MULTIPLIER AND AVERAGING OPERATORS IN THE BANACH SPACES ces(p), $1 < p < \infty$

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Abstract. The Banach sequence spaces $ces(p)$ are generated in a specified way via the classical spaces $\ell_p, 1 < p < \infty$. For each pair $1 < p, q < \infty$ the (p, q)-multiplier operators from $ces(p)$ into $ces(q)$ are known. We determine precisely which of these multipliers is a compact operator. Moreover, for the case of $p = q$ a complete description is presented of those (p, p)-multiplier operators which are mean (resp. uniform mean) ergodic. A study is also made of the linear operator C which maps a numerical sequence to the sequence of its averages. All pairs $1 < p, q < \infty$ are identified for which C maps $ces(p)$ into $ces(q)$ and, amongst this collection, those which are compact. For $p = q$, the mean ergodic properties of C are also treated.

1. Introduction.

For each element $x = (x_n)_n = (x_1, x_2, \dots)$ of $\mathbb{C}^{\mathbb{N}}$ let $|x| := (|x_n|)_n$ and write $x \geq 0$ if $x = |x|$. Of course, $x \leq y$ means that $(y - x) \geq 0$. The Cesàro operator $C : \mathbb{C}^{\mathbb{N}} \longrightarrow \mathbb{C}^{\mathbb{N}}$, defined by

$$C(x) := (x_1, \frac{x_1 + x_2}{2}, \frac{x_1 + x_2 + x_3}{3}, \dots), \quad x \in \mathbb{C}^{\mathbb{N}},$$

satisfies $|C(x)| \leq C(|x|)$ for $x \in \mathbb{C}^{\mathbb{N}}$ and is a vector space isomorphism of $\mathbb{C}^{\mathbb{N}}$ onto itself. It is also a topological isomorphism when $\mathbb{C}^{\mathbb{N}}$ is considered as a (locally convex) Fréchet space with respect to the coordinatewise convergence. For each $1 < p < \infty$ define

$$ces(p) := \Big\{x \in \mathbb{C}^{\mathbb{N}} : \|x\|_{ces(p)} := \|(\frac{1}{n}\sum_{k=1}^{n} |x_k|)_n\|_p = \|C(|x|)\|_p < \infty\Big\}, \tag{1.1}$$

where $\|\cdot\|_p$ denotes the standard norm in ℓ_p. An intensive study of the Banach spaces $ces(p), 1 < p < \infty$, was undertaken in [3]; see also the references therein. In particular, they are reflexive, p-concave Banach lattices (for the order induced by $\mathbb{C}^{\mathbb{N}}$) and the canonical vectors $e_k := (\delta_{nk})_n$, for $k \in \mathbb{N}$, form an unconditional basis, [3], [6]. For any pair $1 < p, q < \infty$ the space $ces(p)$ is known not to be isomorphic to ℓ_q, [3, Proposition 15.13]. It is shown in Proposition 3.3 (for all $p \neq q$) that $ces(p)$ is also not isomorphic to $ces(q)$. It is important to note that the inequality

$$\frac{A_p}{k^{1/p'}} \leq \|e_k\|_{ces(p)} \leq \frac{B_p}{k^{1/p'}}, \quad k \in \mathbb{N}, \tag{1.2}$$

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is valid for strictly positive constants A_p, B_p and with $\frac{1}{p}+\frac{1}{p'}=1$, [3, Lemma 4.7]. It is known, [3, p.26], that $ces(p)=cop(p)$ with equivalent norms, where

$$cop(p) := \Big\{x\in\mathbb{C}^{\mathbb{N}} : \|x\|_{cop(p)} := \|(\sum_{k=n}^{\infty}\frac{|x_k|}{k})_n\|_p < \infty\Big\},\quad 1<p<\infty.$$

The dual Banach spaces $(ces(p))', 1<p<\infty$, are described in Section 12 of [3]. Yet another equivalent norm in $ces(p)$, via the dyadic decomposition of $\mathbb{N}$, is available, [11, Theorem 4.1]. Namely, $x\in\mathbb{C}^{\mathbb{N}}$ belongs to $ces(p)$ if and only if

$$\|x\|_{[p]} := \Big(\sum_{j=0}^{\infty} 2^{j(1-p)}\Big(\sum_{k=2^j}^{2^{j+1}-1}|x_k|\Big)^p\Big)^{1/p} < \infty. \tag{1.3}$$

The spaces $ces(p), 1<p<\infty$, also arise in a very different way. Fix $1<p<\infty$. Since the Cesàro operator $C_{p,p}:\ell_p\longrightarrow\ell_p$, i.e., C restricted to ℓ_p, is a *positive* operator between Banach lattices, it is natural to look for continuous ℓ_p*-valued* extensions of $C_{p,p}$ to Banach lattices $X\subseteq\mathbb{C}^{\mathbb{N}}$ which are larger than ℓ_p and *solid* (i.e., $y\in\mathbb{C}^{\mathbb{N}}$ and $|y|\le|x|$ with $x\in X$ implies that $y\in X$). The *largest* of all those solid Banach lattices in $\mathbb{C}^{\mathbb{N}}$ for which such a continuous, ℓ_p-valued extension of $C_{p,p}:\ell_p\longrightarrow\ell_p$ is possible is precisely $ces(p)$, [6, p.62]. Of course, this "largest extension" $C_{c(p),p}: ces(p)\longrightarrow\ell_p$ is the restriction of C from $\mathbb{C}^{\mathbb{N}}$ to $ces(p)$. Somewhat surprisingly, $C_{c(p),p}$ also possesses an *integral representation.* That is, $ces(p)$ coincides with the L^1-space of an ℓ_p-valued vector measure m_p and $C_{c(p),p}$ is given by

$$C_{c(p),p}(x) = \int_{\mathbb{N}} x(n)\,dm_p(n),\quad x\in L^1(m_p)=ces(p).$$

Here $m_p:\mathcal{R}\longrightarrow\ell_p$ is the σ-additive *vector measure* defined on the δ*-ring* $\mathcal{R}$ of all finite subsets of $\mathbb{N}$ by

$$m_p(A) := C_{p,p}(\chi_A),\quad A\in\mathcal{R}, \tag{1.4}$$

where $\chi_A:\mathbb{N}\longrightarrow\mathbb{C}$ is the element of $\mathbb{C}^{\mathbb{N}}$ given by $\chi_A=\sum_{k\in A}e_k$ for each $A\subseteq\mathbb{N}$. This claim certainly requires a proof. First, the space $L^1(m_p)$ of all m_p*-integrable functions* on $\mathbb{N}$, as defined in [8], [9], is the *optimal domain* for the operator $C_{p,p}$ (in the sense of [9, Corollaries 2.4 and 2.6]) within the class of all Banach function spaces (briefly, B.f.s) over $(\mathbb{N},\mathcal{R},\mu)$ which have *absolutely continuous* norm (briefly, a.c.); here μ denotes counting measure. More precisely, $L^1(m_p)\subseteq\mathbb{C}^{\mathbb{N}}$ contains the domain space ℓ_p of $C_{p,p}$, the integration map $I_{m_p}:L^1(m_p)\longrightarrow\ell_p$ (given by $x\longmapsto\int_{\mathbb{N}}x\,dm_p$ for $x\in L^1(m_p)$) satisfies $I_{m_p}(x)=C_{p,p}(x)$ for each $x\in\ell_p\subseteq L^1(m_p)$, and $L^1(m_p)$ is the *largest* of all B.f.s.' over $(\mathbb{N},\mathcal{R},\mu)$ having a.c.-norm to which $C_{p,p}$ can be extended while still maintaining its values in ℓ_p. To verify this, we observe that an equivalent norm in $L^1(m_p)$ is given by

$$|\!|\!| x |\!|\!|_{L^1(m_p)} := \sup\Big\{\Big\|\int_A x\,dm_p\Big\|_p : A\in\mathcal{R}\Big\},\quad x\in L^1(m_p);$$

see (3) on p.434 of [8]. But, for $x\in L^1(m_p)$ and each $A\in\mathcal{R}$, the function $x\chi_A$ is an $\mathcal{R}$-simple function and so it follows from (1.4) that $\int_A x\,dm_p = C_{p,p}(x\chi_A)$.

Now, for $x \in ces(p)$ fixed, note that

$$\Big\| \int_A x\, dm_p \Big\|_p = \|C_{p,p}(x\chi_A)\|_p = \|C_{c(p),p}(x\chi_A)\|_p \le \|C_{c(p),p}(|x|)\|_p = \|x\|_{ces(p)} < \infty$$

for every $A \in \mathcal{R}$. If we *define* $\int_A x\, dm_p := C_{c(p),p}(x\chi_A) \in \ell_p$ for an *arbitrary* subset $A \subseteq \mathbb{N}$, then x is m_p-integrable in the sense of [8, p.434], [9, p.133], with $\|x\|_{L^1(m_p)} \le \|x\|_{ces(p)}$. Since $ces(p)$ itself is a B.f.s. over $(\mathbb{N}, \mathcal{R}, \mu)$ having an a.c.-norm and containing ℓ_p, we can conclude from the optimality of $L^1(m_p)$ that $ces(p) \subseteq L^1(m_p)$ with a continuous inclusion. On the other hand, recall that $ces(p)$ is the largest solid Banach lattice in $\mathbb{C}^{\mathbb{N}}$ which contains ℓ_p and C maps into ℓ_p. But, the B.f.s. $L^1(m_p)$ *is* such a solid Banach lattice which C maps into ℓ_p. Indeed, since $L^1(m_p) \subseteq \mathbb{C}^{\mathbb{N}}$ with ℓ_p dense in $L^1(m_p)$ (as ℓ_p contains the $\mathcal{R}$-simple functions which are known to be dense in $L^1(m_p)$, [8, p.434]) and C acts in all of $\mathbb{C}^{\mathbb{N}}$, it follows from the fact that norm convergence of a sequence in $L^1(m_p)$ implies the pointwise convergence μ-a.e. of a subsequence, [9, p.134] (in this case meaning coordinatewise convergence in $\mathbb{C}^{\mathbb{N}}$), that the extended operator I_{m_p} is necessarily given by $I_{m_p}(x) = C(x)$ for all $x \in L^1(m_p)$. Accordingly $L^1(m_p) \subseteq ces(p)$ and hence, $L^1(m_p) = ces(p)$ with equivalence of the norms $\|\cdot\|_{L^1(m_p)}$ and $\|\cdot\|_{ces(p)}$. It is an important feature that m_p *cannot* be extended to a more traditional σ-additive, ℓ_p-valued vector measure defined on the σ-*algebra* $2^{\mathbb{N}}$ generated by $\mathcal{R}$. This is because its range $m_p(\mathcal{R})$ is an *unbounded* subset of ℓ_p. Indeed, for $A_n := \{1, 2, ..., N\} \in \mathcal{R}$ we have $m_p(A_N) = \sum_{j=1}^{N} e_j + N \sum_{j=N+1}^{\infty} \frac{1}{j} e_j$ and hence, $\|m_p(A_N)\|_p \ge N^{1/p}$ for all $N \in \mathbb{N}$.

Having presented several equivalent and varied descriptions of the spaces $ces(p)$, $1 < p < \infty$, we now formulate the aim of this note, namely to make a detailed analysis of certain *linear operators* defined on these spaces. Let us be more precise.

Given a pair $1 < p, q < \infty$, an element $a \in \mathbb{C}^{\mathbb{N}}$ is called a (p,q)-*multiplier* if it multiplies $ces(p)$ into $ces(q)$, that is, if $ax \in ces(q)$ for every $x \in ces(p)$, where the product $ax := (a_n x_n)_n$ is defined coordinatewise. The closed graph theorem ensures that the corresponding linear (p,q)-*multiplier operator* $M^a_{p,q} : x \longmapsto ax$ is then necessarily *continuous* from $ces(p)$ into $ces(q)$. If $p = q$, then we denote $M^a_{p,p}$ simply by M^a_p and note that M^a_p is the diagonal operator acting in $ces(p)$ via the matrix having the scalars $\{a_n : n \in \mathbb{N}\}$ in its diagonal. The vector space of all (p,q)-multipliers, denoted by $\mathcal{M}_{p,q}$ (or $\mathcal{M}_p$ if $p = q$), has been completely determined by G. Bennett; see [3, pp.69-70], after recalling that $cop(p) = ces(p)$ for all $1 < p < \infty$.

In Section 2 we investigate various properties of the multiplier operators $M^a_{p,q}$ for all pairs $1 < p, q < \infty$ and $a \in \mathcal{M}_{p,q}$. For instance, those multipliers $a \in \mathcal{M}_{p,q}$ for which $M^a_{p,q}$ is a *compact operator* are characterized; see Propositions 2.2 and 2.5. Also, given $a \in \mathcal{M}_p = \ell_\infty$ it is shown that the *spectrum* of M^a_p is the set

$$\sigma(M^a_p) = \overline{a(\mathbb{N})}, \quad 1 < p < \infty,$$

where $a(\mathbb{N}) := \{a_n : n \in \mathbb{N}\} \subseteq \mathbb{C}$, and that $\|M^a_p\|_{op} = \|a\|_\infty$ with $\|\cdot\|_{op}$ denoting the operator norm of $M^a_p : ces(p) \longrightarrow ces(p)$; see Lemma 2.6 and Proposition 2.7. Furthermore, those $a \in \mathcal{M}_p$ are identified for which the operator M^a_p is *mean* *ergodic* (cf. Proposition 2.8) as well as those for which M_p^a is *uniformly mean ergodic* (cf. Proposition 2.10).

It is clear from (1.1) and the discussions above that the Cesàro operator C is intimately connected to the Banach spaces $ces(p), 1 < p < \infty$. Indeed, Hardy's classical inequality states, for $1 < p < \infty$, that

$$\sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{k=1}^{n} b_k\Big)^p \leq K_p \sum_{n=1}^{\infty} b_n^p$$

for all choices of non-negative numbers $\{b_n\}_{n=1}^{\infty}$ and some constant $K_p > 0$, [12]. Setting $b_n := |x_n|$, for $n \in \mathbb{N}$ and each $x \in \ell_p$, it is immediate that $\|C_{p,p}(|x|)\|_p \leq K_p^{1/p}\|x\|_p$, that is, $\ell_p \subseteq ces(p)$ with a continuous inclusion; Remark 2.2 of [6] shows that this containment is strict. Moreover, the Cesàro operator $C_{c(p),p} : ces(p) \longrightarrow \ell_p$ is continuous; this was already implicitly used above. To see this fix $x \in ces(p)$. Using the fact that $\|\cdot\|_p$ is a Banach lattice norm yields

$$\|C_{c(p),p}(x)\|_p = \|\, |C(x)|\, \|_p \leq \|C(|x|)\|_p = \|x\|_{ces(p)}.$$

The connection between C and $ces(p)$ is further exemplified by the following remarkable result of Bennett, [3, Theorem 20.31].

Proposition 1.1. *Let $1 < p < \infty$ and $x \in \mathbb{C}^{\mathbb{N}}$. Then*

$$x \in ces(p) \quad \textit{if and only if} \quad C(|x|) \in ces(p). \tag{1.5}$$

Further examples of Banach spaces $X \subseteq \mathbb{C}^{\mathbb{N}}$ such that $C(X) \subseteq X$ and for which Proposition 1.1 is valid (with X in place of $ces(p)$) are identified in [5], [6], [7].

In Section 3 it is shown that C maps $ces(p)$ into $ces(q)$, necessarily continuously, if and only if $1 < p \leq q < \infty$; see Proposition 3.5. Furthermore, *all pairs* $1 < p, q < \infty$ are identified for which C maps ℓ_p into $ces(q)$ and for which C maps $ces(p)$ into ℓ_q, as well as the subclass of these continuous operators which are actually *compact*. Two important facts in this regard are that the Cesàro operator $C_{c(p),c(p)} : ces(p) \longrightarrow ces(p)$ has spectrum

$$\sigma(C_{c(p),c(p)}) = \{\lambda \in \mathbb{C} : |\lambda - \tfrac{p'}{2}| \leq \tfrac{p'}{2}\}, \quad 1 < p < \infty, \tag{1.6}$$

[6, Theorem 5.1], and that the natural inclusion map $ces(p) \hookrightarrow ces(q)$ is compact whenever $1 < p < q$; see Proposition 3.4. A consequence of (1.6) is that $C_{c(p),c(p)}$ and $C_{p,p}$ are never mean ergodic.

2. Multiplier operators from $ces(p)$ into $ces(q)$.

According to table 16 on p.69 of [3], given $1 < p \leq q < \infty$ an element $a = (a_n)_n \in \mathbb{C}^{\mathbb{N}}$ belongs to $\mathcal{M}_{p,q}$ *if and only* if the element $(a_n n^{\frac{1}{q}-\frac{1}{p}})_n \in \ell_\infty$. Observe that $(\frac{1}{q} - \frac{1}{p}) \leq 0$. In particular, $\ell_\infty \subseteq \mathcal{M}_{p,q}$ and, if $p = q$, then $\mathcal{M}_p = \ell_\infty$. For fixed $a \in \ell_\infty$, it follows via the inequality $C(|au|) \leq \|a\|_\infty C(|u|)$, for $u \in \mathbb{C}^{\mathbb{N}}$, that $\|M_p^a(x)\|_{ces(p)} = \|C(|ax|)\|_p \leq \|a\|_\infty \|C(|x|)\|_p = \|a\|_\infty \|x\|_{ces(p)}$, for all $x \in ces(p)$. Hence, $M_p^a : ces(p) \longrightarrow ces(p)$ satisfies

$$\|M_p^a\|_{op} \leq \|a\|_\infty, \quad a \in \ell_\infty, \quad 1 < p < \infty. \tag{2.1}$$

Here $\|.\|_{op}$ denotes the operator norm. We begin with a result which is probably known; due to the lack of a reference we include a proof. Let φ be the vector subspace of $\mathbb{C}^{\mathbb{N}}$ consisting of all elements with only finitely many non-zero coordinates.

Lemma 2.1. *Let $T : \mathbb{C}^{\mathbb{N}} \longrightarrow \mathbb{C}^{\mathbb{N}}$ be a continuous linear operator and X, Y be a Banach sequence spaces satisfying $\varphi \subseteq X \subset \mathbb{C}^{\mathbb{N}}$ and $\varphi \subseteq Y \subseteq \mathbb{C}^{\mathbb{N}}$ with continuous inclusions such that $T(X) \subseteq Y$. Then the restriction $T : X \longrightarrow Y$ is a compact operator if and only if it satisfies the following property (K), namely:*

(K) *If a norm bounded sequence $\{x_m\}_{m=1}^{\infty} \subseteq X$ satisfies $\lim_{m\to\infty} x_m = 0$ in the Fréchet space $\mathbb{C}^{\mathbb{N}}$, then $\lim_{m\to\infty} T(x_m) = 0$ in the Banach space Y.*

Proof. By the closed graph theorem $T : X \longrightarrow Y$ is continuous.

Suppose first that $T : X \longrightarrow Y$ is compact. Let $\{x_m\}_{m=1}^{\infty} \subseteq X$ be any sequence in X satisfying $\lim_{m\to\infty} x_m = 0$ in $\mathbb{C}^{\mathbb{N}}$. Assume that the sequence $\{T(x_m)\}_{m=1}^{\infty}$ does not converge to 0 in Y. Select a subsequence $\{x_{m_k}\}_{k=1}^{\infty}$ of $\{x_m\}_{m=1}^{\infty}$ and $r > 0$ such that

$$\|T(x_{m_k})\|_Y \geq r, \quad k \in \mathbb{N}. \tag{2.2}$$

By compactness of T there exists $y \in Y$ and a subsequence $\{x_{m_{k(l)}}\}_{l=1}^{\infty}$ of $\{x_{m_k}\}_{k=1}^{\infty}$ such that $\lim_{l\to\infty} \|T(x_{m_{k(l)}}) - y\|_Y = 0$. Continuity of the inclusion $Y \subseteq \mathbb{C}^{\mathbb{N}}$ implies that also $\lim_{l\to\infty} T(x_{m_{k(l)}}) = y$ in $\mathbb{C}^{\mathbb{N}}$. But, $\lim_{l\to\infty} x_{m_{k(l)}} = 0$ in $\mathbb{C}^{\mathbb{N}}$ and $T : \mathbb{C}^{\mathbb{N}} \longrightarrow \mathbb{C}^{\mathbb{N}}$ is continuous. Accordingly, $\lim_{l\to\infty} T(x_{m_{k(l)}}) = 0$ in $\mathbb{C}^{\mathbb{N}}$ and so $y = 0$; contradiction to (2.2). Hence, necessarily $T(x_m) \longrightarrow 0$ in Y for $m \longrightarrow \infty$. This establishes that T has property (K).

Conversely, suppose that T has property (K). Let $\{x_i\}_{i=1}^{\infty}$ be any bounded sequence in X. To show that T is compact we need to argue that $\{T(x_i)\}_{i=1}^{\infty}$ has a convergent subsequence in Y. Since the inclusion $X \subseteq \mathbb{C}^{\mathbb{N}}$ is continuous, the sequence $\{x_i\}_{i=1}^{\infty}$ is also bounded in the Fréchet-Montel space $\mathbb{C}^{\mathbb{N}}$. Hence, there is a subsequence $u_j := x_{i_j}$, for $j \in \mathbb{N}$, of $\{x_i\}_{i=1}^{\infty}$ and $x \in \mathbb{C}^{\mathbb{N}}$ such that $\lim_{j\to\infty} u_j = x$ in $\mathbb{C}^{\mathbb{N}}$. Suppose that $\{T(u_j)\}_{j=1}^{\infty}$ is *not* convergent in Y. Then $\{T(u_j)\}_{j=1}^{\infty}$ cannot be a Cauchy sequence in Y and hence, there exists $a > 0$ such that, for every $j \in \mathbb{N}$, there exist $k_j, l_j \in \mathbb{N}$ with $j < k_j < l_j$ such that $\|T(u_{k_j}) - T(u_{l_j})\|_Y \geq a$. Via this inequality we can choose for $j = 1$ natural numbers $1 < k_1 < l_1$, then for $j := 1 + l_1$ natural numbers $1 + l_1 < k_2 < l_2$ and so on, such that $1 < k_1 < l_1 < k_2 < l_2 < k_3 < l_3 \ldots$ and, for *these* natural numbers $\{k_n, l_n\}_{n=1}^{\infty}$, we have

$$\|T(u_{k_n}) - T(u_{l_n})\|_Y \geq a, \quad n \in \mathbb{N}. \tag{2.3}$$

Then $z_n := u_{k_n} - u_{l_n}$, for $n \in \mathbb{N}$, is a bounded sequence in X. Since $\lim_{j\to\infty} u_j = x$ in $\mathbb{C}^{\mathbb{N}}$, it follows that $\lim_{n\to\infty} z_n = 0$ in $\mathbb{C}^{\mathbb{N}}$. By property (K), $\lim_{n\to\infty} T(z_n) = 0$ in Y, that is, $\lim_{n\to\infty}(T(u_{k_n}) - T(u_{l_n})) = 0$ in Y which contradicts (2.3). Hence, $\{T(u_j)\}_{j=1}^{\infty}$ *does* converge in Y and is a subsequence of $\{T(x_i)\}_{i=1}^{\infty}$. The compactness of T is thereby verified. □

Proposition 2.2. *Let $1 < p \leq q < \infty$ and $a \in \mathcal{M}_{p,q}$. Then the continuous multiplier operator $M^a_{p,q} : ces(p) \longrightarrow ces(q)$ is compact if and only if $(a_n n^{\frac{1}{q}-\frac{1}{p}})_n \in c_0$.*

Proof. Suppose first that $w = (w_n)_n := (a_n n^{\frac{1}{q}-\frac{1}{p}})_n \in c_0$. Define the element $w_N := (w_1, \ldots, w_N, 0, 0, \ldots)$ for each $N \in \mathbb{N}$ in which case $(w - w_N) \in \ell_\infty$. So, by (2.1), $\|M_p^w - M_p^{w_N}\|_{op} = \|M_p^{w-w_N}\|_{op} \le \|w - w_N\|_\infty$. Since $w \in c_0$, it follows that $\lim_{N\to\infty} \|w - w_N\|_\infty = 0$ and hence, $M_p^w : ces(p) \longrightarrow ces(p)$ is *compact* as each $M_p^{w_N}$, for $N \in \mathbb{N}$, is a finite rank operator. Define $v_n := n^{\frac{1}{p}-\frac{1}{q}}$, for $n \in \mathbb{N}$, in which case $v := (v_n)_n \in \mathcal{M}_{p,q}$ by Bennett's multiplier criterion mentioned above, that is, $M_{p,q}^v : ces(p) \longrightarrow ces(q)$ is *continuous*. Since $M_{p,q}^a = M_{p,q}^v M_p^w$, it follows that $M_{p,q}^a$ is compact.

Conversely, suppose that $M_{p,q}^a$ is a compact operator. According to (1.2), the sequence $f_j := j^{1/p'} e_j$, for $j \in \mathbb{N}$, is bounded in $ces(p)$. Clearly $\{f_j\}_{j=1}^\infty$ converges to 0 in the Fréchet space $\mathbb{C}^\mathbb{N}$. Moreover, $M_{p,q}^a(f_j) = j^{1/p'} a_j e_j$, for $j \in \mathbb{N}$, and $M_{p,q}^a(f_j) \longrightarrow 0$ in $\mathbb{C}^\mathbb{N}$ for $j \longrightarrow \infty$ (as the multiplier operator $M^a : \mathbb{C}^\mathbb{N} \longrightarrow \mathbb{C}^\mathbb{N}$ given by $x \longmapsto ax$ is continuous). Applying Lemma 2.1 to the setting $X := ces(p)$, $Y := ces(q)$ and the continuous multiplier operator $T = M^a : \mathbb{C}^\mathbb{N} \longrightarrow \mathbb{C}^\mathbb{N}$ (whose restriction to X is $M_{p,q}^a$), it follows that $\{M_{p,q}^a(f_j)\}_{j=1}^\infty$ actually converges to 0 in $ces(q)$, that is, $\lim_{j\to\infty} j^{1/p'}|a_j| \cdot \|e_j\|_{ces(q)} = \lim_{j\to\infty} \|j^{1/p'} a_j e_j\|_{ces(q)} = 0$. On the other hand, (1.2) implies that $A_q \le j^{1/p'}\|e_j\|_{ces(q)} \le B_q$ for $j \in \mathbb{N}$. It follows that $\lim_{j\to\infty} j^{1/p'}|a_j|/j^{1/q'} = 0$. Since $\frac{1}{p'} - \frac{1}{q'} = \frac{1}{q} - \frac{1}{p}$ we can conclude that $(a_n n^{\frac{1}{q}-\frac{1}{p}})_n \in c_0$. □

For the case when $p = q$ and $a \in \mathcal{M}_p = \ell_\infty$, Proposition 2.2 implies that the multiplier operator $M_a^p : ces(p) \longrightarrow ces(p)$ is compact if and only if $a \in c_0$.

To treat the cases when $p > q$ we recall, for each $r > 1$, the Banach space

$$d(r) := \{x \in \mathbb{C}^\mathbb{N} : \|x\|_{d(r)} := \|\widehat{x}\|_r < \infty\},$$

where $\widehat{x} = (\widehat{x}_n)_n := (\sup_{k\ge n} |x_k|)_n$ and $\|\widehat{x}\|_r$ is its norm in ℓ_r, [3, pp.3-4].

Lemma 2.3. *Let* $1 < r < \infty$ *and* $x \in d(r)$. *Then* $\lim_{N\to\infty} \|x - x^{(N)}\|_{d(r)} = 0$, *where* $x^{(N)} := (x_1, \ldots, x_N, 0, 0, \ldots)$ *for each* $N \in \mathbb{N}$.

Proof. Given $N \in \mathbb{N}$ observe that $x - x^{(N)} = (0, \ldots, 0, x_{N+1}, x_{N+2}, \ldots)$ and hence, $(x - x^{(N)})\widehat{\ } = (\widehat{x}_{N+1}, \ldots, \widehat{x}_{N+1}, \widehat{x}_{N+2}, \ldots)$ where the first $(N+1)$-coordinates are constantly $\widehat{x}_{N+1}$. It follows that

$$\|x - x^{(N)}\|_{d(r)}^r = (N+1)(\widehat{x}_{N+1})^r + \sum_{n=N+2}^\infty (\widehat{x}_n)^r, \quad N \in \mathbb{N}. \tag{2.4}$$

Since $((\widehat{x}_n)^r)_n$ is a decreasing sequence of non-negative terms which belongs to ℓ_1, it is classical that $\lim_{n\to\infty} n(\widehat{x}_n)^r = 0$, [14, § 3.3 Theorem 1]. Let $\epsilon > 0$. Choose $K \in \mathbb{N}$ such that $n(\widehat{x}_n)^r < \frac{\epsilon^r}{2}$ and $\sum_{n=K}^\infty (\widehat{x}_n)^r < \frac{\epsilon^r}{2}$ for all $n \ge K$. It follows from (2.4) that $\|x - x^{(N)}\|_{d(r)}^r < \epsilon^r$ for all $N \ge K$. The proof is thereby complete. □

Let $1 < q < p < \infty$ and choose r according to $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. Then it follows from table 32 on p.70 of [3] that

$$\mathcal{M}_{p,q} = d(r). \tag{2.5}$$

Lemma 2.4. *Let* $1 < q < p < \infty$ *and* r *satisfy* $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. *Then there exists a constant* $D_{p,q} > 0$ *such that*

$$\|M_{p,q}^a\|_{op} \le D_{p,q}\|a\|_{d(r)}, \quad a \in \mathcal{M}_{p,q} = d(r).$$

Proof. For Banach spaces X, Y let $\mathcal{L}(X,Y)$ denote the Banach space of all continuous linear operators from X into Y, equipped with the operator norm $\|\cdot\|_{op}$. According to (2.5) the linear map $\Phi : d(r) \longrightarrow \mathcal{L}(ces(p), ces(q))$ specified by $\Phi(a) := M^a_{p,q}$ is well defined. To establish the existence of $D_{p,q}$ it suffices to show that Φ has closed graph. This is a standard argument after noting that convergence of a sequence in $d(r)$ implies its coordinatewise convergence. □

The following result shows, for $p > q > 1$, that *every* multiplier operator $M^a_{p,q}$ for $a \in \mathcal{M}_{p,q}$ is compact.

Proposition 2.5. *Let $p > q > 1$. For $a \in \mathbb{C}^{\mathbb{N}}$ the following assertions are equivalent.*

(i) *$a \in \mathcal{M}_{p,q}$, that is, $M^a_{p,q} : ces(p) \longrightarrow ces(q)$ is continuous.*
(ii) *$M^a_{p,q} : ces(p) \longrightarrow ces(q)$ is compact.*
(iii) *$a \in d(r)$ where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.*

Proof. (i) $\Longleftrightarrow$ (iii) is precisely the characterization (2.5) of Bennett.

(ii) $\Longrightarrow$ (i) is clear as every compact linear operator is continuous.

(iii) $\Longrightarrow$ (ii). Let $a^{(N)} := (a_1, \ldots, a_N, 0, 0, \ldots)$ for $N \in \mathbb{N}$. Then $a - a^{(N)} \in d(r)$ for $N \in \mathbb{N}$ and $\lim_{N\to\infty} \|a - a^{(N)}\|_{d(r)} = 0$; see Lemma 2.3. By (2.5) the operators $M^a_{p,q}, M^{a^{(N)}}_{p,q}$ and $M^{a-a^{(N)}}_{p,q} = M^a_{p,q} - M^{a^{(N)}}_{p,q}$ all belong to $\mathcal{L}(ces(p), ces(q))$. Lemma 2.4 yields that $\|M^a_{p,q} - M^{a^{(N)}}_{p,q}\|_{op} \leq D_{p,q}\|a - a^{(N)}\|_{d(r)}$, for $N \in \mathbb{N}$. Hence, $M^a_{p,q}$ is compact as each operator $M^{a^{(N)}}_{p,q}$ has finite rank. □

We now consider further properties of multiplier operators for the case when $p = q$. The space $\mathcal{L}(ces(p), ces(p))$ is simply denoted by $\mathcal{L}(ces(p))$.

Lemma 2.6. *Let $1 < p < \infty$. Then*

$$\|M^a_p\|_{op} = \|a\|_\infty, \quad a \in \ell_\infty = \mathcal{M}_p. \tag{2.6}$$

Proof. Just prior to Proposition 2.2 it was noted that $\|M^a_p\|_{op} \leq \|a\|_\infty$. On the other hand, since $M^a_p(e_j) = a_j e_j$ for $j \in \mathbb{N}$, it is clear that the *point spectrum* $\sigma_{pt}(M^a_p)$, consisting of all the eigenvalues of M^a_p, satisfies

$$a(\mathbb{N}) := \{a_j : j \in \mathbb{N}\} \subseteq \sigma_{pt}(M^a_p) \subseteq \sigma(M^a_p).$$

Then the spectral radius inequality for operators, [10, Ch. VII, Lemma 3.4], yields

$$\|M^a_p\|_{op} \geq r(M^a_p) := \sup\{|\lambda| : \lambda \in \sigma(M^a_p)\} \geq \sup_{j\in\mathbb{N}} |a_j| = \|a\|_\infty.$$

□

The spectrum of multiplier operators in $\mathcal{L}(ces(p))$ can now be determined.

Proposition 2.7. *Let $1 < p < \infty$. Then*

$$\sigma(M^a_p) = \overline{a(\mathbb{N})} = \overline{\{a_j : j \in \mathbb{N}\}}, \quad a \in \mathcal{M}_p. \tag{2.7}$$

Proof. From the proof of Lemma 2.6 we have $a(\mathbb{N}) \subseteq \sigma_{pt}(M^a_p) \subseteq \sigma(M^a_p)$. Since $\sigma(M^a_p)$ is a closed set in $\mathbb{C}$, it follows that $\overline{a(\mathbb{N})} \subseteq \sigma(M^a_p)$.

Suppose that $\lambda \notin \overline{a(\mathbb{N})}$. Then $b = (b_n)_n$ with $b_n := \frac{1}{\lambda - a_n}$ for $n \in \mathbb{N}$ belongs to $\ell_\infty = \mathcal{M}_p$. Using the formula $\lambda I - M^a_p = M^{\lambda\mathbf{1}-a}_p$ (with I the identity operator on $ces(p)$ and $\mathbf{1} := (1,1,1,\ldots))$ it is routine to check that $(\lambda I - M_p^a)M_p^b = I = M_p^b(\lambda I - M_p^a)$. Hence, $\lambda I - M_p^a$ is invertible in $\mathcal{L}(ces(p))$ and so λ lies in the resolvent set of M_p^a. This establishes the inclusion $\sigma(M_p^a) \subseteq \overline{a(\mathbb{N})}$. □

For a Banach space X, an operator $T \in \mathcal{L}(X) := \mathcal{L}(X,X)$ is *mean ergodic* (resp. *uniformly mean ergodic*) if its sequence of Cesàro averages

$$T_{[n]} := \frac{1}{n}\sum_{m=1}^{n} T^m, \quad n \in \mathbb{N}, \tag{2.8}$$

converges to some operator $P \in \mathcal{L}(X)$ in the strong operator topology τ_s, i.e., $\lim_{n\to\infty} T_{[n]}(x) = P(x)$ for each $x \in X$, [10, Ch. VIII] (resp. in the operator norm topology τ_b). According to [10, Ch. VIII, Corollary 5.2] there then exists the direct sum decomposition

$$X = \mathrm{Ker}(I - T) \oplus \overline{(I-T)(X)}. \tag{2.9}$$

Moreover, we have the identities $(I-T)T_{[n]} = T_{[n]}(I-T) = \frac{1}{n}(T - T^{n+1})$, for $n \in \mathbb{N}$, and, setting $T_{[0]} := I$, that

$$\frac{1}{n}T^n = T_{[n]} - \frac{(n-1)}{n}T_{[n-1]}, \quad n \in \mathbb{N}. \tag{2.10}$$

An operator $T \in \mathcal{L}(X)$ is called *power bounded* if $\sup_{n\in\mathbb{N}} \|T^n\|_{op} < \infty$. In this case it is clear that necessarily $\lim_{n\to\infty} \frac{\|T^n\|_{op}}{n} = 0$. A standard reference for mean ergodic operators is [15]. Finally, define $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.

Proposition 2.8. *Let $1 < p < \infty$ and $a \in \mathcal{M}_p = \ell_\infty$. The following statements are equivalent.*

(i) $\|a\|_\infty \le 1$.
(ii) *The multiplier operator $M_p^a \in \mathcal{L}(ces(p))$ is power bounded.*
(iii) *The multiplier operator $M_p^a \in \mathcal{L}(ces(p))$ is mean ergodic.*
(iv) *The spectrum $\sigma(M_p^a) \subseteq \overline{\mathbb{D}}$.*
(v) $\lim_{n\to\infty} \frac{(M_p^a)^n}{n} = 0$ *relative to τ_s in $\mathcal{L}(ces(p))$.*

Proof. (i) $\Longrightarrow$ (ii). Since $\mathcal{M}_p$ is an algebra under coordinatewise multiplication in $\mathbb{C}^{\mathbb{N}}$ we have $(M_p^a)^n = M_p^{a^n}$ (where $a^n := (a_j^n)_j$ for $a = (a_j)_j$) and so, via Lemma 2.6, $\|(M_p^a)^n\|_{op} = \|M_p^{a^n}\|_{op} = \|a^n\|_\infty \le 1, \quad n \in \mathbb{N}$.

(ii) $\Longrightarrow$ (iii). Power bounded operators in reflexive Banach spaces are always mean ergodic, [19].

(i) $\Longrightarrow$ (iv). Since $\|a\|_\infty = \sup\{|\lambda| : \lambda \in a(\mathbb{N})\} \le 1$, (2.7) implies $\sigma(M_p^a) \subseteq \overline{\mathbb{D}}$.

(iv) $\Longrightarrow$ (i). Clear from (2.7).

(iii) $\Longrightarrow$ (i). Suppose that $\|a\|_\infty > 1$. Then there exists $k \in \mathbb{N}$ such that $|a_k| > 1$. Since $(M_p^a)^n(e_k) = a_k^n e_k$ for $n \in \mathbb{N}$, it follows that

$$\frac{\|(M_p^a)^n(e_k)\|_{ces(p)}}{n} = \frac{|a_k|^n}{n}\|e_k\|_{ces(p)}, \quad n \in \mathbb{N},$$

with $|a_k| > 1$. Hence, the sequence $\{\frac{(M_p^a)^n}{n}\}_{n=1}^\infty$ *cannot* converge to $0 \in \mathcal{L}(ces(p))$ in the topology τ_s, thereby violating a necessary condition for M_p^a to be mean ergodic (see (2.10)); contradiction! So, $\|a\|_\infty \le 1$.

(iii) $\Longrightarrow$ (v). This follows from (2.10).
(v) $\Longrightarrow$ (i). See the proof of (iii) $\Longrightarrow$ (i). □

In view of Proposition 2.8 we may assume that $\|a\|_\infty \leq 1$ and M_p^a is power bounded whenever it is mean ergodic. Then $\lim_{n\to\infty} \frac{\|(M_p^a)^n\|_{op}}{n} = 0$ and so, by a well known result of Lin, [17], the *uniform* mean ergodicity of M_p^a is equivalent to the range $(I - M_p^a)(ces(p)) = (M_p^{\mathbf{1}-a})(ces(p))$ of $I - M_p^a$ being a *closed* subspace of $ces(p)$.

Given $w \in \mathbb{C}^{\mathbb{N}}$ define its *support* by $S(w) := \{n \in \mathbb{N} : w_n \neq 0\}$ in which case $w\chi_{S(w)} = w$ as elements of $\mathbb{C}^{\mathbb{N}}$. If $w \in \ell_\infty$, then for each $1 < p < \infty$ we have

$$M_p^w(ces(p)) := \{wx : x \in ces(p)\} = \{w\chi_{S(w)}x : x \in ces(p)\}. \tag{2.11}$$

We will also require the *closed* subspace of $ces(p)$ which is the range of the continuous projection operator $M_p^{\chi_{S(w)}}$, i.e.,

$$X_{w,p} := \{\chi_{S(w)}x : x \in ces(p)\} = M_p^{\chi_{S(w)}}(ces(p)). \tag{2.12}$$

It is routine to check that $X_{w,p}$ is M_p^w*-invariant*. Let $\tilde{M}_p^w : X_{w,p} \longrightarrow X_{w,p}$ be the restriction of M_p^w so that $\tilde{M}_p^w \in \mathcal{L}(X_{w,p})$. Since $w_n \neq 0$ for each $n \in S(w)$, it follows that $\tilde{M}_p^w$ is *injective*. Hence, $\tilde{M}_p^w$ is a vector space isomorphism of $X_{w,p}$ onto its range $\tilde{M}_p^w(X_{w,p})$ in $X_{w,p}$. By (2.11) and (2.12) it is clear that $\tilde{M}_p^w(X_{w,p}) = M_p^w(ces(p))$ *whenever* $M_p^w(ces(p))$ is *closed* in $ces(p)$.

Lemma 2.9. *Let $w \in \ell_\infty$ and $1 < p < \infty$. If the range $M_p^w(ces(p))$ is closed in $ces(p)$, then $0 \notin \overline{(w\chi_{S(w)})(\mathbb{N})}$.*

Proof. By the discussion prior to Lemma 2.9, $\tilde{M}_p^w(X_{w,p})$ is a Banach space for the norm $\|\cdot\|_{ces(p)}$ restricted to the closed subspace $M_p^w(ces(p)) = \tilde{M}_p^w(X_{w,p})$ of $ces(p)$. Via the open mapping theorem $\tilde{M}_p^w : X_{w,p} \longrightarrow X_{w,p}$ is then a Banach space isomorphism. So, there exists $T \in \mathcal{L}(X_{w,p})$ satisfying

$$\tilde{M}_p^w T = I = T\tilde{M}_p^w. \tag{2.13}$$

For each $n \in S(w)$ the basis vector $e_n \in X_{w,p}$. Define $y^{(n)} := T(e_n)$ for $n \in S(w)$. It follows from (2.13) that $e_n = wy^{(n)}$. Since the k-th coordinate of e_n is 0 for $k \in \mathbb{N}\setminus\{n\}$, the same is true of $wy^{(n)}$. Accordingly, $e_n = w_n y^{(n)}$ and so $T(e_n) = y^{(n)} = \frac{1}{w_n}e_n$ for each $n \in S(w)$. But, $\{e_n : n \in S(w)\}$ is a basis for $X_{w,p}$ and $T \in \mathcal{L}(X_{w,p})$ from which we can deduce that $T(x) = w^{-1}x$ for all $x \in X_{w,p}$ (with $w^{-1} := (\frac{1}{w_n})_{n\in S(w)}$). Setting $v := w^{-1}\chi_{S(w)} \in \mathbb{C}^{\mathbb{N}}$, it follows that

$$vx = T(\chi_{S(w)}x) = TM_p^{\chi_{S(w)}}(x) = (jTM_p^{\chi_{S(w)}})(x), \tag{2.14}$$

for each $x \in ces(p)$, with $j : X_{w,p} \longrightarrow ces(p)$ being the natural inclusion map and (2.14) holding as equalities in $\mathbb{C}^{\mathbb{N}}$. But, $jTM_p^{\chi_{S(w)}} \in \mathcal{L}(ces(p))$ if we interpret $M_p^{\chi_{S(w)}} : ces(p) \longrightarrow X_{w,p}$ and hence, (2.14) actually holds in $ces(p)$. That is, $M_v = jTM_p^{\chi_{S(w)}}$ belongs to $\mathcal{L}(ces(p))$ which means that $v \in \mathcal{M}_p$ or, equivalently, that $v \in \ell_\infty$. This implies the desired conclusion. □

Proposition 2.10. *Let $1 < p < \infty$ and $a \in \mathcal{M}_p = \ell_\infty$. The following assertions are equivalent.*

(i) M_p^a *is uniformly mean ergodic.*
(ii) $\|a\|_\infty \leq 1$ *and* $1 \notin \overline{a(\mathbb{N})\setminus\{1\}}$.

Proof. (i) $\Longrightarrow$ (ii). By the discussion immediately after Proposition 2.8 we know that (i) implies $\|a\|_\infty \leq 1$ and the range of $I - M_p^a = M_p^{\mathbf{1}-a}$ is closed in $ces(p)$. Then $w := \mathbf{1} - a$ satisfies the hypothesis of Lemma 2.9. Accordingly, $0 \notin \overline{((\mathbf{1}-a)\chi_{S(1-a)})(\mathbb{N})}$ which is equivalent to $1 \notin \overline{a(\mathbb{N})\setminus\{1\}}$.
(ii) $\Longrightarrow$ (i). The condition $1 \notin \overline{a(\mathbb{N})\setminus\{1\}}$ implies that $u := (\mathbf{1} - a)^{-1}\chi_{S(\mathbf{1}-a)}$ belongs to ℓ_∞. In particular, $M_p^u \in \mathcal{L}(ces(p))$. Moreover, $w := (\mathbf{1} - a) \in \ell_\infty$ satisfies (in $\mathcal{L}(ces(p))$) the identity $M_p^w M_p^u = M_p^{\chi_{S(w)}}$. It follows from (2.11) that $M_p^w(ces(p)) \subseteq M_p^{\chi_{S(w)}}(ces(p)) = X_{w,p}$ (see (2.12)). It is routine to verify the reverse inclusion and so actually $M_p^w(ces(p)) = X_{w,p}$. In particular, the range of $M_p^{\mathbf{1}-a} = I - M_p^a$ is *closed* in $ces(p)$. Since $\|a\|_\infty \leq 1$ implies that M_p^a is power bounded (cf. Proposition 2.8), it follows that $\lim_{n\to\infty} \frac{\|(M_p^a)^n\|_{op}}{n} = 0$. Hence, the criterion of Lin can be applied to conclude that M_p^a is uniformly mean ergodic. □

An example of a multiplier operator which is mean ergodic but not uniformly ergodic is M_p^a with $a := (1 - \frac{1}{n})_n$.
In (2.9), with $X := ces(p)$ and $T := M_p^a$ (for $\|a\|_\infty \leq 1$), note that

$$\mathrm{Ker}(I - M_p^a) = \{x \in ces(p) : x_n = 0 \text{ for all } n \in \mathbb{N} \text{ with } a_n \neq 1\}.$$

Concerning the linear dynamics of a continuous linear operator $T : X \longrightarrow X$ defined on a separable, locally convex Hausdorff space X, recall that T is *hypercyclic* if there exists $x \in X$ whose orbit $\{T^n x : n \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}\}$ is dense in X. If, for some $x \in X$, the *projective orbit* $\{\lambda T^n x : \lambda \in \mathbb{C}, n \in \mathbb{N}_0\}$ is dense in X, then T is called *supercyclic*. Since this projective orbit coincides with $\cup_{n=0}^\infty T^n(\mathrm{span}\{x\})$ we see that supercyclic is the same as 1-supercyclic as defined in [4]. Hypercyclicity always implies supercyclicity but not conversely.

Lemma 2.11. *Let $a = (a_n)_n \in \mathbb{C}^\mathbb{N}$ and define the multiplier operator $M^a : \mathbb{C}^\mathbb{N} \longrightarrow \mathbb{C}^\mathbb{N}$ by $M^a(x) := ax$ for $x \in \mathbb{C}^\mathbb{N}$. Then M^a is not supercyclic in the Fréchet space $\mathbb{C}^\mathbb{N}$.*

Proof. The continuous dual space $(\mathbb{C}^\mathbb{N})'$ of $\mathbb{C}^\mathbb{N}$ is the space φ. Clearly M^a is continuous on $\mathbb{C}^\mathbb{N}$ and its dual operator $(M^a)' : \varphi \longrightarrow \varphi$ is given by $(M^a)'(y) = ay$ for $y \in \varphi$. Moreover, it follows from $(M^a)'(e_j) = a_j e_j$ for $j \in \mathbb{N}$ that each canonical basis vector $e_j \in \varphi$ is an eigenvector of $(M^a)'$. According to Theorem 2.1 of [4] the operator $M^a \in \mathcal{L}(\mathbb{C}^\mathbb{N})$ cannot be supercyclic. □

Given $1 < p < \infty$ and $a \in \mathbb{C}^\mathbb{N}$ the multiplier operator $M^a : \mathbb{C}^\mathbb{N} \longrightarrow \mathbb{C}^\mathbb{N}$ maps ℓ_p into ℓ_p if and only if $a \in \ell_\infty$, [3, table 1, p.69]. Denote this restricted operator by $M_{\{p\}}^a : \ell_p \longrightarrow \ell_p$.

Proposition 2.12. *Let $1 < p < \infty$ and $a \in \ell_\infty$.*
(i) *The multiplier operator $M_{\{p\}}^a \in \mathcal{L}(\ell_p)$ is not supercyclic.*
(ii) *The multiplier operator $M_p^a \in \mathcal{L}(ces(p))$ is not supercyclic.*

Proof. (i) Since ℓ_p is dense in $\mathbb{C}^\mathbb{N}$ (as it contains φ) and the natural inclusion $\ell_p \hookrightarrow \mathbb{C}^\mathbb{N}$ is continuous, the supercyclicity of $M_{\{p\}}^a \in \mathcal{L}(\ell_p)$ would imply the supercyclicity of $M^a \in \mathcal{L}(\mathbb{C}^{\mathbb{N}})$, which is not the case (cf. Lemma 2.11). Hence, $M^a_{\{p\}}$ is not supercyclic.

(ii) Since $ces(p)$ is dense in $\mathbb{C}^{\mathbb{N}}$ and the inclusion $ces(p) \hookrightarrow \mathbb{C}^{\mathbb{N}}$ is continuous, the analogous argument to that of part (i) applies. □

3. The Cesàro operators

Consider a pair $1 < p, q < \infty$. Denote by $C_{c(p),c(q)}$ (resp. $C_{c(p),q}$; $C_{p,c(q)}$; $C_{p,q}$) the Cesàro operator C when it acts from $ces(p)$ into $ces(q)$ (resp. $ces(p)$ into ℓ_q; ℓ_p into $ces(q)$; ℓ_p into ℓ_q), whenever this operator exists. The closed graph theorem then ensures that this operator is continuous. We use the analogous notation for the natural inclusion maps $i_{c(p),c(q)}$; $i_{c(p),q}$; $i_{p,c(q)}$; $i_{p,q}$ whenever they exist. The main aim of this section is to identify all pairs p, q for which these inclusion operators and Cesàro operators *do exist* and, for such pairs, to determine whether or not the operator is *compact*. For each $1 < p < \infty$, the spectrum of $C_{p,p} \in \mathcal{L}(\ell_p)$ is well known, [16, Theorem 2], [20, Theorem 4], and coincides with the spectrum of $C_{c(p),c(p)} \in \mathcal{L}(ces(p))$; see (1.6).

We begin with a preliminary result.

Lemma 3.1. *Let $1 < p < \infty$.*

(i) *The operator $C_{c(p),p} : ces(p) \longrightarrow \ell_p$ exists and satisfies $\|C_{c(p),p}\|_{op} \leq 1$.*

(ii) *The largest amongst the class of spaces ℓ_r, for $1 \leq r < \infty$, which satisfy $\ell_r \subseteq ces(p)$ is the space ℓ_p.*

Proof. (i) Follows from the discussion immediately prior to Proposition 1.1.

(ii) See Remark 2.2(iii) of [6]. □

Proposition 3.2. *Let $1 < p, q < \infty$ be an arbitrary pair.*

(i) *The inclusion map $i_{p,q} : \ell_p \longrightarrow \ell_q$ exists if and only if $p \leq q$, in which case $\|i_{p,q}\|_{op} = 1$.*

(ii) *The inclusion map $i_{p,c(q)} : \ell_p \longrightarrow ces(q)$ exists if and only if $p \leq q$, in which case $\|i_{p,c(q)}\|_{op} \leq q'$.*

(iii) *The inclusion map $i_{c(p),c(q)} : ces(p) \longrightarrow ces(q)$ exists if and only if $p \leq q$, in which case $\|i_{c(p),c(q)}\|_{op} \leq 1$.*

(iv) *$ces(p) \not\subseteq \ell_q$ for all choices of $1 < p, q < \infty$.*

Proof. (i) This is well known.

(ii) Lemma 3.1(ii) shows that $\ell_p \not\subseteq ces(q)$ if $p > q$.

Let $p \leq q$. For $x \in \ell_p$ we have $\|i_{p,c(q)}(x)\|_{ces(q)} = \|x\|_{ces(q)}$ with

$$\|x\|_{ces(q)} := \|C(|x|)\|_q \leq \|C_{q,q}\|_{op}\|x\|_q \leq \|C_{q,q}\|_{op}\|x\|_p,$$

where the last inequality follows via part (i). Since $\|C_{q,q}\|_{op} = p'$, [13, Theorem 326], the desired conclusion is clear.

(iii) If $p > q$, then $ces(p) \not\subseteq ces(q)$. Indeed, by Lemma 3.1(ii) there exists $y \in \ell_p$ with $y \notin ces(q)$. By part (ii), $y \in ces(p)$.

Let $p \leq q$. Fix $x \in ces(p)$. By Lemma 3.1(i) we have $C(|x|) \in \ell_p$ and hence, by part (i), $C(|x|) \in \ell_q$. Accordingly,

$$\|x\|_{ces(q)} := \|C(|x|)\|_q \leq \|C(|x|)\|_p = \|x\|_{ces(p)}.$$

This shows that $i_{c(p),c(q)}$ exists and $\|i_{c(p),c(q)}\|_{op} \leq 1$.

(iv) For arbitrary $1 < p < \infty$ there exists $x \in ces(p)$ with $x \notin \ell_\infty$, [6, Remark 2.2(ii)]. Then also $x \notin \ell_q$ for every $1 < q < \infty$. □

If $1 < p < q < \infty$, then the inclusion $ces(p) \subseteq ces(q)$ as guaranteed by Proposition 3.2(iii) is actually *proper*. Indeed, by Lemma 3.1(ii) there exists $x \in \ell_q$ with $x \notin ces(p)$. Then $y := C(|x|) \in ces(q)$; see Proposition 3.2(ii). But, $x \notin ces(p)$ implies $|x| \notin ces(p)$ and so $y \notin ces(p)$; see Proposition 1.1. That $ces(p) \subsetneqq ces(q)$ also follows from the next result.

Proposition 3.3. *Let $1 < p, q < \infty$ with $p \neq q$. Then $ces(p)$ is not Banach space isomorphic to $ces(q)$.*

Proof. According to (1.3) the closed (sectional) subspace

$$Y := \{x \in ces(p) : x_k = 0 \;\text{ unless }\; k = 2^j \;\text{ for some }\; j = 0, 1, 2, \dots\}$$

is isomorphic to a weighted ℓ_p-space (as $\|x\|_{[p]} = (\sum_{j=0}^{\infty} 2^{j(1-p)} |x_{2^j}|^p)^{1/p}$ for $x \in Y$) and hence, also isomorphic to ℓ_p. Suppose that $ces(p)$ is isomorphic to $ces(q)$. Then ℓ_p is isomorphic to a closed subspace of $ces(q)$. Since $ces(q)$ is isomorphic to a closed subspace of the infinite ℓ_q-sum $\ell_q(E_n)$ with each $E_n, n \in \mathbb{N}$, a finite dimensional space, [21, Theorem 1], it follows that ℓ_p is isomorphic to a closed subspace of $\ell_q(E_n)$. But, $X := \ell_p$ has a shrinking basis (it is reflexive) and so is isomorphic to $\ell_q(D_k)$ with each $D_k, k \in \mathbb{N}$, a finite dimensional space, [18, Theorem 2.d.1]. Since ℓ_q is clearly isomorphic to a closed (sectional) subspace of $\ell_q(D_k)$, it follows that ℓ_q is isomorphic to a closed subspace of ℓ_p with $p \neq q$, which is *not* the case, [18, p.54]. So, $ces(p)$ is not isomorphic to $ces(q)$. □

Via Proposition 3.2 we now determine which inclusion maps are compact.

Proposition 3.4. *Let $1 < p \leq q < \infty$ be arbitrary.*

(i) *The inclusion $i_{p,q} : \ell_p \longrightarrow \ell_q$ is never compact.*

(ii) *The inclusion $i_{c(p),c(q)} : ces(p) \longrightarrow ces(q)$ is compact if and only if $p < q$.*

(iii) *The inclusion $i_{p,c(q)} : \ell_p \longrightarrow ces(q)$ is compact if and only if $p < q$.*

Proof. (i) The image under $i_{p,q}$ of the unit basis vectors $\{e_n : n \in \mathbb{N}\} \subseteq \ell_p$ has no Cauchy subsequence (hence, no convergent subsequence) in ℓ_q because $\|e_n - e_m\|_q = 2^{1/q}$ for all $n \neq m$.

(ii) Since $i_{c(p),c(p)}$ is the identity operator on $ces(p)$ it is surely not compact. So, assume that $p < q$. Then the constant element $a := \mathbf{1}$ satisfies $(a_n n^{\frac{1}{q} - \frac{1}{p}})_n = (n^{\frac{1}{q} - \frac{1}{p}})_n \in c_0$ and hence, by Proposition 2.2 the multiplier operator $M_{p,q}^{\mathbf{1}} \in \mathcal{L}(ces(p), ces(q))$ is compact. But, $M_{p,q}^{\mathbf{1}}$ is precisely the inclusion operator $i_{c(p),c(q)}$.

(iii) Since $C_{p,p}$ is not compact (by (1.6) its spectrum is an uncountable set) and $C_{p,p} = C_{c(p),p}\, i_{p,c(p)}$, also $i_{p,c(p)}$ fails to be compact. So, assume that $p < q$. Then the factorization $i_{p,c(q)} = i_{c(p),c(q)}\, i_{p,c(p)}$ together with the compactness of $i_{c(p),c(q)}$ (see part (ii)) shows that $i_{p,c(q)}$ is compact. □

Now that the continuity and compactness of the various inclusion operators are completely determined we can do the same for the Cesàro operators $C : X \longrightarrow Y$ where $X, Y \in \{\ell_p, ces(q) : p, q \in (1, \infty)\}$. We begin with continuity.

Proposition 3.5. *Let $1 < p, q < \infty$ be an arbitrary pair.*

(i) *$C_{p,q} : \ell_p \longrightarrow \ell_q$ exists if and only if $p \leq q$, in which case $\|C_{p,q}\|_{op} \leq p'$.*

(ii) $C_{p,c(q)} : \ell_p \longrightarrow ces(q)$ *exists if and only if* $p \leq q$, *in which case* $\|C_{p,c(q)}\|_{op} \leq p'q'$.

(iii) $C_{c(p),c(q)} : ces(p) \longrightarrow ces(q)$ *exists if and only if* $p \leq q$, *in which case* $\|C_{c(p),c(q)}\|_{op} \leq q'$.

(iv) $C_{c(p),q} : ces(p) \longrightarrow \ell_q$ *exists if and only if* $p \leq q$, *in which case* $\|C_{c(p),q}\|_{op} \leq 1$.

Proof. (ii) Let $p > q$. According to Lemma 3.1(ii) there exists $x \in \ell_p \setminus ces(q)$, in which case also $|x| \in \ell_p \setminus ces(q)$. *If* $C(|x|) \in ces(q)$, then Proposition 1.1 implies that also $|x| \in ces(q)$; contradiction. So, $|x| \in \ell_p$ but $C(|x|) \notin ces(q)$, i.e., "$C_{p,c(q)}$" does not exist.

Suppose then that $p \leq q$. Then $C_{p,p} \in \mathcal{L}(\ell_p)$ exists with $\|C_{p,p}\|_{op} = p'$ and $i_{p,c(q)} : \ell_p \longrightarrow ces(q)$ exists with $\|i_{p,c(q)}\|_{op} \leq q'$ (cf. Proposition 3.2(ii)). Hence, the composition $C_{p,c(q)} = i_{p,c(q)}\, C_{p,p}$ exists and $\|C_{p,c(q)}\|_{op} \leq p'q'$.

(i) Let $p > q$. If $C_{p,q}$ exists, then by Proposition 3.2(ii) $C_{p,c(q)} = i_{q,c(q)}\, C_{p,q}$ also exists. This contradicts part (ii) which was just proved.

So, assume that $p \leq q$. Then $C_{p,p} \in \mathcal{L}(\ell_p)$ exists with $\|C_{p,p}\|_{op} = p'$ and $i_{p,q}$ exists with $\|i_{p,q}\|_{op} = 1$ (cf. Proposition 3.2(i)). Hence, $C_{p,q} = i_{p,q}\, C_{p,p}$ exists and $\|C_{p,q}\|_{op} \leq p'$.

(iii) Let $p > q$. If $C_{c(p),c(q)}$ exists, then by Proposition 3.2(i) also $C_{p,c(q)} = C_{c(p),c(q)}\, i_{p,c(p)}$ exists. This contradicts part (ii) above.

So, assume that $p \leq q$. Fix $x \in ces(p)$. Then also $|x| \in ces(p)$ and so $C(|x|) \in \ell_p \subseteq \ell_q$; see Lemma 3.1(i) and Proposition 3.2(i). Moreover, $|C(x)| \in \ell_q$ as $|C(x)| \leq C(|x|)$. Hence,

$$\begin{aligned}\|C(x)\|_{ces(q)} &:= \|C(|C(x)|)\|_q \leq \|C_{q,q}\|_{op} \||C(x)|\|_q \leq q'\|C(|x|)\|_q \\ &\leq q'\|C(|x|)\|_p = q'\|x\|_{ces(p)}.\end{aligned}$$

This shows that $C_{c(p),c(q)}$ exists and $\|C_{c(p),c(q)}\|_{op} \leq q'$.

(iv) Let $p > q$. If $C_{c(p),q}$ exists, then also $C_{c(p),c(q)} = i_{q,c(q)}\, C_{c(p),q}$ exists (cf. Proposition 3.2(ii)). This contradicts part (iii).

Assume now that $p \leq q$. Since $C_{c(p),p}$ exists with $\|C_{c(p),p}\|_{op} \leq 1$ (cf. Lemma 3.1(i)) and $i_{p,q}$ exists with $\|i_{p,q}\|_{op} = 1$ (cf. Proposition 3.2(i)), it follows that the composition $C_{c(p),q} = i_{p,q}\, C_{c(p),p}$ exists and $\|C_{c(p),q}\|_{op} \leq 1$. □

Concerning the proof of part (iii) of Proposition 3.5 when $p \leq q$, it is also clear from $C_{c(p),c(q)} = i_{c(p),c(q)}\, C_{c(p),c(p)}$ that $C_{c(p),c(q)}$ exists. However, since $\|i_{c(p),c(q)}\|_{op} \leq 1$ (cf. Proposition 3.2(iii)) and $\|C_{c(p),c(p)}\|_{op} = p'$, this approach only yields $\|C_{c(p),c(q)}\|_{op} \leq p'$ whereas the given proof of (iii) yields $\|C_{c(p),c(q)}\|_{op} \leq q'$ which is a better estimate when $p < q$.

We now have all the facts needed to prove the main result of this section.

Proposition 3.6. *Let* $1 < p \leq q < \infty$ *be arbitrary.*

(i) *The Cesàro operator* $C_{p,q} : \ell_p \longrightarrow \ell_q$ *is compact if and only if* $p < q$.

(ii) *The Cesàro operator* $C_{p,c(q)} : \ell_p \longrightarrow ces(q)$ *is compact if and only if* $p < q$.

(iii) *The Cesàro operator* $C_{c(p),c(q)} : ces(p) \longrightarrow ces(q)$ *is compact if and only if* $p < q$.

(iv) *The Cesàro operator* $C_{c(p),q} : ces(p) \longrightarrow \ell_q$ *is compact if and only if* $p < q$.

Proof. (i) Since $\sigma(C_{p,p})$ is an uncountable set (see the comments prior to Lemma 3.1), it is clear that $C_{p,p}$ is not compact. So, assume that $p < q$. Since $C_{p,q} = C_{c(q),q}\, i_{p,c(q)}$ with $C_{c(q),q} : ces(q) \longrightarrow \ell_q$ continuous (cf. Lemma 3.1(i)) and $i_{p,c(q)} : \ell_p \longrightarrow ces(q)$ compact (by Proposition 3.4(iii)), it follows that $C_{p,q}$ is compact.

(ii) For $p = q$ observe that $(C_{c(p),c(p)})^2 = C_{p,c(p)}\, C_{c(p),p}$. By (1.6) and the spectral mapping theorem, [10, Ch. VII, Theorem 3.11], we see that

$$\sigma((C_{c(p),c(p)})^2) = \{\lambda^2 : |\lambda - \tfrac{p'}{2}| \leq \tfrac{p'}{2}\}$$

is an uncountable set and so $(C_{c(p),c(p)})^2$ is not compact. Hence, also $C_{p,c(p)}$ is not compact.

Assume then that $p < q$. Since the inclusion $i_{c(p),c(q)} : ces(p) \longrightarrow ces(q)$ is compact (cf. Proposition 3.4(ii)), it is clear from the factorization $C_{p,c(q)} = i_{c(p),c(q)}\, C_{p,c(p)}$ that also $C_{p,c(q)}$ is compact.

(iii) For $p = q$ it follows from (1.6) that $\sigma(C_{c(p),c(p)})$ is an uncountable set and so $C_{c(p),c(p)}$ is not compact. Suppose now that $p < q$. Since the inclusion $i_{c(p),c(q)} : ces(p) \longrightarrow ces(q)$ is compact (by Proposition 3.4(ii)), the factorization $C_{c(p),c(q)} = i_{c(p),c(q)}\, C_{c(p),c(p)}$ shows that $C_{c(p),c(q)}$ is compact.

(iv) For $p = q$ we have $C_{c(p),c(p)} = i_{p,c(p)}\, C_{c(p),p}$. By part (iii) the operator $C_{c(p),c(p)}$ is not compact and hence, also $C_{c(p),p}$ is not compact.

Assume now that $p < q$. Select any r satisfying $p < r < q$, in which case we have $C_{c(p),q} = C_{c(r),q}\, i_{c(p),c(r)}$ with $C_{c(r),q}$ continuous (by Proposition 3.5(iv)) and $i_{c(p),c(r)}$ compact (via Proposition 3.4(ii)). Hence, also $C_{c(p),q}$ is compact. □

Our final result concerns the mean ergodicity and linear dynamics of Cesàro operators.

Proposition 3.7. *Let $1 < p < \infty$.*

(i) *The Cesàro operator $C_{p,p} : \ell_p \longrightarrow \ell_p$ is not power bounded, not mean ergodic and not supercyclic.*
(ii) *The Cesàro operator $C_{c(p),c(p)} : ces(p) \longrightarrow ces(p)$ is not power bounded, not mean ergodic and not supercyclic.*

Proof. (i) That $C_{p,p}$ is neither power bounded nor mean ergodic is Proposition 4.2 of [1]. It is known that the Cesàro operator $C : \mathbb{C}^{\mathbb{N}} \longrightarrow \mathbb{C}^{\mathbb{N}}$ is not supercyclic, [2, Proposition 4.3]. Since ℓ_p is dense in $\mathbb{C}^{\mathbb{N}}$ and the natural inclusion $\ell_p \subseteq \mathbb{C}^{\mathbb{N}}$ is continuous, the supercyclicity of $C_{p,p}$ in ℓ_p would imply that $C : \mathbb{C}^{\mathbb{N}} \longrightarrow \mathbb{C}^{\mathbb{N}}$ is supercyclic. Hence, $C_{p,p} \in \mathcal{L}(\ell_p)$ is not supercyclic.

(ii) Suppose that $C_{c(p),c(p)}$ is mean ergodic. According to (2.10) we have $\lim_{n\to\infty} \frac{(C_{c(p),c(p)})^n}{n} = 0$ for τ_s in $\mathcal{L}(ces(p))$ and hence, $\sigma(C_{c(p),c(p)}) \subseteq \overline{\mathbb{D}}$, [10, Ch. VIII, Lemma 8.1]. This contradicts (1.6). Hence, $C_{c(p),c(p)}$ cannot be mean ergodic. Since power bounded operators in reflexive Banach spaces are always mean ergodic, [19], it follows that $C_{c(p),c(p)}$ is not power bounded. Arguing as in part (i), since $ces(p)$ is dense in $\mathbb{C}^{\mathbb{N}}$ and the inclusion $ces(p) \subseteq \mathbb{C}^{\mathbb{N}}$ is continuous, it follows that $C_{c(p),c(p)}$ is not supercyclic. □

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