Integral operators on weighted Banach spaces of entire functions

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Joint work with J. Taskinen

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Aim of the lecture

AIM

Investigate the behaviour of certain integral operators, acting on certain Banach, Fréchet and (LB) spaces of entire functions.

We report on joint work with J. Taskinen (Univ. Helsinki, Finland).

To Volterra operators

The **Volterra operators** are defined by

$$V_g(f)(z) := \int_0^z f(\zeta)g'(\zeta)d\zeta.$$

We consider them on spaces of entire functions and complement work by **Bassallote, Contreras, Hernández-Mancera, Martín and Paul** in 2012 for holomorphic functions on the disc and by **Constantin and Peláez** in 2013 for reflexive weighted Fock spaces.

The case of analytic functions on the disc was considered by **Pommerenke, Aleman, Siskakis, Pau, Peláez and Rättyä**, among others.

Vito Volterra (1860-1940)

We report in this part on joint work with Jari Taskinen (Helsinki (Finland))

Preliminaries

- $H(\mathbb{C})$ and $\mathcal{P}$ denote the space of entire functions and the space of polynomials, respectively. The space $H(\mathbb{C})$ will be endowed with the compact open topology τ_{co}.
- The differentiation operator $Df(z) = f'(z)$ and the integration operator $Jf(z) = \int_0^z f(\zeta)d\zeta$ are continuous on $H(\mathbb{C})$.
- Given an entire function $g \in H(\mathbb{C})$, the **Volterra operator** V_g with symbol g is defined on $H(\mathbb{C})$ by

$$V_g(f)(z) := \int_0^z f(\zeta)g'(\zeta)d\zeta \quad (z \in \mathbb{C}).$$

For $g(z) = z$ this reduces to the integration operator. Clearly V_g defines a continuous operator on $H(\mathbb{C})$.

Definitions

A **weight** v is a continuous function $v : [0,\infty[\to]0,\infty[$, which is non-increasing on $[0,\infty[$ and satisfies $\lim_{r\to\infty} r^m v(r) = 0$ for each $m \in \mathbb{N}$. If necessary, we extend v to $\mathbb{C}$ by $v(z) := v(|z|)$.

The **weighted Banach spaces of entire functions** are defined by

$$H_v^\infty(\mathbb{C}) := \{f \in H(\mathbb{C}) \mid \|f\|_v := \sup_{z\in\mathbb{C}} v(|z|)|f(z)| < \infty\},$$
$$H_v^0(\mathbb{C}) := \{f \in H(\mathbb{C}) \mid \lim_{|z|\to\infty} v(|z|)|f(z)| = 0\},$$

and they are endowed with the weighted sup norm $\|\cdot\|_v$.

Definitions

- $H_v^\infty(\mathbb{C})$ coincides with the **weighted Fock space** $\mathcal{F}_\infty^\phi$ of order infinity when $v(z) = \exp(-\phi(|z|))$, and $\phi : [0, \infty[\to]0, \infty[$ is a twice continuously differentiable increasing function.

- $H_v^0(\mathbb{C})$ is a closed subspace of $H_v^\infty(\mathbb{C})$.

- The polynomials are contained and dense in $H_v^0(\mathbb{C})$ but the monomials do not in general form a Schauder basis (**Lusky**). The Cesàro means of the Taylor polynomials satisfy $\|C_n f\|_v \leq \|f\|_v$ for each $f \in H_v^\infty(\mathbb{C})$ and the sequence $(C_n f)_n$ is $\|\cdot\|_v$-convergent to f when $f \in H_v^0(\mathbb{C})$

Definitions

For a weight v, the associated weight $\tilde{v}$ is defined by

$$\tilde{v}(z) := \left(\sup \left\{ |f(z)| \mid f \in H_v^\infty(\mathbb{C}), \|f\|_v \leq 1 \right\} \right)^{-1} = \left(\|\delta_z\|_v \right)^{-1}, \ z \in \mathbb{C},$$

where δ_z denotes the point evaluation of z.

- $\tilde{v}$ is continuous, radial, $\tilde{v} \geq v > 0$, and for each $z \in \mathbb{D}$ we can find $f_z \in H_v^\infty$, $\|f_z\|_v = 1$ with $|f_z(z)|\tilde{v}(z) = 1$.
- $H_{\tilde{v}}^\infty(\mathbb{C})$ coincides isometrically with $H_v^\infty(\mathbb{C})$, and $H_{\tilde{v}}^0(\mathbb{C})$ with $H_v^0(\mathbb{C})$.

The Volterra operator

The **Volterra operator** V_g with symbol $g \in H(\mathbb{C})$ is defined on $H(\mathbb{C})$ by

$$V_g(f)(z) := \int_0^z f(\zeta)g'(\zeta)d\zeta \quad (z \in \mathbb{C}).$$

Question

How it acts on $H_v^\infty(\mathbb{C})$ or $H_v^0(\mathbb{C})$?

A result

Theorem

Let v be a weight and let $w(r) := \exp(-\alpha r^p)$, where $\alpha > 0, p > 0$ are constants. The following conditions are equivalent for $g \in H(\mathbb{C})$:

(1) $V_g : H_v^\infty(\mathbb{C}) \to H_w^\infty(\mathbb{C})$ is continuous.

(2) $V_g : H_v^0(\mathbb{C}) \to H_w^0(\mathbb{C})$ is continuous.

(3) There exists a constant $C > 0$ such that $|g'(z)| \leq C|z|^{p-1} \exp(\alpha|z|^p)\tilde{v}(|z|)$ for all $z \in \mathbb{C}$, $|z| \geq 1$.

The following conditions are also equivalent for $g \in H(\mathbb{C})$:

(1) $V_g : H_v^\infty(\mathbb{C}) \to H_w^\infty(\mathbb{C})$ is compact.

(2) $V_g : H_v^0(\mathbb{C}) \to H_w^0(\mathbb{C})$ is compact.

(3) $|g'(z)| = o\big(|z|^{p-1} \exp(\alpha|z|^p)\tilde{v}(|z|)\big)$ as $|z| \to \infty$.

Two corollaries

Corollary

If $v(r) = \exp(-\alpha r^p)$, $\alpha > 0, p \geq 1$, then $V_g : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ is continuous if and only if g is a polynomial of degree less than or equal to the integer part of p.

Corollary

If $v(r) = \exp(-\alpha r^p)$, $\alpha > 0, p > 0$, then $V_g : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ is compact if and only if it is weakly compact if and only if g is a polynomial of degree less than or equal to the integer part of $p - 1$.

Multiplication operators

Proposition

Let v and w be weights. The following conditions are equivalent for an entire function $h \in H(\mathbb{C})$:

(1) $M_h : H_v^\infty(\mathbb{C}) \to H_w^\infty(\mathbb{C})$ is continuous.

(2) $M_h : H_v^0(\mathbb{C}) \to H_w^0(\mathbb{C})$ is continuous.

(3) $\sup \frac{w(z)|h(z)|}{\tilde{v}(z)} < \infty$.

(4) $\sup \frac{\tilde{w}(z)|h(z)|}{\tilde{v}(z)} < \infty$.

Multiplication operators

Proposition

Let v and w be weights. The following conditions are equivalent for an entire function $g \in H(\mathbb{C})$:

(1) $M_h : H_v^\infty(\mathbb{C}) \to H_w^\infty(\mathbb{C})$ is compact.

(2) $M_h : H_v^0(\mathbb{C}) \to H_w^0(\mathbb{C})$ is compact.

(3) $\lim_{|z|\to\infty} \frac{w(z)|h(z)|}{\tilde{v}(z)} = 0.$

(4) $\lim_{|z|\to\infty} \frac{\tilde{w}(z)|h(z)|}{\tilde{v}(z)} = 0.$

Volterra operators. Reduction arguments. Definitions

Let $\varphi : [0,\infty[\to]0,\infty[$ be a continuous non-decreasing function, C^1 on $[r_\varphi,\infty[$ for some $r_\varphi \geq 0$. Suppose that φ' is non-decreasing in $[r_\varphi,\infty[$, $\varphi'(r_\varphi) > 0$ and $r^n = O(\varphi'(r))$ as $r \to \infty$ for each $n \in \mathbb{N}$. This implies $r^n = O(\varphi(r))$ as $r \to \infty$ for each $n \in \mathbb{N}$, and that

$$w_\varphi(z) := 1/\varphi(|z|), z \in \mathbb{C},$$

and

$$u_\varphi(z) := 1/\max\{\varphi'(r_\varphi), \varphi'(|z|)\} = \begin{cases} 1/\varphi'(r_\varphi) \ , & |z| \leq r_\varphi \\ 1/\varphi'(|z|) \ , & |z| \geq r_\varphi \end{cases}$$

are weights.

If $\varphi(r) = \exp(\alpha r^p), r \geq 0, \alpha > 0, p > 0$, then $w_\varphi(z) = \exp(-\alpha|z|^p), z \in \mathbb{C}$, and $u_\varphi(z) = \alpha^{-1}p^{-1}|z|^{1-p}\exp(-\alpha|z|^p)$ for $|z|$ large enough.

Volterra operators. Reduction arguments

Proposition

The integration operators $J : H^\infty_{u_\varphi}(\mathbb{C}) \to H^\infty_{w_\varphi}(\mathbb{C})$ and $J : H^0_{u_\varphi}(\mathbb{C}) \to H^0_{w_\varphi}(\mathbb{C})$ are continuous.

Proposition

If the function φ is of smoothness C^2 on $[r_\varphi, \infty[$ for some $r_\varphi > 0$ and it satisfies $\sup_{r \geq r_\varphi} \frac{\varphi''(r)\varphi(r)}{(\varphi'(r))^2} < \infty$ in addition to the general assumptions, then the differentiation operators $D : H^\infty_{w_\varphi}(\mathbb{C}) \to H^\infty_{u_\varphi}(\mathbb{C})$ and $D : H^0_{w_\varphi}(\mathbb{C}) \to H^0_{u_\varphi}(\mathbb{C})$, $Df := f'$, are continuous.

Volterra operators. Reduction arguments

Under the assumptions on φ in the last Proposition, $M(f,r) = O(\varphi(r))$, when $r \to \infty$ if and only if $M(f',r) = O(\varphi'(r))$ for $r \to \infty$. The argument can be traced back, at least, to **Pavlovic** (1999).

Examples of functions φ that satisfy the assumptions of Proposition can be found in the work of Hardy. For example one can take

$$\varphi(r) := r^a (\log r)^b \exp(cr^d + k(\log r)^m),$$

for large r, where $c > 0, d > 0$ or $c = 0, k > 0, m > 1$.

Volterra operators

Theorem

Let φ be of smoothness C^2 on $[r_\varphi, \infty[$ for some $r_\varphi > 0$ and let it satisfy $\sup_{r \geq r_\varphi} \frac{\varphi''(r)\varphi(r)}{(\varphi'(r))^2} < \infty$ in addition to the general assumptions.

The following conditions are equivalent for an entire function $g \in H(\mathbb{C})$:

(1) $V_g : H_v^\infty(\mathbb{C}) \to H_{w_\varphi}^\infty(\mathbb{C})$ is continuous.

(2) $V_g : H_v^0(\mathbb{C}) \to H_{w_\varphi}^0(\mathbb{C})$ is continuous.

(3) $\sup_{|z| \geq r_\varphi} \frac{|g'(z)|}{\varphi'(|z|)\tilde{v}(z)} < \infty$.

Proof.

Assume that condition (1) holds. The differentiation operator $D : H^\infty_{w_\varphi}(\mathbb{C}) \to H^\infty_{u_\varphi}(\mathbb{C})$ is continuous. We can apply (1) and the identity $DV_g = M_{g'}$ to conclude that $M_{g'} : H^\infty_v(\mathbb{C}) \to H^\infty_{u_\varphi}(\mathbb{C})$ is continuous. Now condition (3) follows from the characterization of bounded multiplication operators, since $u_\varphi(z) = 1/\varphi'(|z|), |z| \geq r_\varphi$.

Conversely, if condition (3) holds, the operator $M_{g'} : H^\infty_v(\mathbb{C}) \to H^\infty_{u_\varphi}(\mathbb{C})$ is continuous by the characterization of bounded multiplication operators. We apply the boundedness of $J : H^\infty_{u_\varphi}(\mathbb{C}) \to H^\infty_{w_\varphi}(\mathbb{C})$ to get that $V_g = J \circ M_{g'} : H^\infty_v(\mathbb{C}) \to H^\infty_{w_\varphi}(\mathbb{C})$ is continuous.

Volterra operators

Theorem

Let φ be of smoothness C^2 on $[r_\varphi, \infty[$ for some $r_\varphi > 0$ and let it satisfy $\sup_{r \geq r_\varphi} \frac{\varphi''(r)\varphi(r)}{(\varphi'(r))^2} < \infty$ in addition to the general assumptions.

The following conditions are equivalent for an entire function $g \in H(\mathbb{C})$:

(1) $V_g : H_v^\infty(\mathbb{C}) \to H_{w_\varphi}^\infty(\mathbb{C})$ is compact.

(2) $V_g : H_v^0(\mathbb{C}) \to H_{w_\varphi}^0(\mathbb{C})$ is compact.

(3) $\lim_{|z| \to \infty} \frac{|g'(z)|}{\varphi'(|z|)\tilde{v}(z)} = 0$.

We have already proved

Theorem

Let v be a weight and let $w(r) := \exp(-\alpha r^p)$, where $\alpha > 0, p > 0$ are constants. The following conditions are equivalent for $g \in H(\mathbb{C})$:

(1) $V_g : H_v^\infty(\mathbb{C}) \to H_w^\infty(\mathbb{C})$ is continuous.

(2) $V_g : H_v^0(\mathbb{C}) \to H_w^0(\mathbb{C})$ is continuous.

(3) There exists a constant $C > 0$ such that $|g'(z)| \leq C|z|^{p-1} \exp(\alpha|z|^p)\tilde{v}(|z|)$ for all $z \in \mathbb{C}$, $|z| \geq 1$.

The following conditions are also equivalent for $g \in H(\mathbb{C})$:

(1) $V_g : H_v^\infty(\mathbb{C}) \to H_w^\infty(\mathbb{C})$ is compact.

(2) $V_g : H_v^0(\mathbb{C}) \to H_w^0(\mathbb{C})$ is compact.

(3) $|g'(z)| = o\big(|z|^{p-1} \exp(\alpha|z|^p)\tilde{v}(|z|)\big)$ as $|z| \to \infty$.

Spectrum of Volterra operators

Now we investigate the **spectrum of the Volterra operator** when it acts continuously on a weighted Banach space of entire functions $H_v^\infty(\mathbb{C})$.

Aleman and Constantin in 2009 and **Aleman and Peláez** in 2012 investigated the spectra of Volterra operators on several spaces of of holomorphic functions on the disc. **Constantin** started in 2012 the study of the spectrum of Volterra operator on spaces of entire functions, more precisely on the classical Fock spaces.

We assume that $g \in H(\mathbb{C})$ be a non-constant entire function such that $g(0) = 0$ and V_g is the Volterra operator.

Definitions

X is a Hausdorff locally convex space (lcs).

$\mathcal{L}(X)$ is the space of all continuous linear operators on X.

The **resolvent set** $\rho(T, X)$ of T consists of all $\lambda \in \mathbb{C}$ such that $R(\lambda, T) := (\lambda I - T)^{-1}$ exists in $\mathcal{L}(X)$.

The **spectrum** of T is the set $\sigma(T, X) := \mathbb{C} \setminus \rho(T, X)$. The **point spectrum** is the set $\sigma_{pt}(T, X)$ of those $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not injective.

Preliminary results

Proposition

The operator $V_g - \lambda I : H(\mathbb{C}) \to H(\mathbb{C})$ is injective for each $\lambda \in \mathbb{C}$. In particular $\sigma_{pt}(V_g, H(\mathbb{C})) = \emptyset$. Moreover, $0 \in \sigma(V_g, H(\mathbb{C}))$.

Lemma

Given $\lambda \in \mathbb{C}$, $\lambda \neq 0$, and $h \in H(\mathbb{C})$, the equation $f - (1/\lambda)V_g f = h$ has a unique solution given by

$$f(z) = R_{\lambda,g}h(z) = h(0)e^{\frac{g(z)}{\lambda}} + e^{\frac{g(z)}{\lambda}} \int_0^z e^{-\frac{g(\zeta)}{\lambda}} h'(\zeta)d\zeta, \qquad z \in \mathbb{C}.$$

Preliminary results

Proposition

Let $g \in H(\mathbb{C})$ be a non-constant entire function such that $g(0) = 0$. The Volterra operator V_g satisfies $\sigma(V_g, H(\mathbb{C})) = \{0\}$ and $\sigma_{pt}(V_g, H(\mathbb{C})) = \emptyset$.

Proposition

Let $X \subset H(\mathbb{C})$ be a locally convex space that contains the constants and such that the inclusion $X \subset H(\mathbb{C})$ is continuous. Assume that $V_g : X \to X$ is continuous for some non-constant entire function g such that $g(0) = 0$. Then

$$\{0\} \cup \{\lambda \in \mathbb{C} \setminus \{0\} \mid e^{\frac{g}{\lambda}} \notin X\} \subset \sigma(V_g, X).$$

If X is a Banach space, then

$$\{0\} \cup \overline{\{\lambda \in \mathbb{C} \setminus \{0\} \mid e^{\frac{g}{\lambda}} \notin X\}} \subset \sigma(V_g, X).$$

Preliminary results

Lemma

Let $X \subset H(\mathbb{C})$ be a locally convex space that contains the constants and such that the inclusion $X \to H(\mathbb{C})$ is continuous. Assume that $V_g : X \to X$ is continuous for some non-constant entire function g such that $g(0) = 0$. The following conditions are equivalent:

(i) $\lambda \in \rho(V_g, X)$.

(ii) $R_{\lambda,g} : X \to X$ is continuous.

(iii) (a) $e^{\frac{g}{\lambda}} \in X$, and

(b) $S_{\lambda,g} : X_0 \to X_0,\ S_{\lambda,g}h(z) := e^{\frac{g(z)}{\lambda}} \int_0^z h'(\zeta)e^{-\frac{g(\zeta)}{\lambda}} d\zeta,\ z \in \mathbb{C}$, is continuous on the subspace X_0 of X of all the functions $h \in X$ with $h(0) = 0$.

Preliminary results

Lemma

Let $X \subset H(\mathbb{C})$ be a locally convex space that contains the constants and such that the inclusion $X \to H(\mathbb{C})$ is continuous. Let X_0 be the subspace of X of all the functions $h \in X$ with $h(0) = 0$. The following conditions are equivalent for $\lambda \in \mathbb{C} \setminus \{0\}$.

(i) $S_{\lambda,g} : X_0 \to X_0,\ S_{\lambda,g}h(z) := e^{\frac{g(z)}{\lambda}} \int_0^z h'(\zeta)e^{-\frac{g(\zeta)}{\lambda}} d\zeta,\ z \in \mathbb{C}$, is continuous.

(ii) $T : X_0 \to X_0, Th(z) := e^{\frac{g(z)}{\lambda}} \int_0^z h(\zeta)g'(\zeta)e^{-\frac{g(\zeta)}{\lambda}} d\zeta,\ z \in \mathbb{C}$, is continuous.

The proof is obtained integrating by parts.

Spectra of Volterra operators on $H_v^\infty(\mathbb{C})$

We first deal with the Volterra operator acting on the Banach space $H_v^\infty(\mathbb{C})$, with $v(r) = \exp(-\alpha r^p)$, where $\alpha, p > 0$. Recall:

Proposition

Assume that $v(r) = \exp(-\alpha r^p)$, $\alpha > 0$, $p > 0$.

(i) $V_g : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ is continuous if and only if g is a polynomial of degree less than or equal to the integer part of p.

(ii) $V_g : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ is compact if and only if g is a polynomial of degree less than or equal to the integer part of $p - 1$.

Spectra of Volterra operators on $H_v^\infty(\mathbb{C})$

Lemma

Let v be a weight such that $v(r)e^{\alpha r^n}$ is non-increasing on $[r_0, \infty[$ for some $r_0 > 0$, $\alpha > 0$ and $n \in \mathbb{N}$. The operator $T_\gamma : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ defined by

$$T_\gamma h(z) := e^{\gamma z^n} \int_0^z \zeta^{n-1} h(\zeta) e^{-\gamma \zeta^n} d\zeta, \quad z \in \mathbb{C},$$

is continuous if $|\gamma| < \alpha$.

Spectra of Volterra operators on $H_v^\infty(\mathbb{C})$

Theorem

Assume that $v(r) = \exp(-\alpha r^p)$, $\alpha > 0$, $p > 0$. Let g be a polynomial of degree n less than or equal to the integer part of p with $g(0) = 0$.

(i) If the degree n of g satisfies $n < p$, then $\sigma(V_g, H_v^\infty(\mathbb{C})) = \{0\}$.

(ii) If $p = n \in \mathbb{N}$ and $g(z) = \beta z^n + k(z)$, k a polynomial of degree strictly less than n, then $\sigma(V_g, H_v^\infty(\mathbb{C})) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq |\beta|/\alpha\}$.

Moreover, we have
$\sigma(V_g, H_v^\infty(\mathbb{C})) = \{0\} \cup \overline{\{\lambda \in \mathbb{C} \setminus \{0\} \mid e^{\frac{g}{\lambda}} \notin H_v^\infty(\mathbb{C})\}}$.

In this case we also have $\sigma(V_g, H_v^0(\mathbb{C})) = \sigma(V_g, H_v^\infty(\mathbb{C}))$.

Spectra of Volterra operators on $H_v^\infty(\mathbb{C})$

Idea of the proof.

(i) **If n is less than or equal to the integer part of $p-1$**, then $V_g : H_v^\infty(\mathbb{C}) \to H_v^\infty(\mathbb{C})$ is compact.

Assume now that $p-1 < n < p$. For each $\lambda \neq 0$, $e^{\frac{g}{\lambda}} \in H_v^\infty(\mathbb{C})$.

Suppose first that $g(z) = \beta z^n$ for some $\beta \neq 0$. For $\lambda \neq 0$, take $\gamma > |\beta|/|\lambda|$. Clearly $v(r)e^{\gamma r^n}$ is non-increasing on $[r_0, \infty[$ for some $r_0 > 0$. Our Lemmas above imply $\lambda \in \rho(V_g, H_v^\infty(\mathbb{C}))$.

Suppose now that $g(z) = \beta z^n + k(z)$ for some $\beta \neq 0$ and some polynomial k of degree strictly less than n. Setting $g_1(z) := \beta z^n$, we have $V_g = V_{g_1} + V_k$, and V_k is a compact injective operator on $H_v^\infty(\mathbb{C})$. If $\lambda \neq 0$, we have $V_g - \lambda I = (V_{g_1} - \lambda I) + V_k$. A classical result on operator theory yields $\sigma(V_g, H_v^\infty(\mathbb{C})) = \sigma(V_{g_1}, H_v^\infty(\mathbb{C})) = \{0\}$.

Spectra of Volterra operators on $H_v^\infty(\mathbb{C})$

Idea of the proof continued.

(ii) **We suppose now that** $v(r) = \exp(-\alpha r^n)$**,** $\alpha > 0$, **and that** g **is a polynomial of degree exactly** n**.**

Consider first the case $g(z) = \beta z^n$. For $\lambda \in \mathbb{C} \setminus \{0\}$, we have $e^{\frac{g}{\lambda}} \in H_v^\infty(\mathbb{C})$ if and only if $|\beta|/|\lambda| \leq \alpha$. Therefore, $\{\lambda \mid |\lambda| \leq |\beta|/\alpha\} \subset \sigma(V_g, H_v^\infty(\mathbb{C}))$.

Now take $\lambda \in \mathbb{C}$ with $|\lambda| > |\beta|/\alpha$. Since $v(r)\exp(\alpha r^n) = 1$, our Lemmas above imply $\sigma(V_g, H_v^\infty(\mathbb{C})) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq |\beta|/\alpha\}$ in the present case.

In the general case $g(z) = \beta z^n + k(z), \beta \neq 0$ and some polynomial k of degree strictly less than n, we proceed as in the proof of part (i).

Spectra of Volterra operators on radial Hörmander algebras

Growth functions.

A function $p : \mathbb{C} \to]0, \infty[$ is called a **growth function** if it satisfies:

- (w1) p is continuous and subharmonic.
- (w2) p is radial, that is, $p(z) = p(|z|), z \in \mathbb{C}$.
- (w3) $\log(1 + |z|^2) = o(p(z))$ as $|z| \to \infty$.
- (w4) p is doubling, i.e. $p(2z) = O(p(z))$ as $|z| \to \infty$.

Spectra of Volterra operators on radial Hörmander algebras

Given p, we define the following weighted (LB)-space of entire functions.

$$A_p(\mathbb{C}) := \{f \in \mathcal{H}(\mathbb{C}) : \text{ there is } A > 0 : \sup_{z \in \mathbb{C}} |f(z)| \exp(-Ap(z)) < \infty\},$$

endowed with the inductive limit topology, for which it is a (DFN)-algebra.

Spectra of Volterra operators on radial Hörmander algebras

Given p, we define the following weighted Fréchet space of entire functions.

$$A_p^0(\mathbb{C}) := \{f \in \mathcal{H}(\mathbb{C}) : \text{ for all } \varepsilon > 0 : \sup_{z \in \mathbb{C}} |f(z)| \exp(-\varepsilon p(z)) < \infty\},$$

endowed with the projective topology, for which it is a nuclear Fréchet algebra.

Spectra of Volterra operators on radial Hörmander algebras

- $A_p^0(\mathbb{C}) \subset A_p(\mathbb{C})$.
- Condition (w3) implies that $A_p^0(\mathbb{C})$ contains the polynomials.
- Condition (w4) implies that the spaces are stable under differentiation.
- The differentiation operator D and the integration operator J are continuous on A_p and on A_p^0. The spectrum of these two operators on A_p and on A_p^0 was investigated recently by Beltrán, Bonet and Fernández.

Spectra of Volterra operators on radial Hörmander algebras

Examples:

- When $p(z) = |z|^s$, then $A_p(\mathbb{C})$ consists of all entire functions of order s and finite type or order less than s.
- When $p(z) = |z|^s$, then $A_p^0(\mathbb{C})$ is the space of all entire functions of order at most s and type 0.
- For $s = 1$, $p(z) = |z|$, $A_p(\mathbb{C})$ is the space of all entire functions of exponential type, and $A_p^0(\mathbb{C})$ is the space of entire functions of infraexponential type.

Spectra of Volterra operators on radial Hörmander algebras

Proposition

Let g be an entire function.

(i) $V_g : A_p \to A_p$ is continuous if and only if $g \in A_p$.

(ii) $V_g : A_p^0 \to A_p^0$ is continuous if and only if $g \in A_p^0$.

Spectra of Volterra operators on radial Hörmander algebras

Lemma

Let $p : \mathbb{C} \to [0, \infty[$ be a growth condition and let h be an entire function.

(i) The function e^h belongs to A_p if and only if $M(h, r) = O(p(r))$ as $r \to \infty$. If this is the case, then h is a polynomial.

(ii) The function e^h belongs to A_p^0 if and only if $M(h, r) = o(p(r))$ as $r \to \infty$. If this is the case, then h is a polynomial.

This is a consequence of an inequality of Caratheodory about the behaviour of the real part of entire functions.

Spectra of Volterra operators on radial Hörmander algebras

Theorem

Let $p : \mathbb{C} \to [0, \infty[$ be a growth condition and let $g \in A_p$ be non-constant.

(i) If $M(g, r) = O(p(r)), r \to \infty$, is not satisfied (which happens in particular when $p(r) = o(r), r \to \infty$), then $\sigma(V_g, A_p) = \mathbb{C}$.

(ii) If $M(g, r) = O(p(r)), r \to \infty$, then $\sigma(V_g, A_p) = \{0\}$. In this case g is a polynomial and $r = O(p(r)), r \to \infty$.

Moreover, in both cases we have
$\sigma(V_g, A_p) = \{0\} \cup \overline{\{\lambda \in \mathbb{C} \setminus \{0\} \mid e^{\frac{g}{\lambda}} \notin A_p\}}$.

Spectra of Volterra operators on radial Hörmander algebras

Idea of the proof.

First observe that $M(g/\lambda, r) = (1/|\lambda|)M(g, r)$ for each $\lambda \in \mathbb{C} \setminus \{0\}$ and $r > 0$. Therefore $e^{\frac{g}{\lambda}} \in A_p$ for some (all) $\lambda \neq 0$, if and only if $e^g \in A_p$, that is equivalent to $M(g, r) = O(p(r))$ as $r \to \infty$.

(i) If $M(g, r) = O(p(r))$ as $r \to \infty$ is not satisfied, then $e^{\frac{g}{\lambda}} \notin A_p$ for each $\lambda \neq 0$. We conclude $\sigma(V_g, A_p) = \mathbb{C}$.

Observe that in case $p(r) = o(r), r \to \infty$, then $M(g, r) = O(p(r))$ as $r \to \infty$ is not satisfied, since otherwise we would have $M(g, r) = O(p(r)) = o(r)$ as $r \to \infty$, that implies that g is constant; a contradiction.

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Idea of the proof.

(ii) If $M(g,r) = O(p(r)), r \to \infty$, then $e^{\frac{g}{\lambda}} \in A_p$ for each $\lambda \neq 0$. Given $\lambda \in \mathbb{C} \setminus \{0\}$, setting $G = e^{\frac{g}{\lambda}}$, $1/G = e^{-\frac{g}{\lambda}}$, the associated operator $S_{\lambda,g}$ satisfies $S_{\lambda,g} = M_G \circ J \circ M_{1/G} \circ D$. These four operators are continuous on the algebra A_p. Therefore $\lambda \in \rho(V_g, A_p)$, and $\sigma(V_g, A_p) = \{0\}$.

In this case, since g must be a non constant polynomial, the assumption in (ii) implies $r = O(p(r)), r \to \infty$.

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Theorem

Let $p : \mathbb{C} \to [0, \infty[$ be a growth condition and let $g \in A_p^0$ be non-constant.

(i) If $M(g, r) = o(p(r)), r \to \infty$, is not satisfied (which happens in case $p(r) = O(r), r \to \infty$), then $\sigma(V_g, A_p^0) = \mathbb{C}$.

(ii) If $M(g, r) = o(p(r)), r \to \infty$, then $\sigma(V_g, A_p^0) = \{0\}$. In this case g is a polynomial and $r = o(p(r)), r \to \infty$.

Moreover, in both cases, we have
$\sigma(V_g, A_p^0) = \{0\} \cup \overline{\{\lambda \in \mathbb{C} \setminus \{0\} \mid e^{\frac{g}{\lambda}} \notin A_p^0\}}$.

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