarrow y \) in \( E_0 \) as \( m \rightarrow \infty \), for each \( x \in E_0 \), i.e., \( \{C^m\}_{m \in \mathbb{N}} \) converges in \( L_b(E_0) \). Since \( E_0 \) is a Montel space, \( \{C^m\}_{m \in \mathbb{N}} \) also converges in \( L_b(E_0) \).
This establishes one inclusion in (4.3). For the reverse inclusion let \( x \in E_\alpha \). Then \( x \in c_0(v_k) \) for some \( k \in \mathbb{N} \) and hence, \((I - C)x \in (I - C)(c_0(v_k)) = (I - C)(X_k) \subseteq (I - C)(X(\alpha)) \). Thus, the reverse inclusion in (4.3) is also valid.

Because of (4.3) and the containment \((I - C)(E_\alpha) \subseteq (I - C)(E_\alpha) = X(\alpha)\), which is immediate from Proposition 4.1, to show that \((I - C)(E_\alpha)\) is closed in \( E_\alpha \) it suffices to show that the continuous linear restriction operator \((I - C)|_{X(\alpha)} : X_\alpha \to X_\alpha\) is bijective, actually surjective. Indeed, if \((I - C)(X(\alpha)) = X(\alpha)\), then \((I - C)(E_\alpha) = X(\alpha)\) by (4.3) and hence, \((I - C)(E_\alpha)\) is a closed subspace of \( E_\alpha \).

To establish that \((I - C)|_{X(\alpha)}\) is bijective we require the identity \((X(\alpha), \tau) = \text{ind}_k X_k, \) where \( \tau \) is the relative topology in \( X(\alpha) \) induced from \( E_\alpha \). This identity follows from the general fact that if \((E, \tau) = \text{ind}_n E_n\) is a (LB)-space and \( F \subseteq E \) is a closed subspace with finite codimension, then \((F, \tau|_F) = \text{ind}_n(F \cap E_n)\) is also a (LB)-space, [18, Lemma 6.3.1]. Actually, setting \( \tilde{v}_k(n) := v_k(n+1) \) for all \( k, n \in \mathbb{N} \), we have that \( X(\alpha) \) is topologically isomorphic to \( E(\tilde{\alpha}) := \text{ind}_k c_0(\tilde{v}_k) \).

Indeed, the left-shift operator \( S : X(\alpha) \to E(\tilde{\alpha}) \) given by \( S(x) := (x_2, x_3, \ldots) \) for \( x = (x_n)_{n \in \mathbb{N}} \in X(\alpha) \) is such an isomorphism (because, for each \( k \in \mathbb{N}, \) the left shift operator \( S_k : X_k \to c_0(v_k) \) is a surjective isometry). Consider now the operator \( A := S \circ (I - C)|_{X(\alpha)} \circ S^{-1} \in \mathcal{L}(E(\tilde{\alpha})) \). The claim is that \( A \) is bijective with \( A^{-1} \in \mathcal{L}(E(\tilde{\alpha})) \).

To establish the above claim observe, when interpreted to be acting in the space \( \mathbb{C}^N \), that the operator \( A : \mathbb{C}^N \to \mathbb{C}^N \) is bijective (which is a routine verification) and its inverse \( B := A^{-1} : \mathbb{C}^N \to \mathbb{C}^N \) is determined by the lower triangular matrix \( B = (b_{nm})_{n,m \in \mathbb{N}} \) with entries given as follows: for each \( n \in \mathbb{N} \) we have \( b_{nm} = 0 \) if \( m > n \), \( b_{nm} = \frac{n+1}{n} \) if \( m = n \) and \( b_{nm} = \frac{1}{m} \) if \( 1 \leq m < n \). To show that \( B \) is also the inverse of \( A \) acting on \( E(\tilde{\alpha}) \), we only need to verify that \( B \in \mathcal{L}(E(\tilde{\alpha})) \). To establish this it suffices to show, for each \( k \in \mathbb{N}, \) that there exists \( l \geq k \) such that \( \Phi_{v_k} \circ B \circ \Phi_{v_k}^{-1} \in \mathcal{L}(c_0) \), where for each \( h \in \mathbb{N} \) the operator \( \Phi_{v_h} : c_0(\tilde{v}_h) \to c_0 \) given by \( \Phi_{v_h}(x) = (\tilde{v}_h(n+1)x_n) \) for \( x \in c_0(\tilde{v}_h) \) is a surjective isometry. To this end, given \( k \in \mathbb{N} \) set \( l := k+1, \) say. Then the lower triangular matrix corresponding to \( \Phi_{v_l} \circ B \circ \Phi_{v_l}^{-1} \) is given by \( D := (\frac{v_l(n+1)}{v_k(m+1)}b_{nm})_{n,m \in \mathbb{N}}. \) Moreover, for each fixed \( m \in \mathbb{N}, \) we have

\[
\lim_{n \to \infty} \frac{v_l(n+1)}{v_k(m+1)}b_{nm} = \frac{1}{mv_k(m+1)} \lim_{n \to \infty} v_l(n+1) = 0
\]

and, for each \( n \in \mathbb{N}, \) that

\[
\sum_{m=1}^{\infty} \frac{v_l(n+1)}{v_k(m+1)}b_{nm} = \frac{(n+1)}{n} \frac{v_l(n+1)}{v_k(n+1)} + v_l(n+1) \sum_{m=1}^{n-1} \frac{1}{mv_k(m+1)} \leq 2 + (s_l)^{-\alpha_{n+1}} \sum_{m=1}^{n-1} \frac{s_k^\alpha_{m+1}}{m} \leq 2 + \left( \frac{s_k}{s_l} \right)^{\alpha_{n+1}} \sum_{m=1}^{n-1} \frac{1}{m} \leq 2 + \left( \frac{s_k}{s_l} \right)^{\alpha_{n+1}} (1 + \log(n)) \leq 2 + 2a^{\alpha_{n+1}} \log(n+1),
\]

where \( a := \frac{s_k}{s_l} \in (0, 1) \). Since \( E_\alpha \) is nuclear, there exists \( M \geq 1 \) such that \( \log(n) \leq M\alpha_n \) for all \( n \in \mathbb{N} \) and hence, \( a^{\alpha_n} \log(n) \leq M\alpha_n a^{\alpha_n} \) for \( n \in \mathbb{N} \). Since \( f(x) := xa^x, \)

for $x \in (0, \infty)$, satisfies $f'(x) < 0$ for $x > \frac{1}{\log(\frac{1}{x})}$, the function $f$ is decreasing on $(\frac{1}{\log(\frac{1}{x})}, \infty)$ which implies $\sup_{n \in \mathbb{N}} a^{\alpha_n} \log(n) < \infty$, i.e., $\sum_{n=1}^{\infty} \frac{1}{n^{(n+1)}} < \infty$ for each $n \in \mathbb{N}$. Thus, both the conditions (i), (ii) of [4, Lemma 2.1] are satisfied. Accordingly, $\Phi_{\delta} \circ B \circ \Phi_{\nu_k}^{-1} \in \mathcal{L}(c_0)$. The proof that $(I - C)(E_\alpha)$ is closed is thereby complete.

Since $(I - C)(E_\alpha)$ is closed, (4.1) implies $E_\alpha = \text{Ker}(I - C) \oplus (I - C)(E_\alpha)$. The proof of (2) $\Rightarrow$ (5) in Remark 3.6 of [3] then shows that $(I - C)^m(E_\alpha)$ is closed in $E_\alpha$ for all $m \in \mathbb{N}$.

An operator $T \in \mathcal{L}(X)$, with $X$ a separable lch, is called hypercyclic if there exists $x \in X$ such that the orbit \{ $T^n x : n \in \mathbb{N}$ \} is dense in $X$. If, for some $z \in X$ the projective orbit \{ $\lambda T^n z : n \in \mathbb{N}, \lambda \in \mathbb{C}$ \} is dense in $X$, then $T$ is called hypercyclic. Clearly, hypercyclicity implies supercyclicity.

**Proposition 4.4.** Let $\alpha$ satisfy $\alpha_n \uparrow \infty$. Then $C \in \mathcal{L}(E_\alpha)$ is not supercyclic and hence, also not hypercyclic.

**Proof.** It is known that $C$ is not supercyclic in $\mathcal{C}^\mathbb{N}$, [5, Proposition 4.3]. Since $E_\alpha$ is dense (as it contains $\varphi$) and continuously included in $\mathcal{C}^\mathbb{N}$, the supercyclicity of $C$ in any one of the spaces $E_\alpha$ would imply that $C \in \mathcal{L}(\mathcal{C}^\mathbb{N})$ is supercyclic. $\square$

5. **APPENDIX**

In this section we elaborate on the point raised in Section 1 that the behaviour of the Cesàro operator on the strong dual $(\Lambda^0_1(\alpha))'$ of power series spaces $\Lambda^0_1(\alpha)$ of finite type, is not so relevant in relation to continuity. It turns out that $C$ fails to act in $(\Lambda^0_1(\alpha))'$ for every $\alpha$ with $\alpha_n \uparrow \infty$ such that $(\Lambda^0_1(\alpha))'$ is nuclear. Moreover, there exist $\alpha_n \uparrow \infty$ such that $(\Lambda^0_1(\alpha))'$ is not nuclear and $C \in \mathcal{L}(\Lambda^0_1(\alpha))$ (cf. Example 5.2) as well as other $\alpha_n \uparrow \infty$ such that $(\Lambda^0_1(\alpha))'$ is not nuclear and $C \notin \mathcal{L}(\Lambda^0_1(\alpha))$; see Example 5.3.

In order to be able to formulate the above claims more precisely, let $(v_k)_{k \in \mathbb{N}}$ be a sequence of functions $v_k : \mathbb{N} \to (0, \infty)$ satisfying $v_k(n) \uparrow \infty$, for each $k \in \mathbb{N}$, with $v_k \geq v_{k+1}$ pointwise on $\mathbb{N}$ and $\lim_{n \to \infty} \frac{v_{k+1}(n)}{v_k(n)} = 0$ for all $k \in \mathbb{N}$. Then $\ell_\infty(v_k) \subseteq c_0(v_{k+1})$ continuously for each $k \in \mathbb{N}$ and so

$$k_0(V) := \lim_{k} \ell_\infty(v_k) = \lim_{k} \ell_\infty(v_k).$$

In the notation of Köthe echelon spaces $\lambda_1(\frac{\lambda}{x}) := \text{proj}_k \ell_1(\frac{\lambda}{v_k})$ is a Fréchet-Schwartz space whose strong dual space, i.e., the co-echelon space $(\lambda_1(\frac{\lambda}{x}))'_\beta = \text{ind}_k \ell_\infty(v_k) = k_0(V)$, is a (DFS)-space. It is known that the regular (LB)-space $k_0(V)$ is nuclear if and only if the Fréchet-Schwartz space $\lambda_1(\frac{\lambda}{x})$ is nuclear if and only if the Grothendieck-Pietsch criterion is satisfied: for every $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ with $l > k$ such that the sequence $(\frac{v_l(n)}{v_k(n)})_{n \in \mathbb{N}} \in \ell_1$, [12, Section 21.6]. In case $v_k(n) = e^{\alpha_n/k}$, for $k, n \in \mathbb{N}$, with $\alpha_n \uparrow \infty$, then $k_0(V)$ is the strong dual of the finite type power series space (of order 1) $\Lambda^0_1(\alpha) := \text{proj}_k \ell_1(\frac{1}{v_k})$. This Fréchet space is nuclear if and only if $\lim_{n \to \infty} \frac{\log(n)}{\alpha_n} = 0$, [17, Proposition 29.6]. Whenever this nuclearity condition is satisfied we have $\Lambda^0_1(\alpha) = \text{proj}_k c_0(\frac{1}{v_k})$ which is
precisely the power series space $\Lambda_0(\alpha)$ in which the operator $C$ was investigated in [6].

For the rest of this section, whenever $\alpha_n \uparrow \infty$ we only consider the weights $v_k(n) := e^{\alpha_n/k}$ for $k, n \in \mathbb{N}$.

**Proposition 5.1.** Let the sequence $\alpha_n$ satisfy $\alpha_n \uparrow \infty$ and $\lim_{n \to \infty} \frac{\log(n)}{\alpha_n} = 0$. Then the Cesàro operator $C$ does not act in $k_0(V) = \text{ind}_k c_0(v_k)$.

**Proof.** Since $\lim_{n \to \infty} \frac{\log(n)}{\alpha_n} = 0$, it follows from Lemma 2.2 of [6] that $\lim_{n \to \infty} n^t e^{-\alpha_n} = 0$ for each $t \in \mathbb{N}$, which implies $\lim_{n \to \infty} n e^{-\alpha_n/l} = 0$ for each $l \in \mathbb{N}$. In particular,

$$
\sup_{n \in \mathbb{N}} \frac{e^{\alpha_n/l}}{n} = \infty, \quad \forall l \in \mathbb{N}.
$$

(5.1)

Suppose that $C \in \mathcal{L}(k_0(V))$, i.e., for every $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ with $l > k$ such that $C : c_0(v_k) \to c_0(v_l)$ is continuous. Then, for $k := 1$ there exists $l_1 > 1$ such that $C : c_0(v_1) \to c_0(v_{l_1})$ is continuous, equivalently

$$
M := \sup_{n \in \mathbb{N}} \frac{v_{l_1}(n)}{n} \sum_{m=1}^{n} \frac{1}{v_1(m)} < \infty.
$$

(5.2)

[4, Proposition 2.2(i)]. But, via (5.2), we then have for each $n \in \mathbb{N}$ that

$$
\frac{e^{\alpha_n/l}}{n} = v_1(1) \cdot \frac{v_{l_1}(n)}{nv_1(n)} \leq v_1(1) \cdot \frac{v_{l_1}(n)}{n} \sum_{m=1}^{n} \frac{1}{v_1(m)} \leq M v_1(1).
$$

This contradicts (5.1) for $l := l_1$. Hence, $C$ does not act in $k_0(V).$ \hfill $\Box$

**Example 5.2.** Define $\alpha_n := \log(n+1)$ for $n \in \mathbb{N}$. Since $\lim_{n \to \infty} \frac{\log(n)}{\alpha_n} = 1 \neq 0$, the space $k_0(V)$ is not nuclear. To see that $C \in \mathcal{L}(k_0(V))$ fix any $k \in \mathbb{N}$ and set $l := k+1$. Noting that $v_r(n) = (n+1)^{1/r}$ for $r, n \in \mathbb{N}$, it follows that

$$
\frac{v_l(n)}{n} \sum_{m=1}^{n} \frac{1}{v_k(m)} = \frac{(n+1)^{1/l}}{n} \sum_{m=1}^{n} \frac{1}{(m+1)^{1/k}} \leq 2 \frac{(n+1)^{1/l}}{(n+1)} \sum_{m=1}^{n+1} \frac{1}{m^{1/k}},
$$

(5.3)

for each $n \in \mathbb{N}$. If $k = 1$, then $l = 2$ and it follows from (5.3) and the inequality $\sum_{m=1}^{n+1} \frac{1}{m} \leq 1 + \log(n+1)$ that the left-side of (5.3) is at most $\frac{2(1+\log(n+1))}{(n+1)^{1/2}},$ for $n \in \mathbb{N}$. For $k > 1$, using the inequality $\sum_{m=1}^{n+1} \frac{1}{m} \leq 1 + \frac{(n+1)^{1-\delta}}{1-\delta}$, $\delta \in (0, 1)$, with $\delta := \frac{1}{k}$ it follows from (5.3) (with $l = k+1$) that

$$
\frac{v_l(n)}{n} \sum_{m=1}^{n} \frac{1}{v_k(m)} \leq (n+1)^{\frac{1}{k+1}-1} + \frac{k(n+1)^{\frac{1}{k+1}-1}}{(k-1)}, \quad n \in \mathbb{N}.
$$

In both the cases (i.e., $k = 1$ and $k > 1$) we see that $\sup_{n \in \mathbb{N}} \frac{v_l(n)}{n} \sum_{m=1}^{n} \frac{1}{v_k(m)} < \infty$ and so $C : c_0(v_k) \to c_0(v_l)$ is continuous, [4, Proposition 2.2(i)]. Since this is valid for every $k \in \mathbb{N}$ and with $l := k+1$, it follows that $C \in \mathcal{L}(k_0(V))$.

**Example 5.3.** Let $(j(k))_{k \in \mathbb{N}} \subseteq \mathbb{N}$ be the sequence given by $j(1) := 1$ and $j(k + 1) := 2(k+1)(j(k))^k$, for $k \geq 1.$ Observe that $j(k+1) > k(j(k))^k + 1 > j(k)$ for all $k \in \mathbb{N}$. Define $\beta := (\beta_n)_{n \in \mathbb{N}}$ via $\beta_n := k(j(k))^k$ for $n = j(k), \ldots, j(k+1)-1$. Then $\beta$ is non-decreasing with $\lim_{n \to \infty} \beta_n = \infty$. Let $\gamma := (\gamma_n)_{n \in \mathbb{N}}$ be any strictly increasing sequence satisfying $2 < \gamma_n \uparrow 3$. Then the sequence $\alpha_n := \log(\beta_n + \gamma_n), $
for \(n \in \mathbb{N}\), satisfies \(1 < \alpha_n \uparrow \infty\) and \(\lim_{n \to \infty} \frac{\log(n)}{n} \neq 0\), [6, Remark 2.17]. In particular, \(k_0(V)\) it not nuclear. To establish that \(C\) does not act in \(k_0(V)\) is

suffices to show, for \(k := 1\), that

\[
\sup_{n \in \mathbb{N}} \frac{v_l(n)}{n} \sum_{m=1}^{n} \frac{1}{v_1(m)} = \infty, \quad \forall l \in \mathbb{N}.
\]  

(5.4)

So, fix any \(l \in \mathbb{N}\). Select \(n = j(k)\), for any \(k \in \mathbb{N}\), and observe (for this \(n\)) that

\[
\frac{v_l(n)}{n} \sum_{m=1}^{n} \frac{1}{v_1(m)} = \frac{(\beta_{j(k)} + \gamma_{j(k)})^{1/l}}{j(k)} \sum_{m=1}^{j(k)} \frac{1}{\beta_m + \gamma_m} \geq \frac{(\beta_{j(k)} + \gamma_{j(k)})^{1/l}}{j(k)} \cdot \frac{1}{(\beta_1 + \gamma_1)} \geq (k(j(k))^k + \gamma_{j(k)})^{1/l} \geq \frac{k^{1/l}(j(k))^{\left(\frac{k}{\gamma}\right)}-1}{4} \geq \frac{k^{1/l}(\frac{k}{\gamma})-1}{4},
\]

where we have used \(\frac{1}{\beta_1 + \gamma_1} > \frac{1}{4}\) and \(j(k) \geq k\). Accordingly,

\[
\sup_{n \in \mathbb{N}} \frac{v_l(n)}{n} \sum_{m=1}^{n} \frac{1}{v_1(m)} \geq \sup_{k \in \mathbb{N}} \frac{v_l(j(k))}{j(k)} \sum_{m=1}^{j(k)} \frac{1}{v_1(m)} \geq \sup_{k \in \mathbb{N}} \frac{k^{1/l}(\frac{k}{\gamma})-1}{4} = \infty.
\]

So, (5.4) is satisfied and hence, \(C\) does not act in \(k_0(V)\).

The final two (abstract) results are recorded here in order not to disturb the flow of the text in earlier sections (where these results are needed). We begin with a fact which is surely known; a proof is included for the sake of self-containment.

**Lemma 5.4.** Let \(E = \text{ind}_k(E_k, \| \cdot \|_k)\) be a regular inductive limit of Banach spaces. Then a subset \(\mathcal{H} \subseteq \mathcal{L}(E)\) is equicontinuous if and only if the following condition is satisfied: for every \(k \in \mathbb{N}\) there exists \(l \in \mathbb{N}\) with \(l \geq k\) and \(D > 0\) such that \(\|Tx\|_l \leq D\|x\|_k\), \(\forall T \in \mathcal{H}, x \in E_k\).

**Proof.** First, assume that \(\mathcal{H}\) is equicontinuous. Fix \(k \in \mathbb{N}\), in which case the closed unit ball \(B_k\) of \(E_k\) is bounded in \(E\). The claim is that \(C := \cup_{T \in \mathcal{H}} T(B_k)\) is bounded in \(E\). Indeed, by equicontinuity of \(\mathcal{H}\), given any \(0\)-neighbourhood \(V\) in \(E\) there exists a \(0\)-neighbourhood \(U\) in \(E\) such that \(T(U) \subseteq V\) for all \(T \in \mathcal{H}\). Since \(B_k\) is bounded in \(E\), there exists \(\lambda > 0\) such that \(B_k \subseteq \lambda U\) and hence, \(T(B_k) \subseteq \lambda T(U) \subseteq \lambda V\) for all \(T \in \mathcal{H}\). It follows that \(C \subseteq \lambda V\). Since \(V\) is arbitrary, it follows that \(C\) is bounded in \(E\). But, \(E\) is regular and so there exists \(l \geq k\) such that \(C\) is contained and bounded in \(E_l\). Thus, there exists \(D > 0\) such that \(\|Tx\|_l \leq D\|x\|_k\) for all \(x \in B_k\) and \(T \in \mathcal{H}\). Accordingly, the stated condition (5.5) is satisfied.

Assume that the stated condition (5.5) is satisfied. Since \(E\) is barrelled, the Banach-Steinhaus principle is available and so it suffices to show that the set \(\{Ty : T \in \mathcal{H}\}\) is bounded in \(E\) for each \(y \in E\). So, fix \(y \in E\) in which case \(y \in E_k\) for some \(k \in \mathbb{N}\). Selecting \(l \geq k\) and \(D > 0\) according to condition (5.5), we have \(\|Ty\|_l \leq D\|y\|_k\) for all \(T \in \mathcal{H}\). Hence, the set \(\{Ty : T \in \mathcal{H}\}\) is bounded in \(E_l\) and so, also in \(E\). \(\square\)

The following result occurs in [7, Lemma 5.2].

**Lemma 5.5.** Let \(E = \text{ind}_n(E_n, \| \cdot \|_n)\) be a Hausdorff inductive limit of Banach spaces. Let \(T \in \mathcal{L}(E)\) satisfy the following condition:

\[
\|Ty\|_l \leq D\|y\|_k\]
(A) For each $n \in \mathbb{N}$ the restriction $T_n$ of $T$ to $E_n$ maps $E_n$ into itself and belongs to $\mathcal{L}(E_n)$.

Then the following properties are satisfied.

(i) $\sigma_{pt}(T; E) = \bigcup_{n \in \mathbb{N}} \sigma_{pt}(T_n; E_n)$.

(ii) $\sigma(T; E) \subseteq \bigcap_{m \in \mathbb{N}} \sigma_{\mathbb{N}}(\bigcup_{n=m}^{\infty} \rho(T_n; E_n))$. Moreover, if $\lambda \in \bigcap_{n=m}^{\infty} \rho(T_n; E_n)$ for some $m \in \mathbb{N}$, then $R(\lambda, T_n)$ coincides with the restriction of $R(\lambda, T)$ to $E_n$ for each $n \geq m$.

(iii) If $\bigcup_{n=m}^{\infty} \sigma(T_n; E_n) \subseteq \sigma(T; E)$ for some $m \in \mathbb{N}$, then $\sigma^+(T; E) = \sigma(T; E)$.

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