THE CESÀRO OPERATOR IN WEIGHTED ℓ_1 SPACES

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Abstract. Unlike for ℓ_p, $1 < p \leq \infty$, the discrete Cesàro operator C does *not* map ℓ_1 into itself. We identify precisely those weights w such that C does map $\ell_1(w)$ continuously into itself. For these weights a complete description of the eigenvalues and the spectrum of C are presented. It is also possible to identify all w such that C is a compact operator in $\ell_1(w)$. The final section investigates the mean ergodic properties of C in $\ell_1(w)$. Many examples are presented in order to supplement the results and to illustrate the phenomena that occur.

1. Introduction

The discrete Cesàro operator C is defined on the linear space $\mathbb{C}^{\mathbb{N}}$ (consisting of all scalar sequences) by

$$\mathsf{C}x := \left(x_1, \frac{x_1 + x_2}{2}, \ldots, \frac{x_1 + \ldots + x_n}{n}, \ldots\right), \quad x = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}. \quad (1.1)$$

The operator C is said to *act* in a vector subspace $X \subseteq \mathbb{C}^{\mathbb{N}}$ if it maps X into itself. Of particular interest is the situation when X is a Banach space. Two fundamental questions in this case are: Is $\mathsf{C}\colon X \to X$ continuous and, if so, what is its spectrum? Amongst the classical Banach spaces $X \subseteq \mathbb{C}^{\mathbb{N}}$ where answers are known we mention ℓ_p $(1 < p < \infty)$, [9], [17], and c_0, [17], [21], both c, ℓ_∞, [1], [17], as well as ces_p, $p \in \{0\} \cup (1,\infty)$, [12], the spaces of bounded variation bv_0, [20], and bv_p, $1 \leq p < \infty$, [2], and the Bachelis spaces N^p, $1 < p < \infty$, [11]. For C acting in the *weighted* Banach spaces $\ell_p(w)$, $1 < p < \infty$, and $c_0(w)$ we refer to [4], [5]. There is no claim that this list of spaces (and references) is complete; see also [8].

One aim of this paper is to investigate the two questions mentioned above for C acting in the weighted Banach space $\ell_1(w)$. Unlike for the setting of $\ell_p(w)$, $1 < p < \infty$, where the corresponding paradigm space for C is ℓ_p, $1 < p < \infty$, the "paradigm space" ℓ_1 is *not available* as a guideline for C in $\ell_1(w)$ because C does *not* act in ℓ_1. Hence, it is unclear what to expect when C acts in $\ell_1(w)$.

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So, let $w = (w(n))_{n=1}^{\infty}$ be a sequence, always assumed to be *bounded* and *strictly* positive. Define the vector space

$$\ell_1(w) := \left\{ x = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \colon \sum_{n=1}^{\infty} w(n)|x_n| < \infty \right\},$$

equipped with the norm $\|x\|_{1,w} := \sum_{n\in\mathbb{N}} w(n)|x_n|$, for $x \in \ell_1(w)$. Then $\ell_1(w)$ is isometrically isomorphic to ℓ_1 via the linear multiplication operator $\Phi_w \colon \ell_1(w) \to \ell_1$ given by

$$x = (x_n)_{n\in\mathbb{N}} \mapsto \Phi_w(x) := (w(n)x_n)_{n\in\mathbb{N}}.$$

Accordingly, $\ell_1(w)$ is a weakly sequentially complete Banach space with the Schur property, [19, pp.218–220]. Its dual space $(\ell_1(w))'$ is the Banach space $\ell_\infty(u)$ with the norm $\|x\|_{\infty,u} := \sup_{n\in\mathbb{N}} u(n)|x_n|$, for $x = (x_n)_{n\in\mathbb{N}} \in \ell_\infty(u)$, where $u(n) := w(n)^{-1}$ for $n \in \mathbb{N}$. The closed subspace

$$\left\{ x = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \colon \lim_{n\to\infty} u(n)|x_n| = 0 \right\}$$

of $\ell_\infty(u)$ is denoted by $c_0(u)$ and the restriction of the norm $\|\cdot\|_{\infty,u}$ to $c_0(u)$ is written as $\|\cdot\|_{0,u}$. Of course, the bidual $c_0(u)'' = \ell_\infty(u)$ and the dual $c_0(u)' = \ell_1(w)$. Clearly, the Banach spaces $\ell_\infty(u)$ and $c_0(u)$ are also defined and have the above mentioned properties for *every* strictly positive sequence $u = (u(n))_{n\in\mathbb{N}}$, not just for $u = w^{-1}$. The canonical vectors $e_k := (\delta_{kn})_{n\in\mathbb{N}}$, for $k \in \mathbb{N}$, form an unconditional basis in $\ell_1(w)$. Consequently, whenever C does act in $\ell_1(w)$, then it is necessarily continuous (via the Closed Graph Theorem). If $\inf_{n\in\mathbb{N}} w(n) > 0$, then $\ell_1(w) = \ell_1$ with equivalent norms and so we are in a space in which C does *not* act. Accordingly, we are only interested in the case when $\inf_{n\in\mathbb{N}} w(n) = 0$. Of course, Φ_w is also defined on all of $\mathbb{C}^{\mathbb{N}}$ in which case it is a vector space isomorphism of $\mathbb{C}^{\mathbb{N}}$ onto itself.

For any Banach space X, let I denote the identity operator on X and $\mathcal{L}(X)$ the vector space of all continuous linear operators from X into itself. The *spectrum* and the *resolvent set* of $T \in \mathcal{L}(X)$ are denoted by $\sigma(T)$ and $\rho(T)$, respectively, [13, Ch. VII]. The set of all *eigenvalues* of T, also called the *point spectrum* of T, is denoted by $\sigma_{pt}(T)$. The *spectral radius* $r(T) := \sup\{|\lambda| \colon \lambda \in \sigma(T)\}$ always satisfies $r(T) \le \|T\|$, [13, p.567]. The ideal of compact operators from X into a Banach space Y is denoted by $\mathcal{K}(X, Y)$. If $X = Y$, we simply write $\mathcal{K}(X)$. The dual Banach space of X is denoted by X' and the dual operator of $T \in \mathcal{L}(X)$ by $T' \in \mathcal{L}(X')$.

In Section 2 we identify all weights w such that C acts in $\ell_1(w)$; see Proposition 2.2(i). Necessarily $\inf_{n\in\mathbb{N}} w(n) = 0$ (cf. Remark 2.4(i)) but this condition is far from sufficient; see Examples 2.5(i), (iii). Moreover, this necessary condition cannot be replaced by $w \in c_0$; see Remark 2.4(ii). Even if w is a rapidly decreasing sequence it still need *not* follow that C acts in $\ell_1(w)$; see Remark 2.7(ii). The *compactness* of C in $\ell_1(w)$ is characterized in Proposition 2.2(ii). A useful sufficient condition for w, ensuring the compactness of C in $\ell_1(w)$, is the requirement that

$$\limsup_{n\to\infty} \frac{w(n+1)}{w(n)} \in [0,1); \tag{1.2}$$

see Proposition 2.9. Applications of (1.2) to particular weights are given in parts (i)-(iv) of Examples 2.10. On the other hand, the weights given in (v), (vi) of Examples 2.10 show that the condition (1.2) is not necessary for the compactness of C in $\ell_1(w)$. A comparison type result for compactness (and also for continuity) is presented in Proposition 2.13. The usefulness of this criterion is illustrated via Example 2.14. Somewhat surprisingly, there exist rapidly decreasing weights w for which C acts in $\ell_1(w)$ but, *fails* to be compact; see the weight v in Example 2.12(ii).

Section 3 investigates the *spectrum* of C, provided that C acts in $\ell_1(w)$; for brevity we indicate this by writing $\mathsf{C}^{(1,w)}$ for C or $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$. Relevant for determining $\sigma(\mathsf{C}^{(1,w)})$ are the sets

$$R_w := \{t \in \mathbb{R}\colon \sum_{n=1}^{\infty} n^t w(n) < \infty\} \text{ and } S_w(1) := \{s \in \mathbb{R}\colon \sup_{n\in\mathbb{N}} \frac{1}{n^s w(n)} < \infty\}.$$

Whenever $R_w \neq \mathbb{R}$ (resp. $S_w(1) \neq \emptyset$) we define $t_0 := \sup R_w$ (resp. $s_1 := \inf S_w(1)$). Useful connections between t_0, s_1, the sets R_w, $S_w(1)$ and the condition that w is rapidly decreasing are presented in Propositions 3.4 and 3.5. These propositions are needed to establish the two main results of the section. Theorem 3.7 characterizes $\sigma(\mathsf{C}^{(1,w)})$ and identifies the point spectrum $\sigma_{pt}(\mathsf{C}^{(1,w)})$. It turns out that $\sigma_{pt}(\mathsf{C}^{(1,w)}) = \emptyset$ precisely when $w \notin \ell_1$ (cf. Remark 3.8(iii)). Whenever $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$ and $S_w(1) \neq \emptyset$, necessarily $s_1 > 0$ and

$$\left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{1}{2s_1}\right| \leq \frac{1}{2s_1}\right\} \cup \left\{\frac{1}{n}\colon n \in \mathbb{N}\right\} \subseteq \sigma(\mathsf{C}^{(1,w)});$$

see Proposition 3.9. In particular, $\mathsf{C}^{(1,w)}$ cannot then be compact. This includes such weights as $w_\alpha := (\frac{1}{n^\alpha})_{n\in\mathbb{N}}$ for all $\alpha > 0$ (cf. Example 3.13) and others. On the other hand, if $\mathsf{C}^{(1,w)}$ is compact, then

$$\sigma_{pt}(\mathsf{C}^{(1,w)}) = \left\{\frac{1}{n}\colon n \in \mathbb{N}\right\} \text{ and } \sigma(\mathsf{C}^{(1,w)}) = \{0\} \cup \sigma_{pt}(\mathsf{C}^{(1,w)}) \tag{1.3}$$

and the weight w is necessarily rapidly decreasing. The converse is not valid in general, i.e., there exist rapidly decreasing weights w such that $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$, the spectra of $\mathsf{C}^{(1,w)}$ are given by (1.3) but, $\mathsf{C}^{(1,w)}$ is not compact; see Example 3.15. For each $k \in \mathbb{N}$ there exists a weight w such that $\sigma_{pt}(\mathsf{C}^{(1,w)}) = \{\frac{1}{n}\colon 1 \leq n \leq k\}$; see Example 3.13.

The final section treats various *mean ergodic* properties of $\mathsf{C}^{(1,w)}$. Also relevant is the *power boundedness* of $\mathsf{C}^{(1,w)}$, i.e., $\sup_{n\in\mathbb{N}} \|(\mathsf{C}^{(1,w)})^n\| < \infty$, and the weaker condition of *Cesàro boundedness* (cf. Section 4 for the definition). We record a few sample results. For instance, if $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$, then $\mathsf{C}^{(1,w)}$ is power bounded if and only if the sequence of its iterates $\{(\mathsf{C}^{(1,w)})^n\}_{n\in\mathbb{N}}$ is convergent in the strong operator topology of $\mathcal{L}(\ell_1(w))$ to the projection onto the null space $\mathrm{Ker}(I-\mathsf{C}^{(1,w)})$; see Theorem 4.6(i). Moreover, the power boundedness of $\mathsf{C}^{(1,w)}$implies that $w\in\ell_1$ (cf. Lemma 4.2). It is also established that $\mathsf{C}^{(1,w)}$ is mean ergodic if and only if it is Cesàro bounded (cf. Theorem 4.6(ii)). Such results do not hold for general Banach space operators; see Remark 4.8. Intimately related to the *uniform* mean ergodicity of $\mathsf{C}^{(1,w)}$ (indeed, for any Banach space operator) is the closedness of the range of $I-\mathsf{C}^{(1,w)}$ in $\ell_1(w)$. Under the natural restriction that $w\in\ell_1$, this property is equivalent to the requirement

$$\sup_{m\in\mathbb{N}}\frac{1}{mw(m+1)}\sum_{n=m+1}^{\infty}w(n)<\infty; \tag{1.4}$$

see Proposition 4.11. The condition (1.4) also suffices for $\mathsf{C}^{(1,w)}$ to be both power bounded and uniformly mean ergodic (cf. Proposition 4.12). According to Proposition 4.14, the compactness of $\mathsf{C}^{(1,w)}$ always implies that (1.4) is satisfied; the converse is not true in general (cf. Example 4.16).

An effort has been made to present many and varied examples, both to supplement the results and to illustrate the phenomena that occur.

2. Continuity and compactness of C in $\ell_1(w)$

Given two strictly positive sequences $v=(v(n))_{n=1}^{\infty}$ and $w=(w(n))_{n=1}^{\infty}$, let $T_{v,w}\colon\mathbb{C}^{\mathbb{N}}\to\mathbb{C}^{\mathbb{N}}$ denote the linear operator defined by

$$T_{v,w}x:=\left(\frac{w(n)}{n}\sum_{k=1}^{n}\frac{x_k}{v(k)}\right)_{n\in\mathbb{N}},\quad x=(x_n)_{n\in\mathbb{N}}\in\mathbb{C}^{\mathbb{N}}. \tag{2.1}$$

Observe that $\Phi_w\mathsf{C}=T_{v,w}\Phi_v$ as linear maps on $\mathbb{C}^{\mathbb{N}}$. Hence, the Cesàro operator $\mathsf{C}=\Phi_w^{-1}T_{v,w}\Phi_v$ maps $\ell_1(v)$ continuously (resp., compactly) into $\ell_1(w)$ if and only if the restricted operator $T_{v,w}\in\mathcal{L}(\ell_1)$ (resp., $T_{v,w}\in\mathcal{K}(\ell_1)$). In this regard the following result will be useful, [16, p.11], [21, Lemma 2], [22, p.220].

Lemma 2.1. *Let $A=(a_{mn})_{m,n\in\mathbb{N}}$ be a matrix with entries from $\mathbb{C}$ and $T\colon\mathbb{C}^{\mathbb{N}}\to\mathbb{C}^{\mathbb{N}}$ be the linear operator defined by*

$$Tx:=\left(\sum_{n=1}^{\infty}a_{mn}x_n\right)_{m\in\mathbb{N}},\quad x=(x_n)_{n\in\mathbb{N}},$$

interpreted as $Tx\in\mathbb{C}^{\mathbb{N}}$ exists for $x\in\mathbb{C}^{\mathbb{N}}$. Then $T\in\mathcal{L}(\ell_1)$ if and only if

$$\sup_{n\in\mathbb{N}}\sum_{m=1}^{\infty}|a_{mn}|<\infty.$$

In this case, the operator norm of T is given by $\|T\|=\sup_{n\in\mathbb{N}}\sum_{m=1}^{\infty}|a_{mn}|$.

An immediate application is the following result.

Proposition 2.2. *Let $v = (v(n))_{n=1}^{\infty}$ and $w = (w(n))_{n=1}^{\infty}$ be two bounded, strictly positive sequences.*

(i) C *maps $\ell_1(v)$ continuously into $\ell_1(w)$ if and only if*

$$M_{v,w} := \sup_{n\in\mathbb{N}} \frac{1}{v(n)} \sum_{m=n}^{\infty} \frac{w(m)}{m} < \infty. \tag{2.2}$$

In this case, $\|\mathsf{C}\| = M_{v,w}$.

(ii) C *maps $\ell_1(v)$ compactly into $\ell_1(w)$ if and only if*

$$\lim_{n\to\infty} \frac{1}{v(n)} \sum_{m=n}^{\infty} \frac{w(m)}{m} = 0. \tag{2.3}$$

Proof. (i) By the remark prior to Lemma 2.1 we only need to show that the operator $T_{v,w} \in \mathcal{L}(\ell_1)$ if and only if (2.2) is satisfied.

Now, $T_{v,w} = \Phi_w \mathsf{C} \Phi_v^{-1}$ is defined via the matrix $A := (a_{mn})_{m,n\in\mathbb{N}}$ where, for each $m \in \mathbb{N}$, $a_{mn} := \frac{w(m)}{mv(n)}$ for $1 \le n \le m$ and $a_{mn} := 0$ otherwise. According to Lemma 2.1, $T_{v,w} \in \mathcal{L}(\ell_1)$ if and only if

$$\sup_{n\in\mathbb{N}} \frac{1}{v(n)} \sum_{m=n}^{\infty} \frac{w(m)}{m} < \infty,$$

i.e., if and only if (2.2) is satisfied, in which case $\|T_{v,w}\| = M_{v,w} < \infty$.

So, assume now that $M_{v,w} < \infty$, in which case $\|T_{v,w}\| = M_{v,w}$. Then the identity $\mathsf{C} = \Phi_w^{-1} T_{v,w} \Phi_v$ together with the fact that both Φ_v and Φ_w^{-1} are isometric isomorphisms, implies that $\|\mathsf{C}\| = M_{v,w}$.

(ii) Assume first that $\mathsf{C} \in \mathcal{K}(\ell_1(v), \ell_1(w))$. In particular, C is also continuous and so (2.2) is satisfied with $M_{v,w} < \infty$. The claim is that the operator $A\colon c_0(w^{-1}) \to c_0(v^{-1})$ defined by $Ay := \left(\sum_{m=n}^{\infty} \frac{y_m}{m}\right)_{n\in\mathbb{N}}$, for $y \in c_0(w^{-1})$, is then continuous and its dual operator A' is precisely $\mathsf{C}\colon \ell_1(v) \to \ell_1(w)$. To establish continuity, fix $y \in c_0(w^{-1})$. Let $\varepsilon > 0$. Select $n_0 \in \mathbb{N}$ such that $|y_n| w(n)^{-1} < \varepsilon / M_{v,w}$ for all $n \ge n_0$. It follows, for every $n \ge n_0$, that

$$\frac{1}{v(n)} \left| \sum_{m=n}^{\infty} \frac{y_m}{m} \right| \le \frac{1}{v(n)} \sum_{m=n}^{\infty} \frac{|y_m|}{w(m)} \frac{w(m)}{m} < \frac{\varepsilon}{M_{v,w}} \frac{1}{v(n)} \sum_{m=n}^{\infty} \frac{w(m)}{m} \le \varepsilon.$$

Accordingly, $Ay \in c_0(v^{-1})$. Moreover, for each $n \in \mathbb{N}$ we have that

$$\frac{1}{v(n)} \left| \sum_{m=n}^{\infty} \frac{y_m}{m} \right| \le \frac{1}{v(n)} \sum_{m=n}^{\infty} \frac{|y_m|}{w(m)} \frac{w(m)}{m} \le \|y\|_{0,w^{-1}} \frac{1}{v(n)} \sum_{m=n}^{\infty} \frac{w(m)}{m} \le M_{v,w} \|y\|_{0,w^{-1}},$$

which yields $\|Ay\|_{0,v^{-1}} \le M_{v,w} \|y\|_{0,w^{-1}}$. But, $y \in c_0(w^{-1})$ is arbitrary, and so A is continuous with $\|A\| \le M_{v,w}$. It is routine to check that $A' = \mathsf{C}$.

Since $\mathsf{C} \in \mathcal{K}(\ell_1(v), \ell_1(w))$ and C is the dual operator of A, Schauder's theorem implies that $A \in \mathcal{K}(c_0(w^{-1}), c_0(v^{-1}))$, [19, Theorem 3.4.15], [23, p.282]. In particular, $A \in \mathcal{L}(c_0(w^{-1}), c_0(v^{-1}))$ is necessarily weakly compact. Hence, its bidual operator $A'' = \mathsf{C}' \in \mathcal{L}(\ell_\infty(w^{-1}), \ell_\infty(v^{-1}))$ actually maps $\ell_\infty(w^{-1})$ into $c_0(v^{-1})$, [19, Theorem 3.5.8]. But $w \in \ell_\infty(w^{-1})$ and so $\mathsf{C}'w \in c_0(v^{-1})$, that is, $\lim_{n\to\infty} \frac{(\mathsf{C}'w)(n)}{v(n)} = 0$. Since $\mathsf{C}'w = \left(\sum_{m=n}^\infty \frac{w(m)}{m}\right)_{n\in\mathbb{N}}$, we obtain that $\lim_{n\to\infty} \frac{1}{v(n)} \sum_{m=n}^\infty \frac{w(m)}{m} = 0$, that is, (2.3) is satisfied.

Conversely, suppose that (2.3) holds. Then also (2.2) is valid and so $\mathsf{C} \in \mathcal{L}(\ell_1(v), \ell_1(w))$ by part (i) of this Proposition. Consequently, $\mathsf{C}' \in \mathcal{L}((\ell_\infty(w^{-1}), \ell_\infty(v^{-1}))$. Observe, for every $x \in \ell_\infty(w^{-1})$, that

$$\frac{1}{v(n)}\left|\sum_{m=n}^\infty \frac{x_m}{m}\right| \le \frac{1}{v(n)} \sum_{m=n}^\infty \frac{|x_m|}{w(m)} \frac{w(m)}{m} \le \|x\|_{\infty,w^{-1}} \frac{1}{v(n)} \sum_{m=n}^\infty \frac{w(m)}{m}, \quad n \in \mathbb{N}.$$

Hence, by (2.3) it follows that $\lim_{n\to\infty} \frac{1}{v(n)} \left|\sum_{m=n}^\infty \frac{x_m}{m}\right| = 0$, that is, $\mathsf{C}'x \in c_0(v^{-1})$. Accordingly, C' actually maps $\ell_\infty(w^{-1})$ into $c_0(v^{-1})$. That is, the restriction $A := \mathsf{C}'|_{c_0(w^{-1})}$, which is continuous from $c_0(w^{-1}) \subseteq \ell_\infty(w^{-1})$ into $c_0(v^{-1}) \subseteq \ell_\infty(v^{-1})$, has the property that $A'' = \mathsf{C}'$ is continuous from $\ell_\infty(w^{-1})$ into $\ell_\infty(v^{-1})$ and maps $\ell_\infty(w^{-1})$ into $c_0(v^{-1})$. Accordingly, A is weakly compact, [19, Theorem 3.5.8], and hence, also $\mathsf{C} = A'$ is weakly compact from $\ell_1(v)$ into $\ell_1(w)$, [19, Theorem 3.5.13]. Since the compact and weakly compact subsets of $\ell_1(w) \simeq \ell_1$ coincide, [19, p.255], it follows that C maps $\ell_1(v)$ compactly into $\ell_1(w)$. □

If $v = w$ we denote $M_{v,w}$ simply by M_w. In the event that $M_w < \infty$, the corresponding (continuous) Cesàro operator $\mathsf{C}\colon \ell_1(w) \to \ell_1(w)$ is denoted by $\mathsf{C}^{(1,w)}$. As indicated in Section 1, we also write $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$.

The following simple fact will be used on several occasions.

Lemma 2.3. *Let $\delta > 0$. Then*

$$\sum_{n=m}^\infty \frac{1}{n^{1+\delta}} \le \frac{1}{\delta(m-1)^\delta} \le \frac{2^\delta}{\delta m^\delta}, \quad m \ge 2.$$

Proof. Fix $m \ge 2$. Then

$$\sum_{n=m}^\infty \frac{1}{n^{1+\delta}} \le \int_{m-1}^\infty \frac{1}{x^{1+\delta}}\, dx = \frac{1}{\delta(m-1)^\delta} \le \frac{2^\delta}{\delta m^\delta}.$$

□

Remark 2.4. Let $w = (w(n))_{n\in\mathbb{N}}$ be a bounded, strictly positive weight such that $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$.

(i) Necessarily $\alpha := \inf_{n\in\mathbb{N}} w(n) = 0$. Otherwise, for $n \in \mathbb{N}$, we have

$$\frac{\alpha}{w(1)} \sum_{m=1}^n \frac{1}{m} \le \frac{1}{w(1)} \sum_{m=1}^n \frac{w(m)}{m} \le \frac{1}{w(1)} \sum_{m=1}^\infty \frac{w(m)}{m} \le M_w < \infty,$$

which is impossible.

(ii) The condition $\alpha := \inf_{n\in\mathbb{N}} w(n) = 0$, necessary for the continuity of C in $\ell_1(w)$, cannot be replaced with $w \in c_0$. To see this, define w by $w(n) := 1$ if $n = 2^k$, for $k \in \mathbb{N}$, and $w(n) := \frac{1}{n}$ otherwise. Surely $w \notin c_0$. Set $a_n := \frac{1}{w(n)} \sum_{m=n}^{\infty} \frac{w(m)}{m}$ for $n \in \mathbb{N}$. If $n = 2^k$ for some $k \in \mathbb{N}$, then

$$a_{2^k} = \sum_{m=2^k}^{\infty} \frac{w(m)}{m} \le \sum_{m=1}^{\infty} \frac{1}{m^2} + \sum_{j=k}^{\infty} \frac{1}{2^j} \le \frac{\pi^2}{6} + 1.$$

Clearly, $a_1 = \sum_{m=1}^{\infty} \frac{w(m)}{m} \le \sum_{m=1}^{\infty} \frac{1}{m^2} + \sum_{j=1}^{\infty} \frac{1}{2^j} \le \frac{\pi^2}{6} + 1$. Finally, for fixed $k \in \mathbb{N}$, if $2^k < n < 2^{k+1}$, then Lemma 2.3 implies that

$$a_n \le n \left(\sum_{m=n}^{\infty} \frac{1}{m^2} + \sum_{j=k+1}^{\infty} \frac{1}{2^j} \right) \le n \left(\frac{2}{n} + \frac{1}{2^k} \right) \le 4.$$

So, $\sup_{n\in\mathbb{N}} a_n < \infty$, i.e., $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$; see Proposition 2.2(i).

(iii) Observe that $\|e_1\|_{1,w} = w(1)$ and $\|\mathsf{C}^{(1,w)} e_1\|_{1,w} = \sum_{m=1}^{\infty} \frac{w(m)}{m}$. So,

$$\|\mathsf{C}^{(1,w)}\| \ge \frac{\|\mathsf{C}^{(1,w)} e_1\|_{1,w}}{\|e_1\|_{1,w}} = \frac{1}{w(1)} \sum_{m=1}^{\infty} \frac{w(m)}{m} = 1 + \frac{1}{w(1)} \sum_{m=2}^{\infty} \frac{w(m)}{m} > 1.$$

Fix $1 < p < \infty$. For *every* strictly positive, *decreasing* sequence $w = (w(n))_{n\in\mathbb{N}}$ (i.e., $w(n+1) \le w(n)$ for $n \in \mathbb{N}$) the corresponding Cesàro operator $\mathsf{C}^{(p,w)}$ maps $\ell_p(w)$ continuously into itself and

$$\|\mathsf{C}^{(p,w)}\| \le p', \tag{2.4}$$

where the constant $p' = \frac{p}{p-1}$ is *independent of* w, [4, Proposition 2.2]. Example 2.5(ii) below shows that this is surely not the case for $p = 1$. Here,

$$\ell_p(w) := \left\{ x \in \mathbb{C}^{\mathbb{N}} \colon \|x\|_{p,w} := \left(\sum_{n=1}^{\infty} |x_n|^p w(n) \right)^{1/p} < \infty \right\}$$

which is a Banach space for the norm $\|\cdot\|_{p,w}$ (even if w is not necessarily decreasing). Remark 2.4(i) indicates we only need to consider decreasing weights $w \in c_0$.

Examples 2.5. (i) Fix $\gamma \in (0, 1]$. Define w by $w(1) := 2$ and $w(n) := \frac{1}{(\log n)^\gamma}$ for $n \ge 2$. Then $w \downarrow 0$. Moreover, $\mathsf{C}e_1 = \left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ with

$$\|\mathsf{C}e_1\|_{1,w} = 2 + \sum_{n=2}^{\infty} \frac{1}{n(\log n)^\gamma}. \tag{2.5}$$

By the integral test the series (2.5) is *divergent* and so $\mathsf{C}e_1 \notin \ell_1(w)$. Hence, C does not act in $\ell_1(w)$.

(ii) For $\alpha > 0$ define the decreasing weight $w_\alpha(n) := \frac{1}{n^\alpha}$ for $n \in \mathbb{N}$. Then Lemma 2.3 implies that

$$\begin{aligned} \frac{1}{w_\alpha(n)} \sum_{m=n}^{\infty} \frac{w_\alpha(m)}{m} &= n^\alpha \sum_{m=n}^{\infty} \frac{1}{m^{\alpha+1}} \\ &\leq \frac{n^\alpha}{\alpha(n-1)^\alpha} = \frac{1}{\alpha}\left(\frac{n}{n-1}\right)^\alpha \leq \frac{2^\alpha}{\alpha}, \quad n \geq 2. \end{aligned}$$

Hence, $M_{w_\alpha} = \sup_{n\in\mathbb{N}} \frac{1}{w_\alpha(n)} \sum_{m=n}^{\infty} \frac{w_\alpha(m)}{m} \leq \frac{2^\alpha}{\alpha}$. Via Proposition 2.2(i) we have $\mathsf{C}^{(1,w_\alpha)} \in \mathcal{L}(\ell_1(w_\alpha))$. Observe that $w_\alpha \in \ell_1$ if and only if $\alpha > 1$.

On the other hand, for each fixed $n \in \mathbb{N}$ we have

$$\frac{1}{w_\alpha(n)} \sum_{m=n}^{\infty} \frac{w_\alpha(m)}{m} = n^\alpha \sum_{m=n}^{\infty} \frac{1}{m^{\alpha+1}} \geq n^\alpha \int_n^\infty \frac{1}{s^{\alpha+1}}\, ds = \frac{1}{\alpha}.$$

Accordingly,

$$\|\mathsf{C}^{(1,w_\alpha)}\| = M_{w_\alpha} \geq \frac{1}{\alpha}, \quad \forall \alpha > 0. \tag{2.6}$$

That is, there is no constant $K > 0$ such that $\|\mathsf{C}^{(1,w)}\| \leq K$ for all decreasing weights $w \downarrow 0$ satisfying $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$.

(iii) Let now $\gamma > 1$. Define w by $w(n) := \frac{1}{(\log(n+1))^\gamma}$ for $n \in \mathbb{N}$. Unlike in (i) above, the integral test reveals that now $\sum_{n=1}^{\infty} \frac{1}{n(\log(n+1))^\gamma}$ is convergent. Nevertheless, C is still not continuous from $\ell_1(w)$ into itself. To see this, let $g(x) := x(\log(x+1))^\gamma$, for $x > 0$. Then g is a strictly increasing, positive, differentiable function in $(0,\infty)$ with $g'(x) = (\log(x+1))^\gamma + \gamma\frac{x}{x+1}(\log(x+1))^{\gamma-1} > 0$ for all $x > 0$. Accordingly, $f(x) := \frac{1}{g(x)}$ is strictly decreasing, positive, and continuous in $(0,\infty)$. So, for fixed $n \in \mathbb{N}$, we have

$$\begin{aligned} \sum_{m=n}^{\infty} \frac{1}{m(\log(m+1))^\gamma} &\geq \int_n^\infty \frac{1}{x(\log(x+1))^\gamma}\, dx \geq \int_n^\infty \frac{1}{(x+1)(\log(x+1))^\gamma}\, dx \\ &= \frac{1}{(\gamma-1)(\log(n+1))^{\gamma-1}}. \end{aligned}$$

It follows that

$$\frac{1}{w(n)} \sum_{m=n}^{\infty} \frac{w(m)}{m} \geq (\log(n+1))^\gamma \frac{1}{(\gamma-1)(\log(n+1))^{\gamma-1}} = \frac{\log(n+1)}{\gamma-1}.$$

Accordingly, $\sup_{n\in\mathbb{N}} \frac{1}{w(n)} \sum_{m=n}^{\infty} \frac{w(m)}{m} = \infty$ and so C fails to map $\ell_1(w)$ into itself; see Proposition 2.2(i).

Examples 2.5(i), (iii) show if $w \downarrow 0$ "too slowly", then C may fail to act in $\ell_1(w)$. On the other hand, Example 2.5(ii) indicates if $w \downarrow 0$ "somewhat faster" than in Examples 2.5(i), (iii) (note there that even $w_\alpha \in \ell_1$ for all $\alpha > 1$), then $\mathsf{C}^{(1,w)}$ may be continuous in $\ell_1(w)$. Unfortunately, no rate of decay for $w \downarrow 0$ can be specified apriori to ensure that C acts in $\ell_1(w)$.

Given two bounded, strictly positive sequences v, w satisfying $v(n) \leq w(n)$ for all $n \in \mathbb{N}$ we simply write $v \leq w$.

Proposition 2.6. *Let v be any bounded, strictly positive sequence satisfying $\inf_{n\in\mathbb{N}} v(n) = 0$.*

(i) *There exists a decreasing, strictly positive sequence $w \leq v$ such that $w \in c_0$ and C does not act in $\ell_1(w)$.*
(ii) *There exists a decreasing, strictly positive sequence $u \leq v$ such that $\mathsf{C}^{(1,u)}$ is a compact operator in $\ell_1(u)$.*

Proof. (i) Define $\varphi(n) := \min\{v(k)\colon 1 \leq k \leq n\}$ for $n \in \mathbb{N}$. Then φ is strictly positive, decreasing, satisfies $\varphi \leq v$ and $\varphi \in c_0$.

Since $\lim_{n\to\infty} \sum_{m=k}^{n} \frac{1}{m} = \infty$, for all $k \in \mathbb{N}$, there exists a strictly increasing sequence $(k_j)_{j\in\mathbb{N}}$ in $\mathbb{N}$ (with $k_1 := 1$) satisfying

$$\sum_{m=k_j+1}^{k_{j+1}} \frac{1}{m} > j, \quad j \in \mathbb{N}. \tag{2.7}$$

Define $w(1) := 1$ and $w(n) := \varphi(k_{j+1})$ for $n \in \{k_j + 1, \ldots, k_{j+1}\}$ and each $j \in \mathbb{N}$. Since φ is decreasing, so is w. In addition, for $j \in \mathbb{N}$ and $k_j + 1 \leq n \leq k_{j+1}$ we have $w(n) = \varphi(k_{j+1}) \leq \varphi(n)$, that is, $w \leq \varphi \leq v$ with $w \in c_0$. For each $j \in \mathbb{N}$ we have

$$\frac{1}{w(k_j+1)} \sum_{m=k_j+1}^{\infty} \frac{w(m)}{m} \geq \frac{1}{w(k_j+1)} \sum_{m=k_j+1}^{k_{j+1}} \frac{w(m)}{m} \geq \sum_{m=k_j+1}^{k_{j+1}} \frac{1}{m} > j$$

and hence, $\sup_{n\in\mathbb{N}} \frac{1}{w(k_j+1)} \sum_{m=k_j+1}^{\infty} \frac{w(m)}{m} = \infty$. Then Proposition 2.2(i) shows that C does not act in $\ell_1(w)$.

(ii) Set $u(1) := v(1)$. Inductively, for $n \in \mathbb{N}$ with $u(1), \ldots, u(n)$ already specified, define

$$u(n+1) := \min\left\{v(n+1), \frac{u(n)}{n+1}\right\}.$$

Then u satisfies $0 < u \leq v$ with u decreasing and $u(n+1) \leq \frac{u(n)}{n+1}$ for all $n \in \mathbb{N}$. Accordingly, $\lim_{n\to\infty} \frac{u(n+1)}{u(n)} = 0$ and hence, by Proposition 2.9 below, we have that $\mathsf{C}^{(1,u)} \in \mathcal{K}(\ell_1(u))$. □

Remark 2.7. (i) In the statement of Proposition 2.6 no assumption is made on v as to whether or not C acts in $\ell_1(v)$. The behaviour exhibited in Proposition 2.6 in relation to C acting in $\ell_1(w)$ or not acting in $\ell_1(w)$ (even when $w \downarrow 0$) has no counterpart in the spaces $\ell_p(w)$, $1 < p < \infty$. Indeed, in these spaces, for *every* decreasing sequence w the Cesàro operator C is automatically continuous; see the discussion prior to Examples 2.5. The difference is that for $\ell_1(w)$ the continuity condition (2.2) need not respect existing monotonicity properties of w.

(ii) A sequence $x = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ is *rapidly decreasing* if $(n^k x_n)_{n\in\mathbb{N}} \in \ell_1$ for every $k \in \mathbb{N}$. The space of all such sequences is denoted by s. Let $v \in s$ be arbitrary. Proposition 2.6(ii) implies that there always exists a strictly positive weight $u \leq v$ (hence, $u \in s$) with $\mathsf{C}^{(1,u)} \in \mathcal{K}(\ell_1(u))$. By applying Proposition 2.6(i) to u it follows that there exists another strictly positive sequence $w \leq u$ (hence, also $w \in s$) such that C does *not* act in $\ell_1(w)$.

(iii) The inequality (2.6), together with (2.3) when $v = w$, provides a class of weights $w_\alpha \downarrow 0$, for $\alpha > 0$, such that $\mathsf{C}^{(1,w_\alpha)} \in \mathcal{L}(\ell_1(w_\alpha))$ but, $\mathsf{C}^{(1,w_\alpha)}$ *fails* to be compact.

We now exhibit a large class of weights $w \downarrow 0$ for which $\mathsf{C}^{(1,w)}$ *is* compact.

Lemma 2.8. *Let $r \in (0, 1)$. Then*

$$\lim_{n\to\infty} \frac{1}{r^n} \sum_{m=n}^{\infty} \frac{r^m}{m} = 0. \tag{2.8}$$

Proof. Clearly (2.8) follows from the following inequalities

$$\frac{1}{r^n} \sum_{m=n}^{\infty} \frac{r^m}{m} \leq \frac{1}{nr^n} \sum_{m=n}^{\infty} r^m = \frac{1}{nr^n} \frac{r^n}{(1-r)} = \frac{1}{n(1-r)}, \quad n \in \mathbb{N}.$$

□

Proposition 2.9. *Let w be a bounded, strictly positive sequence such that $\limsup_{n\to\infty} \frac{w(n+1)}{w(n)} =: l \in [0, 1)$. Then $\mathsf{C}^{(1,w)} \in \mathcal{K}(\ell_1(w))$.*

Proof. Let r satisfy $l < r < 1$. Then there exists $n_0 \in \mathbb{N}$ such that $\sup_{n\geq n_0} \frac{w(n+1)}{w(n)} < r$ and hence, $w(n + 1) < rw(n)$ for all $n \geq n_0$. It follows, for a fixed $n \geq n_0$, that $w(m) < r^{m-n} w(n)$ for all $m \geq n$. So, for all $n \geq n_0$, we can conclude that

$$\frac{1}{w(n)} \sum_{m=n}^{\infty} \frac{w(m)}{m} \leq \frac{1}{w(n)} \sum_{m=n}^{\infty} \frac{r^{m-n} w(n)}{m} = \frac{1}{r^n} \sum_{m=n} \frac{r^m}{m}.$$

Then Lemma 2.8 shows that (2.3) holds, i.e., $\mathsf{C}^{(1,w)} \in \mathcal{K}(\ell_1(w))$. □

Examples 2.10. (i) Let $w(n) := n^\beta r^n$, for $r \in (0, 1)$ and $\beta \geq 0$ fixed and for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} \frac{w(n+1)}{w(n)} = r \in (0, 1)$.

(ii) Let $w(n) = \frac{1}{n^n}$ for $n \in \mathbb{N}$. Then

$$\lim_{n\to\infty} \frac{w(n+1)}{w(n)} = \lim_{n\to\infty} \frac{1}{n+1} \left(\frac{n}{n+1}\right)^n = 0.$$

(iii) Fix $a > 0$. Let $w(n) = \frac{a^n}{n!}$ for all $n \in \mathbb{N}$. Then

$$\lim_{n\to\infty} \frac{w(n+1)}{w(n)} = \lim_{n\to\infty} \frac{a}{n+1} = 0.$$

(iv) Let w be the positive sequence defined by $w(1) := 1$ and $w(n+1) := a_n w(n)$ for $n \in \mathbb{N}$, where $a_{2p} := \frac{1}{2}$ and $a_{2p-1} := \frac{1}{p}$ for $p \in \mathbb{N}$. Then, for fixed

$p \in \mathbb{N}$, we have $\frac{w(2p+1)}{w(2p)} = a_{2p} = \frac{1}{2}$ and $\frac{w(2p)}{w(2p-1)} = a_{2p-1} = \frac{1}{p}$. Accordingly, $\limsup_{n\to\infty} \frac{w(n+1)}{w(n)} = \frac{1}{2}$.

According to Proposition 2.9, each of the weights w in (i)-(iv) has the property that $\mathsf{C}^{(1,w)} \in \mathcal{K}(\ell_1(w))$.

(v) Fix $0 < \beta < 1$ and set $w_\beta(n) := e^{-n^\beta}$ for $n \in \mathbb{N}$. Since

$$\lim_{n\to\infty} \frac{w_\beta(n+1)}{w_\beta(n)} = \lim_{n\to\infty} e^{n^\beta-(n+1)^\beta} = \lim_{n\to\infty} e^{-\beta/n^{1-\beta}} = 1,$$

because $n^\beta - (n+1)^\beta = n^\beta(1 - [1 + \frac{\beta}{n} + o(\frac{1}{n})]) \simeq -\beta/n^{1-\beta}$, we see that Proposition 2.9 is not applicable. However,

$$\frac{1}{w_\beta(n)} \sum_{m=n}^{\infty} \frac{w_\beta(m)}{m} = e^{n^\beta} \sum_{m=n}^{\infty} \frac{e^{-m^\beta}}{m^\beta m^{1-\beta}} \le \frac{e^{n^\beta}}{n^\beta} \int_{n-1}^{\infty} \frac{e^{-x^\beta}}{x^{1-\beta}}\, dx$$

as $x \mapsto \frac{e^{-x^\beta}}{x^{1-\beta}} = \frac{1}{x^{1-\beta}e^{x^\beta}}$ is decreasing in $(0,\infty)$. Since $\frac{d}{dx}\left(-\frac{1}{\beta}e^{-x^\beta}\right) = \frac{e^{-x^\beta}}{x^{1-\beta}}$, it follows that $\int_{n-1}^{\infty} \frac{e^{-x^\beta}}{x^{1-\beta}}\, dx = \frac{1}{\beta}e^{-(n-1)^\beta}$ and hence, that

$$\frac{1}{w_\beta(n)} \sum_{m=n}^{\infty} \frac{w_\beta(m)}{m} \le \frac{1}{\beta n^\beta} e^{n^\beta-(n-1)^\beta} \simeq \frac{e^{\beta/n^{1-\beta}}}{\beta n^\beta}.$$

But, $\lim_{n\to\infty} \frac{e^{\beta/n^{1-\beta}}}{\beta n^\beta} = 0$ and so Proposition 2.2(ii), with $v := w_\beta$, implies that $\mathsf{C}^{(1,w_\beta)} \in \mathcal{L}(\ell_1(w_\beta))$ is compact.

If $\beta = 1$, then $w_\beta(n) = e^{-n}$ for $n \in \mathbb{N}$ and so $\lim_{n\to\infty} \frac{w_\beta(n+1)}{w_\beta(n)} = \frac{1}{e} < 1$. For $\beta > 1$, observe from above that $\lim_{n\to\infty} \frac{w_\beta(n+1)}{w_\beta(n)} = \lim_{n\to\infty} e^{-\beta n^{\beta-1}} = 0$. So, for $\beta \ge 1$ the compactness of $\mathsf{C}^{(1,w_\beta)}$ does follow from Proposition 2.9.

(vi) Fix $\gamma > 1$ and set $w_\gamma(n) := e^{-\log^\gamma(n)}$ for $n \in \mathbb{N}$. It is shown in [5, Remark 2.10(ii)] that $\lim_{n\to\infty} \frac{w_\gamma(n+1)}{w_\gamma(n)} = 1$ and so Proposition 2.9 is not applicable. However,

$$\begin{aligned} A_n := \frac{1}{w_\gamma(n)} \sum_{m=n}^{\infty} \frac{w_\gamma(m)}{m} &= e^{\log^\gamma(n)} \sum_{m=n}^{\infty} \frac{e^{-\log^\gamma(m)}}{m} \\ &\le e^{\log^\gamma(n)} \int_{n-1}^{\infty} \frac{e^{-\log^\gamma(x)}}{x}\, dx, \quad n \ge 2, \end{aligned}$$

because $x \mapsto \frac{e^{-\log^\gamma(v)}}{x} = \frac{1}{x \log^\gamma(x)}$ is decreasing in $(1,\infty)$. Accordingly,

$$\begin{aligned} A_n &\le e^{\log^\gamma(n)} \int_{n-1}^{\infty} \frac{e^{-\log^\gamma(x)}}{x}\, dx = e^{\log^\gamma(n)} \int_{n-1}^{\infty} \frac{-1}{\gamma \log^{\gamma-1}(x)} f'(x)\, dx \\ &\le \frac{e^{\log^\gamma(n)}}{\gamma \log^{\gamma-1}(n-1)} \int_{n-1}^{\infty} (-f'(x))\, dx = \frac{e^{\log^\gamma(n)-\log^\gamma(n-1)}}{\gamma \log^{\gamma-1}(n-1)}, \end{aligned}$$

where $f(x) = e^{-\log^\gamma(x)}$ (i.e., $f'(x) = \frac{-\gamma \log^{\gamma-1}(x) e^{-\log^\gamma(x)}}{x}$). But, for $n \geq 2$,

$$\log^\gamma(n) - \log^{\gamma-1}(n-1) = g'(\xi_n), \text{ for some } \xi_n \in ((n-1), n),$$

where $g(t) := \log^\gamma(t)$ satisfies $g'(t) = \frac{\gamma \log^{\gamma-1}(t)}{t} \to 0$ as $t \to \infty$. Hence, $0 \leq \log^\gamma(n) - \log^{\gamma-1}(n-1) \leq 1$ for all $n \geq M$ and some $M \in \mathbb{N}$ with $M \geq 2$. It follows that

$$A_n \leq \frac{e}{\gamma \log^{\gamma-1}(n-1)}, \quad n \geq M,$$

from which we can conclude that $\lim_{n\to\infty} A_n = 0$, i.e., $\mathsf{C}^{(1,w_\gamma)} \in \mathcal{K}(\ell_1(w_\gamma))$ for all $\gamma > 1$; see Proposition 2.2(ii).

Remark 2.11. Examples 2.10(v), (vi) also follow from the following fact.

Let w be a bounded, strictly positive sequence with the property that, for every $k \in \mathbb{N}$ there exists $n(k) \in \mathbb{N}$ such that the sequence $(n^k w(n))_{n=n(k)}^\infty$ is decreasing. Then $\mathsf{C}^{(1,w)} \in \mathcal{K}(\ell_1(w))$.

To see this, set $a_n := \frac{1}{w(n)} \sum_{m=n}^\infty \frac{w(m)}{m}$ for $n \in \mathbb{N}$. Fix $k \in \mathbb{N}$. Then

$$a_n = n^k \sum_{m=n}^\infty \frac{m^k w(m)}{n^k w(n)} \cdot \frac{1}{m^{k+1}} \leq n^k \sum_{m=n}^\infty \frac{1}{m^{k+1}}, \quad n \geq n(k),$$

because $\frac{m^k w(m)}{n^k w(n)} \leq 1$ for all $m \geq n$. But, $\sum_{m=n}^\infty \frac{1}{m^{k+1}} \leq \frac{1}{k(n-1)^k}$ (see Lemma 2.3) and so $a_n \leq \frac{n^k}{k(n-1)^k}$ for $n \geq n(k)$. Since $\sup_{n\geq m(k)} \frac{n^k}{(n-1)^k} \leq 2$ for some $m(k) \geq n(k)$ it follows, for each $k \in \mathbb{N}$, that there exists $m(k) \in \mathbb{N}$ such that $a_n \leq \frac{2}{k}$ for all $n \geq m(k)$. This condition implies that $\lim_{n\to\infty} a_n = 0$ and hence, via Proposition 2.2(ii), that $\mathsf{C}^{(1,w)} \in \mathcal{K}(\ell_1(w))$.

Let v, w be bounded, strictly positive sequences satisfying $v \leq Aw$ for some constant $A > 0$. Then the natural inclusion $\ell_1(w) \subseteq \ell_1(v)$ is continuous because $\|x\|_{1,v} \leq A\|x\|_{1,w}$, for $x \in \ell_1(w)$. Suppose that $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$. Then $\mathsf{C}\colon \ell_1(w) \to \ell_1(v)$ is also continuous with $\|\mathsf{C}\| \leq A\|\mathsf{C}^{(1,w)}\|$. According to Proposition 2.6, C need not have an $\ell_1(v)$-valued continuous linear extension from $\ell_1(w)$ to $\ell_1(v)$. Similarly, if $\mathsf{C}^{(1,w)} \in \mathcal{K}(\ell_1(w))$ and $\mathsf{C}^{(1,v)} \in \mathcal{L}(\ell_1(v))$, then $\mathsf{C}^{(1,v)}$ need not be compact. The following explicit examples illustrate these features.

Example 2.12. (i) Select a strictly increasing sequence $1 =: k_1 < k_2 < \ldots$ in $\mathbb{N}$ satisfying $k_{j+1} > 2k_j$ for each $j \in \mathbb{N}$ and $\lim_{j\to\infty} \sum_{m=1+k_j}^{k_{j+1}} \frac{1}{m} = \infty$ (eg., $k_j := j^j$). Set $v(1) := 1$ and, for each $j \in \mathbb{N}$, define $v(n) := \frac{1}{2^j (k_{j+1} - k_j)}$ for $k_j + 1 \leq n \leq k_{j+1}$. For fixed $j \in \mathbb{N}$ it follows that

$$\frac{1}{v(k_j+1)} \sum_{m=k_j+1}^\infty \frac{v(m)}{m} \geq \frac{1}{v(k_j+1)} \sum_{m=k_j+1}^{k_{j+1}} \frac{v(m)}{m} \geq \sum_{m=k_j+1}^{k_{j+1}} \frac{1}{m}$$

and hence, $\sup_{j\in\mathbb{N}} \frac{1}{v(k_j+1)} \sum_{m=k_j+1}^{\infty} \frac{v(m)}{m} = \infty$. Proposition 2.2(i) implies that C does *not* act in $\ell_1(v)$.

On the other hand, define $w(n) := \frac{2}{n}$ for $n \in \mathbb{N}$. Given $n \geq 2$, select $j \in \mathbb{N}$ such that $k_j + 1 \leq n \leq k_{j+1}$. Then

$$\frac{v(n)}{w(n)} = \frac{n}{2 \cdot 2^j (k_{j+1} - k_j)} \leq \frac{k_{j+1}}{2(k_{j+1} - k_j)} = \frac{1}{2(1 - \frac{k_j}{k_{j+1}})} < 1$$

and so $v \leq w$. Moreover, $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$; see Examples 2.5(ii).

(ii) Define the decreasing sequence v by $v(1) = v(2) := 1$ and

$$v(n) := \frac{1}{2^i 2^{(i+1)2^{i+1}}}, \quad \text{for } \ 2^i + 1 \leq n \leq 2^{i+1} \ \text{ and } \ i \in \mathbb{N},$$

and the sequence $w := (\frac{1}{n^{n+1}})_{n\in\mathbb{N}}$. Given $n \geq 3$ select $i \in \mathbb{N}$ such that $2^i + 1 \leq n \leq 2^{i+1}$. Then

$$\frac{v(n)}{w(n)} = \frac{n^n n}{2^i 2^{(i+1)2^{i+1}}} \leq \frac{(2^{i+1})^{2^{i+1}} 2^{i+1}}{2^i 2^{(i+1)2^{i+1}}} = 2.$$

Since $\frac{v(1)}{w(1)} = 1$ and $\frac{v(2)}{w(2)} = 8$, it follows that $v \leq 8w$. In particular, $v \in s$. According to Proposition 2.9 (as $\lim_{n\to\infty} \frac{w(n+1)}{w(n)} = 0$) $\mathsf{C}^{(1,w)}$ is compact. On the other hand, $\mathsf{C}^{(1,v)}$ is continuous (see Fact 2 in Example 3.15 below) but *not* compact (see Fact 3 in Example 3.15 below).

We now present a positive comparison result where difficulties such as those observed in Example 2.12 do not arise.

Proposition 2.13. *Let v, w be bounded, strictly positive sequences such that $(\frac{v(n)}{w(n)})_{n=n_0}^{\infty}$ is a decreasing sequence for some $n_0 \in \mathbb{N}$.*

(i) *If $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$, then also $\mathsf{C}^{(1,v)} \in \mathcal{L}(\ell_1(v))$.*
(ii) *If $\mathsf{C}^{(1,w)} \in \mathcal{K}(\ell_1(w))$, then also $\mathsf{C}^{(1,v)} \in \mathcal{K}(\ell_1(v))$.*

Proof. (i) Define $\alpha_n := \frac{v(n)}{w(n)}$ for $n \in \mathbb{N}$ in which case $\alpha_n \geq \alpha_{n+1}$ for all $n \geq n_0$. Proposition 2.2(i) implies that $M_w < \infty$; see (2.2). Moreover,

$$\frac{1}{v(n)} \sum_{m=n}^{\infty} \frac{v(m)}{m} = \frac{1}{w(n)} \sum_{m=n}^{\infty} \frac{\alpha_m}{\alpha_n} \cdot \frac{w(m)}{m} \leq \frac{1}{w(n)} \sum_{m=n}^{\infty} \frac{w(m)}{m} \leq M_w, \tag{2.9}$$

for all $n \geq n_0$. In particular, via (2.9),

$$A := \sum_{k=1}^{\infty} \frac{v(k)}{k} = \sum_{k=1}^{n_0-1} \frac{v(k)}{k} + v(n_0) \cdot \frac{1}{v(n_0)} \sum_{k=n_0}^{\infty} \frac{v(k)}{k} < \infty.$$

For each $n \in \{1, \ldots, n_0 - 1\}$ it follows that $\frac{1}{v(n)} \sum_{m=n}^{\infty} \frac{v(m)}{m} \leq \frac{A}{v(n)}$ and hence,

$$M_v = \sup_{n\in\mathbb{N}} \frac{1}{v(n)} \sum_{m=n}^{\infty} \frac{v(m)}{m} \leq \max\left\{ M_w, \max\left\{ \frac{A}{v(1)}, \ldots, \frac{A}{v(n_0 - 1)} \right\} \right\} < \infty.$$

Accordingly, $\mathsf{C}^{(1,v)} \in \mathcal{L}(\ell_1(v))$.

(ii) Let n_0 be as in the statement of the proposition. Let $\epsilon > 0$. Since $\mathsf{C}^{(1,w)}$ is compact, there exists $n_1(\epsilon) > n_0$ such that

$$\frac{1}{w(n)} \sum_{m=n}^{\infty} \frac{w(m)}{m} < \epsilon, \quad n \geq n_1(\epsilon);$$

see (2.3). It then follows from (2.9) that also

$$\frac{1}{v(n)} \sum_{m=n}^{\infty} \frac{v(m)}{m} < \epsilon, \quad n \geq n_1(\epsilon) > n_0.$$

Accordingly, $\mathsf{C}^{(1,v)}$ is also compact; see Proposition 2.2(ii). □

Example 2.14. (i) For $w := (\frac{1}{n^\alpha})_{n\in\mathbb{N}}$ with $\alpha > 0$, Examples 2.5(ii) shows that $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$. Define $v(n) := \frac{1}{n^\alpha \log^\beta(n+1)}$, $n \in \mathbb{N}$, with $\beta > 0$. Then $\frac{v}{w}$ is a decreasing sequence and so Proposition 2.13(i) implies that $\mathsf{C}^{(1,v)} \in \mathcal{L}(\ell_1(v))$.

(ii) Let $w(n) := \frac{1}{n \log^\beta(n+1)}$, $n \in \mathbb{N}$, with $\beta > 1$, in which case $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$ by part (i). Also, via Examples 2.5(ii), $v := (\frac{1}{n^2})_{n\in\mathbb{N}}$ satisfies $\mathsf{C}^{(1,v)} \in \mathcal{L}(\ell_1(v))$. Consider the sequence $\frac{v}{w} = (\frac{\log^\beta(n+1)}{n})_{n\in\mathbb{N}}$. The derivative of the function $f(x) := \frac{\log^\beta(x+1)}{x}$ for $x \geq 1$ is given by

$$f'(x) = \frac{(\beta x - (x+1)\log(x+1))\log^{\beta-1}(x+1)}{x^2(x+1)}$$

and hence, f is decreasing on $((e^\beta - 1), \infty)$. So there exists $n_0 \in \mathbb{N}$ such that $(\frac{v(n)}{w(n)})_{n=n_0}^\infty$ is decreasing. Since $\mathsf{C}^{(1,v)}$ is not compact (by Remark 2.7(iii)), it follows from Proposition 2.13(ii) that $\mathsf{C}^{(1,w)}$ also fails to be compact.

Remark 2.15. Let v, w be bounded, strictly positive sequences satisfying $A_1 v \leq w \leq A_2 v$ for positive constants A_1, A_2. Then $\ell_1(v)$ and $\ell_1(w)$ are equal as vector spaces and the norms $\|\cdot\|_{1,v}$ and $\|\cdot\|_{1,w}$ are equivalent. Accordingly, $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$ (resp. $\mathcal{K}(\ell_1(w))$ if and only if $\mathsf{C}^{(1,v)} \in \mathcal{L}(\ell_1(v))$ (resp. $\mathcal{K}(\ell_1(v))$. For instance, let $v = (\frac{1}{n^\alpha})_{n\in\mathbb{N}}$ with $\alpha > 0$. Consider *any* bounded, strictly positive sequence φ satisfying $\gamma := \inf_{n\in\mathbb{N}} \varphi(n) > 0$. Then $w := (\varphi(n)v(n))_{n\in\mathbb{N}}$ satisfies $\gamma v \leq w \leq \|\varphi\|_\infty v$. Via Examples 2.5(ii), $\mathsf{C}^{(1,v)} \in \mathcal{L}(\ell_1(v))$ and so also $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$. Remark 2.7(iii) shows that $\mathsf{C}^{(1,v)}$ is not compact and hence, also $\mathsf{C}^{(1,w)}$ fails to be compact. Or, suppose that $u \leq v$. Then $v \leq u + v \leq 2v$ and so $\mathsf{C}^{(1,u+v)}$ is continuous (resp. compact) if and only if $\mathsf{C}^{(1,v)}$ is continuous (resp. compact).

3. Spectrum of $\mathsf{C}^{(1,w)}$

The aim of this section is to provide some detailed knowledge of the spectrum of $\mathsf{C}^{(1,w)}$. For $1 < p < \infty$ it is known for *every strictly positive,* *decreasing* weight w that the spectrum of $\mathsf{C}^{(p,w)} \in \mathcal{L}(\ell_p(w))$ satisfies

$$\sigma(\mathsf{C}^{(p,w)}) \subseteq \{\lambda \in \mathbb{C} \colon |\lambda| \leq p'\}$$

with $p' = \frac{p}{p-1}$ a constant *independent* of w; see (2.4) above and [4, Theorem 3.3(i)]. It will be shown, for $p = 1$, that no such constant (independent of w) exists; see Example 3.13. The spectrum of $\mathsf{C}^{(1,w)}$ is characterized in Theorem 3.7. Further properties of $\sigma(\mathsf{C}^{(1,w)})$ are exhibited in Proposition 3.9. Whenever $\mathsf{C}^{(1,w)}$ is a compact operator, a complete description of $\sigma(\mathsf{C}^{(1,w)})$ is given in Proposition 3.11. Several relevant examples are presented.

We begin by recalling the following known fact; see e.g. [3, Proposition 4.1], [6, Propositions 4.3 and 4.4]. For convenience of notation we set $\Sigma := \{\frac{1}{m} \colon m \in \mathbb{N}\}$ and $\Sigma_0 := \Sigma \cup \{0\}$. Recall that $\mathbb{C}^{\mathbb{N}}$ is a Fréchet space for the lc-topology of coordinatewise convergence.

Lemma 3.1. (i) *The spectrum* $\sigma(\mathsf{C}, \mathbb{C}^{\mathbb{N}}) = \sigma_{pt}(\mathsf{C}, \mathbb{C}^{\mathbb{N}}) = \Sigma$.

(ii) *Fix* $m \in \mathbb{N}$. *Let* $x^{(m)} := (x_n^{(m)})_n \in \mathbb{C}^{\mathbb{N}}$ *where* $x_n^{(m)} := 0$ *for* $n \in \{1, \ldots, m-1\}$, $x_m^{(m)} := 1$ *and* $x_n^{(m)} := \frac{(n-1)!}{(m-1)!(n-m)!}$ *for* $n > m$. *Then the* 1*-dimensional eigenspace of* $\frac{1}{m}$ *is given by*

$$\operatorname{Ker}\left(\frac{1}{m} I - \mathsf{C}\right) = \operatorname{span}\{x^{(m)}\} \subseteq \mathbb{C}^{\mathbb{N}}.$$

Remark 3.2. For $\lambda = 1$, the corresponding eigenvector for $\mathsf{C} \colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ is the constant vector $\mathbf{1} := (1)_{n \in \mathbb{N}}$. Accordingly, if w is any bounded, strictly positive weight such that $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$, then $1 \in \sigma_{pt}(\mathsf{C}^{(1,w)})$ if and only if $\mathbf{1} \in \ell_1(w)$, i.e., if and only if $w \in \ell_1$.

The following inequalities, [4, Lemma 3.2], [21, Lemma 7], will be needed.

Lemma 3.3. (i) *Let* $\lambda \in \mathbb{C} \setminus \Sigma_0$ *and set* $\alpha := \operatorname{Re}\left(\frac{1}{\lambda}\right)$. *Then there exist constants* $d > 0$ *and* $D > 0$ (*depending on* α) *such that*

$$\frac{d}{n^{\alpha}} \leq \prod_{k=1}^{n} \left|1 - \frac{1}{k\lambda}\right| \leq \frac{D}{n^{\alpha}}, \quad n \in \mathbb{N}. \tag{3.1}$$

(ii) *For each* $m \in \mathbb{N}$ *we have that*

$$\frac{(n-1)!}{(n-m)!} \simeq n^{m-1}, \quad \textit{for all large } n \in \mathbb{N}. \tag{3.2}$$

For every bounded, strictly positive weight $w = (w(n))_{n \in \mathbb{N}}$ recall that

$$R_w := \{t \in \mathbb{R} \colon \sum_{n=1}^{\infty} n^t w(n) < \infty\}. \tag{3.3}$$

In case $R_w \neq \mathbb{R}$ we define $t_0 := \sup R_w$.

Proposition 3.4. *Let* $w = (w(n))_{n=1}^{\infty}$ *be a bounded, strictly positive sequence. The following conditions are equivalent.*

(i) $(n^m w(n))_n \in \ell_1$ *for all* $m \in \mathbb{N}$.
(ii) $w \in s$.
(iii) $R_w = \mathbb{R}$.

If, in addition, $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w)$*, then* (i)-(iii) *are equivalent to*

(iv) $\Sigma \subseteq \sigma_{pt}(\mathsf{C}^{(1,w)})$.

Proof. (i)⇔(ii) follows from the definition of the space s.

(i)⇔(iii) follows from the definition of R_w; see (3.3).

Assume now that $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$.

(iv)⇒(i) Fix $m \in \mathbb{N}$. Then $\frac{1}{m+1} \in \sigma_{pt}(\mathsf{C}^{(1,w)}) \subseteq \sigma_{pt}(\mathsf{C}, \mathbb{C}^{\mathbb{N}})$ with $x^{(m+1)}$ as its eigenvector in $\mathbb{C}^{\mathbb{N}}$; see Lemma 3.1. So, necessarily $x^{(m+1)} \in \ell_1(w)$, i.e., $(w(n)x_n^{(m+1)})_{n\in\mathbb{N}} \in \ell_1$. But, this happens only if $(n^m w(n))_n \in \ell_1$; see (3.2).

(i)⇒(iv) Fix $m \in \mathbb{N}$. Then $(n^{m-1}w(n))_n \in \ell_1$ and so the sequence $(w(n)x_n^{(m)})_{n\in\mathbb{N}} \in \ell_1$, i.e., $x^{(m)} \in \ell_1(w)$, where $x^{(m)}$ is as in Lemma 3.1. As $x^{(m)}$ is an eigenvector corresponding to the eigenvalue $\frac{1}{m}$ for C acting on $\mathbb{C}^{\mathbb{N}}$, it follows that $\frac{1}{m}$ is also an eigenvalue for $\mathsf{C}^{(1,w)}$. □

Given a strictly positive, bounded sequence $w = (w(n))_{n\in\mathbb{N}}$, recall that $S_w(1) := \{s \in \mathbb{R}\colon \sup_{n\in\mathbb{N}} \frac{1}{n^s w(n)} < \infty\}$. In case $S_w(1) \neq \emptyset$ we defined $s_1 := \inf S_w(1)$. Since $\frac{1}{n^s w(n)} \geq \frac{1}{n^s \|w\|_\infty}$, for $n \in \mathbb{N}$, it follows that $s \notin S_w(1)$ for every $s < 0$ and hence, $S_w(1) \subseteq [0,\infty)$. Accordingly, $s_1 \geq 0$. If $w(n) \geq \alpha$ for all $n \in \mathbb{N}$ and some $\alpha > 0$, then C does not act in $\ell_1(w)$; see Remark 2.4(i). So, we restrict our attention to weights w with $\inf_{n\in\mathbb{N}} w(n) = 0$. In this case $\frac{1}{w} \notin \ell_\infty$. Hence, if $S_w(1) \neq \emptyset$ and $s \in S_w(1)$, then necessarily $s > 0$ with $\frac{M}{n^s} \leq w(n)$ for some $M > 0$ and all $n \in \mathbb{N}$. It follows that $[s,\infty) \subseteq S_w(1)$. So, whenever $S_w(1) \neq \emptyset$ (with $\frac{1}{w} \notin \ell_\infty$) we can conclude that $S_w(1)$ is an interval of the form $[s_1,\infty)$ or (s_1,∞) with $s_1 \geq 0$.

Concerning R_w (see (3.3)), whenever $R_w \neq \mathbb{R}$ the quantity t_0 is finite with $t_0 \geq -1$ and $R_w = (-\infty, t_0)$ or $R_w = (-\infty, t_0]$. Moreover, $R_w = \emptyset$ is impossible as $\sum_{n=1}^\infty n^t w(n) \leq \|w\|_\infty \sum_{n=1}^\infty n^t < \infty$ whenever $t < -1$.

Proposition 3.5. *Let* w *be a bounded, strictly positive sequence.*

(i) *If* $S_w(1) \neq \emptyset$*, then* $t_0 \leq s_1$*. In particular,* $R_w \neq \mathbb{R}$.
(ii) *If* $R_w \neq \mathbb{R}$*, then* $S_w(1) \subseteq [t_0,\infty)$.
(iii) *If* $w \in s$*, then* $S_w(1) = \emptyset$.

Proof. (i) Fix any $s > s_1$. Then, for some $M > 0$, we have $\frac{1}{n^s w(n)} \leq M$ for all $n \in \mathbb{N}$ and hence, $n^s w(n) \geq \frac{1}{M}$ for $n \in \mathbb{N}$. Accordingly, $(n^s w(n))_{n\in\mathbb{N}} \notin \ell_1$, i.e., $s \notin R_w$. This implies that $(s_1,\infty) \subseteq \mathbb{R} \setminus R_w$, i.e., $t_0 \leq s_1$.

(ii) Fix any $t < t_0$ in which case $\lim_{n\to\infty} t^n w(n) = 0$. Hence, there exists $K \in \mathbb{N}$ such that $n^t \leq \frac{1}{w(n)}$ for $n \geq K$. So, for any $s \in \mathbb{R}$ we have (as $\frac{1}{n^s} > 0$ for $n \in \mathbb{N}$) that $\frac{1}{n^s w(n)} \geq \frac{n^t}{n^s}$ for all $n \geq K$. Hence, if $s < t$, then $(\frac{1}{n^s w(n)})_{n\in\mathbb{N}} \notin \ell_\infty$ and so $s \notin S_w(1)$. This implies that $S_w(1) \subseteq [t_0,\infty)$.

(iii) Suppose $r < 0$. Since $w \in c_0$, there exists $L \in \mathbb{N}$ such that $\frac{1}{w(n)} \geq 1$ for $n \geq L$. Hence, $\frac{1}{n^r w(n)} \geq \frac{1}{n^r}$ for all $n \geq L$ and so $r \notin S_w(1)$. For $r \geq 0$ fixed, set $m := 1 + [r]$. Since $\sum_{n=1}^{\infty} n^m w(n) < \infty$ (see Proposition 3.4), there is $J \in \mathbb{N}$ such that $\frac{1}{w(n)} \geq n^m$ for $n \geq J$ and hence, $\frac{1}{n^r w(n)} \geq \frac{n^m}{n^r}$ for $n \geq J$, that is, $r \notin S_w(1)$. □

Remark 3.6. (i) The converse of Proposition 3.5(iii) is not valid. Indeed, let $w = (w(n))_{n\in\mathbb{N}}$ be the strictly positive weight with $w \downarrow 0$ as given in [5, Remark 3.2]. It is shown there that there exists a strictly increasing sequence $(n(k))_{k\in\mathbb{N}}$ in $\mathbb{N}$ with the property: for each $t \in \mathbb{R}$ we have

$$\frac{1}{(n(k))^t w(n(k))} \geq k, \quad k > t.$$

Hence, $t \notin S_w(1)$ and so $S_w(1) = \emptyset$. It is also shown in [5] that $w \notin s$.

(ii) If $v,\ w$ are bounded, strictly positive sequences with $w \leq v$, then $\frac{1}{v} \leq \frac{1}{w}$ from which it follows that $S_1(w) \subseteq S_1(v)$. Hence, $\inf S_1(v) \leq \inf S_1(w)$. Also, it is clear from (3.3) that $R_v \subseteq R_w$ and so $\sup R_v \leq \sup R_w$.

We now come to the main results of this section. The following result characterizes the spectrum of $\mathsf{C}^{(1,w)}$.

Theorem 3.7. *Let $w = (w(n))_{n\in\mathbb{N}}$ be a bounded, strictly positive sequence such that $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$.*

(i) *The following inclusions hold:*

$$\Sigma \subseteq \Sigma_0 \subseteq \sigma(\mathsf{C}^{(1,w)}). \tag{3.4}$$

(ii) *Let $\lambda \notin \Sigma_0$ and set $\alpha := \mathrm{Re}\left(\frac{1}{\lambda}\right)$. Then $\lambda \in \rho(\mathsf{C}^{(1,w)})$ if and only if*

$$\sup_{m\in\mathbb{N}} \frac{1}{m^\alpha w(m)} \sum_{n=m+1}^{\infty} \frac{w(n)}{n^{1-\alpha}} < \infty. \tag{3.5}$$

(iii) *Suppose that $R_w \neq \mathbb{R}$, i.e., $t_0 < \infty$. Then*

$$\left\{\frac{1}{m} \colon m \in \mathbb{N},\ (m-1) \in R_w\right\} = \sigma_{pt}(\mathsf{C}^{(1,w)}) \subseteq \left\{\frac{1}{m} \colon m \in \mathbb{N},\ 1 \leq m \leq t_0 + 1\right\}. \tag{3.6}$$

In particular, $\sigma_{pt}(\mathsf{C}^{(1,w)})$ is a finite subset of Σ (possibly empty).

If $R_w = \mathbb{R}$, then

$$\sigma_{pt}(\mathsf{C}^{(1,w)}) = \Sigma. \tag{3.7}$$

Proof. The proof is via a series of steps.

(i) The dual operator $A := (\mathsf{C}^{(1,w)})' \in \mathcal{L}(\ell_\infty(w^{-1}))$ is given by

$$Ay = \left(\sum_{k=n}^{\infty} \frac{y_k}{k}\right)_{n\in\mathbb{N}}, \quad y = (y_n)_{n\in\mathbb{N}} \in \ell_\infty(w^{-1}). \tag{3.8}$$

Step 1. $0 \notin \sigma_{pt}(A)$.

If $Ay = 0$, for some $y \in \ell_\infty(w^{-1})$, then $z_n := \sum_{k=n}^\infty \frac{y_k}{k} = 0$ for all $n \in \mathbb{N}$. Hence, $y_n = n(z_n - z_{n+1}) = 0$, for $n \in \mathbb{N}$, and so A is injective.

Step 2. $\Sigma \subseteq \sigma_{pt}(A)$.

Let $\lambda \in \Sigma$, i.e., $\lambda = \frac{1}{m}$ for some $m \in \mathbb{N}$. Via (3.16) below, the non-zero vector $y = (y_n)_{n\in\mathbb{N}}$ defined via $y_1 \in \mathbb{C}\setminus\{0\}$ arbitrary, $y_n := y_1 \prod_{k=1}^{n-1}\left(1 - \frac{1}{\lambda k}\right)$ for $1 < n \le m$ and $y_n := 0$ for $n > m$, which clearly belongs to $\ell_\infty(w^{-1})$, satisfies $Ay = \lambda y$.

Step 3. $\Sigma_0 \subseteq \sigma(\mathsf{C}^{(1,w)})$.

For every $T \in \mathcal{L}(X)$ with X a Banach space, we have $\sigma_{pt}(T') \subseteq \sigma(T)$, [13, p.581], with $\sigma(T)$ closed in $\mathbb{C}$. By Step 2 we then have $\Sigma_0 \subseteq \sigma(\mathsf{C}^{(1,w)})$.

(ii) **Step 4.** *Fix $\lambda \notin \Sigma_0$. Then $\lambda \in \rho(\mathsf{C}^{(1,w)})$ if and only if* (3.5) *holds.*

To verify this we argue in a similar way as in [4] or [10]. We recall the formula for $(\mathsf{C} - \lambda I)^{-1} \colon \mathbb{C}^\mathbb{N} \to \mathbb{C}^\mathbb{N}$ whenever $\lambda \notin \Sigma_0$, [21, p.266]. Namely, for $n \in \mathbb{N}$, the n-th row of the matrix for $(\mathsf{C} - \lambda I)^{-1}$ has the entries

$$\frac{-1}{n\lambda^2 \prod_{k=m}^n \left(1 - \frac{1}{\lambda k}\right)}, \quad 1 \le m < n,$$

$$\frac{n}{1 - n\lambda} = \frac{1}{\frac{1}{n} - \lambda}, \quad m = n,$$

and all the other entries in row n are equal to 0. So, we can write

$$(\mathsf{C} - \lambda I)^{-1} = D_\lambda - \frac{1}{\lambda^2} E_\lambda, \tag{3.9}$$

where the diagonal operator $D_\lambda = (d_{nm})_{n,m\in\mathbb{N}}$ is given by $d_{nn} := \frac{1}{\frac{1}{n} - \lambda}$ and $d_{nm} := 0$ if $n \ne m$. The operator $E_\lambda = (e_{nm})_{n,m\in\mathbb{N}}$ is then the lower triangular matrix with $e_{1m} = 0$ for all $m \in \mathbb{N}$ and for every $n \ge 2$, with $e_{nm} := \frac{1}{n \prod_{k=m}^n \left(1 - \frac{1}{\lambda k}\right)}$ if $1 \le m < n$ and $e_{nm} := 0$ if $m \ge n$.

As $\lambda \notin \Sigma_0$, we have $d(\lambda) := \operatorname{dist}(\lambda, \Sigma_0) > 0$ and $|d_{nn}| \le \frac{1}{d(\lambda)}$ for $n \in \mathbb{N}$. Hence, for every $x \in \ell_1(w)$, it follows that

$$\|D_\lambda(x)\|_{1,w} = \sum_{n=1}^\infty |d_{nn} x_n| w(n) \le \frac{1}{d(\lambda)} \sum_{n=1}^\infty |x_n| w(n) = \frac{1}{d(\lambda)} \|x\|_{1,w}.$$

This means that $D_\lambda \in \mathcal{L}(\ell_1(w))$. So, by (3.9) it remains to show that $E_\lambda \in \mathcal{L}(\ell_1(w))$ if and only if (3.5) is satisfied for $\alpha := \operatorname{Re}\left(\frac{1}{\lambda}\right)$.

To this end, we note that $E_\lambda \in \mathcal{L}(\ell_1(w))$ if and only if the operator $\tilde{E}_\lambda \colon \mathbb{C}^\mathbb{N} \to \mathbb{C}^\mathbb{N}$ given by $\tilde{E}_\lambda = \Phi_w E \Phi_w^{-1}$, i.e.,

$$(\tilde{E}_\lambda(x))_n = w(n) \sum_{m=1}^{n-1} \frac{e_{nm}}{w(m)} x_m, \quad x \in \mathbb{C}^\mathbb{N},\ n \in \mathbb{N},$$

defines a continuous linear operator on ℓ_1 (see the comments prior to Lemma 2.1). So, the claim is that $\tilde{E}_\lambda \in \mathcal{L}(\ell_1)$ if and only if (3.5) is satisfied for

$\alpha := \mathrm{Re}\left(\frac{1}{\lambda}\right)$. To establish this claim, observe that (3.1) implies

$$\begin{aligned} &\frac{D^{-1}}{n^{1-\alpha}} \le |e_{n1}| \le \frac{d^{-1}}{n^{1-\alpha}}, \quad n \ge 2, \\ &\frac{d' D^{-1}}{n^{1-\alpha} m^{\alpha}} \le |e_{nm}| \le \frac{d^{-1} D'}{n^{1-\alpha} m^{\alpha}}, \quad 2 \le m < n, \end{aligned} \tag{3.10}$$

for some constants $d' > 0$ and $D' > 0$ depending on λ.

Suppose first that $\tilde{E}_\lambda \in \mathcal{L}(\ell_1)$. Then Lemma 2.1 implies that

$$\sup_{m\in\mathbb{N}} \frac{1}{w(m)} \sum_{n=m+1}^{\infty} w(n)|e_{nm}| < \infty.$$

By (3.10) we have $\sup_{m\in\mathbb{N}} \frac{1}{m^\alpha w(m)} \sum_{n=m+1}^{\infty} \frac{w(n)}{n^{1-\alpha}} < \infty$, i.e., (3.5) is satisfied.

Conversely, if (3.5) is satisfied, then $\sup_{m\in\mathbb{N}} \frac{1}{m^\alpha w(m)} \sum_{n=m+1}^{\infty} \frac{w(n)}{n^{1-\alpha}} < \infty$. By (3.10) this implies that also $\sup_{m\in\mathbb{N}} \frac{1}{w(m)} \sum_{n=m+1}^{\infty} w(n)|e_{nm}| < \infty$. Therefore, via Lemma 2.1, we can conclude that $\tilde{E}_\lambda \in \mathcal{L}(\ell_1)$. The claim is proved.

The proof of part (ii) is thereby complete.

(iii) Suppose first that $R_w \neq \mathbb{R}$.

Step 5. *Both the equality and the inclusion in* (3.6) *are valid.*

The proof of Step 5 is a routine adaption of the proof of Step 7 in the proof of Theorem 3.3 in [4]; just substitute $p = 1$ there. In particular, it follows that $\frac{1}{m} \in \sigma_{pt}(\mathsf{C}^{(1,w)})$ if and only if $(m-1) \in R_w$.

The previous observation also allows us to adapt the argument of Step 8 in the proof of Theorem 3.3 in [4] to establish the following (final)

Step 6. *Assume that* $R_w = \mathbb{R}$. *Then* (3.7) *is valid.* □

Remark 3.8. (i) Step 1 in the proof of Theorem 3.7 implies that $\mathsf{C}^{(1,w)}$ has dense range, i.e., 0 belongs to the continuous spectrum of $\mathsf{C}^{(1,w)}$.

(ii) It is clear from (3.6) that if $\frac{1}{M} \in \sigma_{pt}(\mathsf{C}^{(1,w)})$ for some $M \in \mathbb{N}$, then also $\frac{1}{m} \in \sigma_{pt}(\mathsf{C}^{(1,w)})$ for all $m \in \{1, \dots, M\}$.

(iii) It can happen that $\sigma_{pt}(\mathsf{C}^{(1,w)}) = \emptyset$; see Example 3.13 below. In view of part (i) and Remark 3.2 this is equivalent to $1 \notin \sigma_{pt}(\mathsf{C}^{(1,w)})$, i.e., $w \notin \ell_1$.

(iv) Suppose v, w are bounded, strictly positive sequences such that $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$ and $(\frac{v(n)}{w(n)})_{n=n_0}^{\infty}$ is decreasing for some $n_0 \in \mathbb{N}$. Proposition 2.13(i) implies that $\mathsf{C}^{(1,v)} \in \mathcal{L}(\ell_1(v))$. Let $\lambda \in \rho(\mathsf{C}^{(1,w)}) \setminus \Sigma_0$, i.e., (3.5) holds for $\alpha := \mathrm{Re}\left(\frac{1}{\lambda}\right)$. Setting $\alpha_n := \frac{v(n)}{w(n)}$ for $n \in \mathbb{N}$ it follows, for $m \ge n_0$, that

$$\frac{1}{m^\alpha v(m)} \sum_{n=m+1}^{\infty} \frac{v(n)}{n^{1-\alpha}} = \frac{1}{m^\alpha w(m)} \sum_{n=m+1}^{\infty} \frac{\alpha_m}{\alpha_n} \cdot \frac{w(n)}{n^{1-\alpha}} \le \frac{1}{m^\alpha w(m)} \sum_{n=m+1}^{\infty} \frac{w(n)}{n^{1-\alpha}}.$$

Hence, (3.5) implies that $\gamma := \sup_{m\geq n_0} \frac{1}{m^\alpha v(m)} \sum_{n=m+1}^\infty \frac{v(n)}{n^{1-\alpha}} < \infty$. Set $\delta := \max\{\frac{1}{w(m)} : 1 \leq m < n_0\}$. Then, for $m \in \{1, \dots, n_0 - 1\}$, we have

$$\frac{1}{m^\alpha v(m)} \sum_{n=m+1}^\infty \frac{v(n)}{n^{1-\alpha}} = \frac{1}{m^\alpha v(m)} \sum_{k=m+1}^{n_0} \frac{v(k)}{k^{1-\alpha}} + \frac{n_0^\alpha v(n_0)}{m^\alpha v(m)} \cdot \frac{1}{n_0^\alpha v(n_0)} \sum_{n=n_0+1}^\infty \frac{v(n)}{n^{1-\alpha}}$$

$$\leq \frac{1}{v(m)} \sum_{k=m+1}^{n_0} \frac{v(k)}{k^{1-\alpha}} + \left(\frac{n_0}{m}\right)^\alpha \frac{\gamma v(n_0)}{v(m)} \leq \delta \sum_{k=1}^{n_0} \frac{v(k)}{k^{1-\alpha}} + n_0^\alpha \delta \gamma v(n_0).$$

Accordingly, $\sup_{m\in\mathbb{N}} \frac{1}{m^\alpha v(m)} \sum_{n=m+1}^\infty \frac{v(n)}{n^{1-\alpha}} < \infty$ and so Theorem 3.7(ii), applied to v, shows that $\lambda \in \rho(\mathsf{C}^{(1,v)}) \setminus \Sigma_0$, that is

$$\sigma(\mathsf{C}^{(1,v)}) \subseteq \sigma(\mathsf{C}^{(1,v)}) \cup \Sigma_0 \subseteq \sigma(\mathsf{C}^{(1,w)}) \cup \Sigma_0.$$

Of course, if bounded, strictly positive sequences v and w satisfy $A_1 v \leq w \leq A_2 v$ for positive constants A_1, A_2 and $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$, then Remark 2.15 implies that $\sigma(\mathsf{C}^{(1,w)}) = \sigma(\mathsf{C}^{(1,v)})$. As an application, for fixed $\alpha > 1$ consider the sequence w given by $w(1) = w(2) = 1$ and $w(n) := \frac{1}{i^\alpha 2^{i-1}}$ for $2^i + 1 \leq n \leq 2^{i+1}$ and $i \in \mathbb{N}$. Define $v(n) := \frac{1}{n \log^\alpha(n+1)}$ for $n \in \mathbb{N}$. Then

$$A_1 v \leq w \leq A_2 v \tag{3.11}$$

for positive constants A_1, A_2. To establish (3.11), fix $n \geq 3$ and select $i \in \mathbb{N}$ such that $2^i + 1 \leq n \leq 2^{i+1}$. Then

$$\frac{w(n)}{v(n)} = \frac{n \log^\alpha(n+1)}{i^\alpha 2^{i-1}} \geq \frac{2^i \log^\alpha(2^i)}{i^\alpha 2^{i-1}} = 2 \log^\alpha 2.$$

Since $\frac{w(1)}{v(1)} = \log^\alpha 2$ and $\frac{w(2)}{v(2)} = 2 \log^\alpha 3$ it follows, with $A_1 = 2 \log^\alpha 3$ that the first inequality in (3.11) is satisfied. Concerning the other inequality in (3.11) observe, still with n and i as above, that

$$\frac{w(n)}{v(n)} = \frac{n \log^\alpha(n+1)}{i^\alpha 2^{i-1}} \leq \frac{2^{i+1} \log^\alpha(2^{i+2})}{i^\alpha 2^{i-1}} = 4 \left(\frac{i+2}{i}\right)^\alpha \log^\alpha 2, \; n \geq 3.$$

Since $\lim_{i\to\infty} \frac{i+2}{i} = 1$, there exists $A_2 > 0$ such that $w \leq A_2 v$. This establishes (3.11). It is shown in Example 2.14(ii) that $\mathsf{C}^{(1,v)} \in \mathcal{L}(\ell_1(v))$. According to (3.11) and Remark 2.15 also $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$. The above discussion and (3.11) then imply that

$$\sigma_{pt}(\mathsf{C}^{(1,v)}) = \sigma_{pt}(\mathsf{C}^{(1,w)}) \text{ and } \sigma(\mathsf{C}^{(1,v)}) = \sigma(\mathsf{C}^{(1,w)}). \tag{3.12}$$

Combining Fact 8 of Example 4.17 below with (3.12) yields

$$\sigma_{pt}(\mathsf{C}^{(1,v)}) = \{1\} \text{ and } \sigma(\mathsf{C}^{(1,v)}) = \left\{\lambda \in \mathbb{C} \colon \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\}.$$

It was noted above that always $s_1 \geq 0$. Note, for $w = (\frac{1}{\log(n+1)})_{n\in\mathbb{N}}$, that $s_1 = 0$ and $w \downarrow 0$. But, C does not act in $\ell_1(w)$; see Example 2.5(i). For weights w with $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$ this phenomenon cannot occur.

Proposition 3.9. *Let w be a bounded, strictly positive sequence such that $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$ and $S_w(1) \neq \emptyset$.*

(i) *It is necessarily the case that $s_1 > 0$.*

(ii) *For the dual operator $(\mathsf{C}^{(1,w)})' \in \mathcal{L}(\ell_\infty(w^{-1}))$ of $\mathsf{C}^{(1,w)}$ we have*

$$\left\{\lambda \in \mathbb{C} \colon \left|\lambda - \frac{1}{2s_1}\right| < \frac{1}{2s_1}\right\} \cup \Sigma \subseteq \sigma_{pt}((\mathsf{C}^{(1,w)})') \tag{3.13}$$

and

$$\sigma_{pt}((\mathsf{C}^{(1,w)})') \setminus \Sigma \subseteq \left\{\lambda \in \mathbb{C} \colon \left|\lambda - \frac{1}{2s_1}\right| \leq \frac{1}{2s_1}\right\}. \tag{3.14}$$

For the Cesàro operator $\mathsf{C}^{(1,w)}$ itself we have

$$\left\{\lambda \in \mathbb{C} \colon \left|\lambda - \frac{1}{2s_1}\right| \leq \frac{1}{2s_1}\right\} \cup \Sigma \subseteq \sigma(\mathsf{C}^{(1,w)}). \tag{3.15}$$

Proof. (i) Suppose that $s_1 := \inf S_w(1) = 0$. Fix any $s > 0$. Then $w(n) \geq \frac{c(s)}{n^s}$ for some constant $c(s) > 0$ and all $n \in \mathbb{N}$. Hence, $\sum_{n=1}^\infty \frac{w(n)}{n^{1-s}} \geq c(s) \sum_{n=1}^\infty \frac{1}{n}$ which shows that $\sum_{n=1}^\infty \frac{w(n)}{n^{1-s}}$ *diverges* (for every $s > 0$).

Fix $s > 0$ with $s \notin \Sigma$ and set $\lambda := \frac{1}{s} \in \mathbb{R}$. By the previous paragraph $\sum_{n=1}^\infty \frac{w(n)}{n^{1-s}}$ diverges. Theorem 3.7(ii) implies (put $m = 1$ in (3.5)) that $\lambda \notin \rho(\mathsf{C}^{(1,w)})$, i.e., $\lambda \in \sigma(\mathsf{C}^{(1,w)})$. So, the unbounded set $\{\frac{1}{s} \colon s > 0\} \setminus \Sigma$ is contained in $\sigma(\mathsf{C}^{(1,w)})$; impossible. Hence, $s_1 > 0$.

(ii) We proceed by a series of steps. Denote again by $A \in \mathcal{L}(\ell_\infty(v))$ the dual operator of $\mathsf{C}^{(1,w)}$, where $v := w^{-1}$.

Step 1. $\left\{\lambda \in \mathbb{C} \colon \left|\lambda - \frac{1}{2s_1}\right| < \frac{1}{2s_1}\right\} \subseteq \sigma_{pt}(A)$.

Let $\lambda \in \mathbb{C} \setminus \{0\}$. Then $Ay = \lambda y$ is satisfied for some non-zero $y \in \ell_\infty(v)$ if and only if $\lambda y_n = \sum_{k=n}^\infty \frac{y_k}{k}$ for all $n \in \mathbb{N}$. This yields, for every $n \in \mathbb{N}$, that $\lambda(y_n - y_{n+1}) = \frac{y_n}{n}$ and so $y_{n+1} = \left(1 - \frac{1}{\lambda n}\right) y_n$. It follows that

$$y_{n+1} = y_1 \prod_{k=1}^n \left(1 - \frac{1}{\lambda k}\right), \quad n \in \mathbb{N}, \tag{3.16}$$

with $y_1 \neq 0$. In particular, each eigenvalue of A is simple.

Let now $\lambda \in \mathbb{C} \setminus \Sigma$ satisfy $\left|\lambda - \frac{1}{2s_1}\right| < \frac{1}{2s_1}$ (equivalently, $\alpha := \mathrm{Re}\left(\frac{1}{\lambda}\right) > s_1$); note that $\lambda \neq 0$. For such a λ the vector $y = (y_n)_{n \in \mathbb{N}} \in \mathbb{C}^\mathbb{N}$ defined by (3.16) actually belongs to $\ell_\infty(v)$. Indeed, via Lemma 3.3(i) there exists $c = c(\lambda) > 0$ such that

$$\prod_{k=1}^n \left|1 - \frac{1}{\lambda k}\right| \leq c n^{-\mathrm{Re}(1/\lambda)}, \quad n \in \mathbb{N}.$$

It then follows from (3.16) that

$$|y_n|w(n)^{-1} = |y_1|w(n)^{-1}\prod_{k=1}^{n}\left|1-\frac{1}{\lambda k}\right| \le c|y_1|n^{-\mathrm{Re}(1/\lambda)}w(n)^{-1},$$

where the sequence $(n^{-\mathrm{Re}(1/\lambda)}w(n)^{-1})_{n\in\mathbb{N}}$ is bounded because $\mathrm{Re}(1/\lambda) \in S_w(1)$. That is, $y \in \ell_\infty(v)$. Hence, $\lambda \in \sigma_{pt}(A)$.

Step 2. $\sigma_{pt}(A) \setminus \Sigma_0 \subseteq \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{1}{2s_1}\right| \le \frac{1}{2s_1}\right\}$.

Fix $\lambda \in \sigma_{pt}(A) \setminus \Sigma_0$. According to (3.1) there is $\beta = \beta(\lambda) > 0$ such that

$$\prod_{k=1}^{n}\left|1-\frac{1}{\lambda k}\right| \ge \beta \cdot n^{-\mathrm{Re}(1/\lambda)}, \quad n \in \mathbb{N}. \tag{3.17}$$

But, as argued in Step 1 (for any $y_1 \in \mathbb{C}\setminus\{0\}$) the eigenvector $y = (y_n)_{n\in\mathbb{N}}$ corresponding to the eigenvalue λ of A, which necessarily belongs to $\ell_\infty(v)$, i.e., $\sup_{n\in\mathbb{N}} |y_n|w(n)^{-1} < \infty$, is given by (3.16). Then (3.17) implies that also $\sup_{n\in\mathbb{N}} \frac{1}{n^{\mathrm{Re}(1/\lambda)}w(n)} < \infty$ (i.e., $\mathrm{Re}\left(\frac{1}{\lambda}\right) \in S_w(1)$) and so $\mathrm{Re}\left(\frac{1}{\lambda}\right) \ge s_1$, that is, $\lambda \in \left\{\mu \in \mathbb{C}\colon \left|\mu - \frac{1}{2s_1}\right| \le \frac{1}{2s_1}\right\}$.

It is clear that Steps 1-2 above, together with Steps 1 and 2 in the proof of Theorem 3.7, establish the two containments in (3.13) and (3.14).

For $T \in \mathcal{L}(X)$, with X a Banach space, $\sigma_{pt}(T') \subseteq \sigma(T)$, [13, p.581], with $\sigma(T)$ closed in $\mathbb{C}$. So, (3.15) follows from (3.13). $\square$

Remark 3.10. The converse of Proposition 3.9(i) is not valid. Indeed, for *every* $\alpha > 0$ the exists a weight $w \downarrow 0$ with $s_1 = \alpha$ but, C does *not* act in $\ell_1(w)$. To see this let $(k_j)_{j\in\mathbb{N}} \subseteq \mathbb{N}$ (with $k_1 := 1$) be a strictly increasing sequence satisfying (2.7). Define $w(1) := 1$ and $w(n) := \frac{1}{(k_j+1)^\alpha}$ for each $j \in \mathbb{N}$ and $k_j + 1 \le n \le k_{j+1}$. For $s := \alpha$ observe, if $j \in \mathbb{N}$ and $k_j + 1 \le n \le k_{j+1}$, then

$$\frac{1}{n^s w(n)} = \frac{(k_j+1)^\alpha}{n^\alpha} \le \frac{(k_j+1)^\alpha}{(k_j+1)^\alpha} = 1.$$

Hence, $\sup_{n\in\mathbb{N}} \frac{1}{n^s w(n)} < \infty$ and so $\alpha \in S_w(1)$, i.e., $[\alpha,\infty) \subseteq S_w(1)$. On the other hand, for each $j \in \mathbb{N}$ and $n := k_j + 1$, we have for each $s < \alpha$ that

$$\sup_{j\in\mathbb{N}} \frac{1}{n^s w(n)} = \sup_{j\in\mathbb{N}} \frac{(k_j+1)^\alpha}{(k_j+1)^s} = \infty.$$

It follows that $\sup_{n\in\mathbb{N}} \frac{1}{n^s w(n)} = \infty$ and so $s \notin S_w(1)$. Hence, we have established that $S_w(1) = [\alpha,\infty)$, i.e., $s_1 = \alpha$. Arguing as in the proof of Proposition 2.6(i) it follows that $\sup_{j\in\mathbb{N}} \frac{1}{w(k_j+1)} \sum_{m=k_j+1}^{\infty} \frac{w(m)}{m} = \infty$ and so, via Proposition 2.2(i), C does *not* act in $\ell_1(w)$.

According to Proposition 3.5(iii), such weights w (with $s_1 > 0$) cannot exist in s.

The following result should be compared with [4, Proposition 4.1].

Proposition 3.11. *Let w be a bounded, strictly positive weight such that $\mathsf{C}^{(1,w)} \in \mathcal{K}(\ell_1(w))$. Then the following properties hold.*

(i) $\sigma_{pt}(\mathsf{C}^{(1,w)}) = \Sigma$ *and* $\sigma(\mathsf{C}^{(1,w)}) = \Sigma_0$.

(ii) $w \in s$.

Proof. (i) It is clear that $0 \notin \sigma_{pt}(\mathsf{C}^{(1,w)})$ as $\mathsf{C}\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ is injective. The compactness of $\mathsf{C}^{(1,w)}$ then implies that $\sigma_{pt}(\mathsf{C}^{(1,w)}) = \sigma(\mathsf{C}^{(1,w)}) \setminus \{0\}$, [19, Theorem 3.4.23]. Moreover, Lemma 3.1 reveals that also $\sigma_{pt}(\mathsf{C}^{(1,w)}) \subseteq \sigma_{pt}(\mathsf{C}, \mathbb{C}) = \Sigma$. The two previous facts, together with Theorem 3.7(i), imply the validity of the two equalities in (i).

(ii) By Theorem 3.7(iii) we must have $R_w = \mathbb{R}$. Otherwise, t_0 is finite and so (3.6) implies that $\sigma_{pt}(\mathsf{C}^{(1,w)})$ is a finite set. This is a contradiction to part (i). So, $R_w = \mathbb{R}$ and hence, $w \in s$; see Proposition 3.4. □

Remark 3.12. (i) If w is a bounded, strictly positive weight such that $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$ and $S_w(1) \neq \emptyset$, then (3.15) implies that $\mathsf{C}^{(1,w)} \notin \mathcal{K}(\ell_1(w))$.

(ii) Recall, for $T \in \mathcal{L}(X)$ with X a Banach space, that T is *power bounded* if $\sup_{n\in\mathbb{N}} \|T^n\| < \infty$. If w is as in part (i) and $0 < s_1 < 1$, then $\frac{1}{s_1} > 1$. It follows from (3.15) and the spectral mapping theorem that $[0, \frac{1}{s_1^n}] \subseteq \sigma((\mathsf{C}^{(1,w)})^n)$ for all $n \in \mathbb{N}$. Then the spectral radius inequality implies that $\frac{1}{s_1^n} \leq \|(\mathsf{C}^{(1,w)})^n\|$ for all $n \in \mathbb{N}$. Accordingly, $\mathsf{C}^{(1,w)}$ cannot be power bounded. Since also $\frac{\|(\mathsf{C}^{(1,w)})^n\|}{n} \geq \frac{(s_1^{-1})^n}{n}$ for $n \in \mathbb{N}$, it follows from the Principle of Uniform Boundedness that $\left\{\frac{(\mathsf{C}^{(1,w)})^n}{n}\right\}_{n\in\mathbb{N}}$ cannot converge in $\mathcal{L}(\ell_1(w))$. In particular, $\mathsf{C}^{(1,w)}$ cannot be mean ergodic; see the discussion prior to Lemma 4.1 below.

It is time for some relevant examples.

Example 3.13. (i) Let $w_\alpha(n) = \frac{1}{n^\alpha}$, $n \in \mathbb{N}$, for any fixed $\alpha > 0$. According to Example 2.5(ii) we have $\mathsf{C}^{(1,w_\alpha)} \in \mathcal{L}(\ell_1(w_\alpha))$. It is routine to check that $S_{w_\alpha}(1) = [\alpha, \infty)$ and hence, $s_1 = \alpha > 0$. The claim is that

$$\left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{1}{2\alpha}\right| \leq \frac{1}{2\alpha}\right\} \cup \Sigma = \sigma(\mathsf{C}^{(1,w_\alpha)}). \tag{3.18}$$

Indeed, according to (3.15) we have

$$\left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{1}{2\alpha}\right| \leq \frac{1}{2\alpha}\right\} \cup \Sigma \subseteq \sigma(\mathsf{C}^{(1,w_\alpha)}).$$

To establish the reverse inclusion, fix $\lambda \in \mathbb{C} \setminus \Sigma$ such that $\left|\lambda - \frac{1}{2\alpha}\right| > \frac{1}{2\alpha}$. We show that $\lambda \in \rho(\mathsf{C}^{(1,w_\alpha)})$. To this effect, set $\beta := \operatorname{Re}\left(\frac{1}{\lambda}\right)$. Then $\beta < \alpha$, that is, $(\alpha - \beta) > 0$. Lemma 2.3 implies, for every $m \in \mathbb{N}$, that

$$\sum_{n=m+1}^{\infty} \frac{w_\alpha(n)}{n^{1-\beta}} = \sum_{n=m+1}^{\infty} \frac{1}{n^{1+\alpha-\beta}} \leq \frac{2^{\alpha-\beta}}{(\alpha-\beta)(m+1)^{\alpha-\beta}}$$

and so

$$\sup_{m\in\mathbb{N}} \frac{1}{m^\beta w_\alpha(m)} \sum_{n=m+1}^{\infty} \frac{w_\alpha(n)}{n^{1-\beta}} \le \sup_{m\in\mathbb{N}} \frac{2^{\alpha-\beta}m^{\alpha-\beta}}{(\alpha-\beta)(m+1)^{\alpha-\beta}} \le \frac{2^{\alpha-\beta}}{(\alpha-\beta)}.$$

Hence, Theorem 3.7(ii) implies that $\lambda \in \sigma(\mathsf{C}^{(1,w_\alpha)})$, as claimed.

For $0 < \alpha < 1$ we see that $0 < s_1 < 1$ and so Remark 3.12(ii) implies that $\mathsf{C}^{(1,w_\alpha)}$ is not power bounded.

It is clear from (3.18), as alluded to in the beginning of this section, that there is no constant $K > 0$ such that

$$\sigma(\mathsf{C}^{(1,w)}) \subseteq \{\lambda \in \mathbb{C} \colon |\lambda| \le K\}$$

for all strictly positive, decreasing weights w satisfying $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$.

It is routine to check that $R_{w_\alpha} = (-\infty, (\alpha-1))$ and so $t_0 = (\alpha - 1)$. For $0 < \alpha \le 1$ it follows that $(m-1) \notin R_{w_\alpha}$ for all $m \in \mathbb{N}$, that is, $\sigma_{pt}(\mathsf{C}^{(1,w_\alpha)}) = \emptyset$; see (3.6). This also follows from the fact that $w_\alpha \notin \ell_1$; see Remark 3.2 and Remark 3.8(iii). For $\alpha > 1$ (in which case $w_\alpha \in \ell_1$), it follows from (3.6) that

$$\sigma_{pt}(\mathsf{C}^{(p,w_\alpha)}) = \left\{ \frac{1}{m} \colon m \in \mathbb{N},\ 1 \le m < \alpha \right\}.$$

(ii) Let $w(n) := 1$ if $n = 2^k$, for $k \in \mathbb{N}$, and $w(n) := \frac{1}{n}$ otherwise. It is shown in Remark 2.4(ii) that $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$. The claim is that

$$\sigma(\mathsf{C}^{(1,w)}) = \left\{ \lambda \in \mathbb{C} \colon \left| \lambda - \frac{1}{2} \right| \le \frac{1}{2} \right\}. \tag{3.19}$$

Since $w \notin \ell_1$, Remark 3.8(iii) shows that $\sigma_{pt}(\mathsf{C}^{(1,w)}) = \emptyset$. Let $s > 0$. Then $\frac{1}{n^s w(n)} = \frac{1}{2^{ks}}$ if $n = 2^k$, for $k \in \mathbb{N}$, and $\frac{1}{n^s w(n)} = \frac{n}{n^s}$ if $2^k < n < 2^{k+1}$ for some $k \in \mathbb{N}$. Accordingly, $\sup_{n\in\mathbb{N}} \frac{1}{n^s w(n)} < \infty$ if and only if $s \ge 1$, i.e., $S_w(1) = [1, \infty)$ with $s_1 = 1$. According to (3.15) we have that $\sigma(\mathsf{C}^{(1,w)})$ is contained in the right-side of (3.19). To establish the reverse inclusion, fix $\lambda \in \mathbb{C} \setminus \Sigma$ such that $|\lambda - \frac{1}{2}| > \frac{1}{2}$. To verify $\lambda \in \rho(\mathsf{C}^{(1,w)})$, set $\alpha := \operatorname{Re}\left(\frac{1}{\lambda}\right)$ and $r := \frac{1}{2^{1-\alpha}}$. Then $1 - \alpha > 0$ and $r \in (0,1)$.

For $m = 2^k$, $k \in \mathbb{N}$, it follows from Lemma 2.3 that

$$\begin{aligned}
&\frac{1}{m^\alpha w(m)} \sum_{n=m+1}^{\infty} \frac{w(n)}{n^{1-\alpha}} = \frac{1}{2^{\alpha k}} \sum_{n=m+1}^{\infty} \frac{w(n)}{n^{1-\alpha}} \\
&\le \frac{1}{2^{\alpha k}} \left(\sum_{n=2^k}^{\infty} \frac{1}{n n^{1-\alpha}} + \sum_{j=k}^{\infty} \frac{1}{(2^j)^{1-\alpha}} \right) = \frac{1}{2^{\alpha k}} \left(\sum_{n=2^k}^{\infty} \frac{1}{n^{1+(1-\alpha)}} + \sum_{j=k}^{\infty} r^j \right) \\
&\le \frac{1}{2^{\alpha k}} \left(\frac{2^{1-\alpha}}{(1-\alpha)2^{k(1-\alpha)}} + \frac{r^k}{(1-r)} \right) = \frac{2^{1-\alpha}}{(1-\alpha)2^k} + \frac{1}{(1-r)2^k} \\
&\le \frac{2^{1-\alpha}}{(1-\alpha)} + \frac{1}{(1-r)}.
\end{aligned}$$

On the other hand, if $2^k < m < 2^{k+1}$ for some $k \in \mathbb{N}$, then

$$\begin{aligned}
\frac{1}{m^\alpha w(m)} \sum_{n=m+1}^{\infty} \frac{w(n)}{n^{1-\alpha}} &\leq m^{1-\alpha} \sum_{n=2^k+1}^{\infty} \frac{w(n)}{n^{1-\alpha}} \\
&\leq 2^{(1-\alpha)(k+1)} \left(\sum_{n=2^k}^{\infty} \frac{1}{nn^{1-\alpha}} + \sum_{j=k}^{\infty} \frac{1}{(2^j)^{1-\alpha}} \right) \\
&\leq 2^{(1-\alpha)k} 2^{(1-\alpha)} \left(\frac{2^{1-\alpha}}{(1-\alpha)2^{k(1-\alpha)}} + \frac{1}{(1-r)2^{(1-\alpha)k}} \right) \\
&= \frac{2^{2(1-\alpha)}}{(1-\alpha)} + \frac{2^{1-\alpha}}{(1-r)}.
\end{aligned}$$

So, there exists a constant c (depending on r and α) such that

$$\sup_{m \in \mathbb{N}} \frac{1}{m^\alpha w(m)} \sum_{n=m+1}^{\infty} \frac{w(n)}{n^{1-\alpha}} \leq c.$$

Hence, Theorem 3.7(ii) implies that $\lambda \in \rho(\mathsf{C}^{(1,w)})$. So, the right-side of (3.19) is contained in $\sigma(\mathsf{C}^{(1,w)})$. This completes the argument establishing (3.19). Finally, (3.19) implies that $\mathsf{C}^{(1,w)}$ is *not* compact.

Before presenting the next example we record the following simple fact.

Lemma 3.14. *There exists a constant $c > 0$ such that*

$$c \leq \sum_{j=2^i+1}^{2^{i+1}} \frac{1}{j} \leq 1, \quad i \in \mathbb{N}. \tag{3.20}$$

Proof. Fix $i \in \mathbb{N}$. Then

$$\sum_{j=2^i+1}^{2^{i+1}} \frac{1}{j} \geq \int_{2^i+1}^{2^{i+1}+1} \frac{dx}{x} = \log\left(\frac{2^{i+1}+1}{2^i+1} \right)$$

with $\lim_{i\to\infty} \log\left(\frac{2^{i+1}+1}{2^i+1}\right) = \log(2)$. So, there exists $K \in \mathbb{N}$ with $\sum_{j=2^i+1}^{2^{i+1}} \frac{1}{j} \geq \frac{\log(2)}{2}$, for all $i \geq K$, which implies the existence of $c > 0$ satisfying the first inequality in (3.20). The second inequality in (3.20) follows from

$$\sum_{j=2^i+1}^{2^{i+1}} \frac{1}{j} \leq \sum_{j=2^i+1}^{2^{i+1}} \frac{1}{2^i+1} = \frac{2^i}{2^i+1} \leq 1, \quad i \in \mathbb{N}.$$

□

Proposition 3.11 states if $\mathsf{C}^{(1,w)} \in \mathcal{K}(\ell_1(w))$, then necessarily $w \in s$. The following example shows that the converse is false.

Example 3.15. There exists a strictly positive, decreasing weight $w \in s$ such that $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$, its spectra are given by

$$\sigma(\mathsf{C}^{(1,w)}) = \Sigma_0 \quad \text{and} \quad \sigma_{pt}(\mathsf{C}^{(1,w)})) = \Sigma,$$

but $\mathsf{C}^{(1,w)}$ fails to be compact.

Define the decreasing sequence $w = (w(n))_{n\in\mathbb{N}}$ by $w(1) = w(2) = 1$ and

$$w(n) := \frac{1}{2^i 2^{(i+1)2^{i+1}}}, \quad \text{for } 2^i + 1 \leq n \leq 2^{i+1} \text{ and } i \in \mathbb{N}.$$

Fact 1. The weight $w \in s$.

Since the sequence $(\frac{1}{n^{n+1}})_{n\in\mathbb{N}}$ clearly belongs to s and $w(n) \leq \frac{8}{n^{n+1}}$ for $n \in \mathbb{N}$ (see Example 2.12(ii)), it follows that also $w \in s$.

Fact 2. The operator $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$.

Fix $m \in \mathbb{N}$ with $m \geq 3$. Now choose $i \in \mathbb{N}$ such that $2^i + 1 \leq m \leq 2^{i+1}$. Using the fact, for each $k \in \mathbb{N}$, that $\frac{1}{n} \leq \frac{1}{2^k+1}$ whenever $2^k + 1 \leq n \leq 2^{k+1}$, that each sum of the form $\sum_{n=2^k+1}^{2^{k+1}}(\dots)$ has 2^k terms, and that $\frac{2^k}{2^k+1} \leq 1$, it follows that

$$\begin{aligned}\sum_{n=m}^{\infty} \frac{w(n)}{n} &\leq \sum_{n=2^i+1}^{\infty} \frac{w(n)}{n} = \sum_{k=i}^{\infty} w(2^k+1) \sum_{n=2^k+1}^{2^{k+1}} \frac{1}{n} \\ &\leq \sum_{k=i}^{\infty} w(2^k+1)\frac{2^k}{2^k+1} \leq \sum_{k=i}^{\infty} w(2^k+1).\end{aligned}$$

Due to the definition of $w(2^k + 1)$ for $k \geq i$ we can conclude that

$$\begin{aligned}\sum_{n=m}^{\infty} \frac{w(n)}{n} &\leq \sum_{k=i}^{\infty} \frac{1}{2^k 2^{(k+1)2^{k+1}}} < \sum_{k=i}^{\infty} \frac{1}{2^k 2^{(i+1)2^{i+1}}} \\ &= \frac{1}{2^{(i+1)2^{i+1}}} \sum_{k=i}^{\infty} \frac{1}{2^k} = \frac{1}{2^{(i+1)2^{i+1}}} \cdot \frac{1}{2^{i-1}}.\end{aligned}$$

Since $\frac{1}{w(m)} = \frac{1}{w(2^i+1)} = 2^i 2^{(i+1)2^{i+1}}$, the previous inequality implies that

$$\frac{1}{w(m)} \sum_{n=m}^{\infty} \frac{w(n)}{n} \leq 2.$$

Accordingly, $\sup_{m\geq 3} \frac{1}{w(m)} \sum_{n=m}^{\infty} \frac{w(n)}{n} \leq 2$. Moreover, both $\frac{1}{w(1)} \sum_{n=1}^{\infty} \frac{w(n)}{n} \leq \sum_{n=1}^{\infty} w(n) < \infty$ and $\frac{1}{w(2)} \sum_{n=2}^{\infty} \frac{w(n)}{n} \leq \sum_{n=1}^{\infty} w(n) < \infty$. So, by (2.2) of Proposition 2.2(i) we can conclude that $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$.

Fact 3. The operator $\mathsf{C}^{(1,w)}$ is not compact.

By (2.3) of Proposition 2.2(ii) we need to verify that $\left(\frac{1}{w(m)}\sum_{n=m}^{\infty}\frac{w(n)}{n}\right)_{m\in\mathbb{N}}$ does *not* converge to 0. Fix $i\in\mathbb{N}$. Then, for $m:=2^i+1$, we have

$$\frac{1}{w(2^i+1)}\sum_{n=2^i+1}^{\infty}\frac{w(n)}{n}\geq\frac{1}{w(2^i+1)}\sum_{n=2^i+1}^{2^{i+1}}\frac{w(n)}{n}=\sum_{n=2^i+1}^{2^{i+1}}\frac{1}{n}\geq c$$

with $c>0$ as in Lemma 3.14. Since $\left(\frac{1}{w(2^i+1)}\sum_{n=2^i+1}^{\infty}\frac{w(n)}{n}\right)_{i\in\mathbb{N}}$ is a subsequence of $\left(\frac{1}{w(m)}\sum_{n=m}^{\infty}\frac{w(n)}{n}\right)_{m\in\mathbb{N}}$, we are done.

Fact 4. The spectra are given by $\sigma(\mathsf{C}^{(1,w)})=\Sigma_0$ *and* $\sigma_{pt}(\mathsf{C}^{(1,w)})=\Sigma$.

According to Fact 1 above we have $R_w=\mathbb{R}$ (see Proposition 3.4) and so Theorem 3.7(iii) implies that $\sigma_{pt}(\mathsf{C}^{(1,w)})=\Sigma$.

To verify that $\sigma(\mathsf{C}^{(1,w)})=\Sigma_0$ we need to show that every $\lambda\notin\Sigma_0$ belongs to $\rho(\mathsf{C}^{(1,w)})$. This is achieved by considering the two possible cases. Namely, when $|\lambda-\frac{1}{2}|\leq\frac{1}{2}$ (equivalent to $\alpha:=\operatorname{Re}\left(\frac{1}{\lambda}\right)$ satisfying $\alpha\geq 1$) and when $|\lambda-\frac{1}{2}|>\frac{1}{2}$ (equivalent to $\alpha<1$).

Case (1). Let $\alpha\geq 1$ (i.e., $(\alpha-1)\geq 0$). Then $\lambda\in\rho(\mathsf{C}^{(1,w)})$.

Because $w\in s$ it is clear that $\frac{1}{1^\alpha w(1)}\sum_{n=2}^{\infty}\frac{w(n)}{n^{(1-\alpha)}}=\sum_{n=2}^{\infty}n^{\alpha-1}w(n)<\infty$ and also that $\frac{1}{2^\alpha w(2)}\sum_{n=3}^{\infty}\frac{w(n)}{n^{(1-\alpha)}}=\frac{1}{2^\alpha}\sum_{n=3}^{\infty}n^{\alpha-1}w(n)<\infty$. So, fix $m\in\mathbb{N}$ with $m\geq 3$. Now select $i\in\mathbb{N}$ with $2^i+1\leq m\leq 2^{i+1}$ in which case

$$\sum_{n=m+1}^{\infty}\frac{w(n)}{n^{(1-\alpha)}}\leq\sum_{n=2^i+1}^{\infty}n^{\alpha-1}w(n)=\sum_{k=i}^{\infty}w(2^k+1)\sum_{n=2^k+1}^{2^{k+1}}n^{\alpha-1}.$$

Since $n^{\alpha-1}\leq(2^{k+1})^{\alpha-1}$ for $2^k+1\leq n\leq 2^{k+1}$, with 2^k terms, we have

$$w(2^k+1)\sum_{n=2^k+1}^{2^{k+1}}n^{\alpha-1}\leq w(2^k+1)2^k\cdot(2^{k+1})^{\alpha-1}=\frac{1}{2^k2^{(k+1)2^{k+1}}}\cdot 2^k\cdot(2^{k+1})^{\alpha-1}$$
$$=\frac{1}{2^{(k+1)2^{k+1}}}\cdot\left(\frac{2^{\alpha-1}}{2^\alpha}\right)^{k+1}=\frac{1}{2^{(k+1)2^{k+1}}}\cdot\left(\frac{1}{2}\right)^{k+1}.$$

It follows that

$$\sum_{n=m+1}^{\infty}\frac{w(n)}{n^{(1-\alpha)}}\leq\sum_{k=i}^{\infty}\frac{1}{2^{(k+1)(2^{k+1}-\alpha)}}\cdot\left(\frac{1}{2}\right)^{k+1}.$$

But, for all $k\geq i$ we have $\frac{1}{2^{(k+1)(2^{k+1}-\alpha)}}\leq\frac{1}{2^{(i+1)(2^{i+1}-\alpha)}}$ and so

$$\sum_{n=m+1}^{\infty}\frac{w(n)}{n^{(1-\alpha)}}\leq\frac{1}{2^{(i+1)(2^{i+1}-\alpha)}}\sum_{k=i}^{\infty}\left(\frac{1}{2}\right)^{k+1}=\frac{2^{(i+1)\alpha}}{2^i\cdot 2^{(i+1)2^{i+1}}}.\qquad(3.21)$$

Using $\frac{1}{w(m)} = 2^i \cdot 2^{(i+1)2^{i+1}}$ and $\frac{1}{m^\alpha} \leq \frac{1}{(2^i+1)^\alpha}$ it follows from (3.21) that

$$\frac{1}{m^\alpha w(m)} \sum_{n=m+1}^{\infty} \frac{w(n)}{n^{(1-\alpha)}} \leq \frac{1}{(2^i+1)^\alpha} \cdot 2^i 2^{(i+1)2^{i+1}} \cdot \frac{2^{(i+1)\alpha}}{2^i \cdot 2^{(i+1)2^{i+1}}}$$
$$= 2^\alpha \left(\frac{2^i}{2^i+1}\right)^\alpha \leq 2^\alpha < \infty.$$

Hence, the condition (3.5) in Theorem 3.7(ii) is satisfied, i.e., $\lambda \in \rho(\mathsf{C}^{(1,w)})$.

Case (2). Let $\alpha < 1$. Then $\lambda \in \rho(\mathsf{C}^{(1,w)})$.

Because $w \in s \subseteq \ell_1$ it is clear that $\frac{1}{1^\alpha w(1)} \sum_{n=2}^\infty \frac{w(n)}{n^{(1-\alpha)}} \leq \sum_{n=2}^\infty w(n) < \infty$ and also that $\frac{1}{2^\alpha w(2)} \sum_{n=3}^\infty \frac{w(n)}{n^{(1-\alpha)}} \leq \frac{1}{2^\alpha} \sum_{n=3}^\infty w(n) < \infty$. So, again fix $m \in \mathbb{N}$ with $m \geq 3$ and select $i \in \mathbb{N}$ with $2^i + 1 \leq m \leq 2^{i+1}$. As in Case (1),

$$\sum_{n=m+1}^{\infty} \frac{w(n)}{n^{(1-\alpha)}} \leq \sum_{k=i}^{\infty} w(2^k+1) \sum_{n=2^k+1}^{2^{k+1}} \frac{1}{n^{1-\alpha}}.$$

Since $(1-\alpha) > 0$ and $\frac{1}{n^{1-\alpha}} \leq \frac{1}{(2^k+1)^{1-\alpha}}$ for $2^k + 1 \leq n \leq 2^{k+1}$ it follows that

$$\sum_{n=m+1}^{\infty} \frac{w(n)}{n^{(1-\alpha)}} \leq \sum_{k=i}^{\infty} w(2^k+1) \frac{2^k}{(2^k+1)^{1-\alpha}} \leq \sum_{k=i}^{\infty} w(2^k+1) \frac{2^k}{(2^k)^{1-\alpha}}$$
$$= \sum_{k=i}^{\infty} \frac{1}{2^k \cdot 2^{(k+1)2^{k+1}}} \cdot 2^{k\alpha} \leq \frac{1}{2^{(i+1)2^{i+1}}} \sum_{k=i}^{\infty} \left(\frac{1}{2^{1-\alpha}}\right)^k,$$

where the last inequality uses the fact that $\frac{1}{2^{(k+1)2^{k+1}}} \leq \frac{1}{2^{(i+1)2^{i+1}}}$ for all $k \geq i$. Since, with $A := 1/(1-2^{\alpha-1})$, we have

$$\frac{1}{2^{(i+1)2^{i+1}}} \sum_{k=i}^{\infty} \left(\frac{1}{2^{1-\alpha}}\right)^k = \frac{A}{2^{(i+1)2^{i+1}}} \cdot \frac{1}{2^{(1-\alpha)i}},$$

we can conclude that

$$\sum_{n=m+1}^{\infty} \frac{w(n)}{n^{(1-\alpha)}} \leq \frac{A}{2^{(i+1)2^{i+1}}} \cdot \frac{1}{2^{(1-\alpha)i}}. \tag{3.22}$$

Using $\frac{1}{w(m)} = 2^i \cdot 2^{(i+1)2^{i+1}}$ and the inequality

$$\frac{1}{m^\alpha} = m^{1-\alpha} \cdot \frac{1}{m} \leq (2^{i+1})^{1-\alpha} \cdot \frac{1}{(2^i+1)},$$

it follows from (3.22) and the inequality $\frac{2^i}{2^i+1} < 1$ that

$$\frac{1}{m^\alpha w(m)} \sum_{n=m+1}^{\infty} \frac{w(n)}{n^{(1-\alpha)}} \leq A \cdot \frac{2^i}{(2^i+1)} \cdot \frac{(2^{i+1})^{1-\alpha}}{2^{(1-\alpha)i}} \leq A\, 2^{1-\alpha} < \infty.$$

Again (3.5) in Theorem 3.7(ii) holds, i.e., $\lambda \in \rho(\mathsf{C}^{(1,w)})$. The proof of Fact 4 and hence, the discussion of this example, is thereby complete.

To formulate the final result of this section we require some preliminaries. Let $w \in c_0$ be a decreasing, strictly positive sequence. Then, with a continuous inclusion, we have

$$\ell_p(w) \subseteq c_0(w), \quad 1 \le p < \infty. \tag{3.23}$$

For $p = 1$ this is clear. Fix $1 < p < \infty$. For $x \in \ell_p(w)$ we have

$$|x_n| w(n)^{1/p} = (|x_n|^p w(n))^{1/p} \le \left(\sum_{m=1}^{\infty} |x_m|^p w(m) \right)^{1/p} = \|x\|_{p,w}, \quad n \in \mathbb{N}.$$

Accordingly,

$$0 \le |x_n| w(n) = |x_n| w(n)^{1/p} w(n)^{1/p'} \le w(n)^{1/p'} \|x\|_{p,w}.$$

Since $w \downarrow 0$, it follows that $\lim_{n\to\infty} |x_n| w(n) = 0$, i.e., $x \in c_0(w)$ and

$$\|x\|_{0,w} \le \|w\|_{\infty}^{1/p'} \|x\|_{p,w}, \quad x \in \ell_p(w).$$

For the case $w \in \ell_1$ with $w \downarrow 0$ we have, with a continuous inclusion, that

$$\ell_p(w) \subseteq \ell_1(w), \quad 1 < p < \infty. \tag{3.24}$$

Indeed, define $\mu \colon 2^{\mathbb{N}} \to [0, \infty)$ by $\mu(A) := \sum_{n \in A} w(n)$, for $A \subseteq \mathbb{N}$. Then μ is a *finite*, positive measure and it is well known that $L^p(\mu) \subseteq L^1(\mu)$, for $1 < p < \infty$, with $\|f\|_1 \le \mu(\mathbb{N})^{1/p'} \|f\|_p$. Accordingly,

$$\|x\|_{1,w} \le \left(\sum_{n=1}^{\infty} w(n) \right)^{1/p'} \|x\|_{p,w}, \quad x \in \ell_p(w).$$

The containment (3.24) does not always hold. Indeed, let $w(n) = \frac{1}{\sqrt{n}}$, for $n \in \mathbb{N}$. Fix $p \in (1, \infty)$. Then $x := (\frac{1}{n^\alpha})_{n \in \mathbb{N}}$, for any fixed $\alpha \in (\frac{1}{2p}, \frac{1}{2}]$ satisfies $x \in \ell_p(w)$ but $x \notin \ell_1(w)$. Accordingly, $\ell_p(w) \not\subseteq \ell_1(w)$ for all $1 < p < \infty$.

Proposition 3.16. *Let $w \in c_0$ be decreasing and strictly positive. Then*

$$\cup_{1<p<\infty} \sigma_{pt}(\mathsf{C}^{(p,w)}) \subseteq \sigma_{pt}(\mathsf{C}^{(0,w)}) \subseteq \sigma_{pt}(\mathsf{C}, \mathbb{C}^{\mathbb{N}}) = \Sigma. \tag{3.25}$$

Suppose, in addition, that $w \in \ell_1$ and $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$. Then

$$\cup_{1<p<\infty} \sigma_{pt}(\mathsf{C}^{(p,w)}) \subseteq \sigma_{pt}(\mathsf{C}^{(1,w)}) \subseteq \sigma_{pt}(\mathsf{C}^{(0,w)}) \subseteq \Sigma. \tag{3.26}$$

The proof of the previous result is elementary and is therefore omitted.

4. Iterates of $\mathsf{C}^{(1,w)}$ and mean ergodicity

For X a Banach space, recall that $T \in \mathcal{L}(X)$ is *mean ergodic* (respectively, *uniformly mean ergodic*) if its sequence of Cesàro averages

$$T_{[n]} := \frac{1}{n} \sum_{m=1}^{n} T^m, \quad n \in \mathbb{N}, \tag{4.1}$$

converges to some operator $P \in \mathcal{L}(X)$ in the strong operator topology τ_s, i.e., $\lim_{n\to\infty} T_{[n]} x = Px$ for each $x \in X$, [13, Ch.VIII] (respectively, in the operator norm topology τ_b). According to [13, VIII Corollary 5.2] there then exists the direct decomposition

$$X = \mathrm{Ker}(I - T) \oplus \overline{(I - T)(X)}. \tag{4.2}$$

Moreover, we always have the identities

$$(I - T)T_{[n]} = T_{[n]}(I - T) = \frac{1}{n}(T - T^{n+1}), \qquad n \in \mathbb{N}, \tag{4.3}$$

and, setting $T_{[0]} := I$, that

$$\frac{1}{n}T^n = T_{[n]} - \frac{(n-1)}{n}T_{[n-1]}, \qquad n \in \mathbb{N}. \tag{4.4}$$

An operator $T \in \mathcal{L}(X)$ is *Cesàro bounded* if $\sup_{n\in\mathbb{N}} \|T_{[n]}\| < \infty$. Every mean ergodic operator $T \in \mathcal{L}(X)$ is necessarily Cesàro bounded (by the Principle of Uniform Boundedness) and, via (4.4), also satisfies

$$\tau_s - \lim_{n\to\infty} \frac{1}{n}T^n = 0. \tag{4.5}$$

It is also clear from (4.4) that if T is Cesàro bounded, then $\sup_{n\in\mathbb{N}} \frac{\|T^n\|}{n} < \infty$. If $T \in \mathcal{L}(X)$ is *power bounded* (cf. Remark 3.12(ii)), then T is also Cesàro bounded and $\lim_{n\to\infty} \frac{\|T^n\|}{n} = 0$. Condition (4.5) implies that $\sigma(T) \subseteq \overline{\mathbb{D}}$, [13, p.709, Lemma 1], where $\mathbb{D} := \{\lambda \in \mathbb{C} \colon |\lambda| < 1\}$.

To characterize the mean ergodicity of $\mathsf{C}^{(1,w)}$ we require some preliminary facts.

Lemma 4.1. *Let w be a bounded, strictly positive sequence such that $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$. The following properties are satisfied.*

(i) *Each basis vector $e_r \in (I - \mathsf{C}^{(1,w)})(\ell_1(w))$ for $r \geq 2$.*

(ii) *We have the equalities*

$$\overline{(I - \mathsf{C}^{(1,w)})(\ell_1(w))} = \{x \in \ell_1(w) \colon x_1 = 0\} = \overline{\mathrm{span}\{e_r \colon r \geq 2\}}. \tag{4.6}$$

(iii) *The range of $I - \mathsf{C}^{(1,w)}$ is closed if and only if it coincides with*

$$\{x \in \ell_1(w) \colon x_1 = 0\}.$$

(iv) *The following three conditions are equivalent.*

(a) $\mathrm{Ker}(I - \mathsf{C}^{(1,w)}) \neq \{0\}$.

(b) $\mathrm{Ker}(I - \mathsf{C}^{(1,w)}) = \mathrm{span}\{\mathbf{1}\}$.

(c) $\mathbf{1} \in \ell_1(w)$, *that is,* $w \in \ell_1$.

If $\mathbf{1} \notin \ell_1(w)$, *then* $\mathrm{Ker}(I - \mathsf{C}^{(1,w)}) = \{0\}$.

Proof. (i) This follows from the identities

$$e_{r+1} = (I - \mathsf{C}^{(1,w)})(e_{r+1} - \frac{1}{r}\sum_{k=1}^{r} e_k), \quad r \in \mathbb{N},$$

which can be verified by direct calculation.

(ii) Clearly, $\{x \in \ell_1(w) \colon x_1 = 0\} = \overline{\text{span}\{e_r \colon r \geq 2\}}$. Part (i) implies

$$\{x \in \ell_1(w) \colon x_1 = 0\} \subseteq \overline{(I - \mathsf{C}^{(1,w)})(\ell_1(w))}.$$

On the other hand, since the 1-st coordinate of $\mathsf{C}^{(1,w)}x$ is x_1 for all $x \in \ell_1(w)$, we see that

$$(I - \mathsf{C}^{(1,w)})(\ell_1(w)) \subseteq \{x \in \ell_1(w) \colon x_1 = 0\}.$$

The previous two containments imply (4.6).

(iii) This is a direct consequence of part (ii) and the fact that the subspace $\{x \in \ell_1(w) \colon x_1 = 0\}$ of $\ell_1(w)$ is closed.

(iv) The Cesàro operator $\mathsf{C} \colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ satisfies $\text{Ker}(I - \mathsf{C}) = \text{span}\{\mathbf{1}\}$. Hence, $\text{Ker}(I - \mathsf{C}^{(1,w)}) = \text{span}\{\mathbf{1}\}$ if and only if $\mathbf{1} \in \ell_1(w)$.

If $\mathbf{1} \notin \ell_1(w)$, then $(I - \mathsf{C}^{(1,w)})$ is injective, i.e., $\text{Ker}(I - \mathsf{C}^{(1,w)}) = \{0\}$. □

Lemma 4.2. *Let w be a bounded, strictly positive sequence such that $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$. If $\mathsf{C}^{(1,w)}$ is Cesàro bounded, then necessarily $w \in \ell_1$. In particular, this is the case whenever $\mathsf{C}^{(1,w)}$ is power bounded or mean ergodic.*

Proof. It is known that $\mathsf{C} \colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ is power bounded, uniformly mean ergodic and satisfies both $\text{Ker}(I - \mathsf{C}) = \text{span}\{\mathbf{1}\}$ and

$$(I - \mathsf{C})(\mathbb{C}^{\mathbb{N}}) = \{x \in \mathbb{C}^{\mathbb{N}} \colon x_1 = 0\} = \overline{\text{span}\{e_r\}_{r \geq 2}}; \tag{4.7}$$

see [3, Proposition 4.1], [6, Proposition 4.3].

Observe that the sequence $\{\mathsf{C}_{[n]}e_1\}_{n \in \mathbb{N}}$ converges to $\mathbf{1}$ in $\mathbb{C}^{\mathbb{N}}$. Indeed, we have $e_1 = \mathbf{1} - (0, 1, 1, 1, \ldots)$ and, since $\mathsf{C} \in \mathcal{L}(\mathbb{C}^{\mathbb{N}})$ is power bounded, that

$$(I - \mathsf{C})(\mathbb{C}^{\mathbb{N}}) = \{x \in \mathbb{C}^{N} \colon \lim_{n \to \infty} \mathsf{C}_{[n]}x = 0\},$$

[23, Chap.VIII, §3, Theorem 1]. Hence, the sequence

$$\mathsf{C}_{[n]}e_1 = \mathsf{C}_{[n]}\mathbf{1} - \mathsf{C}_{[n]}(0, 1, 1, 1, \ldots) = \mathbf{1} - \mathsf{C}_{[n]}(0, 1, 1, 1, \ldots), \quad n \in \mathbb{N},$$

converges to $\mathbf{1}$ in $\mathbb{C}^{\mathbb{N}}$ as $n \to \infty$ because $(0, 1, 1, \ldots) \in (I - \mathsf{C})(\mathbb{C}^{\mathbb{N}})$ by (4.7).

We now proceed to verify that $w \in \ell_1$. By assumption $\mathsf{C}^{(1,w)}$ is Cesàro bounded and so $\{\mathsf{C}^{(1,w)}_{[n]}e_1\}_{n \in \mathbb{N}}$ is a bounded subset of $\ell_1(w)$. By Alaoglu's theorem all norm closed balls of $\ell_1(w)$ are $\sigma(\ell_1(w), c_0(w^{-1}))$-compact (i.e., weakly* compact) and, equipped with the topology $\sigma(\ell_1(w), c_0(w^{-1}))$, they are metrizable because $c_0(w^{-1})$ is a separable Banach space, [19, Corollary 2.6.20]. Therefore, there is a subsequence $\{\mathsf{C}^{(1,w)}_{[n(k)]}e_1\}_{k \in \mathbb{N}}$ of $\{\mathsf{C}^{(1,w)}_{[n]}e_1\}_{n \in \mathbb{N}}$ and a vector $u \in \ell_1(w)$ such that $\mathsf{C}^{(1,w)}_{[n(k)]}e_1 \to u$ for the topology $\sigma(\ell_1(w), c_0(w^{-1}))$ as $k \to \infty$. Since the topology $\sigma(\ell_1(w), c_0(w^{-1}))$ is finer than the topology of coordinatewise convergence in $\ell_1(w)$, we can conclude that $\mathsf{C}^{(1,w)}_{[n(k)]}e_1 = \mathsf{C}_{[n(k)]}e_1 \to u$ in $\mathbb{C}^{\mathbb{N}}$ as $k \to \infty$. The previous paragraph then implies that $u = \mathbf{1}$ and so $\mathbf{1} \in \ell_1(w)$. In other words, $w \in \ell_1$. □

Remark 4.3. If $0 < \alpha \leq 1$, then the weight $w_\alpha := \left(\frac{1}{n^\alpha}\right)_{n \in \mathbb{N}}$ satisfies $w_\alpha \notin \ell_1$. By Lemma 4.2, $\mathsf{C}^{(1,w_\alpha)}$ is not Cesàro bounded. The same is true for the weight w in Remark 2.4(ii).

Lemma 4.4. *Let w be a bounded, strictly positive sequence such that $w \in \ell_1$ and $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$. Then*

$$\ell_1(w) = \mathrm{Ker}(I - \mathsf{C}^{(1,w)}) \oplus \overline{(I - \mathsf{C}^{(1,w)})(\ell_1(w))}. \tag{4.8}$$

Proof. Set $f_1 := \mathbf{1}$ and define $f_j := f_1 - \sum_{k=1}^{j-1} e_k$ for $j \geq 2$. Since $w \in \ell_1$, we have $\{f_j\}_{j\in\mathbb{N}} \subseteq \ell_1(w)$. Moreover, (4.6) reveals that

$$\{f_j\}_{j\geq 2} \subseteq \{x \in \ell_1(w) \colon x_1 = 0\} = \overline{\mathrm{span}\{e_r \colon r \geq 2\}}.$$

In particular, this implies that

$$e_1 = (f_1 - f_2) \in \mathrm{span}\{f_1\} \oplus \overline{\mathrm{span}\{e_r \colon r \geq 2\}}.$$

Since $\{e_r\}_{r\in\mathbb{N}}$ is a basis for $\ell_1(w)$, it follows that

$$\ell_1(w) = \mathrm{span}\{f_1\} \oplus \overline{\mathrm{span}\{e_r \colon r \geq 2\}}.$$

The conclusion now follows from Lemma 4.1(ii), (iv). □

Let $m \in \mathbb{N}$. According to [15, Sect. 11.12], C^m is the moment difference operator for the measure on $[0,1]$ given by $d\mu = f_m(t)\,dt$, with

$$f_m(t) := \frac{1}{(m-1)!} \log^{m-1}\left(\frac{1}{t}\right), \quad t \in (0,1].$$

Therefore, the identities

$$(\mathsf{C}^m x)_n = \sum_{k=1}^{n} \binom{n-1}{k-1} x_k \int_0^1 t^{k-1}(1-t)^{n-k} f_m(t)\,dt, \quad n \in \mathbb{N}, \tag{4.9}$$

hold for all $x \in \mathbb{C}^{\mathbb{N}}$; see also [17, p.125].

Lemma 4.5. *Let w be a bounded, strictly positive sequence such that $w \in \ell_1$ and $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$. Then, for each $r \geq 2$, the sequence $\{(\mathsf{C}^{(1,w)})^m e_r\}_{m\in\mathbb{N}}$ converges to 0 in $\ell_1(w)$.*

Proof. Fix $r \geq 2$. By (4.9), for each $m \in \mathbb{N}$, we have $((\mathsf{C}^{(1,w)})^m e_r)_n = 0$ for $1 \leq n < r$ and

$$((\mathsf{C}^{(1,w)})^m e_r)_n = \binom{n-1}{r-1} \int_0^1 t^{r-1}(1-t)^{n-r} f_m(t)\,dt, \quad n \geq r. \tag{4.10}$$

Proceeding as in the proof of [14, Theorem 1], define $g_m(0) := 0$, $g_m(t) := t f_m(t)$ for $0 < t \leq 1$ and

$$a_m := \sup\{g_m(t) \colon t \in [0,1]\}, \quad m \in \mathbb{N}.$$

For each $m \in \mathbb{N}$ we obtain that $|((\mathsf{C}^{(1,w)})^m e_r)_n| \leq \frac{a_m}{r-1}$ for all $n \in \mathbb{N}$. Hence,

$$w(n)|((\mathsf{C}^{(1,w)})^m e_r)_n| \leq \frac{w(n) a_m}{r-1}, \quad n \in \mathbb{N},$$

from which it follows that

$$\|(\mathsf{C}^{(1,w)})^m e_r\|_{1,w} \leq \|w\|_1 \frac{a_m}{r-1}, \quad m \in \mathbb{N}.$$

According to [14, Lemma 1] we have $\lim_{m\to\infty} a_m = 0$, which implies the desired conclusion. □

We can now establish the first main result of this section.

Theorem 4.6. *Let w be a bounded, strictly positive sequence such that $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$.*

(i) *$\mathsf{C}^{(1,w)}$ is power bounded if and only if $\{(\mathsf{C}^{(1,w)})^m\}_{m\in\mathbb{N}}$ converges in $\mathcal{L}_s(\ell_1(w))$ to the projection onto $\operatorname{Ker}(I-\mathsf{C}^{(1,w)})$ along $\overline{(I-\mathsf{C}^{(1,w)})(\ell_1(w))}$. In this case, $\mathsf{C}^{(1,w)}$ is necessarily mean ergodic.*

(ii) *$\mathsf{C}^{(1,w)}$ is mean ergodic if and only if $\mathsf{C}^{(1,w)}$ is Cesàro bounded.*

Proof. (i) Assume that $\mathsf{C}^{(1,w)}$ is power bounded. Then $w \in \ell_1$ by Lemma 4.2. It follows from Lemma 4.4 that (4.8) holds and from Lemma 4.1 that $\operatorname{Ker}(I - \mathsf{C}^{(1,w)}) = \operatorname{span}\{\mathbf{1}\}$ and $\overline{(I - \mathsf{C}^{(1,w)})(\ell_1(w))} = \overline{\operatorname{span}}\{e_r\}_{r\geq 2}$. So, by (4.8), for $x \in \ell_1(w)$ we have $x = y + z$ with $y \in \operatorname{Ker}(I - \mathsf{C}^{(1,w)})$ and $z \in \overline{(I - \mathsf{C}^{(1,w)})(\ell_1(w))}$. Then, for each $m \in \mathbb{N}$, it follows that

$$(\mathsf{C}^{(1,w)})^m x = (\mathsf{C}^{(1,w)})^m y + (\mathsf{C}^{(1,w)})^m z = y + (\mathsf{C}^{(1,w)})^m z. \tag{4.11}$$

Moreover, for each $r \geq 2$, $\lim_{m\to\infty}(\mathsf{C}^{(1,w)})^m e_r = 0$ in $\ell_1(w)$; see Lemma 4.5. Since $\mathsf{C}^{(1,w)}$ is power bounded and $\operatorname{span}\{e_r\}_{r\geq 2}$ is dense in $\overline{(I - \mathsf{C}^{(1,w)})(\ell_1(w))}$ (cf. Lemma 4.1(ii)), it follows that $\lim_{m\to\infty}(\mathsf{C}^{(1,w)})^m z = 0$ in $\ell_1(w)$ for each $z \in \overline{(I - \mathsf{C}^{(1,w)})(\ell_1(w))}$. Hence, $\lim_{m\to\infty}(\mathsf{C}^{(1,w)})^m x = y$ in $\ell_1(w)$; see (4.11).

The assumption of the reverse implication implies, in particular, that $\{(\mathsf{C}^{(1,w)})^m\}_{m\in\mathbb{N}}$ converges in $\mathcal{L}_s(\ell_1(w))$ and so, by the Principle of Uniform Boundedness, $\mathsf{C}^{(1,w)}$ is power bounded.

If a sequence in a locally convex Hausdorff space (briefly, lcHs) is convergent, so is its sequence of averages (to the same limit). Hence, the convergence of $\{(\mathsf{C}^{(1,w)})^m\}_{m\in\mathbb{N}}$ in the lcHs $\mathcal{L}_s(\ell_1(w)) := (\mathcal{L}(\ell_1(w)), \tau_s)$ implies the convergence of $\{\mathsf{C}^{(1,w)}_{[n]}\}_{n\in\mathbb{N}}$ in $\mathcal{L}_s(\ell_1(w))$, i.e., $\mathsf{C}^{(1,w)}$ is mean ergodic.

(ii) If $\mathsf{C}^{(1,w)}$ is mean ergodic then, as noted before, $\mathsf{C}^{(1,w)}$ is also Cesàro bounded.

Assume now that $\mathsf{C}^{(1,w)}$ is Cesàro bounded, in which case $w \in \ell_1$ (cf. Lemma 4.2). Again by Lemma 4.4 we see that (4.8) holds. We need to verify that $\{\mathsf{C}^{(1,w)}_{[n]}\}_{n\in\mathbb{N}}$ is a convergent sequence in $\mathcal{L}_s(\ell_1(w))$. This follows from an argument similar to the one in part (i). □

The following result should be compared with Lemma 4.5.

Corollary 4.7. *Let w be a bounded, strictly positive sequence such that $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$ is power bounded. Then $\lim_{m\to\infty}(\mathsf{C}^{(1,w)})^m e_1 = \mathbf{1}$.*

Proof. Lemma 4.2 implies that $\mathbf{1} \in \ell_1(w)$. According to Theorem 4.6(i) there exists $u \in \operatorname{Ker}(I - \mathsf{C}^{(1,w)})$ such that

$$\lim_{m\to\infty}(\mathsf{C}^{(1,w)})^m e_1 = u \ \text{ in } \ \ell_1(w). \tag{4.12}$$

Since $\mathrm{Ker}(I - \mathsf{C}^{(1,w)}) = \mathrm{span}\{\mathbf{1}\}$, there exists $\lambda \in \mathbb{C}$ such that $u = \lambda \mathbf{1}$. But, the 1-st coordinate of $(\mathsf{C}^{(1,w)})^m e_1$ equals 1 for every $m \in \mathbb{N}$ and so it follows from (4.12) that $\lambda = 1$. □

Remark 4.8. Theorem 4.6 is special for the Cesàro operator acting in $\ell_1(w)$ and is not valid for a general Banach space operator $T \in \mathcal{L}(X)$.

Indeed, concerning part (i) of Theorem 4.6, the proof shows that whenever $\{T^m\}_{m\in\mathbb{N}}$ converges in $\mathcal{L}_s(X)$, then T is necessarily power bounded. To see that the converse is false in general, consider the Banach space $X = C([0,1])$ equipped with the sup-norm $\|\cdot\|_\infty$ and define $T \in \mathcal{L}(X)$ by $Tf := \varphi f$ for $f \in X$, where $\varphi(t) := t$ for $t \in [0,1]$. Since $T^m f = \varphi^m f$ for $f \in X$, with $\|\varphi^m\|_\infty \leq 1$ for all $m \in \mathbb{N}$, it is clear that T is power bounded. However, if $\mathbf{1}$ is the function constantly equal to 1 in $[0,1]$, then the sequence $T^m \mathbf{1} = \varphi^m$, $m \in \mathbb{N}$, converges pointwise on $[0,1]$ to the discontinuous function $\chi_{\{1\}}$. In particular, $\{T^m \mathbf{1}\}_{m\in\mathbb{N}}$ cannot be a convergent sequence in X.

Concerning part (ii) of Theorem 4.6, the mean ergodicity of an operator always implies its Cesàro boundedness. To see that the converse is false in general, let X and T be as in the previous paragraph. Since T is power bounded, it is Cesàro bounded. But, T is not mean ergodic. Indeed,

$$T_{[n]}\mathbf{1} = \frac{1}{n}\sum_{m=1}^{n} \varphi^m, \quad n \in \mathbb{N}.$$

Since $\left(\frac{1}{n}\sum_{m=1}^{n}\varphi^m\right)(t) = \frac{t-t^{n+1}}{n(1-t)}$ for $t \in [0,1)$ and $\left(\frac{1}{n}\sum_{m=1}^{n}\varphi^m\right)(1) = 1$, for all $n \in \mathbb{N}$, it is clear that $\{T_{[n]}\mathbf{1}\}_{n\in\mathbb{N}}$ converges pointwise on $[0,1]$ to the discontinuous function $\chi_{\{1\}}$. In particular, $\{T_{[n]}\mathbf{1}\}_{n\in\mathbb{N}}$ cannot be a convergent sequence in X and so T is not mean ergodic.

Given a bounded, strictly positive sequence w, for the remainder of this section we use the notation

$$X_1(w) := \{x \in \ell_1(w) \colon x_1 = 0\},$$

which is always a closed subspace of $\ell_1(w)$. In the event that $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$, the subspace $X_1(w)$ is clearly invariant for $\mathsf{C}^{(1,w)}$; see (1.1).

Lemma 4.9. *Let w be a bounded, strictly positive sequence such that $w \in \ell_1$ and $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$. Then*

$$(I - \mathsf{C}^{(1,w)})(\ell_1(w)) = (I - \mathsf{C}^{(1,w)})(X_1(w)). \tag{4.13}$$

Proof. Clearly, $(I - \mathsf{C}^{(1,w)})(X_1(w)) \subseteq (I - \mathsf{C}^{(1,w)})(\ell_1(w))$.

To verify the reverse inclusion, we proceed as in the proof of [5, Lemma 4.5]. First observe, via (1.1), that for each $x \in \ell_1(w)$ we have

$$(I - \mathsf{C}^{(1,w)})x = \left(0, x_2 - \frac{x_1+x_2}{2}, x_3 - \frac{x_1+x_2+x_3}{3}, \ldots\right), \tag{4.14}$$

and, in particular, for each $y \in X_1(w)$ that

$$(I - \mathsf{C}^{(1,w)})y = \left(0, \frac{y_2}{2}, y_3 - \frac{y_2 + y_3}{3}, y_4 - \frac{y_2 + y_3 + y_4}{4}, \dots\right). \tag{4.15}$$

Fix $x \in \ell_1(w)$. We apply (4.14) to conclude that

$$x_j - \frac{1}{j}\sum_{k=1}^{j} x_k = \frac{1}{j}\left((j-1)x_j - \sum_{k=1}^{j-1} x_k\right), \quad j \geq 2, \tag{4.16}$$

is the j-th coordinate of the vector $(I - \mathsf{C}^{(1,w)})x$. Set $y_i := x_i - x_1$ for all $i \in \mathbb{N}$. Then the vector $y := (y_i)_{i\in\mathbb{N}}$ belongs to $X_1(w)$ because $w \in \ell_1$ implies that $(0, 1, 1, 1, \dots) \in \ell_1(w)$. We apply (4.15) to conclude that the j-th coordinate of $(I - \mathsf{C}^{(1,w)})y$ is given by (4.16) for $j \geq 2$. Hence,

$$(I - \mathsf{C}^{(1,w)})x = (I - \mathsf{C}^{(1,w)})y \in (I - \mathsf{C}^{(1,w)})(X_1(w)).$$

□

Remark 4.10. The equality (4.13) *fails* whenever $w \notin \ell_1$ and $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$. Indeed, in this case Lemma 4.1(iv) implies that $(I - \mathsf{C}^{(1,w)})$ is injective. This implies that $x := (I - \mathsf{C}^{(1,w)})e_1$ cannot belong to $(I - \mathsf{C}^{(1,w)})(X_1(w))$. So, the containment

$$(I - \mathsf{C}^{(1,w)})(X_1(w)) \subsetneqq (I - \mathsf{C}^{(1,w)})(\ell_1(w))$$

is proper whenever $w \notin \ell_1$. For the existence of weights $w \notin \ell_1$ such that $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$ see Remark 2.4(ii) and also Examples 2.5(ii) with $0 < \alpha \leq 1$.

Given a bounded, strictly positive weight $w = (w(n))_{n\in\mathbb{N}}$, we introduce the associated quantity

$$\mathcal{U}_w := \sup_{m\in\mathbb{N}} \frac{1}{mw(m+1)} \sum_{n=m+1}^{\infty} w(n) = \sup_{r\geq 2} \frac{1}{(r-1)w(r)} \sum_{n=r}^{\infty} w(n). \tag{4.17}$$

It turns out that $\mathcal{U}_w$ is useful for determining certain mean ergodic and related properties of $\mathsf{C}^{(1,w)}$. As a sample, it is clear that $\mathcal{U}_w < \infty$ implies $w \in \ell_1$. Moreover, $\mathcal{U}_w < \infty$ also implies that $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$. This follows directly from Proposition 2.2(i) and the inequality

$$\frac{1}{w(m+1)} \sum_{n=m+1}^{\infty} \frac{w(n)}{n} \leq \frac{1}{mw(m+1)} \sum_{n=m+1}^{\infty} w(n), \quad m \in \mathbb{N}.$$

The following result characterizes the condition $\mathcal{U}_w < \infty$.

Proposition 4.11. *Let w be a bounded, strictly positive sequence such that $w \in \ell_1$ and $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$. The following conditions are equivalent.*

(i) *The range of $I - \mathsf{C}^{(1,w)}$ is closed in $\ell_1(w)$.*
(ii) *$(I - \mathsf{C}^{(1,w)})(\ell_1(w)) = X_1(w)$.*
(iii) *$(I - \mathsf{C}^{(1,w)})(X_1(w)) = X_1(w)$.*

(iv) *The quantity* $\mathcal{U}_w < \infty$.

Proof. (i)$\Leftrightarrow$(ii) follows from Lemma 4.1(iii) and the definition of $X_1(w)$.

(ii)$\Leftrightarrow$(iii) is clear from Lemma 4.9.

(iii)$\Leftrightarrow$(iv). First observe (via (1.1)) that $(I - \mathsf{C}^{(1,w)})$ maps $X_1(w)$ into itself, and that the restriction $(I-\mathsf{C}^{(1,w)})\colon X_1(w) \to X_1(w)$ is both continuous and injective. The injectivity follows from Lemma 4.1(iv) as $\mathbf{1} \notin X_1(w)$.

According to the previous paragraph, condition (iii) is equivalent to the restricted operator $(I - \mathsf{C}^{(1,w)})\colon X_1(w) \to X_1(w)$ being bijective (i.e., surjective). By the Open Mapping Theorem this, in turn, is equivalent to $(I - \mathsf{C}^{(1,w)})\colon X_1(w) \to X_1(w)$ having a continuous inverse.

So, (iii)$\Leftrightarrow$(iv) is equivalent to showing that (iv) holds if and only if the operator $(I-\mathsf{C}^{(1,w)})\colon X_1(w) \to X_1(w)$ is bijective with a continuous inverse. To do this we first note, with $\tilde{w}(n) := w(n+1)$ for $n \in \mathbb{N}$, that the linear shift operator $S\colon X_1(w) \to \ell_1(\tilde{w})$ defined by

$$S(x) := (x_2, x_3, \dots), \quad x \in X_1(w),$$

is an isometric isomorphism of $X_1(w)$ onto $\ell_1(\tilde{w})$. So, it suffices to verify

$$A := S \circ (I - \mathsf{C}^{(1,w)})|_{X_1(w)} \circ S^{-1} \in \mathcal{L}(\ell_1(\tilde{w})),$$

which is given by the formula

$$Ax = \left(\frac{1}{n+1}\left(nx_n - \sum_{k=1}^{n-1} x_k\right)\right)_{n\in\mathbb{N}}, \quad x \in \ell_1(\tilde{w}), \tag{4.18}$$

with $x_0 := 0$ (see the purely algebraic calculations in the proof of Lemma 4.5 in [5]), is bijective with a continuous inverse if and only if (iv) holds.

Now the operator A given by (4.18), when considered from $\mathbb{C}^{\mathbb{N}}$ to $\mathbb{C}^{\mathbb{N}}$, is bijective and routine calculations show that its inverse map $B\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ is determined by the lower triangular matrix $B = (b_{nm})_{n,m\in\mathbb{N}}$ with entries given by $b_{nm} = 0$ if $m > n$, $b_{nm} = \frac{n+1}{n}$ if $m = n$ and $b_{nm} = \frac{1}{m}$ if $1 \leq m < n$. The restriction of the linear map B acts continuously from $\ell_1(\tilde{w})$ into itself if and only if $D := \Phi_{\tilde{w}} \circ B \circ \Phi_{\tilde{w}}^{-1}$ belongs to $\mathcal{L}(\ell_1)$, where $\Phi_{\tilde{w}}\colon \ell_1(\tilde{w}) \to \ell_1$ is the isometric isomorphism given by

$$\Phi_{\tilde{w}}(x) := (w(n+1)x_n)_{n\in\mathbb{N}}, \quad x \in \ell_1(\tilde{w}).$$

Of course, both $\Phi_{\tilde{w}}$ and $\Phi_{\tilde{w}}^{-1}$ can be extended to isomorphisms between $\mathbb{C}^{\mathbb{N}}$ (which we denote by the same symbol as no confusion can occur). The linear operator D (considered from $\mathbb{C}^{\mathbb{N}}$ into itself) is associated with the lower triangular matrix $(\frac{w(n+1)}{w(m+1)} b_{nm})_{n,m\in\mathbb{N}}$. By Lemma 2.1, $D \in \mathcal{L}(\ell_1)$ if and only if $\sup_{m\in\mathbb{N}} \sum_{n=1}^{\infty} \frac{w(n+1)b_{nm}}{w(m+1)} < \infty$. Since $w \in \ell_1$ and $\lim_{m\to\infty} \frac{m+1}{m} = 1$, this condition is equivalent to $\mathcal{U}_w < \infty$ (see (4.17)). This completes the proof of (iii)$\Leftrightarrow$(iv). □

Proposition 4.12. *Let w be a bounded, strictly positive sequence such that $\mathcal{U}_w < \infty$. Then $\mathsf{C}^{(1,w)}$ is power bounded and uniformly mean ergodic.*

Proof. It was already noted that $\mathcal{U}_w < \infty$ implies $w \in \ell_1$ and $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$. From the proof of Lemma 4.5 (and its notation) recall that

$$|(\mathsf{C}^{(1,w)})^m e_r)_n| \leq \frac{a_m}{r-1}, \quad n \in \mathbb{N}, \tag{4.19}$$

for $m \in \mathbb{N}$ and $r \geq 2$. Moreover, $a_m \geq \frac{1}{2} f_m(\frac{1}{2}) > 0$ and $\lim_{m\to\infty} a_m = 0$.

Since $(\mathsf{C}^{(1,w)})^m \mathbf{1} = \mathbf{1}$ for all $m \in \mathbb{N}$ and $\ell_1(w) = \mathrm{span}\{\mathbf{1}\} \oplus X_1(w)$ (by Lemma 4.1 and Lemma 4.4), to show that $\mathsf{C}^{(1,w)}$ is power bounded it suffices to show that $\sup_{m\in\mathbb{N}} \|(\mathsf{C}^{(1,w)})^m x\|_{1,w} < \infty$ for each $x = (0, x_2, x_3, \ldots) \in X_1(w)$. So, fix such an $x \in X_1(w)$, in which case $x = \sum_{r=2}^{\infty} x_r e_r$. Recall from the proof Lemma 4.5, for each $m \in \mathbb{N}$, that $(\mathsf{C}^{(1,w)})^m e_r)_n = 0$ if $r \geq 2$ and $1 \leq n < r$. Accordingly, for each $n,\ m \in \mathbb{N}$ we have

$$\begin{aligned} w(n)|((\mathsf{C}^{(1,w)})^m x)_n| &\leq w(n) \sum_{r=2}^{\infty} |x_r| \cdot |((\mathsf{C}^{(1,w)})^m e_r)_n| \\ &= w(n) \sum_{r=2}^{n} |x_r| \cdot |((\mathsf{C}^{(1,w)})^m e_r)_n|. \end{aligned}$$

Hence, for every $m,\ n \in \mathbb{N}$ and $x \in X_1(w)$ it follows that

$$\begin{aligned} \|(\mathsf{C}^{(1,w)})^m x\|_{1,w} &= \sum_{n=2}^{\infty} w(n)|((\mathsf{C}^{(1,w)})^m x)_n| \\ &\leq \sum_{n=2}^{\infty} w(n) \sum_{r=2}^{n} |x_r| \cdot |((\mathsf{C}^{(1,w)})^m e_r)_n| \\ &= \sum_{r=2}^{\infty} w(r)|x_r| \frac{1}{w(r)} \sum_{n=r}^{\infty} w(n)|((\mathsf{C}^{(1,w)})^m e_r)_n| \\ &\leq \|(a_m)_{m\in\mathbb{N}}\|_\infty \|x\|_{1,w} \sup_{r\geq 2} \frac{1}{(r-1)w(r)} \sum_{n=r}^{\infty} w(n), \end{aligned}$$

where the last inequality relies on (4.19). An examination of (4.17) now shows that $\mathcal{U}_w < \infty$ implies that $\sup_{m\in\mathbb{N}} \|(\mathsf{C}^{(1,w)})^m x\|_{1,w} < \infty$ for each $x \in X_1(w)$. As already noted, this yields that $\mathsf{C}^{(1,w)}$ is power bounded.

Using now the fact that $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$ is power bounded, we have $\lim_{m\to\infty} \frac{\|\mathsf{C}^{(1,w)})^m\|}{m} = 0$. Since also the range of $I - \mathsf{C}^{(1,w)}$ is a closed subspace of $\ell_1(w)$ (cf. Proposition 4.11), we can apply a result of Lin, [18, Theorem], to conclude that $\mathsf{C}^{(1,w)}$ is uniformly mean ergodic. □

Remark 4.13. Let w be a bounded, strictly positive sequence such that $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$ and $\lim_{m\to\infty} \frac{\|(\mathsf{C}^{(1,w)})^m\|}{m} = 0$. Then $\mathsf{C}^{(1,w)}$ is uniformly mean ergodic if and only if $\mathcal{U}_w < \infty$. Indeed, by Proposition 4.12 the condition $\mathcal{U}_w < \infty$ implies uniform mean ergodicity. On the other hand, if $\mathsf{C}^{(1,w)}$ is uniformly mean ergodic (in which case $w \in \ell_1$ by Lemma 4.2), then Lin's theorem, [18], ensures that $I - \mathsf{C}^{(1,w)}$ has closed range in $\ell_1(w)$. Hence, $\mathcal{U}_w < \infty$; see Proposition 4.11.

Proposition 4.14. *Let w be a bounded, strictly positive sequence such that $\mathsf{C}^{(1,w)} \in \mathcal{K}(\ell_1(w))$. Then necessarily $\mathcal{U}_w < \infty$.*

In particular, $\mathsf{C}^{(1,w)}$ is both power bounded and uniformly mean ergodic.

Proof. Proposition 3.11(ii) shows $w \in \ell_1$. Moreover, the compactness of $\mathsf{C}^{(1,w)}$ implies that $(I - \mathsf{C}^{(1,w)})(\ell_1(w))$ is closed in $\ell_1(w)$, [19, Lemma 3.4.20]. Now apply Proposition 4.11 to conclude that $\mathcal{U}_w < \infty$. Hence, $\mathsf{C}^{(1,w)}$ is power bounded and uniformly mean ergodic by Proposition 4.12. □

Remark 4.15. According to Proposition 2.9, $\mathsf{C}^{(1,w)}$ is compact whenever $\limsup_{n\to\infty} \frac{w(n+1)}{w(n)} \in [0,1)$. In particular, this is the case for $w = (n^\beta r^n)_{n\in\mathbb{N}}$ with $r \in (0,1)$ and $\beta \geq 0$, for $w = (\frac{1}{n^n})_{n\in\mathbb{N}}$ and for $w = (\frac{a^n}{n!})_{n\in\mathbb{N}}$ with $a > 0$; see Examples 2.10(i)-(iii). Proposition 4.14 implies in all cases that $\mathsf{C}^{(1,w)}$ is power bounded and uniformly mean ergodic. By the same reasoning the Cesàro operator corresponding to each of the weights in (iv), (v), (vi) of Example 2.10 is power bounded and uniformly mean ergodic.

Example 4.16. (i) Consider $w_\alpha(n) = (\frac{1}{n^\alpha})_{n\in\mathbb{N}}$ for fixed $\alpha > 0$. For $\alpha \in (0,1]$, Remark 4.3 implies that $\mathsf{C}^{(1,w_\alpha)} \in \mathcal{L}(\ell_1(w_\alpha))$ is *not* Cesàro bounded and hence, is *neither* mean ergodic nor power bounded. The same is true for the weight w in Remark 2.4(ii).

On the other hand if $\alpha > 1$, then it follows from Lemma 2.3 that $\sum_{n=m}^\infty \frac{1}{n^\alpha} \leq \frac{1}{(\alpha-1)(m-1)^{\alpha-1}}$. Thus, for each $m \geq 2$,

$$\frac{1}{(m-1)w_\alpha(m)} \sum_{n=m}^\infty w_\alpha(n) \leq \frac{m^\alpha}{(\alpha-1)(m-1)^\alpha} \leq \frac{2^\alpha}{(\alpha-1)}$$

and so $\mathcal{U}_{w_\alpha} < \infty$. Hence, Proposition 4.12 implies that $\mathsf{C}^{(1,w_\alpha)}$ is power bounded and uniformly mean ergodic. However, $\mathsf{C}^{(1,w_\alpha)}$ is *not* compact; see Remark 2.7(iii). Observe that $w_\alpha \in \ell_1 \setminus s$ for $\alpha > 1$.

(ii) Let $w \in s$ be the weight considered in Example 3.15. The claim is that $\mathcal{U}_w < \infty$. To see this fix $n \in \mathbb{N}$ with $n \geq 2$ and choose $i \in \mathbb{N}$ such that $2^i + 1 \leq n \leq 2^{i+1}$. Then

$$\sum_{m=n}^\infty w(m) \leq \sum_{m=2^i+1}^\infty w(m) = \sum_{j=i}^\infty \sum_{m=2^j+1}^{2^{j+1}} w(m).$$

Since each sum $\sum_{m=2^j+1}^{2^{j+1}}(\ldots)$ has 2^j terms and $w(m) = \frac{1}{2^j 2^{(j+1)2^{j+1}}}$ for all $2^j + 1 \leq m \leq 2^{j+1}$, it follows that

$$\sum_{m=n}^\infty w(m) \leq \sum_{j=i}^\infty 2^j \frac{1}{2^j 2^{(j+1)2^{j+1}}} = \sum_{j=i}^\infty \frac{1}{2^{(j+1)2^{j+1}}}.$$

As $\frac{1}{(n-1)} \le \frac{1}{2^i}$ and $\frac{1}{w(n)} = 2^i 2^{(i+1)2^{i+1}}$, the previous inequality implies

$$\begin{aligned}\frac{1}{(n-1)w(n)}\sum_{m=n}^{\infty} w(m) &\le \frac{1}{2^i}\cdot 2^i 2^{(i+1)2^{i+1}}\sum_{j=i}^{\infty}\frac{1}{2^{(j+1)2^{j+1}}}\\ &= 2^{(i+1)2^{i+1}}\sum_{j=i}^{\infty}\frac{1}{2^{j+1}}\cdot\frac{1}{2^{(j+1)(2^{j+1}-1)}}.\end{aligned}$$

But, $\frac{1}{2^{(j+1)(2^{j+1}-1)}} \le \frac{1}{2^{(i+1)(2^{i+1}-1)}}$ for all $j \ge i$ and so

$$\frac{1}{(n-1)w(n)}\sum_{m=n}^{\infty} w(m) \le 2^{(i+1)2^{i+1}}\cdot\frac{1}{2^{(i+1)(2^{i+1}-1)}}\sum_{j=i}^{\infty}\frac{1}{2^{j+1}} = 2.$$

According to (4.17) we have $\mathcal{U}_w \le 2$. Then Proposition 4.12 shows that $\mathsf{C}^{(1,w)}$ is power bounded and uniformly mean ergodic. But, $\mathsf{C}^{(1,w)}$ is *not* compact; see Fact 3 in Example 3.15.

The final example exhibits features different to the previous examples (eg. $\mathcal{U}_w = \infty$). Its spectrum is also precisely determined.

Example 4.17. Let $\alpha > 1$. Define the bounded, strictly positive weight w by $w(1) = w(2) := 1$ and $w(n) := \frac{1}{i^\alpha 2^{i-1}}$ for $2^i + 1 \le n \le 2^{i+1}$ and $i \in \mathbb{N}$. We record various properties of w.

Fact 1. $w \in \ell_1$, but $w \notin s$.

Define $v := \left(\frac{1}{n\log^\alpha(n+1)}\right)_{n\in\mathbb{N}}$. It is shown in (3.11) of Remark 3.8(iv) that $A_1 v \le w \le A_2 v$ for positive constants A_1, A_2. The integral test for convergence of series implies that $v \in \ell_1$ and hence, also $w \in \ell_1$. Clearly, $v \notin s$ and so also $w \notin s$.

Fact 2. $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$.

This was established in Remark 3.8(iv).

Fact 3. $\mathcal{U}_w = \infty$.

Fix $m \ge 3$ and choose $i \in \mathbb{N}$ to satisfy $2^i + 1 \le m \le 2^{i+1}$. Then

$$\frac{1}{(m-1)w(m)}\sum_{n=m}^{\infty} w(n) \ge \frac{1}{(m-1)w(m)}\sum_{n=2^{i+1}+1}^{\infty} w(n)$$
$$= \frac{1}{(m-1)w(m)}\sum_{j=i+1}^{\infty}\sum_{n=2^j+1}^{2^{j+1}}\frac{1}{j^\alpha 2^{j-1}} = \frac{1}{(m-1)w(m)}\sum_{j=i+1}^{\infty}\frac{2^j}{j^\alpha 2^{j-1}}.$$

Since $\frac{1}{(m-1)} \ge \frac{1}{2^{i+1}-1}$ and $\frac{1}{w(m)} = i^\alpha 2^{i-1}$, it follows that

$$\frac{1}{(m-1)w(m)}\sum_{n=m}^{\infty} w(n) \ge \frac{1}{(2^{i+1}-1)} i^\alpha 2^{i-1} 2\sum_{j=i+1}^{\infty}\frac{1}{j^\alpha} = \frac{i^\alpha 2^i}{(2^{i+1}-1)}\sum_{j=i+1}^{\infty}\frac{1}{j^\alpha}.$$

But, $\sum_{j=i+1}^{\infty}\frac{1}{j^\alpha}\geq\int_{i+1}^{\infty}\frac{dx}{x^\alpha}=\frac{1}{(\alpha-1)(i+1)^{\alpha-1}}$ and so

$$\frac{1}{(m-1)w(m)}\sum_{n=m}^{\infty}w(n)\geq\frac{i^\alpha 2^i}{(\alpha-1)(i+1)^{\alpha-1}(2^{i+1}-1}$$
$$=\frac{i}{(\alpha-1)}\cdot\left(\frac{i}{i+1}\right)^{\alpha-1}\frac{2^i}{2^{i+1}-1}.$$

Since $\lim_{i\to\infty}\left(\frac{i}{i+1}\right)^{\alpha-1}=1$ and $\frac{2^i}{2^{i+1}-1}=\frac{1}{2-2^{-i}}>\frac{1}{2}$, it follows from the previous inequality that

$$\mathcal{U}_w=\sup_{m\geq 2}\frac{1}{(m-1)w(m)}\sum_{n=m}^{\infty}w(n)=\infty.$$

Fact 4. $\mathsf{C}^{(1,w)}$ *is not compact.*

This is immediate from Proposition 4.14.

Fact 5. The range of $I-\mathsf{C}^{(1,w)}$ *is not closed in* $\ell_1(w)$.

See Facts 1 and 2 and Proposition 4.11.

Fact 6. $\ell_1(w)=\mathrm{span}\{\mathbf{1}\}\oplus\overline{(I-\mathsf{C}^{(1,w)})(\ell_1(w))}$.

Follows from Facts 1 and 2 and Lemma 4.4.

Fact 7. $S_w(1)=(1,\infty)$ *and* $s_1=1$.

Fix $s>0$. From the definition of w we have

$$\begin{aligned}\sup_{n\in\mathbb{N}}\frac{1}{n^s w(n)} &= \max\left\{1,\frac{1}{2^s},\sup_{i\in\mathbb{N}}\left(\max_{n=2^i+1,\dots,2^{i+1}}\frac{1}{n^s w(n)}\right)\right\}\\ &= \max\left\{1,\frac{1}{2^s},\sup_{i\in\mathbb{N}} i^\alpha 2^{i-1}\left(\max_{n=2^i+1,\dots,2^{i+1}}\frac{1}{n^s}\right)\right\}.\end{aligned}$$

Since $\frac{1}{2^{s(i+1)}}\leq\frac{1}{n^s}\leq\frac{1}{2^{si}}$ for all $2^i+1\leq n\leq 2^{i+1}$ and $i\in\mathbb{N}$, it follows that

$$\max\left\{1,\frac{1}{2^s},\sup_{i\in\mathbb{N}}\frac{1}{2^{s+1}}\frac{i^\alpha}{2^{i(s-1)}}\right\}\leq\sup_{n\in\mathbb{N}}\frac{1}{n^s w(n)}\leq\max\left\{1,\frac{1}{2^s},\sup_{i\in\mathbb{N}}\frac{1}{2}\cdot\frac{i^\alpha}{2^{i(s-1)}}\right\}.$$

Accordingly, $\sup_{n\in\mathbb{N}}\frac{1}{n^s w(n)}<\infty$ if and only if $s>1$, i.e., $S_w(1)=(1,\infty)$. Hence, $s_1=\inf S_w(1)=1$.

Fact 8. $\sigma_{pt}(\mathsf{C}^{(1,w)})=\{1\}$ *and* $\sigma(\mathsf{C}^{(1,w)})=\{\lambda\in\mathbb{C}\colon|\lambda-\frac{1}{2}|\leq\frac{1}{2}\}$.

Since $w\in\ell_1$, we have $1\in\sigma_{pt}(\mathsf{C}^{(1,w)})$; see Remark 3.2. Moreover, $s_1=1$ and so Proposition 3.5(i) implies that $t_0\leq 1$. Then (3.6) shows that $\sigma_{pt}(\mathsf{C}^{(1,w)})\subseteq\{1,\frac{1}{2}\}$. Hence, to establish that $\sigma_{pt}(\mathsf{C}^{(1,w)})=\{1\}$ it suffices to show that $\frac{1}{2}\notin\sigma_{pt}(\mathsf{C}^{(1,w)})$, i.e., that $1\notin R_w$ (see (3.6)). That this is indeed so follows from the inequalities

$$\begin{aligned}\sum_{n=1}^{\infty}n^1 w(n) &= 2+\sum_{i=1}^{\infty}\sum_{n=2^i+1}^{2^{i+1}}\frac{n}{i^\alpha 2^{i-1}}\geq 2+\sum_{i=1}^{\infty}\sum_{n=2^i+1}^{2^{i+1}}\frac{2^i}{i^\alpha 2^{i-1}}\\ &= 2+2\sum_{i=1}^{\infty}\sum_{n=2^i+1}^{2^{i+1}}\frac{1}{i^\alpha}\geq\sum_{i=1}^{\infty}\frac{2^i}{i^\alpha}=\infty.\end{aligned}$$

Hence, the point spectrum $\sigma_{pt}(\mathsf{C}^{(1,w)}) = \{1\}$.

Since $s_1 = 1$, it follows from Proposition 3.9(ii) that

$$\left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\} \subseteq \sigma(\mathsf{C}^{(1,w)}). \tag{4.20}$$

For the reverse inclusion, let $\lambda \in \mathbb{C}$ satisfy $\left|\lambda - \frac{1}{2}\right| > \frac{1}{2}$ and set $\beta := \mathrm{Re}\left(\frac{1}{\lambda}\right)$. Then $\beta < 1$, i.e., $(1-\beta) > 0$. Fix $m \geq 3$ and select $i \in \mathbb{N}$ such that $(2^i + 1) < m + 1 < 2^{i+1}$ (note that also $(2^i + 1) \leq m < 2^{i+1}$). Then

$$\sum_{n=m+1}^{\infty} \frac{w(n)}{n^{1-\beta}} \leq \sum_{n=2^i+1}^{\infty} \frac{w(n)}{n^{1-\beta}} = \sum_{j=i}^{\infty} \sum_{n=2^j+1}^{2^{j+1}} \frac{1}{j^\alpha 2^{j-1}} \cdot \frac{1}{n^{1-\beta}}.$$

Since $\frac{1}{w(m)} = i^\alpha 2^{i-1}$ with $\frac{1}{m^\beta} \leq (\frac{1}{2^i})^\beta$ and $\frac{1}{n^{1-\beta}} \leq \frac{1}{(2^j)^{1-\beta}}$ for $(2^i + 1) \leq n \leq 2^{i+1}$, it follows that

$$\frac{1}{m^\beta w(m)} \sum_{n=m+1}^{\infty} \frac{w(n)}{n^{1-\beta}} \leq \frac{i^\alpha 2^{i-1}}{2^{i\beta}} \sum_{j=i}^{\infty} \frac{1}{j^\alpha 2^{j-1}} \cdot \frac{1}{(2^j)^{1-\beta}} \cdot 2^j.$$

But, $\frac{i^\alpha}{j^\alpha} \leq 1$ for all $j \geq i$ and so, for all $m \geq 3$, we have

$$\frac{1}{m^\beta w(m)} \sum_{n=m+1}^{\infty} \frac{w(n)}{n^{1-\beta}} \leq 2^{i(1-\beta)} \sum_{j=i}^{\infty} \left(\frac{1}{2^{1-\beta}}\right)^j = \frac{2^{1-\beta}}{2^{1-\beta} - 1}.$$

On the other hand, recalling that $w \in \ell_1$, we also have $\frac{1}{w(1)} \sum_{n=2}^{\infty} \frac{w(n)}{n^{1-\beta}} \leq \sum_{n=2}^{\infty} w(n) < \infty$ and $\frac{1}{2^\beta w(2)} \sum_{n=3}^{\infty} \frac{w(n)}{n^{1-\beta}} \leq \frac{1}{2^\beta} \sum_{n=1}^{\infty} w(n) < \infty$. Accordingly,

$$\sup_{m \in \mathbb{N}} \frac{1}{m^\beta w(m)} \sum_{n=m+1}^{\infty} \frac{w(n)}{n^{1-\beta}} < \infty$$

and so Theorem 3.7(ii) implies that $\lambda \in \rho(\mathsf{C}^{(1,w)})$. Hence, $\{\lambda \in \mathbb{C}\colon |\lambda - \frac{1}{2}| > \frac{1}{2}\} \subseteq \rho(\mathsf{C}^{(1,w)})$ which implies that (4.20) is an equality.

It would be interesting to know whether or not $\mathsf{C}^{(1,w)}$ (equivalently, $\mathsf{C}^{(1,v)}$; see Fact 1) is power bounded.

Concerning the dynamics of a continuous linear operator T defined on a separable Banach space X, recall that T is *hypercyclic* if there exists $x \in X$ such that the orbit $\{T^n x\colon n \in \mathbb{N}_0\}$ is dense in X. If, for some $x \in X$, the projective orbit $\{\lambda T^n x\colon \lambda \in \mathbb{C},\ n \in \mathbb{N}_0\}$ is dense in X, then T is called *supercyclic*. Clearly, hypercyclicity implies supercyclicity.

Proposition 4.18. *Let w be a bounded, strictly positive sequence such that $\mathsf{C}^{(1,w)} \in \mathcal{L}(\ell_1(w))$. Then $\mathsf{C}^{(1,w)}$ is not supercyclic and so, not hypercyclic.*

Proof. By Step 2 in the proof of Theorem 3.7 the *infinite* set $\Sigma \subseteq \sigma_{pt}((\mathsf{C}^{(1,w)})')$. By Theorem 3.2 of [7], $\mathsf{C}^{(1,w)}$ cannot be *supercyclic*. □

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