OPERATORS ON THE FRÉCHET SEQUENCE SPACES $\mathbf{ces(p+), 1 \le p < \infty}$

ANGELA A. ALBANESE, JOSÉ BONET, WERNER J. RICKER

ABSTRACT. The Fréchet sequence spaces $ces(p+)$ are very different to the Fréchet sequence spaces $\ell_{p+}, 1 \le p < \infty$, that generate them, [3]. The aim of this paper is to investigate various properties (eg. continuity, compactness, mean ergodicity) of certain linear operators acting in and between the spaces $ces(p+)$, such as the Cesàro operator, inclusion operators and multiplier operators. Determination of the spectra of such classical operators is an important feature. It turns out that both the space of multiplier operators $\mathcal{M}(ces(p+))$ and its subspace $\mathcal{M}_c(ces(p+))$ consisting of the compact multiplier operators are independent of p. Moreover, $\mathcal{M}_c(ces(p+))$ can be topologized so that it is the strong dual of the Fréchet-Schwartz space $ces(1+)$ and $(\mathcal{M}_c(ces(p+))'_\beta \simeq ces(1+)$ is a *proper* subspace of the Köthe echelon Fréchet space $\mathcal{M}(ces(p+)) = \lambda^\infty(A), 1 \le p < \infty$, for a suitable matrix A.

1. Introduction

Given an element $x = (x_n)_n = (x_1, x_2, \dots)$ of $\mathbb{C}^\mathbb{N}$ let $|x| := (|x_n|)_n$ and write $x \ge 0$ if $x = |x|$. By $x \le y$ we mean that $(y - x) \ge 0$. The sequence space $\mathbb{C}^\mathbb{N}$ is a (locally convex) Fréchet space with respect to the coordinatewise convergence. For each $1 < p < \infty$ define

$$ces(p) := \{x \in \mathbb{C}^\mathbb{N} : \|x\|_{ces(p)} := \|(\tfrac{1}{n} \textstyle\sum_{k=1}^n |x_k|)_n\|_p < \infty\}, \qquad (1.1)$$

where $\| \cdot \|_p$ denotes the standard norm in ℓ_p. An intensive study of the Banach spaces $ces(p), 1 < p < \infty$, was undertaken in [6], [12]; see also the references therein. They are reflexive, p-concave Banach lattices (for the order induced by the positive cone of $\mathbb{C}^\mathbb{N}$) and the canonical vectors $e_k := (\delta_{nk})_n$, for $k \in \mathbb{N}$, form an unconditional basis, [6], [9]. For every pair $1 < p, q < \infty$ the space $ces(p)$ is *not* isomorphic to ℓ_q, [6, Proposition 15.13]; it is also *not* isomorphic to $ces(q)$ if $p \ne q$, [4, Proposition 3.3].

The Cesàro operator $\mathsf{C} : \mathbb{C}^\mathbb{N} \longrightarrow \mathbb{C}^\mathbb{N}$, defined by

$$\mathsf{C}(x) := (x_1, \tfrac{x_1+x_2}{2}, \dots, \tfrac{x_1+x_2+\dots+x_n}{n}, \dots), \quad x \in \mathbb{C}^\mathbb{N}, \qquad (1.2)$$

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satisfies $|\mathsf{C}(x)| \le \mathsf{C}(|x|)$ for $x \in \mathbb{C}^{\mathbb{N}}$ and is a topological, linear isomorphism of $\mathbb{C}^{\mathbb{N}}$ onto itself. It is clear from (1.1) that

$$\|x\|_{ces(p)} = \|\mathsf{C}(|x|)\|_p, \quad x \in ces(p), \tag{1.3}$$

for each $1 < p < \infty$. Hardy's inequality, [14, Theorem 326], ensures that $\ell_p \subseteq ces(p)$ with $\|x\|_{ces(p)} \le p'\|x\|_p$ for $x \in \ell_p$, where the conjugate index p' of p is given by $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, the containment $\ell_p \subseteq ces(p)$ is *proper*, [9, Remark 2.2]. It is routine to verify that C maps $ces(p)$ continuously into ℓ_p.

For each $1 \le p < \infty$ define the vector space $\ell_{p+} := \bigcap_{q>p} \ell_q$; it is a Fréchet space (and Fréchet lattice for the order induced by the positive cone of $\mathbb{C}^{\mathbb{N}}$) with respect to the increasing sequence of *lattice norms*

$$q_k : x \longmapsto \|x\|_{p_k}, \quad x \in \ell_{p+}, \quad k \in \mathbb{N}, \tag{1.4}$$

for any sequence $p < p_{k+1} < p_k$ with $p_k \downarrow p$. It is known that each $\ell_{p+} \subseteq \mathbb{C}^{\mathbb{N}}$ (with a continuous inclusion) is a reflexive, quasinormable, non-Montel, countably normed Fréchet space which is solid in $\mathbb{C}^{\mathbb{N}}$ and contains no isomorphic copy of any infinite dimensional Banach space, [10], [18] . Clearly, for each $1 < p < \infty$, the Banach space $\ell_p \subseteq \ell_{p+}$ continuously and with a proper inclusion. Since $\mathsf{C} : \ell_p \longrightarrow \ell_p$ is continuous for each $1 < p < \infty$ (with operator norm p'), [14, Theorem 326]), it follows that $\mathsf{C} : \ell_{p+} \longrightarrow \ell_{p+}$ is also continuous, [2, Section 2]. The class of Fréchet spaces $ces(p+) := \bigcap_{p<q} ces(q)$, for $1 \le p < \infty$, where $ces(p+)$ is equipped with the increasing sequence of *lattice norms*

$$r_k : x \longmapsto \|x\|_{ces(p_k)} = \|\mathsf{C}(|x|)\|_{p_k}, \quad x \in ces(p+), \quad k \in \mathbb{N}, \tag{1.5}$$

for any sequence $p < p_{k+1} < p_k$ satisfying $\lim_{k\to\infty} p_k = p$ (i.e., $ces(p+) = \mathrm{proj}_k\, ces(p_k)$), has been studied in the recent article [3]. It was shown there that $ces(p+)$ coincides with the power series space of order one and finite type $\Lambda_{-1/p'}(\alpha)$ with $\alpha := (\log(k))_{k\in\mathbb{N}}$. The spaces $ces(p+)$ are generated by the spaces ℓ_{p+} in the sense that the *largest solid* Fréchet lattice in $\mathbb{C}^{\mathbb{N}}$ which contains ℓ_{p+} and which C maps continuously into ℓ_{p+} is precisely $ces(p+)$, [3].

The aim of this note is to investigate the behaviour of several natural operators defined in the spaces $ces(p+)$ for $p \ge 1$. We point out that a detailed investigation of the Cesàro operator C acting on the Banach spaces $ces(p), 1 < p < \infty$, was carried out in [4], [9], and on the Fréchet spaces ℓ_{p+} for $p \ge 1$ was undertaken in [2]. Here we treat C when it is acting in the Fréchet spaces $ces(p+)$ for $p \ge 1$. Its spectrum is determined in Theorem 2.3 and its mean ergodic properties are presented in Proposition 2.6. The properties of multiplier operators on $ces(p+)$ are studied in Section 3, especially their spectrum, compactness and mean ergodicity. Curiously, when acting in $ces(p+)$, the multipliers (resp. the subclass of compact ones) and their spectrum are *independent* of p; see Proposition 3.1 and Remark 3.2(ii) (resp. Proposition 3.7). The same is true for multipliers acting in ℓ_{p+};

see (3.9) and Proposition 3.15. Actually, the space of compact multipliers for $ces(p+)$ (resp. ℓ_{p+}) can be topologized so that it is isomorphic to the strong dual space of the Fréchet-Schwartz space $ces(1+)$ (resp. the reflexive Fréchet space ℓ_{1+}); see Proposition 3.11 (resp. Proposition 3.17). This is reminiscent of the dual space description of the space of compact (Fourier) multiplier operators from $L^p(G)$ to $L^q(G)$ when G is a compact group and $1 < p, q < \infty$, [5]. Section 4 is devoted to various operators acting between *different* spaces. For instance, it is shown that C maps $ces(p+)$ continuously into $ces(q+)$ if and only if $1 \leq p \leq q < \infty$ (see Proposition 4.4), whereas it is a compact operator if and only if $p < q$ (cf. Proposition 4.5). These results rely on a knowledge of the continuity/compactness properties of various inclusion maps between the family of spaces $\{\ell_{p+}, ces(q+) : 1 \leq p, q < \infty\}$; see Propositions 4.2 and 4.3.

We now introduce a few relevant notions needed in the sequel. Let X and Y be locally convex Hausdorff spaces (briefly, lcHs). The identity operator on X is denoted by I and $\mathcal{L}(X,Y)$ denotes the space of all continuous linear operators from X into Y. If $X = Y$, we denote $\mathcal{L}(X,Y)$ simply by $\mathcal{L}(X)$. Denote by Γ_X any system of continuous seminorms determining the topology of X. Let $\mathcal{L}_s(X)$ denote $\mathcal{L}(X)$ endowed with the strong operator topology τ_s which is determined by the seminorms $T \to q_x(T) := q(Tx)$, for each $x \in X$ and $q \in \Gamma_X$. Moreover, $\mathcal{L}_b(X)$ denotes $\mathcal{L}(X)$ equipped with the topology τ_b of uniform convergence on the bounded subsets of X which is determined by the seminorms $\gamma_{B,q} : T \to q_B(T) := \sup_{x\in B} q(Tx)$, for each bounded set $B \subseteq X$ and $q \in \Gamma_X$. For unexplained notation and standard concepts from functional analysis and lcHs , we refer to [17]; see also [15].

Given a lcHs X and $T \in \mathcal{L}(X)$, the *resolvent set* $\rho(T)$ of T consists of all $\lambda \in \mathbb{C}$ such that $R(\lambda, T) := (\lambda I - T)^{-1}$ exists in $\mathcal{L}(X)$. The set $\sigma(T) := \mathbb{C} \setminus \rho(T)$ is called the *spectrum* of T. The *point spectrum* $\sigma_{pt}(T)$ of T consists of all $\lambda \in \mathbb{C}$ such that $(\lambda I - T)$ is not injective. If we need to stress the space X, then we write $\sigma(T;X)$, $\sigma_{pt}(T;X)$ and $\rho(T;X)$. Given $\lambda, \mu \in \rho(T)$ the *resolvent identity* $R(\lambda,T) - R(\mu,T) = (\mu-\lambda)R(\lambda,T)R(\mu,T)$ holds. Unlike for Banach spaces, it may happen that $\rho(T) = \emptyset$ or that $\rho(T)$ is not open. This is why some authors prefer the subset $\rho^*(T)$ of $\rho(T)$ consisting of all $\lambda \in \mathbb{C}$ for which there exists $\delta > 0$ such that $B(\lambda, \delta) := \{z \in \mathbb{C} \colon |z - \lambda| < \delta\} \subseteq \rho(T)$ and $\{R(\mu,T)\colon \ \mu \in B(\lambda,\delta)\}$ is equicontinuous in $\mathcal{L}(X)$. Define $\sigma^*(T) := \mathbb{C} \setminus \rho^*(T)$, which is a closed set containing $\sigma(T)$. If $T \in \mathcal{L}(X)$ with X a Banach space, then $\sigma(T) = \sigma^*(T)$. The range $\{Sx : x \in X\}$ of $S \in \mathcal{L}(X)$ is denoted by Im(S) and its closure by $\overline{\text{Im(S)}}$.

2. The Cesàro operator on $ces(p+)$

The aim of this section is to investigate certain properties of C when it is acting on the Fréchet spaces $ces(p+)$ for $p \geq 1$. Given $1 < p < \infty$, it follows from Hardy's inequality that

$$\|\mathsf{C}(x)\|_{ces(p)} := \|\mathsf{C}(|\mathsf{C}(x)|)\|_p \leq p'\|\mathsf{C}(x)\|_p \leq p'\|\mathsf{C}(|x|)\|_p = p'\|x\|_{ces(p)},$$

for each $x \in ces(p)$, i.e., $\mathsf{C} \in \mathcal{L}(ces(p))$ with operator norm $\|\mathsf{C}\|_{op} \leq p'$. In view of the fact that the sequence of norms (1.5) generate the topology of $ces(p+)$ it is clear that $\mathsf{C} : ces(p+) \longrightarrow ces(p+)$ is *necessarily continuous*, i.e., $\mathsf{C} \in \mathcal{L}(ces(p+))$. The continuity of C from $ces(p+)$ into $ces(q+)$ with $p \neq q$ will be treated in Section 4. We will require the following description of the spectrum of $\mathsf{C} \in \mathcal{L}(ces(p))$; see [9, Theorem 5.1] and its proof. The dual operator of a Fréchet space operator $T \in \mathcal{L}(X)$ is denoted by $T' : X' \longrightarrow X'$, where X' is the space of all continuous linear functionals on X. When X' is equipped with its strong topology β we write X'_β.

Theorem 2.1. *Let $1 < p < \infty$. Then $\mathsf{C} \in \mathcal{L}(ces(p))$ satisfies $\|\mathsf{C}\|_{op} = p'$ and has spectra given by*

$$\sigma(\mathsf{C}; ces(p)) = \{\lambda \in \mathbb{C} : |\lambda - \tfrac{p'}{2}| \leq \tfrac{p'}{2}\} \quad \text{and} \quad \sigma_{pt}(\mathsf{C}; ces(p)) = \emptyset. \tag{2.1}$$

Moreover, if $|\lambda - \frac{p'}{2}| \leq \frac{p'}{2}$, then $\overline{\mathrm{Im}(\lambda \mathrm{I} - \mathsf{C})} \neq ces(p)$. In addition,

$$\sigma_{pt}(\mathsf{C}'; (ces(p))') = \{\lambda \in \mathbb{C} : |\lambda - \tfrac{p'}{2}| \leq \tfrac{p'}{2}\}. \tag{2.2}$$

The following result concerning the spectrum of certain operators on Fréchet spaces will be needed in the sequel, [2, Lemma 2.1].

Lemma 2.2. *Let $X = \cap_{n\in\mathbb{N}} X_n$ be a Fréchet space which is the intersection of a sequence of Banach spaces $((X_n, \|\cdot\|_n))_{n\in\mathbb{N}}$ satisfying $X_{n+1} \subseteq X_n$ with $\|x\|_n \leq \|x\|_{n+1}$ for each $n \in \mathbb{N}$ and $x \in X_{n+1}$. Let $T \in \mathcal{L}(X)$ satisfy the following condition:*

(A) *For each $n \in \mathbb{N}$ there exists $T_n \in \mathcal{L}(X_n)$ such that the restriction of T_n to X (resp. of T_n to X_{n+1}) coincides with T (resp. with T_{n+1}).*

Then $\sigma(T; X) \subseteq \cup_{n\in\mathbb{N}} \sigma(T_n; X_n)$ and $R(\lambda, T)$ coincides with the restriction of $R(\lambda, T_n)$ to X for each $n \in \mathbb{N}$ and each $\lambda \in \cap_{n\in\mathbb{N}} \rho(T_n; X_n)$.

Moreover, if $\cup_{n\in\mathbb{N}} \sigma(T_n; X_n) \subseteq \overline{\sigma(T; X)}$, then $\sigma^*(T; X) = \overline{\sigma(T; X)}$.

The spectra of the Cesàro operator acting in $ces(p+), p \geq 1$, can now be determined.

Theorem 2.3. (i) *Let $1 < p < \infty$. The following statements are valid.*

(a1) $\sigma_{pt}(\mathsf{C}; ces(p+)) = \emptyset$.
(a2) $\sigma(\mathsf{C}; ces(p+)) = \{\lambda \in \mathbb{C} : |\lambda - \frac{p'}{2}| < \frac{p'}{2}\} \cup \{0\}$.
(a3) $\sigma^*(\mathsf{C}; ces(p+)) = \overline{\sigma(\mathsf{C}; ces(p+))} = \{\lambda \in \mathsf{C} : |\lambda - \frac{p'}{2}| \leq \frac{p'}{2}\}$.
(ii) *For $p = 1$ the following statements are valid.*
(b1) $\sigma_{pt}(\mathsf{C}; ces(1+)) = \emptyset$.
(b2) $\sigma(\mathsf{C}; ces(1+)) = \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > 0\} \cup \{0\}$.
(b3) $\sigma^*(\mathsf{C}; ces(1+)) = \overline{\sigma(\mathsf{C}; ces(1+))} = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \geq 0\}$.

Proof. The statements (a1) and (b1) follow from (2.1) since $ces(p+) \subseteq ces(q)$, for $1 \leq p < q$, implies that $\sigma_{pt}(\mathsf{C}; ces(p+)) \subseteq \sigma_{pt}(\mathsf{C}; ces(q))$.

For the remainder of the proof fix $1 \leq p < p_{n+1} < p_n$ satisfying $p_n \downarrow p$ (i.e., $p'_n \uparrow p'$) and denote by $\mathsf{C}_n : ces(p_n) \longrightarrow ces(p_n)$ the Cesàro operator $\mathsf{C} \in \mathcal{L}(\mathbb{C}^{\mathbb{N}})$ restricted to the Banach space $ces(p_n)$, for $n \in \mathbb{N}$.

We now prove statements (a2) and (b2). First of all, for $p > 1$, apply Lemma 2.2 (with $X := ces(p+)$ and $X_n := ces(p_n), n \in \mathbb{N}$) and (2.1) for each $p_n, n \in \mathbb{N}$, to conclude that

$$\sigma(\mathsf{C}; ces(p+)) \subseteq \cup_{n\in\mathbb{N}}\sigma(\mathsf{C}_n; ces(p_n)) \subseteq \{\lambda \in \mathbb{C} : |\lambda - \tfrac{p'}{2}| < \tfrac{p'}{2}\} \cup \{0\}.$$

Similarly, for $p = 1$, in which case $p' = \infty$ and $p'_n \uparrow \infty$, we can conclude that

$$\sigma(\mathsf{C}; ces(1+)) \subseteq \cup_{n\in\mathbb{N}}\sigma(C_n; ces(p_n)) \subseteq \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > 0\} \cup \{0\}.$$

Next we verify that $0 \in \sigma(\mathsf{C}; ces(p+))$, for which it suffices to show that $\mathsf{C} \in \mathcal{L}(ces(p+))$ is not surjective. To this effect, define $y := \sum_{i=1}^{\infty} \frac{1}{2i-1} e_{2i-1} \in \ell_{p+} \subseteq ces(p+)$. We point out that $e_n := (\delta_{nk})_k$, for $n \in \mathbb{N}$, are an unconditional basis for $ces(p+)$, [3, Proposition 3.5]. Then the element $x := \mathsf{C}^{-1}(y) = (1, -1, 1, -1, \dots)$ in $\mathbb{C}^{\mathbb{N}}$ satisfies $|x| = (1, 1, 1, 1, \dots)$ with $\mathsf{C}(|x|) = |x| \notin ces(q)$ for every $q > 1$. Accordingly $x \notin ces(p+)$ for all $p \geq 1$. Since x is the unique element in $\mathbb{C}^{\mathbb{N}}$ satisfying $y = \mathsf{C}(x)$ (as $\mathsf{C} \in \mathcal{L}(\mathbb{C}^{\mathbb{N}})$ is an isomorphism), it follows that $y \in ces(p+)$ is *not* in the range of $\mathsf{C} \in \mathcal{L}(ces(p+))$ for every $p \geq 1$. So, we have shown that $0 \in \sigma(\mathsf{C}; ces(p+))$ for $p \geq 1$.

Now fix $\lambda \in \mathbb{C}$ with $|\lambda - \frac{p'}{2}| < \frac{p'}{2}$ if $p > 1$, and $\mathrm{Re}(\lambda) > 0$ if $p = 1$. In both cases there is $n_0 \in \mathbb{N}$ such that $|\lambda - \frac{p'_n}{2}| < \frac{p'_n}{2}$ for all $n \geq n_0$. We prove that $\mathrm{Im}(\lambda I - \mathsf{C})$ is not dense in $ces(p+)$, which implies that $\lambda \in \sigma(\mathsf{C}; ces(p+))$. Proceeding by contradiction, assume that $\mathrm{Im}(\lambda I - \mathsf{C})$ is dense in $ces(p+)$. Since the natural inclusion map of $ces(p+)$ into $ces(p_{n_0})$ is continuous with dense range, also $\mathrm{Im}(\lambda I - \mathsf{C}_{n_0})$ is dense in $ces(p_{n_0})$. However, this contradicts Theorem 2.1. So, we have established that $\{\lambda \in \mathbb{C} : |\lambda - \frac{p'}{2}| < \frac{p'}{2}\} \subseteq \sigma(\mathsf{C}; ces(p+))$. This completes the proof of parts (a2) and (b2).

It remains to show that statements (a3) and (b3) hold. In the proof of parts (a2) and (b2) it was established that

$$\cup_{n\in\mathbb{N}}\sigma(\mathsf{C}_n; ces(p_n)) = \sigma(\mathsf{C}; ces(p+)) \subseteq \overline{\sigma(\mathsf{C}; ces(p+))}.$$

Then Lemma 2.2 implies that $\sigma^*(\mathsf{C}; ces(p+)) = \overline{\sigma(\mathsf{C}; ces(p+))}$. □

Proposition 2.4. (i) *If* $1 < p < \infty$, *then*

$$\{\lambda \in \mathbb{C} : |\lambda - \tfrac{p'}{2}| < \tfrac{p'}{2}\} \subseteq \sigma_{pt}(\mathsf{C}', (ces(p+))'_\beta).$$

(ii) *If* $p = 1$, *then*

$$\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > 0\} \subseteq \sigma_{pt}(\mathsf{C}', (ces(1+))'_\beta).$$

Proof. (i) Let $\lambda \in \mathbb{C}$ satisfy $|\lambda - \frac{p'}{2}| < \frac{p'}{2}$, in which case $|\lambda - \frac{r'}{2}| < \frac{r'}{2}$ for some $p < r < \infty$. Since $ces(p+) = \Lambda_{-1/p'}(\alpha)$, with $\alpha = (\log(k))_{k\in\mathbb{N}}$, is a power series space, its dual space $(ces(p+))'_\beta \subseteq \mathbb{C}^{\mathbb{N}}$ is a sequence space, [17, p.

357] . It is then routine to verify that the dual operator $\mathsf{C}' : (ces(p+))'_\beta \longrightarrow (ces(p+))'_\beta$ is given by

$$\mathsf{C}'(y) := (\sum_{k=n}^{\infty} \tfrac{y_n}{n})_n, \quad y = (y_n)_n \in (ces(p+))'_\beta.$$

Since $(ces(r))' \subseteq (ces(p+))'_\beta$, Theorem 2.1 implies that

$$\{\mu \in \mathbb{C} : |\mu - \tfrac{r'}{2}| < \tfrac{r'}{2}\} = \sigma_{pt}(\mathsf{C}; (ces(r))') \subseteq \sigma_{pt}(\mathsf{C}'; (ces(p+))'_\beta).$$

But, $\lambda \in \sigma_{pt}(\mathsf{C}'; (ces(r))')$ via (2.2) and hence, $\lambda \in \sigma_{pt}(\mathsf{C}'; (ces(p+))'_\beta)$.

(ii) Fix $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > 0$ and select $r' > 1$ such that $|\lambda - \frac{r'}{2}| < \frac{r'}{2}$. By (2.2) there exists $0 \neq u \in (ces(r))' \subseteq (ces(1+))'_\beta$ such that $\mathsf{C}(u) = \lambda u$. The conclusion then follows as in part (i). □

Remark 2.5. (i) An operator $T \in \mathcal{L}(X)$, with X a lcHs, is called *bounded* (resp. *compact*) if there exists a neighbourhood U of $0 \in X$ such that $T(U)$ is a bounded (resp. relatively compact) subset of X. If X is Montel (i.e., each bounded set is relatively compact), then T is compact if and only if it is bounded. We point out that each $ces(p+), 1 \leq p < \infty$, is a Fréchet-Schwartz space, [3, Proposition 3.5], hence, a Montel space. Accordingly, there is no distinction between $\mathsf{C} : ces(p+) \longrightarrow ces(p+)$ being compact or bounded. Since the spectrum of a compact operator is necessarily a compact subset of $\mathbb{C}$, [11, Theorem 9.10.2], it follows from parts (a2) and (b2) of Theorem 2.3 that $\mathsf{C} : ces(p+) \longrightarrow ces(p+)$ *fails* to be a compact operator for every $p \geq 1$.

(ii) For the Banach spaces $\ell_p, 1 < p < \infty$, it is known that $\sigma(\mathsf{C}; \ell_p) = \sigma(\mathsf{C}; ces(p)) = \{\lambda \in \mathbb{C} : |\lambda - \frac{p'}{2}| \leq \frac{p'}{2}\}$ and, moreover, that $ces(p)$ is the largest solid Banach lattice $X \subseteq \mathbb{C}^{\mathbb{N}}$ which contains ℓ_p and such that $\mathsf{C} : \ell_p \longrightarrow \ell_p$ has a continuous linear extension $\mathsf{C} : X \longrightarrow \ell_p$, [9, Section 5]. Analogous to the Banach space case it turns out that also $\sigma(\mathsf{C}; \ell_{p+}) = \sigma(\mathsf{C}; ces(p+))$ for $p \geq 1$; see Theorem 2.3 above and Theorems 2.2 and 2.4 in [2].

An operator $T \in \mathcal{L}(X)$, with X a lcHs space, is called *power bounded* if $\{T^n\}_{n=1}^\infty$ is an equicontinuous subset of $\mathcal{L}(X)$. Given $T \in \mathcal{L}(X)$, the averages

$$T_{[n]} := \tfrac{1}{n} \sum_{m=1}^{n} T^m, \qquad n \in \mathbb{N},$$

of the iterates of T are called the Cesàro means of T. It is routine to verify that $\frac{T^n}{n} = T_{[n]} - \frac{(n-1)}{n} T_{[n-1]}$ for $n \geq 2$. The operator T is said to be *mean ergodic* (resp., *uniformly mean ergodic*) if $\{T_{[n]}\}_{n=1}^\infty$ is a convergent sequence in $\mathcal{L}_s(X)$ (resp., in $\mathcal{L}_b(X)$), [16]. A lcHs operator $T \in \mathcal{L}(X)$, with X separable, is called *hypercyclic* if there exists $x \in X$ such that the orbit $\{T^n x \colon n \in \mathbb{N}_0\}$ is dense in X, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. If, for some $z \in X$, the projective orbit $\{\lambda T^n z \colon \lambda \in \mathbb{C},\ n \in \mathbb{N}_0\}$ is dense in X, then T is called *supercyclic*. Clearly, hypercyclicity implies supercyclicity.

Proposition 2.6. *The Cesàro operator* $\mathsf{C} \in \mathcal{L}(ces(p+))$, *for* $1 \leq p < \infty$, *is not power bounded, not mean ergodic and not supercyclic.*

Proof. First let $1 < p < \infty$. By Proposition 2.4(i) $\lambda := (1 + p')/2$ belongs to $\sigma_{pt}(\mathsf{C}', (ces(p+))'_\beta)$ and so there exists a non-zero vector $u \in (ces(p+))'_\beta$ satisfying $\mathsf{C}'(u) = \lambda u$. Choose $x \in ces(p+)$ such that $\langle x, u\rangle \neq 0$. Then

$$\langle \tfrac{1}{n}\mathsf{C}^n x, u\rangle = \tfrac{1}{n}\langle x, (\mathsf{C}')^n(u)\rangle = \tfrac{1}{n}\lambda^n\langle x, u\rangle, \quad n \in \mathbb{N},$$

with $\lambda > 1$. Hence, the set $\{\frac{1}{n}\mathsf{C}^n(x) \ : \ n \in \mathbb{N}\}$ is unbounded in $ces(p+)$ and so the sequence $\{\frac{1}{n}\mathsf{C}^n\}_{n=1}^\infty$ cannot converge to 0 in $\mathcal{L}_s(ces(p+))$. This implies that C is not mean ergodic as $\frac{\mathsf{C}^n}{n} = \mathsf{C}_{[n]} - \frac{(n-1)}{n}\mathsf{C}_{[n-1]}$ for $n \geq 2$. Since each space $ces(p+), 1 \leq p < \infty$, is reflexive and power bounded operators in reflexive spaces are necessarily mean ergodic, [1, Corollary 2.7], it follows that $\mathsf{C} \in \mathcal{L}(ces(p+))$ cannot be power bounded.

Suppose that $\mathsf{C} \in \mathcal{L}(ces(p+))$ is supercyclic. As $ces(p+)$ is dense in $\mathbb{C}^\mathbb{N}$, it follows that $\mathsf{C} : \mathbb{C}^\mathbb{N} \longrightarrow \mathbb{C}^\mathbb{N}$ is supercyclic; a contradiction to [2, Proposition 4.3].

Consider now $p = 1$. By Proposition 2.4(ii), there is a non-zero vector $u \in (ces(1+))'_\beta$ satisfying $\mathsf{C}'(u) = 2u$. Choose $x \in ces(p+)$ such that $\langle x, u\rangle \neq 0$. Then $\langle \frac{1}{n}\mathsf{C}^n x, u\rangle = \frac{2^n}{n}\langle x, u\rangle$, for each $n \in \mathbb{N}$. The proof can now be completed as for the case $p > 1$ above. □

3. Multiplier operators on $ces(p+)$

Given $\varphi = (\varphi_i)_i \in \mathbb{C}^\mathbb{N}$, the multiplication operator $M_\varphi : \mathbb{C}^\mathbb{N} \longrightarrow \mathbb{C}^\mathbb{N}$ is defined coordinatewise by $M_\varphi(x) := (\varphi_i x_i)_i$ for each $x = (x_i)_i \in \mathbb{C}^\mathbb{N}$. We will also write φx for $(\varphi_i x_i)_i$. According to entry 16 in the table on p.69 of [6], given $1 < p \leq q < \infty$ an element $\varphi = (\varphi_i)_i \in \mathbb{C}^\mathbb{N}$ satisfies $M_\varphi(ces(p)) \subseteq ces(q)$ if and only if $(i^{\frac{1}{q}-\frac{1}{p}}\varphi_i)_i \in \ell_\infty$; see also [6, p.71]. Observe that $(\frac{1}{q} - \frac{1}{p}) \leq 0$.

The aim of this section is to investigate the class

$$\mathcal{M}(ces(p+)) := \{\varphi \in \mathbb{C}^\mathbb{N} : M_\varphi \in \mathcal{L}(ces(p+))\}, \quad 1 \leq p < \infty,$$

of all multipliers acting in $ces(p+)$ as well as its subclass

$$\mathcal{M}_c(ces(p+)) := \{\varphi \in \mathcal{M}(ces(p+)) : M_\varphi \in \mathcal{L}(ces(p+)) \text{ is compact}\},$$

for $1 \leq p < \infty$, consisting of all the compact multipliers. Both of these classes will be explicity identified and the spectra of their members determined. Recall that $ces(p+)$ is generated, via a certain averaging process, by the space ℓ_{p+}, for $p \geq 1$. Accordingly, it is of some interest to also identify the corresponding multipliers spaces $\mathcal{M}(\ell_{p+})$ and $\mathcal{M}_c(\ell_{p+})$ of ℓ_{p+} and to compare the situation with $ces(p+)$. It will be shown that both $\mathcal{M}_c(ces(p+))$ and $\mathcal{M}_c(\ell_{p+})$ are the *strong dual spaces* of suitable Fréchet spaces (independent of p). The mean ergodic properties of multiplier operators in ℓ_{p+} and $ces(p+)$ will also described.

Proposition 3.1. *Let $1 \le p < \infty$ and $\varphi = (\varphi_i)_i \in \mathbb{C}^{\mathbb{N}}$. The following conditions are equivalent .*

(i) $M_\varphi(ces(p+)) \subseteq ces(p+)$.
(ii) $\varphi \in \mathcal{M}(ces(p+))$, *i.e.*, $M_\varphi \in \mathcal{L}(ces(p+))$.
(iii) *For each* $r > p$ *there exist* $s \in (p, r]$ *and* $C > 0$ *such that* $|\varphi_i| \le C i^{\frac{1}{s}-\frac{1}{r}}$ *for all* $i \in \mathbb{N}$.
(iv) *For each* $\eta \in (0, 1)$ *there exists* $K > 0$ *such that* $|\varphi_i| \le K i^\eta$, *for* $i \in \mathbb{N}$.

Proof. Conditions (i) and (ii) are equivalent by the closed graph theorem for Fréchet spaces.

Since $ces(p+) = \mathrm{proj}_{p<q} ces(q)$ is the projective limit of the Banach spaces $(ces(q), \| \cdot \|_{ces(q)})$ for $q > p$, condition (ii) is equivalent to the following requirement:

$$\forall r > p \;\; \exists s \in (p, r] \; \exists A > 0 \;\; \text{such that} \;\; ||M_\varphi(x)||_{ces(r)} \le A||x||_{ces(s)},$$

for $x \in ces(s)$. This requirement is equivalent to the fact that for each $r > p$ there exists $s \in (p, r]$ such that $M_\varphi : ces(s) \longrightarrow ces(r)$ is continuous. By entry 16 in the table on p.69 of [6], this is precisely the condition (iii).

(iii) $\Longrightarrow$ (iv). Fix $\eta \in (0, 1)$. Set $r := (p + \eta) > p$ and select $s \in (p, r)$ according to (iii). Then $\delta := (s - p)$ satisfies $0 < \delta < \eta$ and $s = p + \delta$. Moreover,

$$\tfrac{1}{s} - \tfrac{1}{r} = \tfrac{(n-\delta)}{(p+\eta)(p+\delta)} < \tfrac{(\eta-\delta)}{p^2} \le (\eta - \delta) < \eta. \tag{3.1}$$

Since $|\varphi_i| \le C i^{\frac{1}{s}-\frac{1}{r}}$ for $i \in \mathbb{N}$ (by (iii)), condition (iv) follows from (3.1).

(iv) $\Longrightarrow$ (iii). Let $r > p$ be given. Select $\varepsilon \in (0, 1)$ with $\varepsilon < (r - p)$. Then $s := p + \frac{\varepsilon}{2}$ satisfies $p < s < r$. By (iv) applied to $\eta := \frac{\varepsilon}{(p+1)(2p+1)} \in (0, 1)$ there exists $K > 0$ such that $|\varphi_i| \le K i^\eta$ for all $i \in \mathbb{N}$. Since $(p+1)(2p+1) > (p + \varepsilon)(2p + \varepsilon)$, it follows that

$$0 < \eta < \tfrac{\varepsilon}{(p+\varepsilon)(2p+\varepsilon)} = \tfrac{1}{p+(\varepsilon/2)} - \tfrac{1}{(p+\varepsilon)} < \left(\tfrac{1}{s} - \tfrac{1}{r}\right).$$

Accordingly, $|\varphi_i| \le K i^\eta < K i^{\frac{1}{s}-\frac{1}{r}}$ for $i \in \mathbb{N}$, i.e., (iii) is satisfied. □

Remark 3.2. (i) It is clear from (iv) of Proposition 3.1 that $\ell_\infty \subseteq \mathcal{M}(ces(p+))$ for every $1 \le p < \infty$. Note that $\varphi := (\log(i + 1))_i \notin \ell_\infty$ also satisfies condition (iv) of Proposition 3.1. So, $\ell_\infty \subsetneqq \mathcal{M}(ces(p+))$ is a *proper* containment for all $p \ge 1$.

(ii) Given $0 < \eta < 1$, define the weight $w_\eta := (i^{-\eta})_i \in \mathbb{C}^{\mathbb{N}}$ and write $u \in \ell_\infty(w_\eta)$ if and only if the coordinatewise product $uw_\eta \in \ell_\infty$. Then the characterization given in Proposition 3.1(iv) can be formulated as

$$\mathcal{M}(ces(p+)) = \bigcap_{0<\eta<1} \ell_\infty(w_\eta), \quad p \ge 1. \tag{3.2}$$

In particular, $\mathcal{M}(ces(p+))$ is *independent* of $p \in [1, \infty)$ and, by part (i), properly contains ℓ_∞. Observe that $0 < \eta_1 < \eta_2 < 1$ implies that $\ell_\infty(w_{\eta_1}) \subseteq \ell_\infty(w_{\eta_2})$. It follows that the vector space (3.2) is also an *algebra* relative to coordinatewise multiplication (which corresponds to composition of operators in $\mathcal{L}(ces(p+))$). It is routine to check from (3.2) that $\mathcal{M}(ces(p+))$ is also *solid*, that is, if $\varphi \in \mathcal{M}(ces(p+))$ and $\psi \in \mathbb{C}^{\mathbb{N}}$ satisfies $|\psi| \leq |\varphi|$, then also $\psi \in \mathcal{M}(ces(p+))$. Let $A = (a_n)_{n=1}^{\infty}$ be the Köthe matrix corresponding to the increasing sequence of functions $a_n : \mathbb{N} \longrightarrow \mathbb{C}$ given by $a_n := w_{1/n}$ for $n \in \mathbb{N}$. Then, as a linear subspace of $\mathbb{C}^{\mathbb{N}}$, the right-side of (3.2) is precisely the Köthe echelon space of infinite order corresponding to A, which we denote symbolically by $\lambda^{\infty}(A)$, [17, Ch.27]. That is,

$$\mathcal{M}(ces(p+)) = \lambda^{\infty}(A), \quad p \geq 1. \tag{3.3}$$

We now turn our attention to the compactness of multipliers. For this we first require a description of the spectra of multiplier operators in $\mathcal{L}(ces(p+))$.

Proposition 3.3. *Let $1 \leq p < \infty$ and $\varphi \in \mathcal{M}(ces(p+))$.*

(i) $\sigma_{pt}(M_{\varphi}; ces(p+)) = \{\varphi_i : i \in \mathbb{N}\}$, *where* $\varphi = (\varphi_i)_i$.

(ii) *The complex number $\lambda \in \rho(M_{\varphi}; ces(p+))$ if and only if for each $r > p$ there exist $s \in (p, r]$ and $\varepsilon > 0$ such that*

$$|\lambda - \varphi_i| \geq \varepsilon \cdot i^{\frac{1}{r} - \frac{1}{s}}, \quad i \in \mathbb{N}.$$

In this case the inverse operator $(M_{\lambda-\varphi})^{-1} = M_{1/(\lambda-\varphi)}$ in $\mathcal{L}(ces(p+))$ and so $1/(\lambda - \varphi) \in \mathcal{M}(ces(p+))$.

(iii) $\overline{\sigma(M_{\varphi}; ces(p+))} = \sigma^*(M_{\varphi}; ces(p+)) = \overline{\{\varphi_i : i \in \mathbb{N}\}}$.

Proof. (i) Since $M_{\varphi}(e_i) = \varphi_i e_i$ for $i \in \mathbb{N}$, it is clear that $\{\varphi_i : i \in \mathbb{N}\} \subseteq \sigma_{pt}(M_{\varphi}; ces(p+))$. Moreover, if $\lambda \in \mathbb{C}$ satisfies $M_{\varphi}(x) = \lambda x$, that is, $(\varphi_i x_i)_i = (\lambda x_i)_i$ for some $x \in ces(p+) \setminus \{0\}$, then $\lambda \in \{\varphi_i : i \in \mathbb{N}\}$.

(ii) It is clear (from (i)) that $\lambda \in \mathbb{C}$ belongs to $\rho(M_{\varphi}; ces(p+))$ if and only if $\lambda \notin \{\varphi_i : i \in \mathbb{N}\}$ and the element $\psi_{\lambda} := (\frac{1}{\lambda - \varphi_i})_i \in \mathbb{C}^{\mathbb{N}}$ belongs to $\mathcal{M}(ces(p+))$. Accordingly, the stated condition follows from Proposition 3.1 applied to ψ_{λ}. Clearly, $(M_{\lambda-\varphi})^{-1} = M_{\psi_{\lambda}}$ whenever $\psi_{\lambda} \in \mathcal{M}(ces(p+))$.

(iii) Part (i) implies that $\{\varphi_i : i \in \mathbb{N}\} \subseteq \sigma(M_{\varphi}; ces(p+))$. Hence,

$$\overline{\{\varphi_i : i \in \mathbb{N}\}} \subseteq \overline{\sigma(M_{\varphi}; ces(p+))} \subseteq \sigma^*(M_{\varphi}; ces(p+)). \tag{3.4}$$

Suppose that $\lambda \notin \overline{\{\varphi_i : i \in \mathbb{N}\}}$. Then there exists $\varepsilon > 0$ such that $|\lambda - \varphi_i| \geq \varepsilon$ for each $i \in \mathbb{N}$. If $\mu \in \mathbb{C}$ satisfies $|\mu - \lambda| < \varepsilon/2$, then $|\mu - \varphi_i| \geq \varepsilon/2$ for each $i \in \mathbb{N}$. Via part (ii) we can conclude (by choosing $s := r$ for each $r > p$) that $\mu \in \rho(M_{\varphi}; ces(p+))$ and $(M_{\mu-\varphi})^{-1} = M_{1/(\mu-\varphi)}$. Moreover, for each $r > p$, we have

$$||M_{1/(\mu-\varphi)}(x)||_{ces(r)} = \|(\tfrac{x_i}{\mu - \varphi_i})_i\|_{ces(r)} \leq (2/\varepsilon)||x||_{ces(r)}, \quad x \in ces(p+).$$

Since $ces(p+) = \text{proj}_{p<r} ces(r)$, it follows that $\{(M_{\mu-\varphi})^{-1} : |\mu - \lambda| < \varepsilon/2\}$ is an equicontinuous subset of $\mathcal{L}(ces(p+))$, that is, $\lambda \in \rho^*(M_{\varphi}; ces(p+))$. So, we have established that $(\mathbb{C} \setminus \overline{\{\varphi_i : i \in \mathbb{N}\}}) \subseteq \rho^*(M_{\varphi}; ces(p+))$. Combined with (3.4) this yields the desired conclusion. $\square$

Remark 3.4. (i) Remark 3.2(i) and Proposition 3.3(iii) show that the spectrum of $\varphi \in \mathcal{M}(ces(p+))$ need not be a compact subset of $\mathbb{C}$. This may occur because $\varphi \notin \ell_\infty$; consider $\varphi := (\log(i+1))_i$ as in Remark 3.2(i). The phenomenon also occurs for certain bounded multipliers. For instance, $\psi := (1/\varphi) \in \ell_\infty \subseteq \mathcal{M}(ces(p+))$ has bounded spectrum $\sigma(M_\psi; ces(p+))$ but, $\sigma(M_\psi; ces(p+))$ is *not* compact in $\mathbb{C}$ as it fails to contain the limit point 0 (because $(M_\psi)^{-1} = M_\varphi \in \mathcal{L}(ces(p+))$ and so $0 \in \rho(M_\psi; ces(p+))$).

(ii) If $T \in \mathcal{L}(X)$, with X a lcHs, is compact, then $\sigma(T;X)$ is a compact subset of $\mathbb{C}$ and every non-zero point of $\sigma(T;X)$ is isolated, [11, Theorem 9.10.2], [13, p.204]. This implies, via Proposition 3.3(iii), that if $\varphi \in \mathcal{M}_c(ces(p+))$, then necessarily $\varphi \in c_0$. That is,

$$\mathcal{M}_c(ces(p+) \subseteq c_0, \quad p \geq 1.$$

(iii) For each $p \geq 1$, Proposition 3.3(ii) implies that the unital, commutative subalgebra $\{M_\varphi : \varphi \in \mathcal{M}(ces(p+))\}$ of $\mathcal{L}(ces(p+))$ is *inverse closed* in $\mathcal{L}(ces(p+))$. That is, if $T \in \{M_\varphi : \varphi \in \mathcal{M}(ces(p+))\}$ is invertible in $\mathcal{L}(ces(p+))$, then $T^{-1} = M_\psi$ for some $\psi \in \mathcal{M}(ces(p+))$.

Lemma 3.5. *Let $1 \leq p < \infty$ and $\varphi \in \mathcal{M}(ces(p+))$. The operator $M_\varphi \in \mathcal{L}(ces(p+))$ is compact if and only if there exists $q > p$ such that for all $r \in (p,q)$ the operator $M_\varphi \in \mathcal{L}(\mathbb{C}^\mathbb{N})$ maps the Banach space $ces(q)$ continuously into the Banach space $ces(r)$ (denoted simply by $M_\varphi : ces(q) \to ces(r)$).*

Proof. Assume first that there exists $q > p$ such that $M_\varphi : ces(q) \to ces(r)$ is continuous for all $r \in (p,q)$. Set $U_{p,q} := \{x \in ces(p+) : ||x||_{ces(q)} \leq 1\}$, which is a neighbourhood of 0 in $ces(p+)$. Since $U_{p,q}$ is contained in the closed unit ball of the Banach space $ces(q)$, the continuity assumption on M_φ implies that $M_\varphi(U_{p,q})$ is a bounded set in $ces(r)$ for all $r \in (p,q)$ Accordingly, $M_\varphi(U_{p,q})$ is a bounded set in $ces(p+)$ and hence, is relatively compact as $ces(p+)$ is a Montel space.

Conversely, if $M_\varphi \in \mathcal{L}(ces(p+))$ is compact, then by definition there exists $q > p$ such that $M_\varphi(U_{p,q})$ is a relatively compact (i.e., bounded) set in $ces(p+)$. Thus, $M_\varphi(U_{p,q})$ is a bounded set in $ces(r)$ for all $r \in (p,q)$. Hence, for each $r \in (p,q)$, the operator $M_\varphi : (ces(p+), \|\cdot\|_{ces(q)}) \to ces(r)$ acting between these two *normed* spaces is continuous. The density of $ces(p+)$ in $ces(q)$ ensures that there is a unique continuous, linear extension $T : ces(q) \to ces(r)$ of M_φ. By considering the canonical basis vectors $\{e_n : n \in \mathbb{N}\}$ it is routine to check that this unique extension T is precisely $M_\varphi : ces(q) \to ces(r)$. □

Given $x = (x_i)_i \in \ell_\infty$, its *least decreasing majorant* $\widehat{x} \in \ell_\infty$ is defined by

$$\widehat{x} := (\sup_{i \geq k} |x_i|)_k,$$

[6, p.9]. For each $r \in (1,\infty)$ the space

$$d(r) := \{x \in \ell_\infty : \widehat{x} \in \ell_r\}$$

is a Banach space when endowed with the norm

$$\|x\|_{d(r)} := \|\widehat{x}\|_r, \quad x \in d(r),$$

[6, p.3 & p.9]. Since $|x| \leq \widehat{x}$ whenever $x \in \ell_\infty$, it is clear that $d(r) \subseteq \ell_r \subseteq c_0$ for $1 < r < \infty$. If $1 < p < \infty$, then $d(p')$ is isomorphic to the dual Banach space $(ces(p))'$ of $ces(p)$, [6, p.61 & Corollary 12.17].

The following result is Proposition 2.5 of [4].

Proposition 3.6. *Let $1 < r < q$. For $\varphi \in \mathbb{C}^{\mathbb{N}}$ the following assertions are equivalent.*

(i) *The operator $M_\varphi : ces(q) \longrightarrow ces(r)$ is continuous.*
(ii) *The operator $M_\varphi : ces(q) \longrightarrow ces(r)$ is compact.*
(iii) *$\varphi \in d(s)$, where $s > r$ is given by $\frac{1}{s} := \frac{1}{r} - \frac{1}{q}$.*

It is now possible to characterize the compact multipliers for $ces(p+)$.

Proposition 3.7. *Let $1 \leq p < \infty$ and $\varphi \in \mathcal{M}(ces(p+))$. The following conditions are equivalent.*

(i) *$\varphi \in \mathcal{M}_c(ces(p+))$, that is, $M_\varphi \in \mathcal{L}(ces(p+))$ is a compact operator.*
(ii) *$\widehat{\varphi} \in \ell(\infty-) := \cup_{t>1} \ell_t \subseteq c_0$.*
(iii) *$\varphi \in d(\infty-) := \cup_{s>1} d(s) \subseteq c_0$.*

Proof. Recall from Remark 3.4(ii) that $\mathcal{M}_c(ces(p+)) \subseteq c_0$.

(ii) $\Longrightarrow$ (i). Since $\widehat{\varphi} \in \ell(\infty-)$, there exists $t > p$ such that $\widehat{\varphi} \in \ell_t$. Select $\varepsilon > 0$ according to $p + \frac{p^2}{\varepsilon} = t$ and set $q := p + \varepsilon$. Now fix any $r \in (p, q)$ and define $s > 0$ by $\frac{1}{s} = \frac{1}{r} - \frac{1}{q}$. Since $p < r$, it follows that $\frac{1}{s} < \frac{\varepsilon}{p^2 + \varepsilon p} = \frac{1}{t}$. Hence, $s > t$. Therefore $\widehat{\varphi} \in \ell_s$ as $\ell_t \subseteq \ell_s$, that is, $\varphi \in d(s)$. Proposition 3.6 implies that $M_\varphi : ces(q) \longrightarrow ces(r)$ is continuous. Since $q > p$ with $r \in (p, q)$ arbitrary, we can apply Lemma 3.5 to conclude that $M_\varphi \in \mathcal{L}(ces(p+))$ is compact.

(i) $\Longrightarrow$ (ii). Since the operator $M_\varphi \in \mathcal{L}(ces(p+))$ is compact, there exists $q > p$ such that $M_\varphi : ces(q) \longrightarrow ces(r)$ is continuous for each $r \in (p, q)$; see Lemma 3.5. Define $r := (p + q)/2$ and $s := \frac{(p+q)q}{q-p}$. Then $\frac{1}{s} = \frac{1}{r} - \frac{1}{q}$. Proposition 3.6 yields that $\widehat{\varphi} \in \ell_s \subseteq \ell_{\infty-}$ as $s > 1$.

(ii) $\Longleftrightarrow$ (iii). Observe, that $\varphi \in \cup_{s>1} d(s)$ if and only if $\varphi \in d(t)$ for some $t > 1$, that is, if and only if $\widehat{\varphi} \in \ell_t$ for some $t > 1$, which is equivalent to $\widehat{\varphi} \in \ell(\infty-)$. □

Remark 3.8. (i) Proposition 3.7 shows that $\mathcal{M}_c(ces(p+)) = \cup_{s>1} d(s)$ is independent of $p \in [1, \infty)$. Moreover, the containment $\cup_{s>1} d(s) \subseteq c_0$ is *proper*. To see this consider $\psi := (\frac{1}{\log(1+i)})_i$. Since $\psi \in \ell_\infty$, Remark 3.2 implies that $\psi \in \lambda^\infty(A)$. However, ψ is *not* a compact multiplier. Indeed, by Remark 3.2(i) also $\varphi := \frac{1}{\psi} \in \lambda^\infty(A)$. So, if ψ were compact, then the identity operator $I = M_\varphi M_\psi$ would be compact.

(ii) Via the description of the dual space $(ces(1+))'_\beta$ of $ces(1+)$ as given in Proposition 4.6 of [3] it follows from Proposition 3.7(iii) that also

$$\mathcal{M}_c(ces(p+)) = (ces(1+))'_\beta, \quad 1 \le p < \infty, \tag{3.5}$$

as an equality of linear spaces.

(iii) It is immediate from Proposition 3.7(iii) that $\mathcal{M}_c(ces(p+))$, for $p \ge 1$, is a *solid* sublattice of $\mathbb{C}^{\mathbb{N}}$.

Fix any $p \ge 1$. There are two natural ways to topologize the linear space $\lambda^\infty(A)$ of all p-multipliers for $ces(p+)$. Indeed, it is clear from (3.2) and (3.3) that $\lambda^\infty(A) = \mathrm{proj}_{0<\eta<1}\ell_\infty(w_\eta)$ is the Köthe echelon Fréchet space of infinite order (see Chapter 27 of [17]) when it is equipped with the topology κ_∞ determined by the increasing sequence of norms

$$\|\varphi\|^{(\infty)}_{1/k} := \sup_{i\in\mathbb{N}} |\varphi_i| i^{-1/k}, \quad \varphi \in \lambda^\infty(A), \quad k \in \mathbb{N}.$$

On the other hand, Proposition 3.1 implies that the map $J_p : \lambda^\infty(A) \longrightarrow \mathcal{L}_b(ces(p+))$ given by

$$J_p(\varphi) := M_\varphi, \quad \varphi \in \lambda^\infty(A),$$

is well defined, linear and injective. Accordingly, the topology τ_b can be transferred from $\mathcal{L}_b(ces(p+))$ to $\lambda^\infty(A)$. Namely, each continuous seminorm ν on $\mathcal{L}_b(ces(p+))$ defines a corresponding seminorm on $\lambda^\infty(A)$ via the formula $\varphi \longmapsto \nu(J_p(\varphi))$ for $\varphi \in \lambda^\infty(A)$. Denote this lcH-topology on $\lambda^\infty(A)$ by $\tau_b(\infty)$. Of course, $\tau_b(\infty)$ is generated by the family of seminorms $\tilde{\gamma}_{B,q}$ given by

$$\tilde{\gamma}_{B,q}(\varphi) := \gamma_{B,\|\cdot\|_{ces(q)}}(J_p(\varphi)) = \sup_{x\in B} \|M_\varphi(x)\|_{ces(q)}, \quad \varphi \in \lambda^\infty(A), \tag{3.6}$$

for all bounded sets $B \subseteq ces(p+)$ and all $q > p$; see Section 1. Somewhat surprisingly, $\tau_b(\infty)$ coincides with the Fréchet space topology κ_∞ on $\lambda^\infty(A)$. In order to establish this we require a preliminary result.

Lemma 3.9. (i) *For each $p > 1$ there exist positive constants A_p, B_p such that*

$$\frac{A_p}{i^{1/p'}} \le \|e_i\|_{ces(p)} \le \frac{B_p}{i^{1/p'}}, \quad i \in \mathbb{N}.$$

(ii) *For each $s > 0$, the canonical basis vectors $\{e_i\}_{i\in\mathbb{N}} \subseteq d(s)$ satisfy $\|e_i\|_{d(s)} = i^{1/s}$ for $i \in \mathbb{N}$.*

(iii) *For each $q > 1$ there exist positive constants C_q, D_q such that*

$$C_q i^{1/q'} \le \|e_i\|_{(ces(q))'} \le D_q i^{1/q'}, \quad i \in \mathbb{N}. \tag{3.7}$$

(iv) *For all pairs $1 < p \le q < \infty$ and with $\eta := (\frac{1}{p} - \frac{1}{q}) \ge 0$ the space $\ell_\infty(w_\eta)$ as defined in Remark 3.2(ii) is a Banach space relative to the norm*

$$\|\varphi\|^{(\infty)}_{p,q} := \sup_{i\in\mathbb{N}} |\varphi_i| i^{-\eta}, \quad \varphi \in \ell_\infty(w_\eta).$$

Given $\varphi \in \ell_\infty(w_\eta)$ the operator M_φ is continuous and so belongs to the Banach space $\mathcal{L}_b(ces(p), ces(q))$. Moreover, there exists a constant $M(p,q) > 0$ satisfying

$$\|M_\varphi\|_{op} := \sup_{\|u\|_{ces(p)\leq 1}} \|M_\varphi(u)\|_{ces(q)} \leq M(p,q)\|\varphi\|^{(\infty)}_{p,q}, \quad \varphi \in \ell_\infty(w_\eta).$$

Proof. (i) See Lemma 4.7 in [6].

(ii) Since $\widehat{e_i} = (1, \ldots, 1, 0, 0, \ldots)$ with 1 in the first i coordinates, it follows that $\|e_i\|_{d(s)} := \|\widehat{e_i}\|_s = i^{1/s}$, for $i \in \mathbb{N}$.

(iii) Since the dual Banach space $(ces(q))'$ is isomorphic to $d(q')$, there exist positive constants C_q, D_q such that

$$C_q\|x\|_{d(q')} \leq \|x\|_{(ces(q))'} \leq D_q\|x\|_{d(q')}, \quad x \in d(q').$$

Part (ii) and the choice $x := e_i$, for each $i \in \mathbb{N}$, then imply (3.7).

(iv) Since $(\ell_\infty(w_\eta), \|\cdot\|^{(\infty)}_{p,q})$ is a weighted ℓ_∞-space, it is surely a Banach space. The discussion prior to Proposition 3.1 shows that $M_\varphi \in \mathcal{L}(ces(p), ces(q))$ for each $\varphi \in \ell_\infty(w_\eta)$. It follows that the map $\Phi_{p,q} : \ell_\infty(w_\eta) \longrightarrow \mathcal{L}_b(ces(p), ces(q))$, acting between Banach spaces and given by $\Phi_{p,q}(\varphi) := M_\varphi$ is well defined, linear and injective. The existence of $M(p,q) > 0$ is equivalent to the continuity of $\Phi_{p,q}$ which we establish via the closed graph theorem.

Consider any sequence $\{\varphi^{(k)}\}_{k\in\mathbb{N}} \subseteq \ell_\infty(w_\eta)$ which converges to 0 for the norm $\|\cdot\|^{(\infty)}_{p,q}$ and such that $\{\Phi_{p,q}(\varphi^{(k)})\}_{k\in\mathbb{N}}$ converges for the operator norm to some element $T \in \mathcal{L}_b(ces(p), ces(q))$. The aim is to show that $T = 0$, for which it suffices to show that $T(e_i) = 0$ for each basis vector $e_i \in ces(p), i \in \mathbb{N}$. This is achieved by verifying that $M_{\varphi^{(k)}}(e_i) \longrightarrow 0$ in $ces(q)$ for $k \longrightarrow \infty$. So, fix $i_0 \in \mathbb{N}$. Since $M_{\varphi(k)}(e_{i_0}) = \varphi^{(k)}_{i_0}\tilde{e}_{i_0}$ (where $\tilde{e}_{i_0}$ (resp. e_{i_0}) indicates the i_0-th canonical basis vector considered as an element of $ces(q)$ (resp. of $ces(p)$)), it follows that

$$\begin{aligned}\|M_{\varphi^{(k)}}(e_{i_0})\|_{ces(q)} &= |\varphi^{(k)}_{i_0}| \cdot \|\tilde{e}_{i_0}\|_{ces(q)} = i_0^\eta\|\tilde{e}_{i_0}\|_{ces(q)}|\varphi^{(k)}_{i_0}|i_0^{-\eta} \\ &\leq i_0^\eta\|\tilde{e}_{i_0}\|_{ces(q)}\|\varphi^{(k)}\|^{(\infty)}_{p,q},\end{aligned}$$

for each $k \in \mathbb{N}$. Letting $k \longrightarrow \infty$ it is clear that $M_{\varphi^{(k)}}(e_{i_0}) \longrightarrow 0$ in $ces(q)$. □

Proposition 3.10. *Let $1 \leq p < \infty$. The linear map $J_p : (\lambda^\infty(A), \kappa_\infty) \longrightarrow \mathcal{L}_b(ces(p+))$ is a bicontinuous isomorphism of $\lambda^\infty(A)$ onto its image $\mathrm{Im}(J_p) = J_p(\lambda^\infty(A))$ in $\mathcal{L}_b(ces(p+))$. In particular, the lcH-topology $\tau_b(\infty)$ and the metrizable lc-topology κ_∞ coincide on $\lambda^\infty(A)$.*

Proof. To verify that J_p is continuous fix any seminorm $\tilde{\gamma}_{B,q}$ of the form (3.6) with $q > p$ and $B \subseteq ces(p+)$ a bounded set. Since each $r \in (\frac{q}{q+1}, q)$ satisfies $\alpha_r := (\frac{1}{r} - \frac{1}{q}) < 1$, it is possible to choose r close enough to q so that also $r \in (p,q)$. Fix such an r, in which case $0 < \alpha_r < 1$. Since

$\|\cdot\|_{ces(r)}$ is a continuous norm on $ces(p+)$, there exists $R(r) > 0$ such that $\|x\|_{ces(r)} \leq R(r)$ for all $x \in B$, that is,

$$\|\tfrac{x}{R(r)}\|_{ces(r)} \leq 1, \quad x \in B. \tag{3.8}$$

It follows from (3.6), (3.8) and Lemma 3.9(iv) that, for each $\varphi \in \lambda^\infty(A)$, we have

$$\gamma_{B,q}(J_p(\varphi)) = R(r) \sup_{x \in B} \|M_\varphi(\tfrac{x}{R(r)})\|_{ces(q)} \leq R(r) \sup_{\|u\|_{ces(r)} \leq 1} \|M_\varphi(u)\|_{ces(q)}$$

$$\leq R(r) M(r,q) \|\varphi\|^{(\infty)}_{r,q},$$

where the norm $\|\cdot\|^{(\infty)}_{r,q}$ of $\ell_\infty(w_{\alpha_r})$ restricted to $(\lambda^\infty(A), \kappa_\infty)$ is continuous. The continuity of J_p is thereby established.

To establish the continuity of $J_p^{-1} : (\mathrm{Im}(J_p), \tau_b) \longrightarrow (\lambda^\infty(A), \kappa_\infty)$ we need to consider two separate cases.

Case (i). Let $p > 1$. Fix $0 < \alpha < 1$ and choose $s \in (0, \frac{1}{p})$ such that $0 < (\frac{1}{p} - s) < \alpha$. Then $q(\alpha) := \frac{1}{s}$ satisfies $p < q(\alpha)$ and $(\frac{1}{p} - \frac{1}{q(\alpha)}) \in (0, \alpha)$. Let $\varphi \in \lambda^\infty(A) \subseteq \ell_\infty(w_\alpha)$. For each $i \in \mathbb{N}$ we have $|\varphi_i| = \langle M_\varphi(e_i), \tilde{e_i} \rangle$, with $e_i \in ces(p)$ and where $\tilde{e_i}$ indicates the i-th canonical basis vector considered as an element of $(ces(q(\alpha)))'$. Since $\varphi \in \ell_\infty(w_\alpha)$ implies that $M_\varphi \in \mathcal{L}(ces(p), ces(q(\alpha)))$, it follows from parts (i), (iii) of Lemma 3.9 that

$$|\varphi_i| \leq \|M_\varphi\|_{op} \|e_i\|_{ces(p)} \|\tilde{e_i}\|_{(ces(q(\alpha)))'} \leq B_p D_{q(\alpha)} \|M_\varphi\|_{op} i^{\frac{1}{q(\alpha)'} - \frac{1}{p'}}$$

$$= B_p D_{q(\alpha)} \|M_\varphi\|_{op} i^{\frac{1}{p} - \frac{1}{q(\alpha)}},$$

for each $i \in \mathbb{N}$. Because $0 < (\frac{1}{p} - \frac{1}{q(\alpha)}) < \alpha$ implies $i^{\frac{1}{p} - \frac{1}{q(\alpha)}} < i^\alpha$, it follows that

$$\|\varphi\|^{(\infty)}_\alpha := \sup_{i \in \mathbb{N}} |\varphi_i| i^{-\alpha} \leq B_p D_{q(\alpha)} \|M_\varphi\|_{op}.$$

But, continuity of the inclusion $ces(p) \subseteq ces(p+)$ implies that the unit ball U_p of $ces(p)$ is a bounded subset of $ces(p+)$. Hence,

$$\|M_\varphi\|_{op} = \sup_{x \in U_p} \|M_\varphi(x)\|_{ces(q(\alpha))} = \gamma_{U_p, \|\cdot\|_{ces(q(\alpha))}}(J_p(\varphi))$$

which, together with the previous inequality, yields

$$\|\varphi\|^{(\infty)}_\alpha \leq B_p D_{q(\alpha)} \gamma_{U_p, \|\cdot\|_{ces(q(\alpha))}}(J_p(\varphi)), \quad \varphi \in \lambda^\infty(A).$$

Since $0 < \alpha < 1$ is arbitrary, this shows that $J_p^{-1} : (\mathrm{Im}(J_p), \tau_b) \longrightarrow (\lambda^\infty(A), \kappa_\infty)$ is continuous.

Case (ii). Suppose now that $p = 1$. Since there is no Banach space "ces(1)", the proof of Case (i) is not applicable. The continuous inclusions $\ell_1 \subseteq \ell_{1+} \subseteq ces(1+)$ imply that the unit ball V_1 of ℓ_1 is a bounded subset of $ces(1+)$.

Fix any $0 < \alpha < 1$ and choose $q(\alpha) > 1$ such that $0 < (1 - \frac{1}{q(\alpha)}) < \alpha$. Since $\ell_1 \subseteq ces(1+) \subseteq ces(q(\alpha))$ and $J_1(\varphi) \in \mathcal{L}_b(ces(1+))$, for each $\varphi \in \lambda^\infty(A)$, imply that $M_\varphi(ces(1+)) \subseteq ces(1+)$, it follows that $M_\varphi(\ell_1) \subseteq ces(q(\alpha))$.

Via the closed graph theorem it can be shown that $M_\varphi \in \mathcal{L}(\ell_1, ces(q(\alpha)))$ for all $\varphi \in \lambda^\infty(A)$. Fix $i \in \mathbb{N}$. As for Case (i), now with $e_i \in \ell_1$ (so that $\|e_i\|_1 = 1$) and $\tilde{e}_i \in ces(q(\alpha))$ the canonical basis vectors, we have

$$|\varphi_i| \leq \|M_\varphi\|_{op} \|\tilde{e}_i\|_{ces(q(\alpha))} \leq \|M_\varphi\|_{op} D_{q(\alpha)} i^{\frac{1}{q(\alpha)'}}$$

$$= \|M_\varphi\|_{op} D_{q(\alpha)} i^{1-\frac{1}{q(\alpha)}} < \|M_\varphi\|_{op} D_{q(\alpha)} i^{\alpha}.$$

Moreover,

$$\|M_\varphi\|_{op} = \sup_{x \in V_1} \|M_\varphi(x)\|_{ces(q(\alpha))} = \gamma_{V_1, \|\cdot\|_{ces(q(\alpha))}}(J_1(\varphi))$$

which, together with the previous inequality, yields

$$\|\varphi\|_\alpha^{(\infty)} := \sup_{i \in \mathbb{N}} |\varphi_i| i^{-\alpha} \leq D_{q(\alpha)} \gamma_{V_1 \|\cdot\|_{ces(q(\alpha))}}(J_1(\varphi)), \quad \varphi \in \lambda^\infty(A).$$

So, also $J_1^{-1} : (\mathrm{Im}(J_1), \tau_b) \longrightarrow (\lambda^\infty(A), \kappa_\infty)$ is continuous. □

Proposition 3.7 shows that the compact multipliers for $ces(p+)$ satisfy

$$\mathcal{M}_c(ces(p+)) = d(\infty-) := \cup_{s>1} d(s), \quad 1 \leq p < \infty,$$

where the union is formed with respect to the increasing family of Banach spaces $\{d(s)\}_{s>1}$. Hence, we can endow $d(\infty-)$ with the *inductive limit topology* τ_{ind}, [17, Ch.24]. In particular, $(d(\infty-), \tau_{\mathrm{ind}})$ is a regular (LB)-space, [17, p.291]. According to Proposition 4.6 of [3] the space $(d(\infty-), \tau_{\mathrm{ind}})$ is linearly isomorphic to the strong dual space $(ces(1+))'_\beta$ of the Fréchet space $ces(1+)$. So, we have established the following fact.

Proposition 3.11. *For each $p \geq 1$, the space of compact multipliers $\mathcal{M}_c(ces(p+))$ is linearly isomorphic to the strong dual space of the fixed Fréchet-Schwartz space (independent of p)*

$$\mathcal{M}_c(ces(p+)) = (d(\infty-), \tau_{\mathrm{ind}}) \simeq (ces(1+))'_\beta, \quad 1 \leq p < \infty.$$

Since $\mathcal{M}_c(ces(p+)) \subseteq \mathcal{M}(ces(p+))$ for $1 \leq p < \infty$, that is, $d(\infty-) \subseteq \lambda^\infty(A)$, one can also equip $d(\infty-)$ with the relative topology κ_∞ from $\lambda^\infty(A)$. The relationship between the lcHs' $(d(\infty-), \tau_{\mathrm{ind}})$ and $(d(\infty-), \kappa_\infty)$ is clarified by the following result.

Proposition 3.12. *The identity map $j : (d(\infty-), \tau_{\mathrm{ind}}) \longrightarrow (d(\infty-), \kappa_\infty)$ is continuous and the topology τ_{ind} on $d(\infty-)$ is strictly stronger than the lc-metrizable topology κ_∞.*

Proof. To establish the continuity of j it suffices to show that $j : (d(\infty-), \tau_{\mathrm{ind}}) \longrightarrow (\lambda^\infty(A), \kappa_\infty)$ is continuous. By [17, Proposition 24.7] this reduces to showing, for each fixed $s > 1$, that the identity inclusion $j_s : (d(s), \|\cdot\|_{d(s)}) \longrightarrow (\lambda^\infty(A), \kappa_\infty)$ between Fréchet spaces is continuous. Since $j_s = j_2 \circ j_1$, with both $j_1 : (d(s), \|\cdot\|_{d(s)}) \longrightarrow (\ell_\infty, \|\cdot\|_\infty)$ and $j_2 : (\ell_\infty, \|\cdot\|_\infty) \longrightarrow (\lambda^\infty(A), \kappa_\infty)$ being the natural inclusion maps, it suffices to show that both j_1, j_2 are continuous.

Concerning j_1, note that $\|\varphi\|_\infty = \|\widehat{\varphi}\|_\infty$ for every $\varphi \in \ell_\infty$ and that $\|\psi\|_\infty \le \|\psi\|_s$ for every $\psi \in \ell_s$. Since $\varphi \in d(s) \subseteq \ell_\infty$ implies that $\widehat{\varphi} \in \ell_s$, it follows that

$$\|j_1(\varphi)\|_\infty = \|\varphi\|_\infty = \|\widehat{\varphi}\|_\infty \le \|\widehat{\varphi}\|_s := \|\varphi\|_{d(s)}, \quad \varphi \in d(s).$$

Accordingly, j_1 is continuous.

To establish the continuity of j_2 fix $0 < \alpha < 1$. Then

$$\|j_2(\varphi)\|_\alpha^{(\infty)} = \|\varphi\|_\alpha^{(\infty)} := \sup_{i\in\mathbb{N}} \frac{|\varphi_i|}{i^\alpha} \le \|\varphi\|_\infty, \quad \varphi \in \ell_\infty.$$

Hence, j_2 is also continuous. The continuity of j is thereby established.

The continuity of j implies that τ_{ind} is certainly a stronger topology on $d(\infty-)$ than κ_∞. If these two topologies were equal, then the identity map $j^{-1} : (d(\infty-), \kappa_\infty) \longrightarrow (d(\infty-), \tau_{\text{ind}})$ would also be continuous and so τ_{ind} would be a lc-metrizable topology on $d(\infty-)$. By the Grothendieck-Floret factorization theorem this is impossible, [19, Proposition 8.5.38]. Hence, τ_{ind} is *strictly* stronger than κ_∞. □

Because of significant differences, it is worthwhile to compare the spaces $\mathcal{M}_c(\ell_{p+}), \mathcal{M}(\ell_{p+})$ with the corresponding spaces $\mathcal{M}_c(ces(p+)), \mathcal{M}(ces(p+))$, where $\mathcal{M}(\ell_{p+}) := \{\varphi \in \mathbb{C}^{\mathbb{N}} : M_\varphi \in \mathcal{L}(\ell_{p+})\}$ and $\mathcal{M}_c(\ell_{p+}) := \{\varphi \in \mathcal{M}(\ell_{p+}) : M_\varphi \in \mathcal{L}(\ell_{p+})$ is compact$\}$, for $1 \le p < \infty$. The multiplier space $\mathcal{M}(\ell_{p+})$ is already known, namely

$$\mathcal{M}(\ell_{p+}) = \ell_\infty, \quad 1 \le p < \infty, \tag{3.9}$$

[7, Corollary 5.3]. This space is obviously independent of $p \ge 1$ and different to $\mathcal{M}(ces(p+)) = \lambda^\infty(A)$, which contains ℓ_∞ as a *proper* subspace; see Remark 3.2(i). In order to identify $\mathcal{M}_c(\ell_{p+})$ we require some preliminaries. The following fact is immediate from the entries 1 and 17 in the tables on pp. 69–70 of [6].

Lemma 3.13. *Let $1 \le p, q < \infty$ be an arbitrary pair and $\varphi \in \mathbb{C}^{\mathbb{N}}$.*

(i) *If $p \le q$, then $M_\varphi : \ell_p \longrightarrow \ell_q$ is continuous if and only if $\varphi \in \ell_\infty$.*
(ii) *If $p > q$, then $M_\varphi : \ell_p \longrightarrow \ell_q$ is continuous if and only if $\varphi \in \ell_r$ where r satisfies $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.*

Concerning compactness of the Banach space operators M_φ from ℓ_p to ℓ_q we have the following characterization.

Lemma 3.14. *Let $1 \le p, q < \infty$ be an arbitrary pair and $\varphi \in \mathbb{C}^{\mathbb{N}}$.*

(i) *If $p \le q$, then $M_\varphi : \ell_p \longrightarrow \ell_q$ is compact if and only if $\varphi \in c_0$.*
(ii) *If $p > q$, then $M_\varphi : \ell_p \longrightarrow \ell_q$ is compact if and only if it is continuous if and only if $\varphi \in \ell_r$ with $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.*

Proof. (i) Suppose that $\varphi \in c_0$. Then $M_\varphi \in \mathcal{L}(\ell_p, \ell_q)$ by Lemma 3.13(i). Define $\varphi^{(N)} := (\varphi_1, \dots, \varphi_N, 0, 0, \dots)$ for $N \in \mathbb{N}$. Then $M_{\varphi^{(N)}} : \ell_p \longrightarrow \ell_q$ is a finite rank operator and hence, is compact for each $N \in \mathbb{N}$. Since $p \le q$, it is routine to verify that each $\varphi - \varphi^{(N)}$ satisfies (relative to the operator norm in $\mathcal{L}(\ell_p, \ell_q)$)

$$\|M_{\varphi-\varphi^{(N)}}\|_{op} \le \|\varphi - \varphi^{(N)}\|_\infty, \quad N \in \mathbb{N}.$$

Hence, as $\|\varphi - \varphi^{(N)}\|_\infty \longrightarrow 0$ for $N \longrightarrow \infty$, it follows that M_φ is a compact operator.

Conversely, suppose that M_φ is compact. In particular, $M_\varphi \in \mathcal{L}(\ell_p, \ell_q)$. By a standard compactness criterion for continuous linear operators between Banach sequence spaces, it follows that $\lim_{m\to\infty} M_\varphi(x_m) = 0$ in ℓ_q whenever $\{x_m\}_{m=1}^\infty$ is a bounded sequence in ℓ_p satisfying $\lim_{m\to\infty} x_m = 0$ in the Fréchet space $\mathbb{C}^\mathbb{N}$, [4, Lemma 2.1]. Since the canonical basis vectors $x_m := e_m$, for $m \in \mathbb{N}$, are norm bounded in ℓ_p and satisfy $e_m \longrightarrow 0$ in $\mathbb{C}^\mathbb{N}$ for $m \longrightarrow \infty$, it follows that $|\varphi_m| = \|M_\varphi(e_m)\|_q \longrightarrow 0$ for $m \longrightarrow \infty$. Accordingly, $\varphi \in c_0$.

(ii) This follows from the facts that compact operators are necessarily continuous, Lemma 3.13(ii), and Pitt's theorem, which states that every operator $T \in \mathcal{L}(\ell_p, \ell_q)$ is necessarily compact whenever $p > q$, [20]. □

The following result for ℓ_{p+} is the analogue of Proposition 3.7 for $ces(p+)$.

Proposition 3.15. *Let $1 \le p < \infty$ and $\varphi \in \mathcal{M}(\ell_{p+}) = \ell_\infty$. The following conditions are equivalent.*

(i) *$\varphi \in \mathcal{M}_c(\ell_{p+})$, that is, $M_\varphi \in \mathcal{L}(\ell_{p+})$ is a compact operator.*
(ii) *$\varphi \in \ell(\infty-) = \cup_{s>1}\ell_s$.*

Proof. (i) $\Longrightarrow$ (ii). Let $\varphi \in \mathcal{M}_c(\ell_{p+})$. By definition there exists $q > p$ such that the Banach space operator $M_\varphi : \ell_q \longrightarrow \ell_r$ is compact (hence, continuous) for every $r \in (p, q)$. Choose $r := \frac{p+q}{2} \in (p, q)$ in which case $q > r$. Then $s := \frac{(p+q)q}{(q-p)} > 0$ satisfies $\frac{1}{s} = \frac{1}{r} - \frac{1}{q}$ and so Lemma 3.14(ii) implies that $\varphi \in \ell_s \subseteq \ell(\infty-)$.

(ii) $\Longrightarrow$ (i). Fix $\varphi \in \ell(\infty-)$. Then there exists $s > p$ such that $\varphi \in \ell_s$. Set $\varepsilon := \frac{p^2}{(s-p)} > 0$ (in which case $s = p + \frac{p^2}{\varepsilon}$) and define $q > p$ by $q := p + \varepsilon$. Now, fix any $r \in (p, q)$ and define t via $\frac{1}{t} = \frac{1}{r} - \frac{1}{q}$. Since $p < r$, we have $\frac{1}{t} < \frac{\varepsilon}{p^2+\varepsilon p} = \frac{1}{s}$ and $t > s$, that is, $\varphi \in \ell_t$ (as $\ell_s \subseteq \ell_t$). Then Lemma 3.14(iv) implies that $M_\varphi : \ell_q \longrightarrow \ell_r$ is compact. So, we have shown that there exists $q > p$ such that $M_\varphi : \ell_q \longrightarrow \ell_r$ is compact for all $r \in (p, q)$, that is, $M_\varphi \in \mathcal{L}(\ell_{p+})$ is a compact operator. □

Given any $p \ge 1$, we know that $\varphi \in \mathbb{C}^\mathbb{N}$ satisfies $\varphi \in \mathcal{M}(\ell_{p+})$ if and only if $\varphi \in \ell_\infty$ and that $\varphi \in \mathcal{M}_c(\ell_{p+})$ if and only if $\varphi \in \ell(\infty-)$; see Proposition 3.15.

Proposition 3.16. *Let $1 \le p < \infty$. The linear map $\Lambda_p : (\ell_\infty, \|\cdot\|_\infty) \longrightarrow (\mathcal{L}_b(\ell_{p+}), \tau_b)$ defined by $\Lambda_p(\varphi) := M_\varphi$, for $\varphi \in \ell_\infty$, is a bicontinuous isomorphism of the Banach space $(\ell_\infty, \|\cdot\|_\infty)$ onto its image $\mathrm{Im}(\Lambda_p)$ in $\mathcal{L}_b(\ell_{p+})$.*

In particular, the lcH-topology on ℓ_∞ induced by τ_b (from $\mathcal{L}_b(\ell_{p+})$) coincides with the norm topology on ℓ_∞ induced by $\|\cdot\|_\infty$.

Proof. It is clear from (3.9) that Λ_p is well defined, linear and injective. The seminorms $\overline{\gamma}_{B,q}$ determined by transfering the topology τ_b from $\mathcal{L}_b(\ell_{p+})$ to ℓ_∞ are given by

$$\overline{\gamma}_{B,q}(\varphi) := \gamma_{B,\|\cdot\|_q}(\Lambda_p(\varphi)) = \sup_{x\in B} \|M_\varphi(x)\|_q, \quad \varphi \in \ell_\infty,$$

for all bounded sets $B \subseteq \ell_{p+}$ and all $q > p$; see Section 1. Fix such a set B and $q > p$. It follows that

$$\overline{\gamma}_{B,q}(\varphi) \leq \|\varphi\|_\infty \sup_{x\in B} \|x\|_q = \delta_{B,q}\|\varphi\|_\infty, \quad \varphi \in \ell_\infty,$$

with $\delta_{B,q} := \sup_{x\in B} \|x\|_q < \infty$ because B is bounded in ℓ_{p+} and hence, is also bounded in ℓ_q as $\ell_{p+} \subseteq \ell_q$ continuously. This establishes the continuity of Λ_p.

To verify the continuity of $\Lambda_p^{-1} : (\mathrm{Im}(\Lambda_p), \tau_b) \longrightarrow (\ell_\infty, \|\cdot\|_\infty)$, let B_p denote the unit ball of ℓ_p and let $q > p$ be arbitrary. Since $\ell_p \subseteq \ell_{p+}$ continuously, B_p is also a bounded set in ℓ_{p+}. Note that the canonical basis vectors $\{e_k\}_{k\in\mathbb{N}}$ of ℓ_p lie in B_p. Let $\tilde{e}_i$, for $i \in \mathbb{N}$, denote the basis vector $e_i \in \mathbb{C}^{\mathbb{N}}$ considered as an element of the dual Banach space $\ell_{q'}$ of ℓ_q, in which case $\|\tilde{e}_i\|_{q'} = 1$. For each $\varphi \in \ell_\infty$ it follows from Lemma 3.13(i) that the Banach space operator $M_\varphi : \ell_p \longrightarrow \ell_q$ is continuous. Since $\{e_k\}_{k\in\mathbb{N}} \subseteq B_p$, it follows that

$$\begin{aligned} |\varphi_i| &= |\langle M_\varphi(e_i), \tilde{e}_i\rangle| \leq \|M_\varphi(e_i)\|_q \|\tilde{e}_i\|_{q'} = \|M_\varphi(e_i)\|_q \leq \sup_{x\in B_p} \|M_\varphi(x)\|_q \\ &= \gamma_{B_p,q}(\Lambda_p(\varphi)), \end{aligned}$$

for each $i \in \mathbb{N}$. Forming the supremum with respect to $i \in \mathbb{N}$ yields

$$\|\varphi\|_\infty \leq \gamma_{B_p,q}(\Lambda_p(\varphi)), \quad \varphi \in \ell_\infty,$$

that is, Λ_p^{-1} is continuous from $(\mathrm{Im}(\Lambda_p), \tau_b) \longrightarrow (\ell_\infty, \|\cdot\|_\infty)$. □

Proposition 3.15 shows that

$$\mathcal{M}_c(\ell_{p+}) = \ell(\infty-) = \cup_{s>1}\ell_s, \quad 1 \leq p < \infty.$$

Equip $\ell(\infty-)$ with the inductive limit topology τ_{ind}, in which case $(\ell(\infty-), \tau_{\mathrm{ind}})$ is a regular (LB)-space. It is known that $(\ell(\infty-), \tau_{\mathrm{ind}})$ is precisely the strong dual space $(\ell_{1+})'_\beta$ of the reflexive Fréchet space ℓ_{1+}. Summarizing yields the following result.

Proposition 3.17. *For each $p \geq 1$, the compact multipliers for ℓ_{p+} are given by*

$$\mathcal{M}_c(\ell_{p+}) = (\ell(\infty-), \tau_{\mathrm{ind}}) = (\ell_{1+})'_\beta,$$

a space which is independent of p.

Since $\ell(\infty-) \subseteq \ell_\infty$, one can also equip $\ell(\infty-)$ with the topology induced by the norm $\|\cdot\|_\infty$.

Proposition 3.18. *The identity map $\rho : (\ell(\infty-), \tau_{\text{ind}}) \longrightarrow (\ell(\infty-), \|\cdot\|_\infty)$ is continuous and the topology τ_{ind} is strictly stronger than the topology induced by the norm $\|\cdot\|_\infty$.*

Proof. For each $s > 1$, the identity inclusion $\rho_s : (\ell_s, \|\cdot\|_s) \longrightarrow (\ell_\infty, \|\cdot\|_\infty)$ is well known to be continuous. This implies that ρ is continuous; see the proof of Proposition 3.12.

The analogous argument as in the last paragraph of the proof of Proposition 3.12 shows that τ_{ind} is a *strictly* stronger topology on $\ell(\infty-)$ than the topology induced by $\|\cdot\|_\infty$. □

Remark 3.19. (i) The identity inclusion $\ell_{1+} \subseteq ces(1+)$ is continuous between reflexive Fréchet spaces and is *proper*, [3, Lemma 2.1], which implies for their strong dual spaces that

$$d(\infty-) = (ces(1+))'_\beta \subsetneqq (\ell_{1+})'_\beta = \ell(\infty-).$$

Hence, for each $1 \le p < \infty$ we have

$$\mathcal{M}_c(ces(p+)) \subsetneqq \mathcal{M}_c(\ell_{p+})$$

in contrast to

$$\mathcal{M}(ces(p+)) = \lambda^\infty(A) \supsetneqq \ell_\infty = \mathcal{M}(\ell_{p+}).$$

(ii) Due to the reflexivity of the Fréchet space $ces(1+)$, the strong dual of the *compact* multipliers $(\mathcal{M}_c(ces(p+)))'_\beta = (d(\infty-), \tau_{\text{ind}})'_\beta = ces(1+)$ is identifiable with a *proper* linear subspace of the Fréchet space $\mathcal{M}(ces(p+)) = (\lambda^\infty(A), \kappa_\infty)$ of *all* p-multipliers , for $p \ge 1$. Similarly, $(\mathcal{M}_c(\ell_{p+}))'_\beta = (\ell(\infty-), \tau_{\text{ind}})'_\beta = \ell_{1+}$ is a *proper* linear subspace of the Banach space $\mathcal{M}(\ell_{p+}) = (\ell_\infty, \|\cdot\|_\infty)$ of *all* multipliers for ℓ_{p+}, for $p \ge 1$. This is in contrast to the situation for the space of all compact (Fourier) multiplier operators acting in $L^p(G)$, for G a compact group, where equality occurs, [5].

The final part of this section examines the mean ergodic properties of multiplier operators. We begin with the spaces $ces(p+)$.

Proposition 3.20. *Let $1 \le p < \infty$. The following conditions are equivalent for $\varphi \in \mathcal{M}(ces(p+))$.*

(i) *M_φ is power bounded.*
(ii) *M_φ is mean ergodic.*
(iii) *M_φ is uniformly mean ergodic.*
(iv) *$\varphi \in \ell_\infty$ and $||\varphi||_\infty \le 1$.*
(v) *$\sigma(M_\varphi; ces(p+)) \subseteq \overline{\mathbb{D}}$.*

Proof. (i) $\Longrightarrow$ (ii) because $ces(p+)$ is a reflexive Fréchet space, [1, Corollary 2.7].

(ii) $\Longleftrightarrow$ (iii). Clearly (iii) $\Longrightarrow$ (ii). Concerning (ii) $\Longrightarrow$ (iii), note that the convergence of the sequence $\{(M_\varphi)_{[n]}\}_{n=1}^\infty$ in $\mathcal{L}_s(ces(p+))$ implies that it is an *equicontinuous* subset of $\mathcal{L}(ces(p+))$. Since $ces(p+)$ is a Montel space, it follows that τ_b restricted to $\{(M_\varphi)_{[n]}\}_{n=1}^\infty$ agrees with τ_s and so $\{(M_\varphi)_{[n]}\}_{n=1}^\infty$ also convergences in $\mathcal{L}_b(ces(p+))$, that is, M_φ is uniformly mean ergodic.

(ii) $\Longrightarrow$ (iv). By condition (ii), for each $x \in ces(p+)$ the sequence $\{\frac{1}{k}(M_\varphi)^k(x)\}_{k=1}^\infty$ converges to 0 in $ces(p+)$. In particular, for each basis vector $x = e_j$ it follows that $\lim_{k\to\infty} \frac{\varphi_j^k}{k} = 0$ and hence, $|\varphi_j| \leq 1$. Since $j \in \mathbb{N}$ is arbitrary, $||\varphi||_\infty \leq 1$ and so condition (iv) is satisfied.

(iv) $\Longrightarrow$ (i). Recall that a sequence $\{r_k\}_{k=1}^\infty$ of lattice norms generating the topology of $ces(p+)$ is given by (1.5). Suppose that $\varphi \in \ell_\infty$ and $||\varphi||_\infty \leq 1$. Then, for each $n \in \mathbb{N}$, we have $\|\varphi^n\|_\infty \leq 1$ and so

$$r_k((M_\varphi)^n(x)) = r_k(\varphi^n x) \leq r_k(x), \quad x \in ces(p+), \quad n, k \in \mathbb{N},$$

which implies that $\{(M_\varphi)^n : n \in \mathbb{N}\}$ is an equicontinuous subset of $\mathcal{L}(ces(p+))$. Hence, condition (i) is satisfied.

(iv) $\Longleftrightarrow$ (v). This is a direct consequence of Proposition 3.3(iii). □

In order to determine the mean ergodic properties of multiplier operators in ℓ_{p+} we first require a knowledge of their spectra.

Proposition 3.21. *Fix $1 \leq p < \infty$ and let $\varphi \in \mathcal{M}(\ell_{p+}) = \ell_\infty$.*

(i) $\sigma_{pt}(M_\varphi; \ell_{p+}) = \{\varphi_i : i \in \mathbb{N}\}$, *where* $\varphi = (\varphi_i)_i$.

(ii) $\sigma(M_\varphi; \ell_{p+}) = \sigma^*(M_\varphi; \ell_{p+}) = \overline{\{\varphi_i : i \in \mathbb{N}\}}$.

Proof. (i) The analogous argument used in the proof of Proposition 3.3(i) applies here.

(ii) Let $P : 2^\mathbb{N} \longrightarrow \mathcal{L}_s(\ell_{p+})$ denote the *canonical spectral measure* defined via $P(A) := M_{\chi_A}$ for $A \subseteq \mathbb{N}$; see [7, Section 1]. Then the lcHs $L^1(P)$ consisting of all the P-integrable functions is precisely

$$L^1(P) = \mathcal{M}(\ell_{p+}) = \ell_\infty$$

and $\int_\mathbb{N} f\, dP = M_f$ for all $f \in L^1(P)$, [7, Proposition 5.1 & Corollary 5.3]. Part (i) implies that

$$\{\varphi_i : i \in \mathbb{N}\} \subseteq \sigma(M_\varphi; \ell_{p+}) \subseteq \sigma^*(M_\varphi; \ell_{p+}). \tag{3.10}$$

Keeping in mind that

(I) the τ_s-bounded subsets of $\mathcal{L}(\ell_{p+})$ are precisely the equicontinuous subsets of $\mathcal{L}(\ell_{p+})$,

(II) if $f : \mathbb{N} \longrightarrow \mathbb{C}^* := \mathbb{C}\cup\{\infty\}$ is a P-integrable function, then $f^{-1}(\{\infty\})$ is a P-null set, [21, p.279], and

(III) the only set $A \subseteq \mathbb{N}$ satisfying $P(A) = I$ is the set $A = \mathbb{N}$,

it follows that the spectrum $\sigma(\int_\mathbb{N} \varphi\, dP)$ of $M_\varphi = \int_\mathbb{N} \varphi\, dP \in \mathcal{L}(\ell_{p+})$, for $\varphi \in L^1(P)$, as defined in [21] coincides with the spectrum $\sigma^*(M_\varphi; \ell_{p+})$ as defined in Section 1 above. It then follows from the identity (2) in Theorem 1 of [21] that

$$\sigma^*(M_\varphi; \ell_{p+}) = \overline{\{\varphi_i : i \in I\}}, \tag{3.11}$$

where the closure of $\{\varphi_i : i \in \mathbb{N}\}$ in $\mathbb{C}$ coincides with its closure in $\mathbb{C}^*$ as $\varphi \in \ell_\infty$. Combining (3.10) and (3.11) yields the desired equalities. $\square$

Proposition 3.22. *Fix $1 \leq p < \infty$ and let $\varphi \in \mathcal{M}(\ell_{p+}) = \ell_\infty$. The following conditions are equivalent.*

(i) *$M_\varphi \in \mathcal{L}(\ell_{p+})$ is power bounded.*
(ii) *M_φ is mean ergodic.*
(iii) $\|\varphi\|_\infty \leq 1$.
(iv) $\sigma(M_\varphi; \ell_{p+}) \subseteq \overline{\mathbb{D}}$.

Proof. (i) $\implies$ (ii). This follows from the reflexitivity of the Fréchet space ℓ_{p+}, [1, Corollary 2.7].

(ii) $\implies$ (iii). Adapt the proof of (ii) $\implies$ (iv) in Proposition 3.20 by replacing $ces(p+)$ with ℓ_{p+}.

(iii) $\implies$ (iv). A sequence $\{q_k\}_{k=1}^\infty$ of lattice norms generating the topology of ℓ_{p+} is given by (1.4). Now, $\|\varphi\|_\infty \leq 1$ implies that $\|\varphi^n\|_\infty \leq 1$ for all $n \in \mathbb{N}$ and so

$$q_k((M_\varphi)^n(x)) = q_k(\varphi^n x) \leq \|\varphi^n\|_\infty \cdot q_k(x), \quad x \in \ell_{p+},$$

for all $n, k \in \mathbb{N}$. Hence, $\{(M_\varphi)^n : n \in \mathbb{N}\}$ is an equicontinuous subset of $\mathcal{L}(\ell_{p+})$, that is, M_φ is power bounded. $\square$

There is no analogue for Proposition 3.22 of the condition (iii) in Proposition 3.20. This is due to the fact that $ces(p+)$ is a Montel space whereas ℓ_{p+} is not and is illustrated by the following example.

Remark 3.23. Fix $1 \leq p < \infty$. Let $0 \leq \varphi = (\varphi_i)_i$ be any increasing sequence satisfying $\varphi_i \uparrow 1$. Then $\|\varphi\|_\infty \leq 1$ and so M_φ is mean ergodic by Proposition 3.22. However, M_φ is *not* uniformly mean ergodic; see the proof of Proposition 2.15 in [1] and note that the operator $T^{(\mu)}$ defined there is precisely M_φ.

Given $1 < p < \infty$, the multipliers $\mathcal{M}(ces(p)) = \ell_\infty$ (see entry 16 in the table on p. 69 of [6]). Similarly, for $1 \leq p < \infty$, also $\mathcal{M}(\ell_p) = \ell_\infty$ (see entry 1 in the table on p. 69 of [6]). In both cases, if $\varphi \in \ell_\infty$, then it is known that the corresponding multiplier operators $M_\varphi \in \mathcal{L}(\ell_p)$ and $M_\varphi \in \mathcal{L}(ces(p))$ are *not* supercyclic, [4, Proposition 2.12]. For the Fréchet space $\ell_{p+}, 1 \leq p < \infty$, we know that $\mathcal{M}(\ell_{p+}) = \ell_\infty$. It is routine to check that if $\varphi \in \ell_\infty$, then the dual operator $(M_\varphi)' \in \mathcal{L}((\ell_{p+})'_\beta)$ of $M_\varphi \in \mathcal{L}(\ell_{p+})$ is precisely $M_\varphi \in \mathcal{L}(\mathbb{C}^\mathbb{N})$ acting continuously in $(\ell_{p+})'_\beta = \ell_{p'-} := \cup_{s>p'} \ell_s$. On the other hand, let $\varphi \in \mathcal{M}(ces(p+)) = \lambda^\infty(A)$. Recall that $(ces(p+))'_\beta = d(p'-) = \cup_{r>p'} d(r)$, [3, Proposition 4.6]. Let $T := (M_\varphi)' \in \mathcal{L}((ces(p+))'_\beta)$ and consider the canonical vectors $\{e_i\}_{i\in\mathbb{N}} \subseteq (ces(p+))'_\beta = d(p'-)$. Fix $i \in \mathbb{N}$. Then

$$\langle x, T(e_i)\rangle = \langle M_\varphi(x), e_i\rangle = \varphi_i x_i = \langle x, \varphi_i e_i\rangle, \quad x \in ces(p+),$$

which implies that $T(e_i) = \varphi_i e_i$, that is, $(M_\varphi)'(e_i) = \varphi_i e_i$ for all $i \in \mathbb{N}$. So, in both cases, we have shown that

$$\{\varphi_i : i \in \mathbb{N}\} \subseteq \sigma_{pt}((M_\varphi)'; Z), \tag{3.12}$$

where $Z = (\ell_{p+})'_\beta$ if $\varphi \in \mathcal{M}(\ell_{p+})$ and $Z = (ces(p+))'_\beta$ if $\varphi \in \mathcal{M}(ces(p+))$.

Proposition 3.24. *Let $1 \leq p < \infty$.*

(i) *For each $\varphi \in \mathcal{M}(\ell_{p+})$, the operator $M_\varphi \in \mathcal{L}(\ell_{p+})$ is not supercyclic.*

(ii) *For each $\varphi \in \mathcal{M}(ces(p+))$, the operator $M_\varphi \in \mathcal{L}(ces(p+))$ is not supercyclic.*

Proof. In both cases, if φ is non-constant, then (3.12) implies that $(M_\varphi)'$ has at least two linearly independent eigenvectors. Since supercyclic is the same as being 1-supercyclic in the sense of [8], it follows from Theorem 2.1 of [8] that M_φ is not supercyclic.

In the event that $\varphi = \alpha(1, 1, 1, \ldots)$ for some $\alpha \in \mathbb{C}$ is constant it follows, again in both cases, that $M_\varphi = \alpha I$ and so $(M_\varphi)^n = \alpha^n I$, for $n \in \mathbb{N}$. This implies that $\{\lambda (M_\varphi)^n(x) : \lambda \in \mathbb{C}, n \in \mathbb{N}\} \subseteq \text{span}\{x\}$ for each x (in the appropriate space ℓ_{p+} or $ces(p+)$). Accordingly, M_φ is not supercyclic. □

4. Inclusions and operators from $ces(p+)$ into $ces(q+)$

Consider a pair $1 \leq p, q < \infty$. Denote by $\mathsf{C}_{c(p),c(q)}$ (resp. $\mathsf{C}_{c(p),q}$; $\mathsf{C}_{p,c(q)}$; $\mathsf{C}_{p,q}$) the Cesàro operator C when it acts from $ces(p+)$ into $ces(q+)$ (resp. $ces(p+)$ into ℓ_{q+}; resp. ℓ_{p+} into $ces(q+)$; resp. ℓ_{p+} into ℓ_{q+}), whenever this operator exists. The closed graph theorem then ensures that this operator is continuous. We use the analogous notation for the natural inclusion maps $i_{c(p),c(q)}$; $i_{c(p),q}$; $i_{p,c(q)}$; $i_{p,q}$ whenever they exist. The main aim of this section is to identify all pairs p, q for which these inclusion operators and Cesàro operators *do exist* and, for such pairs, to determine whether or not the operator is *bounded* or *compact*.

The following known result is needed below. Due to the difficulty in finding a precise reference, we include a proof for the sake of self-containment.

Lemma 4.1. *Let $E := \text{proj}_m E_m$ and $F := \text{proj}_n F_n$ be Fréchet spaces such that $E = \cap_{m\in\mathbb{N}} E_m$ with each $(E_m, \|\cdot\|_m)$ a Banach space (resp. $F = \cap_{n\in\mathbb{N}} F_n$ with each $(F_n, \|\cdot\|)$ a Banach space). Moreover, it is assumed that E is dense in E_m and that $E_{m+1} \subseteq E_m$ with a continuous inclusion for each $m \in \mathbb{N}$ (resp. $F_{n+1} \subseteq F_n$ with a continuous inclusion for each $n \in \mathbb{N}$). Let $T : E \longrightarrow F$ be a linear operator.*

(i) *T is continuous if and only if for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that T has a unique continuous linear extension $T_{m,n} : E_m \longrightarrow F_n$.*

(ii) *Assume that T is continuous. Then T is bounded if and only if there exists $m_0 \in \mathbb{N}$ such that, for every $n \in \mathbb{N}$, the operator T has a unique continuous linear extension $T_{m_0,n} : E_{m_0} \longrightarrow F_n$.*

Proof. (i) Suppose that $T : E \longrightarrow F$ is continuous. According to [22, V Proposition 4.12] the operator $T : E \longrightarrow F_n$ is continuous for each $n \in \mathbb{N}$. That is, for each $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $T : (E, \|\cdot\|_m) \longrightarrow F_n$ is continuous. By the density of E in E_m there exists a unique continuous linear map $T_{m,n} \in \mathcal{L}(E_m, F_n)$ whose restriction $T_{m,n}|_E = T$.

On the other hand, suppose that for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $T : E \longrightarrow F$ has a unique continuous linear extension $T_{m,n} \in \mathcal{L}(E_m, F_n)$. Then $T : (E, \|\cdot\|_m) \longrightarrow F_n$ is continuous and hence, so is $T : E \longrightarrow F_n$ as the inclusion map $E \hookrightarrow (E_m, \|\cdot\|_m)$ is continuous. Accordingly, $T : E \longrightarrow F$ is continuous, [22, V Proposition 4.12].

(ii) Assume that $T \in \mathcal{L}(E, F)$ is a bounded map. Then there exists $m_0 \in \mathbb{N}$ such that the neighbourhood $U_{m_0} := \{x \in E : \|x\|_{m_0} \leq 1\}$ of 0 in E has the property that $T(U_{m_0})$ is a bounded subset of F. That is, $T(U_{m_0})$ is a bounded set in the Banach space $(F_n, \|\cdot\|_n)$ for all $n \in \mathbb{N}$. In particular, $T : (E, \|\cdot\|_{m_0}) \longrightarrow (F_n, \|\cdot\|_n)$ is also continuous for each $n \in \mathbb{N}$. By the density of E in E_{m_0} there exists a unique continuous linear extension $T_{m_0,n} : (E_{m_0}, \|\cdot\|_{m_0}) \longrightarrow (F_n, \|\cdot\|_n)$ for $n \in \mathbb{N}$.

Conversely, suppose there exists $m_0 \in \mathbb{N}$ such that, for each $n \in \mathbb{N}$, the operator T has a unique continuous linear extension $T_{m_0,n} : E_{m_0} \longrightarrow F_n$. The 0-neighbourhood $U_{m_0} \subseteq E$ satisfies $U_{m_0} \subseteq \{x \in E_{m_0} : \|x\|_{m_0} \leq 1\}$ and hence, $T(U_{m_0}) = T_{m_0,n}(U_{m_0})$ is a bounded set in F_n for $n \in \mathbb{N}$. Since $F = \text{proj}_n F_n$, it follows that $T(U_{m_0})$ is a bounded set in F, i.e., $T \in \mathcal{L}(E, F)$ is a bounded operator. □

For the rest of this section, given pairs $1 \leq p, q < \infty$, we define $p_n := p + \frac{1}{n}$ and $q_n := q + \frac{1}{n}$, for $n \in \mathbb{N}$. In this case $\ell_{p+} = \text{proj}_n \ell_{p_n}$ and $\ell_{q+} = \text{proj}_n \ell_{q_n}$ as well as $ces(p+) = \text{proj}_n ces(p_n)$ and $ces(q+) = \text{proj}_n ces(q_n)$.

Proposition 4.2. *Let $1 \leq p, q < \infty$ be an arbitrary pair.*

(i) *The inclusion map $i_{p,q} : \ell_{p+} \longrightarrow \ell_{q+}$ exists if and only if $p \leq q$, in which case the inclusion is continuous.*
(ii) *The inclusion map $i_{p,c(q)} : \ell_{p+} \longrightarrow ces(q+)$ exists if and only if $p \leq q$, in which case the inclusion is continuous.*
(iii) *The inclusion map $i_{c(p),c(q)} : ces(p+) \longrightarrow ces(q+)$ exists if and only if $p \leq q$, in which case the inclusion is continuous.*
(iv) *$ces(p+) \nsubseteq \ell_{q+}$ for all choices of $1 \leq p, q < \infty$.*

Proof. (i) Let $p > q$. Then $\ell_{p+} \nsubseteq \ell_{q+}$ since $x := (n^{-1/p})_n$ belongs to ℓ_{p+} but $x \notin \ell_r$ with $q < r := (p+q)/2$, i.e., $x \notin \ell_{q+}$.

On the other hand, if $p \leq q$, then $p_n \leq q_n$ for each $n \in \mathbb{N}$ and so $\ell_{p+} \subseteq \ell_{q+}$ continuously (by Lemma 4.1(i) with $E_m = \ell_{q_m}, F_n = \ell_{p_n}$ and $T = i_{p,q}$).

(ii) If $p > q$, then there exists $n \in \mathbb{N}$ such that $q_n < p$. Hence, $\ell_p \nsubseteq ces(q_n)$ by [9, Remark 2.2(ii)]. Since $\ell_p \subseteq \ell_{p+}$ and $ces(q+) \subseteq ces(q_n)$, it follows that $\ell_{p+} \nsubseteq ces(q+)$.

Let $p \leq q$. Then $\ell_{p_n} \subset ces(q_n)$ with a continuous inclusion, for each $n \in \mathbb{N}$, [4, Proposition 3.2(ii)]. Hence, $\ell_{p+} \subseteq ces(q+)$ and via Lemma 4.1(i), with

$E_m = \ell_{q_m}, F_n = ces(q_n)$ and $T = i_{p,c(q)}$, we can conclude that $i_{p,c(q)}$ exists and is continuous.

(iii) If $p \leq q$, then $ces(p_n) \subset ces(q_n)$ with a continuous inclusion for each $n \in \mathbb{N}$, [4, Proposition 3.2(iii)]. Again we can apply Lemma 4.1(i) to conclude that the inclusion map $i_{c(p),c(q)} : ces(p+) \longrightarrow ces(q+)$ exists and is continuous.

In case $p > q$, we have $\ell_{p+} \nsubseteq ces(q+)$ by part (ii). But, $\ell_{p+} \subseteq ces(p+)$ and so $ces(p+) \nsubseteq ces(q+)$.

(iv) This is a direct consequence of [4, Proposition 3.2(iv)]. □

Proposition 4.3. *Let $1 \leq p \leq q < \infty$ be arbitrary.*

(i) *The inclusion map $i_{p,q} : \ell_{p+} \longrightarrow \ell_{q+}$ is bounded if and only if $p < q$. Moreover, $i_{p,q}$ is never compact.*
(ii) *The inclusion map $i_{c(p),c(q)} : ces(p+) \longrightarrow ces(q+)$ is bounded if and only if $p < q$. In this case, $i_{c(p),c(q)}$ is also compact.*
(iii) *The inclusion map $i_{p,c(q)} : \ell_{p+} \longrightarrow ces(q+)$ is bounded and compact whenever $p < q$.*
(iv) *The inclusion map $i_{p,c(p)} : \ell_{p+} \longrightarrow ces(p+)$ is not bounded.*

Proof. (i) The inclusion map $i_{p,p}$ is precisely the identity operator on ℓ_{p+}. Since ℓ_{p+} is not normable, it follows that $i_{p,p}$ cannot be a bounded map. On the other hand, if $p < q$, then we have the factorization $i_{p,q} = B \circ A$, for the continuous inclusion maps $A : \ell_{p+} \longrightarrow \ell_r$ and $B : \ell_r \longrightarrow \ell_{q+}$, with $r := \frac{(p+q)}{2} \in (p,q)$. But, A is a bounded map as $x \mapsto \|x\|_r$ is a continuous norm on ℓ_{p+} and so $A^{-1}(B_r)$ is a neighbourhood of 0 in ℓ_{p+} which is mapped into the bounded set $B_r := \{x \in \ell_r : \|x\| \leq 1\} \subseteq \ell_r$. Hence, $i_{p,q}$ is also bounded. On the other hand, still for $p < q$, if $i_{p,q}$ were compact, then the inclusion $\ell_p \subseteq \ell_{2q}$ would also be compact as it is the composition of the continuous inclusions

$$\ell_p \subseteq \ell_{p+} \xrightarrow{i_{p,q}} \ell_{q+} \subseteq \ell_{2q}.$$

But, it is known that the inclusion $\ell_p \subseteq \ell_{2q}$ is not compact, [4, Proposition 3.4(i)].

(ii) The inclusion $i_{c(p),c(p)}$ is not bounded (hence, not compact) since $ces(p+)$ is an infinite dimensional Fréchet-Montel space. However, if $p < q$, then the inclusion map $i_{c(p),c(q)}$ is bounded (hence, compact as $ces(p+)$ is Montel) because it has the factorization (via the continuous natural inclusions)

$$ces(p+) \subseteq ces(r) \subseteq ces(q+),$$

with $r := \frac{p+q}{2} \in (p,q)$ and the inclusion $ces(p+) \subseteq ces(r)$ is bounded.

For $p > q$ there is no inclusion of $ces(p+)$ into $ces(q+)$; see Proposition 4.2(iii).

(iii) This follows from part (ii) because of the continuous inclusions $\ell_{p+} \subseteq ces(p+) \subseteq ces(q+)$.

(iv) Proceeding by contradiction, assume that the inclusion $i_{p,c(p)} : \ell_{p+} \longrightarrow ces(p+)$ is bounded. By Lemma 4.1(ii), with $E_m = \ell_{p_m}$ and $F_n = ces(p_n)$

for $m, n \in \mathbb{N}$, there exists $m_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ the natural inclusion $\ell_{p_{m_0}} \subseteq ces(p_n)$ is continuous. However, for $n := m_0 + 1$ we have $p_{m_0} > p_n$ which yields a contradiction to [4, Proposition 3.2(ii)]. □

Now that the relevant properties of the inclusion operators are completely determined we can do the same for the Cesàro operators $\mathsf{C} : X \longrightarrow Y$ where $X, Y \in \{\ell_{p+}, ces(q+) : p, q \in [1, \infty)\}$. We begin with continuity.

Proposition 4.4. *Let $1 \leq p, q < \infty$ be an arbitrary pair.*

(i) $\mathsf{C}_{p,q} : \ell_{p+} \longrightarrow \ell_{q+}$ *exists if and only if $p \leq q$, in which case $\mathsf{C}_{p,q}$ is continuous.*
(ii) $\mathsf{C}_{p,c(q)} : \ell_{p+} \longrightarrow ces(q+)$ *exists if and only if $p \leq q$, in which case $\mathsf{C}_{p,c(q)}$ is continuous.*
(iii) $\mathsf{C}_{c(p),c(q)} : ces(p+) \longrightarrow ces(q+)$ *exists if and only if $p \leq q$, in which case $\mathsf{C}_{c(p),c(q)}$ is continuous.*
(iv) $\mathsf{C}_{c(p),q} : ces(p+) \longrightarrow \ell_{q+}$ *exists if and only if $p \leq q$, in which case $\mathsf{C}_{c(p),q}$ is continuous.*

Proof. The proof of each of the four statements is similar and depends on [4, Proposition 3.5]. We only present the details of part (i).

If $p \leq q$, then $p_n \leq q_n$ for each $n \in \mathbb{N}$. Proposition 3.7(i) of [4] implies that $\mathsf{C} : \ell_{p_n} \to \ell_{q_n}$ is continuous for each $n \in \mathbb{N}$. The continuity of $\mathsf{C}_{p,q}$ now follows from Lemma 4.1(i).

Assume now that $p > q$. Set $r := \frac{(p+q)}{2} \in (p, q)$ and select $n \in \mathbb{N}$ such that $q_n < r$. If $\mathsf{C}_{p,q} : \ell_{p+} \longrightarrow \ell_{q+}$ exists (and is necessarily continuous by the closed graph theorem for Fréchet spaces), then Lemma 4.1(i) ensures the existence of $m_0 \in \mathbb{N}$ such that $\mathsf{C} : \ell_{p_{m_0}} \to \ell_{q_n}$ is continuous. But, this contradicts [4, Proposition 3.5(i)] as $q_n < r < p < p_{m_0}$. □

Proposition 4.5. *Let $1 \leq p \leq q < \infty$ be arbitrary.*

(i) *The Cesàro operator $\mathsf{C}_{p,q} : \ell_{p+} \longrightarrow \ell_{q+}$ is bounded if and only if it is compact if and only if $p < q$.*
(ii) *The Cesàro operator $\mathsf{C}_{p,c(q)} : \ell_{p+} \longrightarrow ces(q+)$ is bounded if and only if it is compact if and only if $p < q$.*
(iii) *The Cesàro operator $\mathsf{C}_{c(p),c(q)} : ces(p+) \longrightarrow ces(q+)$ is bounded if and only if it is compact if and only if $p < q$.*
(iv) *The Cesàro operator $\mathsf{C}_{c(p),q} : ces(p+) \longrightarrow \ell_{q+}$ is bounded if and only if it is compact if and only if $p < q$.*

Proof. The argument that the Cesàro operator is not bounded for the case $p = q$ is similar for each of the four cases. So, we only verify one of these cases. For example, in the statement (i), assume that $\mathsf{C}_{p,p} : \ell_{p+} \longrightarrow \ell_{p+}$ is bounded. By Lemma 4.1(ii) there is $m_0 \in \mathbb{N}$ such that $\mathsf{C} : \ell_{p_{m_0}} \to \ell_{p_n}$ is continuous for all $n \in \mathbb{N}$. If we choose $n := m_0 + 1$ say, then $p_{m_0} > p_n$ and we have a contradiction to [4, Proposition 3.5(i)].

Now assume that $p < q$, set $r := \frac{(p+q)}{2}$ and consider each of the four cases.

(i) Since $p < r < q$, Propositions 4.2(i) and 4.4(i) imply that the Cesàro operator $\mathsf{C}_{p,q} : \ell_{p+} \longrightarrow \ell_{q+}$ factors continuously as $\mathsf{C}_{p,q} = \mathsf{C}_{r,q} \circ i_{p,r}$. Moreover, the operator $i_{p,r} : \ell_{p+} \longrightarrow \ell_{r+}$ is bounded by Proposition 4.3(i). Hence, $\mathsf{C}_{p,q}$ is also bounded.

The operator $\mathsf{C}_{p,q}$ is even compact. Indeed, $\mathsf{C}_{p,q} = \mathsf{C} \circ j_1$ with j_1 the continuous inclusion of ℓ_{p+} into ℓ_r and $\mathsf{C} : \ell_r \longrightarrow \ell_{q+}$ the Cesàro operator, whose compactness follows from the fact that $\mathsf{C} : \ell_r \longrightarrow \ell_{q_n}$ is compact for each $n \in \mathbb{N}$, [4, Proposition 3.6(i)].

(ii) In this case $\mathsf{C}_{p,c(q)} : \ell_{p+} \longrightarrow ces(q+)$ is bounded because it factors continuously as $\mathsf{C}_{p,c(q)} = \mathsf{C}_{r,c(q)} \circ i_{p,r}$ (see Propositions 4.2(i) and 4.4(ii)) and $i_{p,r}$ is bounded by Proposition 4.3(i). The operator $\mathsf{C}_{p,c(q)}$ is then also compact since $ces(q+)$ is a Montel space.

(iii) The operator $\mathsf{C}_{c(p),c(q)} : ces(p+) \longrightarrow ces(q+)$ is compact because it factors continuously as $\mathsf{C}_{c(p),c(q)} = \mathsf{C}_{c(r),c(q)} \circ i_{c(p),c(r)}$ (see Propositions 4.2(iii) and 4.4(iii)) and $i_{c(p),c(r)}$ is compact by Proposition 4.3(ii).

(iv) The Cesàro operator $\mathsf{C}_{c(p),q} : ces(p+) \longrightarrow \ell_{q+}$ is compact since it factors continuously as $\mathsf{C}_{c(p),q} = \mathsf{C}_{c(r),q} \circ i_{c(p),c(r)}$ (see Propositions 4.2(iii) and 4.4(iv)) and $i_{c(p),c(r)}$ is compact by Proposition 4.3(ii).

□

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References

[1] A.A. Albanese, J. Bonet, W.J. Ricker, *Mean ergodic operators in Fréchet spaces,* Ann. Acad. Sci. Fenn. Math. **34** (2009), 401–436.

[2] A.A. Albanese, J. Bonet, W.J. Ricker, *The Cesàro operator in the Fréchet spaces ℓ^{p+} and L^{p-},* Glasgow Math. J. **59** (2017), 273–287.

[3] A.A. Albanese, J. Bonet, W.J. Ricker, *The Fréchet spaces $ces(p+), 1 < p < \infty$*, J. Math. Anal. Appl. **458** (2018), 1314–1323.

[4] A.A. Albanese, J. Bonet, W.J. Ricker, *Multiplier and averaging operators in the Banach spaces $ces(p), 1 < p < \infty$*, Preprint 2018.

[5] G.F. Bachelis, J.E. Gilbert, *Banach spaces of compact multipliers and their dual spaces*, Math. Z. **125** (1972), 285–297. MR 0338693.

[6] G. Bennett, *Factorizing the classical inequalities*, Mem. Amer. Math. Soc. **120** (1996), no 576, viii + 130 pp.

[7] J. Bonet, W.J. Ricker, *The canonical spectral measure in Köthe echelon spaces*, Integral Equations Operator Theory, **53** (2005), 477–496.

[8] P.S. Bourdon, N.S. Feldman, J.H. Shapiro, *Some properties of N-supercyclic operators*, Studia Math. **165** (2004), 135–157.

[9] G.P. Curbera, W.J. Ricker, *Solid extensions of the Cesàro operator on ℓ^p and c_0,* Integral Equations Operator Theory **80** (2014), 61–77.

[10] J.-C. Díaz, *An example of a Fréchet space, not Montel, without infinite-dimensional normable subspaces*, Proc. Amer. Math. Soc. **96** (1986), 721.

[11] R.E. Edwards, *Functional Analysis, Theory and Applications*, Holt, Rinehart and Winston, New York Chicago San Francisco, 1965.
[12] K.-G. Grosse-Erdmann, *The Blocking Technique, Weighted Mean Operators and Hardy's Inequality*, Lecture Notes in Mathematics, **1679**, Springer Verlag, Berlin Heidelberg, 1998.
[13] A. Grothendieck, *Topological Vector Spaces*, Gordon and Breach, London, 1973.
[14] G.H. Hardy, J.E. Littlewood, G. Pólya, *Inequalities,* Cambridge University Press, Cambridge, reprinted, 1964.
[15] H. Jarchow, *Locally Convex Spaces,* Teubner, Stuttgart, 1981.
[16] U. Krengel, *Ergodic Theorems.* de Gruyter Studies in Mathematics, **6**. Walter de Gruyter Co., Berlin, 1985.
[17] R. Meise, D. Vogt, *Introduction to Functional Analysis*, Clarendon Press, Oxford, 1997.
[18] G. Metafune, V.B. Moscatelli, *On the space* $\ell^{p+} = \cap_{q>p}\ell^q$, Math. Nachr. **147** (1990), 7–12.
[19] P. Pérez Carreras, J. Bonet, *Barrelled Locally Convex Spaces*, North Holland, Amsterdam, 1987.
[20] H.R. Pitt, *A note on bilinear forms,* J. London Math. Soc. **11** (1936), 171–174.
[21] W.J. Ricker, *A spectral mapping theorem for scalar-type spectral operators in locally convex spaces*, Integral Equations Operator Theory, **8** (1985), 276–288.
[22] A.P. Robertson, W. Robertson, *Topological Vector Spaces*, Cambridge University Press, Cambridge, 1973

Angela A. Albanese, Dipartimento di Matematica e Fisica "E. De Giorgi", Università del Salento- C.P.193, I-73100 Lecce, Italy
E-mail address: angela.albanese@unisalento.it

José Bonet, Instituto Universitario de Matemática Pura y Aplicada IUMPA, Universitat Politècnica de València, E-46071 Valencia, Spain
E-mail address: jbonet@mat.upv.es

Werner J. Ricker, Math.-Geogr. Fakultät, Katholische Universität Eichstätt-Ingolstadt, D-85072 Eichstätt, Germany
E-mail address: werner.ricker@ku.de