THE FRÉCHET SPACES $\mathbf{ces(p+)}, 1 < p < \infty$

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Abstract. The Banach spaces $ces(p), 1 < p < \infty$, were intensively studied by G. Bennett and others. The *largest* solid Banach lattice in $\mathbb{C}^{\mathbb{N}}$ which contains ℓ_p and which the Cesàro operator $\mathsf{C} : \mathbb{C}^{\mathbb{N}} \longrightarrow \mathbb{C}^{\mathbb{N}}$ maps into ℓ_p is $ces(p)$. For each $1 \leq p < \infty$, the (positive) operator C also maps the Fréchet space $\ell_{p+} = \bigcap_{q>p} \ell_q$ into itself. It is shown that the *largest* solid Fréchet lattice in $\mathbb{C}^{\mathbb{N}}$ which contains ℓ_{p+} and which C maps into ℓ_{p+} is precisely $ces(p+) := \bigcap_{q>p} ces(q)$. Although the spaces ℓ_{p+} are well understood, it seems that the spaces $ces(p+)$ have not been considered at all. A detailed study of the Fréchet spaces $ces(p+), 1 \leq p < \infty$, is undertaken. They are very different to the Fréchet spaces ℓ_{p+} which generate them in the above sense. We prove that each $ces(p+)$ is a power series space of finite type and order one, and that all the spaces $ces(p+), 1 \leq p < \infty$, are isomorphic.

1. Introduction

Given an element $x = (x_n)_n = (x_1, x_2, \ldots)$ of $\mathbb{C}^{\mathbb{N}}$ let $|x| := (|x_n|)_n$ and write $x \geq 0$ if $x = |x|$. By $x \leq y$ we mean that $(y - x) \geq 0$. The sequence space $\mathbb{C}^{\mathbb{N}}$ is a (locally convex) Fréchet space with respect to the coordinatewise convergence. For each $1 < p < \infty$ define

$$ces(p) := \{x \in \mathbb{C}^{\mathbb{N}} : \|x\|_{ces(p)} := \|(\frac{1}{n}\sum_{k=1}^{n} |x_k|)_n\|_p < \infty\}, \qquad (1.1)$$

where $\|\cdot\|_p$ denotes the standard norm in ℓ_p. An intensive study of the Banach spaces $ces(p), 1 < p < \infty$, was undertaken in [6],[13]; see also the references therein. They are reflexive, p-concave Banach lattices (for the order induced by the positive cone of the Fréchet lattice $\mathbb{C}^{\mathbb{N}}$) and the canonical vectors $e_k := (\delta_{nk})_n$, for $k \in \mathbb{N}$, form an unconditional basis, [6], [8]. For every pair $1 < p, q < \infty$ the space $ces(p)$ is *not* isomorphic to ℓ_q, [6, Proposition 15.13], and is also *not* isomorphic to $ces(q)$ if $p \neq q$, [4, Proposition 3.3].

The Cesàro operator $\mathsf{C} : \mathbb{C}^{\mathbb{N}} \longrightarrow \mathbb{C}^{\mathbb{N}}$, defined by

$$\mathsf{C}(x) := (x_1, \frac{x_1 + x_2}{2}, \ldots, \frac{x_1 + x_2 + \ldots + x_n}{n}, \ldots), \quad x \in \mathbb{C}^{\mathbb{N}}, \qquad (1.2)$$

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satisfies $|\mathsf{C}(x)| \leq \mathsf{C}(|x|)$ for $x \in \mathbb{C}^{\mathbb{N}}$ and is a topological isomorphism of $\mathbb{C}^{\mathbb{N}}$ onto itself. It is clear from (1.1) that $\|x\|_{ces(p)} = \|\mathsf{C}(|x|)\|_p$ for $x \in ces(p)$. Hardy's inequality, [15, Theorem 326], ensures that $\ell_p \subseteq ces(p)$ with $\|x\|_{ces(p)} \leq p'\|x\|_p$ for $x \in \ell_p$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, $\ell_p \subseteq ces(p)$ is a *proper* containment, [8, Remark 2.2]. It is routine to verify that C maps $ces(p)$ continuously into ℓ_p. The following remarkable fact (due to Bennett, [6, Theorem 20.31]) reveals a special feature of $ces(p)$.

Proposition 1.1. *Let $1 < p < \infty$ and $x \in \mathbb{C}^{\mathbb{N}}$. Then*

$$x \in ces(p) \ \textit{ if and only if } \ \mathsf{C}(|x|) \in ces(p). \tag{1.3}$$

The spaces $ces(p)$ also arise in a different way. Fix $1 < p < \infty$. Since the Cesàro operator $\mathsf{C}_p : \ell_p \longrightarrow \ell_p$ (i.e., C restricted to ℓ_p) is a *positive* operator between Banach lattices, it is natural to seek continuous, ℓ_p*-valued* extensions of C_p to Banach lattices $X \subseteq \mathbb{C}^{\mathbb{N}}$ which are larger than ℓ_p and *solid* (i.e., $y \in \mathbb{C}^{\mathbb{N}}$ and $|y| \leq |x|$ with $x \in X$ implies that $y \in X$). The *largest* of all those solid Banach lattices in $\mathbb{C}^{\mathbb{N}}$ which contain ℓ_p and for which such a continuous, ℓ_p-valued extension of C_p is possible is precisely $ces(p)$, [8, p.62]. Of course, this "largest extension" $\mathsf{C} : ces(p) \longrightarrow \ell_p$ is the restriction of C from $\mathbb{C}^{\mathbb{N}}$ to $ces(p)$.

For each $1 \leq p < \infty$ define the vector space $\ell_{p+} := \bigcap_{q>p} \ell_q$; it is a Fréchet space (and lattice for the order induced by the positive cone of $\mathbb{C}^{\mathbb{N}}$) with respect to the increasing sequence of *lattice norms*

$$x \longmapsto \|x\|_{p_k}, \quad x \in \ell_{p+}, \quad k \in \mathbb{N}, \tag{1.4}$$

for any sequence $p < p_{k+1} < p_k$ with $p_k \downarrow p$. Moreover, each $\ell_{p+} \subseteq \mathbb{C}^{\mathbb{N}}$ (with a continuous inclusion) is a reflexive, quasinormable, non-Montel, countably normed Fréchet space which is solid in $\mathbb{C}^{\mathbb{N}}$ and contains no isomorphic copy of any infinite dimensional Banach space, [9], [18] . Clearly, for each $1 < p < \infty$ the Banach space $\ell_p \subseteq \ell_{p+}$ continuously and with a proper inclusion. Since C_p is continuous for each $1 < p < \infty$ (with operator norm p' where $\frac{1}{p}+\frac{1}{p'} = 1$, [15, Theorem 326]), it follows that $\mathsf{C} : \ell_{p+} \longrightarrow \ell_{p+}$ is also continuous, [3, Section 2]. The natural question is: To what extent do the properties and interrelations between the Banach spaces ℓ_p and $ces(p), 1 < p < \infty$, alluded to above reflect themselves in the connections between the corresponding Fréchet spaces ℓ_{p+} and $ces(p+) := \bigcap_{q>p} ces(q)$ which they generate? Although the Fréchet spaces $\ell_{p+}, 1 \leq p < \infty$, are well understood (see eg. [1], [9], [10], [18], [19] and the references therein), it seems that the Fréchet spaces $ces(p+), 1 \leq p < \infty$, which are equipped with the *lattice norms*

$$x \longmapsto \|x\|_{ces(p_k)}, \quad x \in ces(p+), \quad k \in \mathbb{N},$$

for any sequence $p < p_{k+1} < p_k$ satisfying $\lim_{k\to\infty} p_k = p$ (i.e., $ces(p+) = \mathrm{proj}_k \, ces(p_k)$), have not been considered at all. The aim of this note is to make a detailed study of these spaces and to expose some of their striking features. Let us describe some sample results.

First, just like for $\mathsf{C}_p : \ell_p \longrightarrow \ell_p$, for $1 < p < \infty$, the Cesàro operator $\mathsf{C}_{p+} : \ell_{p+} \longrightarrow \ell_{p+}$, for $1 \leq p < \infty$, is also a *positive* operator, albeit now between Fréchet lattices. It turns out that the *largest* of all those solid Fréchet lattices in $\mathbb{C}^{\mathbb{N}}$ which contain ℓ_{p+} and C maps into ℓ_{p+} (necessarily continuously) is precisely $ces(p+)$; see Proposition 2.5. Although each Fréchet space ℓ_{p+}, for $1 \leq p < \infty$, *fails* to have the property (1.3) of Proposition 1.1 (with ℓ_{p+} in place of $ces(p)$), the space $ces(p+)$ that it generates in the above sense *does* have this remarkable property; see Propositions 2.2 and 2.4. A further contrast to ℓ_{p+} is that each $ces(p+), 1 \leq p < \infty$, is a Fréchet-Schwartz space (but, not nuclear) and the canonical vectors $\{e_k : k \in \mathbb{N}\}$ form an unconditional basis (cf. Proposition 3.5). In particular, $ces(p+)$ cannot be isomorphic to any of the non-Montel spaces $\ell_{q+}, 1 \leq q < \infty$. Since, for $p \neq q$, the spaces ℓ_{p+} and ℓ_{q+} are also not isomorphic (cf. Proposition 3.3), it is rather surprising that $ces(p+)$ and $ces(q+)$ *are* isomorphic Fréchet spaces for *all pairs* $1 \leq p, q < \infty$. These results are obtained as a consequence of the main result of this paper showing, remarkably, that $ces(p+)$ coincides with the power series space of order one and finite type $\Lambda_{-1/p'}(\alpha)$ with $\alpha := (\log(k))_{k\in\mathbb{N}}$; see Theorem 3.1. Accordingly, all these spaces are diagonally isomorphic. We mention two further consequences. The Fréchet spaces ℓ_{p+}, for $1 \leq p < \infty$, all *fail* to be (FBa)-spaces, [19], whereas every Fréchet space $ces(p+)$ *is* an (FBa)-space, since it is a Köthe echelon space of order one; see Proposition 4.1. It is known that ℓ_{p+} has the property that every ℓ_{p+}-valued vector measure has relatively compact range if and only if $1 \leq p < 2$. This property also holds for $ces(p+)$, but for *every* $1 \leq p < \infty$.

2. Optimal solid lattice properties of $ces(p+)$

We begin by noting, for each $1 \leq p < \infty$, that $ces(p+)$ is reflexive, [17, Proposition 25.15], since each Banach space $ces(q)$, $q > p$, is reflexive, [6, p.61].

Lemma 2.1. *For each $1 \leq p < \infty$, the space $ces(p+)$ is a solid Fréchet lattice subspace of $\mathbb{C}^{\mathbb{N}}$ and $\ell_{p+} \subseteq ces(p+)$ with a continuous and proper inclusion.*

Proof. Clearly $ces(p+)$ is a solid Fréchet lattice subspace of $\mathbb{C}^{\mathbb{N}}$. Since $\ell_q \subseteq ces(q)$ with a continuous inclusion for each $q > p > 1$, it follows that $\ell_{p+} \subseteq ces(p+)$ continuously.

Fix $1 < p < \infty$. By [8, Remark 2.2(ii)] there exists $x \in ces(p) \backslash \ell_\infty$. Since $ces(p) \subseteq ces(p+)$ and $\ell_{p+} \subseteq \ell_\infty$, it follows that $x \in ces(p+) \backslash \ell_{p+}$.

For $p = 1$ we know that $\ell_{1+} \subseteq ces(1+)$. If this containment was an equality , then the open mapping theorem for Fréchet spaces, [17, Theorem 24.30], implies that the identity map from ℓ_{1+} onto $ces(1+)$ is an isomorphism. This is impossible as ℓ_{1+} is non-Montel whereas $ces(1+)$ is Montel (see Proposition 3.5(ii) below). So, $\ell_{1+} \subsetneqq ces(1+)$. □

The following observation is a direct consequence of the striking property of $ces(q)$, $1 < p < \infty$, exhibited in Proposition 1.1 and the definition of $ces(p+) = \bigcap_{q>p} ces(q)$.

Proposition 2.2. *Let $1 \leq p < \infty$ and $x \in \mathbb{C}^{\mathbb{N}}$. Then*

$$x \in ces(p+) \ \text{ if and only if } \ \mathsf{C}(|x|) \in ces(p+). \tag{2.1}$$

We will require the following fact.

Lemma 2.3. *For each $1 \leq p < \infty$, the Cesàro operator $\mathsf{C} : ces(p+) \longrightarrow \ell_{p+}$ is continuous.*

Proof. Fix $1 \leq p < \infty$. If $x \in ces(p+)$, then $|x| \in ces(q)$ for all $q > p$ and so $\mathsf{C}(|x|) \in \ell_q$ for all $q > p$. This is because $\mathsf{C} : ces(q) \longrightarrow \ell_q$ is continuous as

$$\|\mathsf{C}(x)\|_q = \|\,|\mathsf{C}(x)|\,\|_q \leq \|\mathsf{C}(|x|)\|_q = \|x\|_{ces(q)}, \quad x \in ces(q).$$

Hence, $\mathsf{C}(|x|) \in \ell_{p+}$. This shows that C maps $ces(p+)$ into ℓ_{p+}, necessarily continuously by the closed graph theorem for Fréchet spaces, [17, Theorem 24.31]. □

The next result, in combination with Proposition 2.2, shows that $ces(p+), 1 \leq p < \infty$, exhibits a very desirable property which ℓ_{p+} *fails* to possess.

Proposition 2.4. *For each $1 \leq p < \infty$, the Fréchet space ℓ_{p+} fails to have the property* (2.1) *in Proposition* 2.2 (*with ℓ_{p+} in place of $ces(p+)$*).

Proof. Fix $1 \leq p < \infty$. Assume that ℓ_{p+} *does* have the property (2.1) in Proposition 2.2. By Lemma 2.1 there exists $x \in ces(p+)\setminus\ell_{p+}$. Hence, also $|x| \in ces(p+)\setminus\ell_{p+}$. Then Lemma 2.3 implies that $\mathsf{C}(|x|) \in \ell_{p+}$ and hence, by the assumption on ℓ_{p+}, also $|x| \in \ell_{p+}$; contradiction. So, ℓ_{p+} fails the property. □

The following result should be compared with its Banach lattice counterpart, [8, p.62].

Proposition 2.5. *The space $ces(p+)$, $1 \leq p < \infty$, is the largest solid Fréchet lattice X in $\mathbb{C}^{\mathbb{N}}$ which contains ℓ_{p+} such that $\mathsf{C}(X) \subseteq \ell_{p+}$.*

Proof. Let $[\mathsf{C}, \ell_{p+}]_s$ denote the largest solid Fréchet lattice X in $\mathbb{C}^{\mathbb{N}}$ which contains ℓ_{p+} such that $\mathsf{C}(X) \subseteq \ell_{p+}$. Since $\mathsf{C}(ces(p+)) \subseteq \ell_{p+}$ (cf. Lemma 2.3), it follows that $ces(p+)$ itself is a solid Fréchet lattice which contains ℓ_{p+} such that $\mathsf{C}(ces(p+)) \subseteq \ell_{p+}$. Accordingly, $ces(p+) \subseteq [\mathsf{C}, \ell_{p+}]_s$.

Let X be any solid Fréchet lattice in $\mathbb{C}^{\mathbb{N}}$ which contains ℓ_{p+} such that $\mathsf{C}(X) \subseteq \ell_{p+}$. Given $x \in X$ also $|x| \in X$ and hence, $\mathsf{C}(|x|) \in \ell_{p+} \subseteq ces(p+)$. Proposition 2.2 implies that $x \in ces(p+)$. Accordingly, $X \subseteq ces(p+)$. This implies that $[\mathsf{C}, \ell_{p+}]_s \subseteq ces(p+)$. □

Since $\ell_{p+} \subseteq ces(p+)$ continuously (cf. Lemma 2.1), in addition to $\mathsf{C} : \ell_{p+} \longrightarrow \ell_{p+}$ one may also consider the positive Cesàro operator $\mathsf{C} : ces(p+) \longrightarrow$

$ces(p+)$. Even though the target space $ces(p+)$ is now genuinely larger than ℓ_{p+} (cf. Lemma 2.1), no further solid extension occurs!

Proposition 2.6. *The space $ces(p+)$, $1 \le p < \infty$, is also the largest solid Fréchet lattice X in $\mathbb{C}^{\mathbb{N}}$ which contains $ces(p+)$ such that $\mathsf{C}(X) \subseteq ces(p+)$.*

Proof. Denote by $[\mathsf{C}, ces(p+)]_s$ the largest solid Fréchet lattice X in $\mathbb{C}^{\mathbb{N}}$ which contains $ces(p+)$ such that $\mathsf{C}(X) \subseteq ces(p+)$. Clearly $ces(p+) \subseteq [\mathsf{C}, ces(p+)]_s$ because $\mathsf{C}(ces(p+)) \subseteq \ell_{p+} \subseteq ces(p+)$; see Lemma 2.3.

Let X be any solid Fréchet lattice in $\mathbb{C}^{\mathbb{N}}$ which contains $ces(p+)$ such that $\mathsf{C}(X) \subseteq ces(p+)$. Given $x \in X$ also $|x| \in X$. Hence, $\mathsf{C}(|x|) \in ces(p+)$. By Proposition 2.2, $x \in ces(p+)$. So, $X \subseteq ces(p+)$. This implies $[\mathsf{C}, ces(p+)]_s \subseteq ces(p+)$. □

For the Banach lattice counterpart of Proposition 2.6 see [8, Theorem 2.5].

3. $ces(p+)$ as a power series space of finite type and order 1

A power series Fréchet space of *finite type* $r \in \mathbb{R}$ *and order* 1 is defined, for any given strictly increasing sequence $\alpha = (\alpha_k)_k \subseteq (0, \infty)$ satisfying $\lim_{k\to\infty} \alpha_k = \infty$, by

$$\Lambda_r(\alpha) := \{x \in \mathbb{C}^{\mathbb{N}} : |\!|\!| x |\!|\!|_t := \sum_{k=1}^{\infty} |x_k| e^{t\alpha_k} < \infty, \quad \forall t < r\};$$

see [17, Ch.29], also for the definition of the norms generating the Fréchet topology of $\Lambda_r(\alpha)$.

Our main result is rather remarkable and surprising. We require the following inequality

$$\frac{A_p}{k^{1/p'}} \le \|e_k\|_{ces(p)} \le \frac{B_p}{k^{1/p'}}, \quad k \in \mathbb{N}, \tag{3.1}$$

valid for strictly positive constants A_p, B_p and with $\frac{1}{p} + \frac{1}{p'} = 1$, [6, Lemma 4.7], where $e_k := (\delta_{k,n})_n$ for each $k \in \mathbb{N}$.

Theorem 3.1. *The Fréchet space $ces(p+)$, $1 \le p < \infty$, is isomorphic to the power series space $\Lambda_{-1/p'}(\alpha)$, $\frac{1}{p} + \frac{1}{p'} = 1$, of finite type $-1/p'$ and order 1, where $\alpha = (\log k)_k$.*

Proof. Fix $1 \le p < \infty$. Observe that

$$\Lambda_{-1/p'}((\log k)_k) = \{x \in \mathbb{C}^{\mathbb{N}} : ||x||_t = \sum_{j=1}^{\infty} |x_j| j^t \text{ for all } \ t < -1/p'\}.$$

Let $1 < q < \infty$. For $x \in ces(q)$ we have

$$||x - \sum_{j=1}^{m} x_j e_j||_{ces(q)} \le \Big(\sum_{n=m+1}^{\infty} (\frac{1}{n} \sum_{k=1}^{n} |x_k|)^q \Big)^{1/q} \to 0, \quad \text{as} \quad m \to \infty.$$

Therefore $(e_k)_k$ are a basis of $ces(q)$ for $1 < q < \infty$, [8, Proposition 2.1], hence, also of $ces(p+)$ for each $1 \leq p < \infty$. Consequently, via (3.1) we have, with $\frac{1}{q} + \frac{1}{q'} = 1$, that

$$||x||_{ces(q)} \leq \sum_{j=1}^{\infty} |x_j| \cdot ||e_j||_{ces(q)} \leq B_q \sum_{j=1}^{\infty} |x_j| j^{-\frac{1}{q'}}.$$

By [6, Lemma 4.7], for each $\beta > 0$ we have

$$\frac{1}{\beta}\frac{1}{j^\beta} \leq \sum_{n=j}^{\infty} \frac{1}{n^{1+\beta}}, \qquad j \in \mathbb{N}. \tag{3.2}$$

Let $p < q < \infty$. Given $p < s < q$ it is clear that $y := (n^{-\frac{1}{q'}})_n \in \ell_{s'}$. So we can apply (3.2) with $\beta = \frac{1}{q'}$ and Hölder's inequality to get

$$q' \sum_{j=1}^{\infty} |x_j| j^{-\frac{1}{q'}} \leq \sum_{j=1}^{\infty} |x_j| \sum_{n=j}^{\infty} \frac{1}{n} n^{-\frac{1}{q'}} =$$

$$= \sum_{n=1}^{\infty} \big(\frac{1}{n} \sum_{j=1}^{n} |x_j|\big) n^{-\frac{1}{q'}} \leq ||y||_{s'} ||x||_{ces(s)}.$$

This proves the result.

□

Corollary 3.2. *Each of the Fréchet spaces ces(p+), for* $1 \leq p < \infty$, *is isomorphic to the power series space* $\Lambda_0(\alpha)$ *of finite type* 0 *and order* 1, *where* $\alpha = (\log k)_k$.

Proof. This follows directly from the fact that all finite type power series spaces $\Lambda_r(\alpha)$, with α fixed, are diagonally isomorphic; see [17], page 358, lines 1-5. □

In view of Corollary 3.2 the following observation is relevant.

Proposition 3.3. *For every distinct pair* $1 \leq p, q < \infty$ *the Fréchet spaces* ℓ_{p+} *and* ℓ_{q+} *are not isomorphic.*

Proof. We may assume that $p < q$. Suppose that there exists an isomorphism $T : \ell_{q+} \longrightarrow \ell_{p+}$. Since the natural inclusion $\ell_q \subseteq \ell_{q+}$ is continuous, the restricted operator $T|_{\ell_q} : \ell_q \longrightarrow \ell_{p+}$ is continuous. Consequently, since the inclusion $\ell_{p+} \subseteq \ell_r$ is continuous for each $r > p$, it follows that $T|_{\ell_q} : \ell_q \longrightarrow \ell_r$ is continuous for each $r \in (p, q)$. By Pitt's theorem, [20], $T : \ell_q \longrightarrow \ell_r$ is *compact* (we denote $T|_{\ell_q}$ simply by T again as no confusion can occur). Choose now any $r \in (p, q)$. Since $\{e_j\}_{j=1}^{\infty}$ is a bounded set in ℓ_q and $T : \ell_q \longrightarrow \ell_r$ is compact, the image $\{T(e_j)\}_{j=1}^{\infty}$ is a relatively compact subset of ℓ_r. Consequently, as $\ell_{p+} = \bigcap_{p<r<q} \ell_r = \mathrm{proj}_{p<r<q} \ell_r$, it follows that $\{T(e_j)\}_{j=1}^{\infty}$ is also a relatively compact subset of ℓ_{p+}. Hence, there exists $y \in \ell_{p+}$ and a subsequence $\{T(e_{j(k)})\}_{k=1}^{\infty}$ of $\{T(e_j)\}_{j=1}^{\infty}$ such

that $T(e_{j(k)}) \longrightarrow y$ in ℓ_{p+} for $k \longrightarrow \infty$. By continuity of the inverse operator $T^{-1} : \ell_{p+} \longrightarrow \ell_{q+}$ it follows that $e_{j(k)} \longrightarrow T^{-1}(y)$ in ℓ_{q+}. Choose any $s > q$, in which case $\ell_{q+} \subseteq \ell_s$ continuously, then also $e_{j(k)} \longrightarrow T^{-1}(y)$ in the Banach space ℓ_s. This is impossible as $\|e_{j(k)} - e_{j(l)}\|_{\ell_s} = 2^{1/s} \geq 1$ for all $k \neq l$. Hence, no such isomorphism T of ℓ_{q+} onto ℓ_{p+} can exist. □

Remark 3.4. For each pair $1 \leq p < q < \infty$ it is clear that

$$ces(p+) \subseteq ces(q+). \tag{3.3}$$

Even though $ces(p+)$ and $ces(q+)$ are isomorphic as Fréchet spaces (cf. Corollary 3.2), the containment (3.3) is *proper*. Indeed, if it were an equality, then for any fixed $r \in (p, q)$ it would follow from the (continuous) inclusions $ces(p+) \subseteq ces(r) \subseteq ces(q) \subseteq ces(q+)$ that $ces(r) = ces(q)$. Consequently, the Banach spaces $ces(r)$ and $ces(q)$ would be isomorphic (with $r < q$) which is not the case, [4, Proposition 3.3].

We now collect some further consequences of Theorem 3.1.

Proposition 3.5. *For each $1 \leq p < \infty$ the following assertions hold.*

- (i) *The Fréchet space $ces(p+)$ is a Köthe echelon space of order 1 and the canonical vectors $(e_j)_{j\in\mathbb{N}}$ form an unconditional basis of $ces(p+)$.*
- (ii) *$ces(p+)$ is a Fréchet-Schwartz space but, it is not nuclear.*
- (iii) *$ces(p+)$ is not isomorphic to ℓ_{q+} for each $1 \leq q < \infty$.*

Proof. (i) The space $ces(p+)$ is a Köthe echelon space of order 1 (by Theorem 3.1). The canonical vectors are an unconditional basis for every Köthe echelon space of order 1. Even stronger, they form an absolute basis, [16, pp.314–315].

(ii) Every power series space is Schwartz by [17, Proposition 27.10]. The non-nuclearity of $ces(p+) = \Lambda_{-1/p'}(\alpha)$, $\alpha = (\log k)_k$ is a direct consequence of Corollary 3.2 and [17, Proposition 29.6 (2)].

(iii) The space ℓ_{q+} is not Montel for each $1 \leq q < \infty$, [18], and hence, it cannot be isomorphic to $ces(p+)$, [17, Lemma 24.19]. □

Definition 3.6. Let X be a Fréchet space whose topology is generated by a fundamental sequence $(\|\cdot\|_n)_{n\in\mathbb{N}}$ of seminorms. The space X has the property $(\overline{\Omega})$ if: $\forall l \in \mathbb{N},\ c \in (0,1)\ \exists\ k > l, \forall\ h > k, \exists\, C > 0\ :$

$$\|y\|_k'^{1+c} \leq C\|y\|_h'\|y\|_l'^{c}, \quad \forall\ y \in X',$$

where $\|y\|_k' := \sup\{|\langle x, y\rangle| \colon \|x\|_k \leq 1\}$ is the dual norm $\|\cdot\|_k'$ of $\|\cdot\|_k$ in X'.

It is a consequence of [17, Lemma 29.16] that the condition in Definition 3.6 coincides with property $(\overline{\Omega})$.

Every space ℓ_{p+}, for $1 \leq p < \infty$, satisfies property $(\overline{\Omega})$, [18, proof of Proposition 2.4]. Since all power series spaces of finite type have property $(\overline{\Omega})$, [17, Proposition 29.12], we deduce the following result.

Proposition 3.7. *For each $1 \leq p < \infty$ the Fréchet space $ces(p+)$ has property $(\overline{\Omega})$.*

4. Further properties of $ces(p+)$

According to Taskinen, [23], a Fréchet space X is called an (FBa)-*space* if, for every Banach space Y, every bounded subset of the complete projective tensor product $X\widehat{\otimes}Y$ is contained in the closed convex hull $\overline{\mathrm{co}(C \otimes D)}$ of bounded sets $C \subseteq X$ and $D \subseteq Y$. In 1986 Taskinen constructed Fréchet spaces which are not (FBa)-spaces, thereby solving the "problem of topologies of Grothendieck", [22]. Peris proved that the Fréchet spaces ℓ_{p+}, for $1 \leq p < \infty$, are *not* (FBa) spaces, thus providing a natural and concrete class of spaces of this type, [19].

Proposition 4.1. *Each Fréchet space $ces(p+)$, for $1 \leq p < \infty$, is an (FBa)-space.*

Proof. Each space $ces(p+), 1 \leq p < \infty$, is isomorphic to a Köthe echelon space of order 1 (cf. Proposition 3.5 (i)). Hence, it is necessarily an (FBa)-space, [14, p.70], [22, Section 3]. □

Now we consider some features of $ces(p+)$ of a somewhat different nature. First, $ces(p+)$ is the complexification of the corresponding real Riesz space $ces_{\mathbb{R}}(p+) := \{x \in ces(p+) : x = (x_n)_n \in \mathbb{R}^{\mathbb{N}}\}$, in the sense of [26, pp.187–201]. Since $ces_{\mathbb{R}}(p+)$ is solid in $\mathbb{R}^{\mathbb{N}}$ it follows that $ces_{\mathbb{R}}(p+)$ is *Dedekind complete* (i.e., every subset of $ces_{\mathbb{R}}(p+)$ which is bounded from above in the order has a least upper bound) and hence, (per definition) also its complexification $ces(p+)$ is Dedekind complete. Moreover, being reflexive, each of the (separable) Fréchet lattices $ces(p+), 1 \leq p < \infty$, has a *Lebesgue topology*, [2, Theorems 10.3 and 10.9], that is, if $x^{(\alpha)} \downarrow 0$ is a decreasing net in the order of $ces(p+)$, then $\lim_\alpha x^{(\alpha)} = 0$ in the topology of $ces(p+)$.

For every $1 \leq p < \infty$, the Fréchet space ℓ_{p+} has the property that *every* ℓ_{p+}-valued vector measure (always assumed to be countably additive and defined on a σ-algebra) has relatively compact range if and only if $p \in [1, 2)$, [7, Proposition 2.8]. Once again the optimal solid extension $ces(p+)$ of ℓ_{p+} exhibits better behaviour in this regard.

Proposition 4.2. *Let $p \in [1, \infty)$. Then every $ces(p+)$-valued vector measure necessarily has relatively compact range.*

Proof. The range of every $ces(p+)$-valued vector measure is a relatively weakly compact set, [24], and hence, is also relatively compact as $ces(p+)$ is a Fréchet-Montel space (by Proposition 3.5(ii)). □

Since $ces(p+), 1 \leq p < \infty$, is *not* nuclear (cf. Proposition 3.5(ii)), there exist $ces(p+)$-valued vector measures which *fail* to have finite variation, [11, Corollary and Theorem 2].

A Fréchet space X is said to have the *Rybakov property*, [12], if for every X-valued vector measure ν there exists $x' \in X'$ such that $\nu \ll |\langle \nu, x' \rangle|$ (i.e., $\nu(F) = 0$ for every measurable set $F \subseteq E$ whenever $|\langle \nu, x' \rangle|(E) = 0$). Here $\langle \nu, x' \rangle$ is the complex measure $E \longmapsto \langle \nu(E), x' \rangle$ and $|\langle \nu, x' \rangle|$ denotes its total variation measure. A Fréchet space X has Rybakov's property if and only if it admits a continuous norm; see [12, Theorem 2.2] for real spaces and [21, Proposition 2.2] for complex spaces. Since both ℓ_{p+} and $ces(p+)$ admit a continuous norm, we have the following fact.

Proposition 4.3. *For each $1 \leq p < \infty$ the Fréchet spaces ℓ_{p+} and $ces(p+)$ have the Rybakov property.*

A classical result of Bade, [5, Theorem 3.1], states: Given a σ -complete Boolean of projections $\mathcal{M}$ in a Banach space X, for each $x_0 \in X$ there exists $x' \in X'$ (called a Bade functional for x_0 with respect to $\mathcal{M}$) satisfying

(i) $\langle P(x_0), x' \rangle \geq 0$ for all $P \in \mathcal{M}$, and

(ii) if $\langle P(x_0), x' \rangle = 0$ for some $P \in \mathcal{M}$, then $P(x_0) = 0$.

A Fréchet space X is said to have the *Bade property*, [21], if every σ-complete Boolean algebra of projections $\mathcal{M}$ in X satisfies (i), (ii) above (for every $x_0 \in X$). This is the case if and only if X admits a continuous norm, [21, Corollary 2.1], which yields the following result.

Proposition 4.4. *For each $1 \leq p < \infty$ the Fréchet spaces ℓ_{p+} and $ces(p+)$ have the Bade property.*

Our final result presents a description of the dual of $ces(p+)$.

Recall that the *Köthe dual* $X^\times$ (or the associate space) of a Banach sequence space $(X, \|\cdot\|_X)$, with $\varphi \subseteq X \subseteq \mathbb{C}^\mathbb{N}$, is defined by

$$X^\times := \left\{ x \in \mathbb{C}^\mathbb{N} \colon \sum_{k=1}^{\infty} |x_k y_k| < \infty, \ \forall y \in X \right\},$$

endowed with the norm

$$\|x\|_{X^\times} := \sup \left\{ \sum_{k=1}^{\infty} |x_k y_k| \colon \|y\|_X \leq 1 \right\},$$

[17, Ch.27]. Here φ is the subspace of $\mathbb{C}^\mathbb{N}$ consisting of those vectors having finite support. Every $v \in X^\times$ defines an element of the dual Banach space X' of $(X, ||.||_X)$ via $u \to \sum_{n=1}^\infty u_n v_n$ for $u \in X$, and $||v||_{X^\times} = ||v||_{X'}$.

For $1 < q < \infty$ and $\frac{1}{q} + \frac{1}{q'} = 1$, the space $d(q')$ is defined as

$$d(q') := \left\{ x \in \mathbb{C}^\mathbb{N} \colon \sum_{n=1}^{\infty} \sup_{k \geq n} (|x_k|^{q'}) < \infty \right\},$$

which is a Banach space when endowed with the norm

$$\|x\|_{d(q')} := \left(\sum_{n=1}^{\infty} \sup_{k \geq n} (|x_k|^{q'}) \right)^{1/q'}, \quad x \in d(q').$$

Observe that

$$x \in d(q') \text{ if and only if } \widehat{x} := (\sup_{k\geq n} |x_k|)_{n\in\mathbb{N}} \in \ell_{q'} \tag{4.1}$$

and that

$$\|x\|_{d(q')} = \|\widehat{x}\|_{\ell_{q'}};$$

see [6, p.3 & p.9]. We will require the following result of Bennett [6, p.61 & Corollary 12.17].

Lemma 4.5. *Let* $1 < q < \infty$ *and* $\frac{1}{q}+\frac{1}{q'} = 1$. *The map* $\Phi_q\colon (ces(q))' \to d(q')$ *defined by* $\Phi_q(f) := (\langle e_j, f\rangle)_{j\in\mathbb{N}}$, *for* $f \in (ces(q))'$, *is a linear isomorphism of the dual Banach space* $(ces(q))'$ *onto the Banach space* $d(p')$ *and*

$$\frac{1}{q'}\|\Phi_q(f)\|_{d(q')} \leq \|f\|_{(ces(q))'} \leq (q-1)^{1/q}\|\Phi_q(f)\|_{d(q')}, \quad f \in (ces(q))'.$$

Moreover, $(ces(q))^\times = d(q')$ *and* $(d(q'))^\times = ces(q)$, *with equivalent norms.*

Fix $1 \leq p < \infty$ and any sequence $p < p_{n+1} < p_n$ for $n \in \mathbb{N}$ satisfying $\lim_{n\to\infty} p_n = p$. Then $p'_n < p'_{n+1} < p'$ for $n \in \mathbb{N}$. Since $\|x\|_{d(r)} = \|\widehat{x}\|_{\ell_r}$ for all $x \in d(r)$ and any $1 < r < \infty$, it follows that $d(p'_n) \subseteq d(p'_{n+1})$ with a continuous inclusion for each $n \in \mathbb{N}$. We endow the vector space $d(p'-) := \cup_{n\in\mathbb{N}} d(p'_n)$, which is an increasing union, with the inductive limit topology, i.e., $d(p'-) := \operatorname{ind}_n d(p'_n)$, [17, Ch.24].

Proposition 4.6. *Let* $1 \leq p < \infty$. *The map* $\Lambda : (ces(p+))' \to d(p'-)$ *given by* $\Lambda(f) := (\langle e_j, f\rangle)_{j\in\mathbb{N}}$, *for* $f \in (ces(p+))'$, *defines a linear bijection which is a topological isomorphism of the strong dual space* $(ces(p+))'_\beta$ *onto* $d(p'-) = \operatorname{ind}_n d(p'_n)$.

Proof. First observe that φ is dense in the Fréchet space $ces(p+)$ as it is dense in each Banach space $ces(q)$ for $q > p$.

Fix $u \in (ces(p+))'$. Select $n \in \mathbb{N}$ and a constant $K > 0$ such that

$$|\langle x, u\rangle| \leq K\|x\|_{ces(p_n)}, \quad x \in ces(p+).$$

So, there exists a unique $\tilde{u} \in (ces(p_n))'$ whose restriction to $ces(p+) \subseteq ces(p_n)$ coincides with u. By Lemma 4.5 the element $(\langle e_j, u\rangle)_{j\in\mathbb{N}} = (\langle e_j, \tilde{u}\rangle)_{j\in\mathbb{N}}$ belongs to $d(p'_n) \subseteq d(p'-)$.

The previous argument implies that Λ is well defined. It is clearly linear. Moreover, Λ is injective by the density of $\varphi = \operatorname{span}\{e_j : j \in \mathbb{N}\}$ in $ces(p+)$. To show that Λ is also surjective let $y = (y_j)_{j\in\mathbb{N}} \in d(p'-)$. Then there exists $m \in \mathbb{N}$ such that $y \in d(p'_m)$. Via Lemma 4.5 we can find $f \in (ces(p_m))'$ with $y = (\langle e_j, f\rangle)_{j\in\mathbb{N}}$. Then the restriction v of f to $ces(p+)$ belongs to $(ces(p+))'$ and $\Lambda(v) = y$.

Define the injection $J_n\colon (ces(p_n))' \to (ces(p+))'$, for $n \in \mathbb{N}$, by setting $J_n(f)$ to be the restriction of $f \in (ces(p_n))'$ to $ces(p+)$. By the earlier part of the proof $(ces(p+))' = \cup_{n\in\mathbb{N}} J_n((ces(p_n))')$ and so we may consider in $(ces(p+))'$ the inductive limit topology $\operatorname{ind}_n(ces(p_n))'$. Since $ces(p+)$ is reflexive, its strong dual $(ces(p+))'_\beta$ coincides with $\operatorname{ind}_n(ces(p_n))'$.

By Lemma 4.5, for each $n \in \mathbb{N}$ the restriction Λ_{p_n} of Λ to $(ces(p_n))'$ is continuous from $(ces(p_n))'$ onto $d(p'_n)$. This implies that $\Lambda : (ces(p+))'_\beta \to \operatorname{ind}_n d(p'_n)$ is a continuous bijection. By the closed graph theorem for (LB)-spaces, [17, Theorem 24.31 & Remark 24.36], Λ is also a topological isomorphism. □

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