MULTIPLIER AND AVERAGING OPERATORS IN THE
BANACH SPACES $\text{ces}(p)$, $1 < p < \infty$

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Abstract. The Banach sequence spaces $\text{ces}(p)$ are generated in a specified way via the classical spaces $\ell_p$, $1 < p < \infty$. For each pair $1 < p, q < \infty$ the $(p,q)$-multiplier operators from $\text{ces}(p)$ into $\text{ces}(q)$ are known. We determine precisely which of these multipliers is a compact operator. Moreover, for the case of $p = q$ a complete description is presented of those $(p,p)$-multiplier operators which are mean (resp. uniform mean) ergodic. A study is also made of the linear operator $C$ which maps a numerical sequence to the sequence of its averages. All pairs $1 < p, q < \infty$ are identified for which $C$ maps $\text{ces}(p)$ into $\text{ces}(q)$ and, amongst this collection, those which are compact. For $p = q$, the mean ergodic properties of $C$ are also treated.

1. Introduction.

For each element $x = (x_n)_n = (x_1,x_2,\ldots)$ of $\mathbb{C}^\mathbb{N}$ let $|x| := (|x_n|)_n$ and write $x \geq 0$ if $x = |x|$. Of course, $x \leq y$ means that $(y - x) \geq 0$. The Cesàro operator $C : \mathbb{C}^\mathbb{N} \to \mathbb{C}^\mathbb{N}$, defined by

$$C(x) := (x_1, \frac{x_1 + x_2}{2}, \frac{x_1 + x_2 + x_3}{3}, \ldots), \quad x \in \mathbb{C}^\mathbb{N},$$

satisfies $|C(x)| \leq C(|x|)$ for $x \in \mathbb{C}^\mathbb{N}$ and is a vector space isomorphism of $\mathbb{C}^\mathbb{N}$ onto itself. It is also a topological isomorphism when $\mathbb{C}^\mathbb{N}$ is considered as a (locally convex) Fréchet space with respect to the coordinatewise convergence. For each $1 < p < \infty$ define

$$\text{ces}(p) := \left\{ x \in \mathbb{C}^\mathbb{N} : \|x\|_{\text{ces}(p)} := \| \frac{1}{n} \sum_{k=1}^{n} |x_k| |n\|_p = \|C(|x|)\|_p < \infty \right\}, \quad (1.1)$$

where $\| \cdot \|_p$ denotes the standard norm in $\ell_p$. An intensive study of the Banach spaces $\text{ces}(p)$, $1 < p < \infty$, was undertaken in [3]; see also the references therein. In particular, they are reflexive, $\rho$-concave Banach lattices (for the order induced by $\mathbb{C}^\mathbb{N}$) and the canonical vectors $e_k := (\delta_{nk})_n$, for $k \in \mathbb{N}$, form an unconditional basis, [3], [6]. For any pair $1 < p, q < \infty$ the space $\text{ces}(p)$ is known not to be isomorphic to $\ell_q$, [3, Proposition 15.13]. It is shown in Proposition 3.3 (for all $p \neq q$) that $\text{ces}(p)$ is also not isomorphic to $\text{ces}(q)$. It is important to note that the inequality

$$\frac{A_p}{k^{1/p'}} \leq \|e_k\|_{\text{ces}(p)} \leq \frac{B_p}{k^{1/p'}}, \quad k \in \mathbb{N}, \quad (1.2)$$

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is valid for strictly positive constants \( A_p, B_p \) and with \( \frac{1}{p} + \frac{1}{p'} = 1 \), [3, Lemma 4.7]. It is known, [3, p.26], that \( ces(p) = cop(p) \) with equivalent norms, where

\[
cop(p) := \left\{ x \in \mathbb{C}^N : \|x\|_{cop(p)} := \left\| \left( \sum_{k=n}^{\infty} |x_k| \right)_n \right\|_p < \infty \right\}, \quad 1 < p < \infty.
\]

The dual Banach spaces \( (ces(p))', 1 < p < \infty, \) are described in Section 12 of [3]. Yet another equivalent norm in \( ces(p) \), via the dyadic decomposition of \( \mathbb{N} \), is available, [11, Theorem 4.1]. Namely, \( x \in \mathbb{C}^N \) belongs to \( ces(p) \) if and only if

\[
\|x\|_p := \left( \sum_{j=0}^{\infty} 2^{j(1-p)} \left( \sum_{k=2^j}^{2^{j+1}-1} |x_k| \right)^p \right)^{1/p} < \infty. \tag{1.3}
\]

The spaces \( ces(p), 1 < p < \infty, \) also arise in a very different way. Fix \( 1 < p < \infty \). Since the Cesàro operator \( C_{p,p} : \ell_p \to \ell_p \), i.e., \( C \) restricted to \( \ell_p \), is a positive operator between Banach lattices, it is natural to look for continuous \( \ell_p \)-valued extensions of \( C_{p,p} \) to Banach lattices \( X \subseteq \mathbb{C}^N \) which are larger than \( \ell_p \) and solid (i.e., \( y \in \mathbb{C}^N \) and \( |y| \leq |x| \) with \( x \in X \) implies that \( y \in X \)). The largest of all those solid Banach lattices in \( \mathbb{C}^N \) for which such a continuous, \( \ell_p \)-valued extension of \( C_{p,p} : \ell_p \to \ell_p \) is possible is precisely \( ces(p) \), [6, p.62]. Of course, this "largest extension" \( C_{c(p),p} : ces(p) \to \ell_p \) is the restriction of \( C \) from \( \mathbb{C}^N \) to \( ces(p) \). Somewhat surprisingly, \( C_{c(p),p} \) also possesses an integral representation. That is, \( ces(p) \) coincides with the \( L^1 \)-space of an \( \ell_p \)-valued vector measure \( m_p \), and \( C_{c(p),p} \) is given by

\[
C_{c(p),p}(x) = \int_{\mathbb{N}} x(n) \, dm_p(n), \quad x \in L^1(m_p) = ces(p).
\]

Here \( m_p : \mathcal{R} \to \ell_p \) is the \( \sigma \)-additive vector measure defined on the \( \delta \)-ring \( \mathcal{R} \) of all finite subsets of \( \mathbb{N} \) by

\[
m_p(A) := C_{p,p}(\chi_A), \quad A \in \mathcal{R}, \tag{1.4}
\]

where \( \chi_A : \mathbb{N} \to \mathbb{C} \) is the element of \( \mathbb{C}^N \) given by \( \chi_A = \sum_{k \in A} e_k \) for each \( A \subseteq \mathbb{N} \). This claim certainly requires a proof. First, the space \( L^1(m_p) \) of all \( m_p \)-integrable functions on \( \mathbb{N} \), as defined in [8], [9], is the optimal domain for the operator \( C_{p,p} \) (in the sense of [9, Corollaries 2.4 and 2.6]) within the class of all Banach function spaces (briefly, B.f.s) over \( (\mathbb{N}, \mathcal{R}, \mu) \) which have absolutely continuous norm (briefly, a.c.); here \( \mu \) denotes counting measure. More precisely, \( L^1(m_p) \subseteq \mathbb{C}^N \) contains the domain space \( \ell_p \) of \( C_{p,p} \), the integration map \( I_{m_p} : L^1(m_p) \to \ell_p \) (given by \( x \to \int_{\mathbb{N}} x \, dm_p \) for \( x \in L^1(m_p) \)) satisfies \( I_{m_p}(x) = C_{p,p}(x) \) for each \( x \in \ell_p \subseteq L^1(m_p) \), and \( L^1(m_p) \) is the largest of all B.f.s \( \ell_p \) over \( (\mathbb{N}, \mathcal{R}, \mu) \) having a.c.-norm to which \( C_{p,p} \) can be extended while still maintaining its values in \( \ell_p \). To verify this, we observe that an equivalent norm in \( L^1(m_p) \) is given by

\[
\|x\|_{L^1(m_p)} := \sup \left\{ \left. \left\| \int_A x \, dm_p \right\|_p : A \in \mathcal{R} \right\}, \quad x \in L^1(m_p);
\]

see (3) on p.434 of [8]. But, for \( x \in L^1(m_p) \) and each \( A \in \mathcal{R} \), the function \( x\chi_A \) is a \( \mathcal{R} \)-simple function and so it follows from (1.4) that \( \int_A x \, dm_p = C_{p,p}(x\chi_A) \).
Now, for \( x \in ces(\ell) \) fixed, note that

\[
\left\| \int_A x \, dm_p \right\|_p = \left\| C_p,\ell(x \chi_A) \right\|_p = \left\| C_{(p),\ell}(x \chi_A) \right\|_p \leq \left\| C_{(p),\ell}(|x|) \right\|_p = \| x \|_{ces(\ell)} < \infty
\]

for every \( A \in \mathcal{R} \). If we define \( \int_A x \, dm_p := C_{(p),\ell}(x \chi_A) \in \ell_p \) for an arbitrary subset \( A \subseteq \mathbb{N} \), then \( x \) is \( m_p \)-integrable in the sense of [8, p.434], [9, p.133], with \( \| x \|_{L^p(m_p)} \leq \| x \|_{ces(\ell)} \). Since \( ces(p) \) itself is a B.f.s. over \((\mathbb{N}, \mathcal{R}, \mu)\) having an a.c.-norm and containing \( \ell_p \), we can conclude from the optimality of \( L^1(m_p) \) that \( ces(p) \subseteq L^1(m_p) \) with a continuous inclusion. On the other hand, recall that \( ces(p) \) is the largest solid Banach lattice in \( \mathbb{C}^\mathbb{N} \) which contains \( \ell_p \) and \( C \) maps into \( \ell_p \). But, the B.f.s. \( L^1(m_p) \) is such a solid Banach lattice which \( C \) maps into \( \ell_p \). Indeed, since \( L^1(m_p) \subseteq \mathbb{C}^\mathbb{N} \) with \( \ell_p \) dense in \( L^1(m_p) \) (as \( \ell_p \) contains the \( \mathcal{R} \)-simple functions which are known to be dense in \( L^1(m_p) \); [8, p.434]) and \( C \) acts in all of \( \mathbb{C}^\mathbb{N} \), it follows from the fact that norm convergence of a sequence in \( L^1(m_p) \) implies the pointwise convergence \( \mu \)-a.e. of a subsequence, [9, p.134] (in this case meaning coordinatewise convergence in \( \mathbb{C}^\mathbb{N} \)), that the extended operator \( I_{m_p} \) is necessarily given by \( I_{m_p}(x) = C(x) \) for all \( x \in L^1(m_p) \). Accordingly \( L^1(m_p) \subseteq ces(p) \) and hence, \( L^1(m_p) = ces(p) \) with equivalence of the norms \( \| \cdot \|_{L^1(m_p)} \) and \( \| \cdot \|_{ces(p)} \).

It is an important feature that \( m_p \) cannot be extended to a more traditional \( \sigma \)-additive, \( \ell_p \)-valued vector measure defined on the \( \sigma \)-algebra \( 2^{\mathbb{N}} \) generated by \( \mathcal{R} \). This is because its range \( m_p(\mathcal{R}) \) is an unbounded subset of \( \ell_p \). Indeed, for \( (A_n := \{1, 2, ..., N\}) \in \mathcal{R} \) we have \( m_p(A_N) = \sum_{j=1}^N e_j + N \sum_{j=N+1}^\infty \frac{1}{2} e_j \) and hence, \( \| m_p(A_N) \|_{\ell_p} \geq N^{1/p} \) for all \( N \in \mathbb{N} \).

Having presented several equivalent and varied descriptions of the spaces \( ces(p) \), \( 1 < p < \infty \), we now formulate the aim of this note, namely to make a detailed analysis of certain linear operators defined on these spaces. Let us be more precise.

Given a pair \( 1 < p, q < \infty \), an element \( a \in \mathbb{C}^\mathbb{N} \) is called a \( (p, q) \)-multiplier if it multiplies \( ces(p) \) into \( ces(q) \), that is, if \( ax \in ces(q) \) for every \( x \in ces(p) \), where the product \( ax := (a_n x_n)_n \) is defined coordinatewise. The closed graph theorem ensures that the corresponding linear \( (p, q) \)-multiplier operator \( M^a_{pq} : x \mapsto ax \) is then necessarily continuous from \( ces(p) \) into \( ces(q) \). If \( p = q \), then we denote \( M^a_{pp} \) simply by \( M^a_p \) and note that \( M^a_p \) is the diagonal operator acting in \( ces(p) \) via the matrix having the scalars \( \{a_n : n \in \mathbb{N}\} \) in its diagonal. The vector space of all \( (p, q) \)-multipliers, denoted by \( \mathcal{M}_{pq} \) (or \( \mathcal{M}_p \) if \( p = q \)), has been completely determined by G. Bennett; see [3, pp.69-70], after recalling that \( cop(p) = ces(p) \) for all \( 1 < p < \infty \).

In Section 2 we investigate various properties of the multiplier operators \( M^a_{pq} \) for all pairs \( 1 < p, q < \infty \) and \( a \in \mathcal{M}_{pq} \). For instance, those multipliers \( a \in \mathcal{M}_{pq} \) for which \( M^a_{pq} \) is a compact operator are characterized; see Propositions 2.2 and 2.5. Also, given \( a \in \mathcal{M}_p = \ell_\infty \) it is shown that the spectrum of \( M^a_p \) is the set

\[ \sigma(M^a_p) = \overline{\{a(n) : n \in \mathbb{N}\}}, \quad 1 < p < \infty, \]

where \( a(\mathbb{N}) := \{a_n : n \in \mathbb{N}\} \subseteq \mathbb{C} \), and that \( \| M^a_p \|_{op} = \| a \|_\infty \) with \( \| \cdot \|_{op} \) denoting the operator norm of \( M^a_p : ces(p) \to ces(p) \); see Lemma 2.6 and Proposition 2.7. Furthermore, those \( a \in \mathcal{M}_p \) are identified for which the operator \( M^a_p \) is mean
ergodic (cf. Proposition 2.8) as well as those for which \( M_p^a \) is uniformly mean ergodic (cf. Proposition 2.10).

It is clear from (1.1) and the discussions above that the Cesàro operator \( C \) is intimately connected to the Banach spaces \( ces(p) \), \( 1 < p < \infty \). Indeed, Hardy's classical inequality states, for \( 1 < p < \infty \), that

\[
\sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} b_k \right)^p \leq K_p \sum_{n=1}^{\infty} b_n^p
\]

for all choices of non-negative numbers \( \{b_n\}_{n=1}^{\infty} \) and some constant \( K_p > 0 \). Setting \( b_n := |x_n| \), for \( n \in \mathbb{N} \) and each \( x \in \ell_p \), it is immediate that \( ||C_p(|x|)||_p \leq K_p^{1/p} ||x||_p \), that is, \( \ell_p \subseteq ces(p) \) with a continuous inclusion; Remark 2.2 of [6] shows that this containment is strict. Moreover, the Cesàro operator \( C_{(p),p} : ces(p) \rightarrow \ell_p \) is continuous; this was already implicitly used above. To see this fix \( x \in ces(p) \). Using the fact that \( || \cdot ||_p \) is a Banach lattice norm yields

\[
||C_{(p),p}(x)||_p = ||C(x)||_p \leq ||C(|x|)||_p = ||x||_{ces(p)}.
\]

The connection between \( C \) and \( ces(p) \) is further exemplified by the following remarkable result of Bennett, [3, Theorem 20.31].

**Proposition 1.1.** Let \( 1 < p < \infty \) and \( x \in \mathbb{C}^N \). Then

\[
x \in ces(p) \quad \text{if and only if} \quad C(|x|) \in ces(p).
\]

Further examples of Banach spaces \( X \subseteq \mathbb{C}^N \) such that \( C(X) \subseteq X \) and for which Proposition 1.1 is valid (with \( X \) in place of \( ces(p) \)) are identified in [5], [6], [7].

In Section 3 it is shown that \( C \) maps \( ces(p) \) into \( ces(q) \), necessarily continuously, if and only if \( 1 < p \leq q < \infty \); see Proposition 3.5. Furthermore, all pairs \( 1 < p,q < \infty \) are identified for which \( C \) maps \( \ell_p \) into \( ces(q) \) and for which \( C \) maps \( ces(p) \) into \( \ell_q \), as well as the subclass of these continuous operators which are actually compact. Two important facts in this regard are that the Cesàro operator \( C_{(p),c(p)} : ces(p) \rightarrow ces(p) \) has spectrum

\[
\sigma(C_{(p),c(p)}) = \{ \lambda \in \mathbb{C} : |\lambda - \frac{p}{2}| \leq \frac{p}{2} \}, \quad 1 < p < \infty,
\]

[6, Theorem 5.1], and that the natural inclusion map \( ces(p) \hookrightarrow ces(q) \) is compact whenever \( 1 < p < q \); see Proposition 3.4. A consequence of (1.6) is that \( C_{(p),c(p)} \) and \( C_{p,p} \) are never mean ergodic.

2. Multiplier operators from \( ces(p) \) into \( ces(q) \).

According to Table 16 on p.69 of [3], given \( 1 < p \leq q < \infty \) an element \( a = (a_n)_n \in \mathbb{C}^N \) belongs to \( M_{p,q} \) if and only if the element \( (a_n, n^{\frac{1}{q} - \frac{1}{p}})_n \in \ell_\infty \). Observe that \( (\frac{1}{q} - \frac{1}{p}) \leq 0 \). In particular, \( \ell_\infty \subseteq M_{p,q} \) and, if \( p = q \), then \( M_p = \ell_\infty \). For fixed \( a \in \ell_\infty \), via the inequality \( C(|au|) \leq ||u||_\infty C(|u|) \), for \( u \in \mathbb{C}^N \), that \( ||M_p^a(x)||_{ces(p)} = ||C(|ax|)||_p \leq ||a||_\infty ||C(|x|)||_p = ||a||_\infty ||x||_{ces(p)} \), for all \( x \in ces(p) \). Hence, \( M_p^a : ces(p) \rightarrow ces(p) \) satisfies

\[
||M_p^a||_{op} \leq ||a||_\infty, \quad a \in \ell_\infty, \quad 1 < p < \infty.
\]

Here \( || \cdot ||_{op} \) denotes the operator norm. We begin with a result which is probably known; due to the lack of a reference we include a proof. Let \( \varphi \) be the
vector subspace of $\mathbb{C}^N$ consisting of all elements with only finitely many non-zero coordinates.

**Lemma 2.1.** Let $T : \mathbb{C}^N \to \mathbb{C}^N$ be a continuous linear operator and $X, Y$ be a Banach sequence spaces satisfying $\varphi \subseteq X \subseteq \mathbb{C}^N$ and $\varphi \subseteq Y \subseteq \mathbb{C}^N$ with continuous inclusions such that $T(X) \subseteq Y$. Then the restriction $T : X \to Y$ is a compact operator if and only if it satisfies the following property (K), namely:

(K) If a norm bounded sequence $\{x_m\}_{m=1}^\infty \subseteq X$ satisfies $\lim_{m \to \infty} x_m = 0$ in the Fréchet space $\mathbb{C}^N$, then $\lim_{m \to \infty} T(x_m) = 0$ in the Banach space $Y$.

**Proof.** By the closed graph theorem $T : X \to Y$ is continuous.

Suppose first that $T : X \to Y$ is compact. Let $\{x_m\}_{m=1}^\infty \subseteq X$ be any sequence in $X$ satisfying $\lim_{m \to \infty} x_m = 0$ in $\mathbb{C}^N$. Assume that the sequence $\{T(x_m)\}_{m=1}^\infty$ does not converge to 0 in $Y$. Select a subsequence $\{x_{m_k}\}_{k=1}^\infty$ of $\{x_m\}_{m=1}^\infty$ and $r > 0$ such that

$$\|T(x_{m_k})\|_Y \geq r, \quad k \in \mathbb{N}. \quad (2.2)$$

By compactness of $T$ there exists $y \in Y$ and a subsequence $\{x_{m_{k(i)}}\}_{i=1}^\infty$ of $\{x_{m_k}\}_{k=1}^\infty$ such that $\lim_{i \to \infty} \|T(x_{m_{k(i)}}) - y\|_Y = 0$. Continuity of the inclusion $Y \subseteq \mathbb{C}^N$ implies that also $\lim_{i \to \infty} T(x_{m_{k(i)}}) = y$ in $\mathbb{C}^N$. But, $\lim_{m \to \infty} x_{m_k} = 0$ in $\mathbb{C}^N$ and $T : \mathbb{C}^N \to \mathbb{C}^N$ is continuous. Accordingly, $\lim_{i \to \infty} T(x_{m_{k(i)}}) = 0$ in $\mathbb{C}^N$ and so $y = 0$; contradiction to (2.2). Hence, necessarily $T(x_m) \to 0$ in $Y$ for $m \to \infty$.

This establishes that $T$ has property (K).

Conversely, suppose that $T$ has property (K). Let $\{x_i\}_{i=1}^\infty$ be any bounded sequence in $X$. To show that $T$ is compact we need to argue that $\{T(x_i)\}_{i=1}^\infty$ has a convergent subsequence in $Y$. Since the inclusion $X \subseteq \mathbb{C}^N$ is continuous, the sequence $\{x_i\}_{i=1}^\infty$ is also bounded in the Fréchet-Montel space $\mathbb{C}^N$. Hence, there is a subsequence $u_j := x_{i_j}$, for $j \in \mathbb{N}$, of $\{x_i\}_{i=1}^\infty$ and $x \in \mathbb{C}^N$ such that $\lim_{j \to \infty} u_j = x$ in $\mathbb{C}^N$. Suppose that $\{T(u_j)\}_{j=1}^\infty$ is not convergent in $Y$. Then $\{T(u_j)\}_{j=1}^\infty$ cannot be a Cauchy sequence in $Y$ and hence, there exists $a > 0$ such that, for every $j \in \mathbb{N}$, there exist $k_j, l_j \in \mathbb{N}$ with $j < k_j < l_j$ such that $\|T(u_{k_j}) - T(u_{l_j})\|_Y < a$. Via this inequality we can choose for $j = 1$ natural numbers $1 < k_1 < l_1$, then for $j := 1 + l_1$ natural numbers $1 + l_1 < k_2 < l_2$ and so on, such that $1 < k_1 < l_1 < k_2 < l_2 < k_3 < l_3 \ldots$ and, for these natural numbers $\{k_n, l_n\}_{n=1}^\infty$, we have

$$\|T(u_{k_n}) - T(u_{l_n})\|_Y \geq a, \quad n \in \mathbb{N}. \quad (2.3)$$

Then $z_n := u_{k_n} - u_{l_n}$, for $n \in \mathbb{N}$, is a bounded sequence in $X$. Since $\lim_{j \to \infty} u_j = x$ in $\mathbb{C}^N$, it follows that $\lim_{n \to \infty} z_n = 0$ in $\mathbb{C}^N$. By property (K), $\lim_{n \to \infty} T(z_n) = 0$ in $Y$, that is, $\lim_{n \to \infty} (T(u_{k_n}) - T(u_{l_n})) = 0$ in $Y$ which contradicts (2.3). Hence, $\{T(u_j)\}_{j=1}^\infty$ does converge in $Y$ and is a subsequence of $\{T(x_i)\}_{i=1}^\infty$. The compactness of $T$ is thereby verified. \qed

**Proposition 2.2.** Let $1 < p \leq q < \infty$ and $a \in M_{p,q}$. Then the continuous multiplier operator $M_p^a : ces(p) \to ces(q)$ is compact if and only if $(a_n \frac{1}{n^{\frac{1}{q}-\frac{1}{p}}})_n \in c_0$.

**Proof.** Suppose first that $w = (w_n)_n := (a_n \frac{1}{n^{\frac{1}{q}-\frac{1}{p}}})_n \in c_0$. Define the element $w_N := (w_1, \ldots, w_N, 0, 0, \ldots)$ for each $N \in \mathbb{N}$ in which case $(w - w_N) \in \ell_{\infty}$. So,
by (2.1), \( \| M_p^w - M_p^{w_N} \|_{op} = \| M_p^{w - w_N} \|_{op} \leq \| w - w_N \|_{\infty} \). Since \( w \in c_0 \), it follows that \( \lim_{N \to \infty} \| w - w_N \|_{\infty} = 0 \) and hence, \( M_p^w : ces(p) \to ces(p) \) is compact as each \( M_p^{w_N} \), for \( N \in \mathbb{N} \), is a finite rank operator. Define \( v_n := n^{\frac{p}{q} - \frac{1}{q}} \), for \( n \in \mathbb{N} \), in which case \( v := (v_n)_n \in \mathcal{M}_{p,q} \) by Bennett’s multiplier criterion mentioned above, that is, \( M_p^v : ces(p) \to ces(p) \) is continuous. Since \( M_{p,q}^a = M_p^v \), it follows that \( M_{p,q}^a \) is compact.

Conversely, suppose that \( M_{p,q}^a \) is a compact operator. According to (1.2), the sequence \( f_j := j^{1/p'} e_j \), for \( j \in \mathbb{N} \), is bounded in \( ces(p) \). Clearly \( \{f_j\}_{j=1}^{\infty} \) converges to 0 in the Fréchet space \( \mathbb{C}^N \). Moreover, \( M_{p,q}^a(f_j) = j^{1/p'} a_j e_j \), for \( j \in \mathbb{N} \), and \( M_{p,q}^a(f_j) \to 0 \) in \( \mathbb{C}^N \) for \( j \to \infty \) (as the multiplier operator \( M^a : \mathbb{C}^N \to \mathbb{C}^N \) given by \( x \mapsto ax \) is continuous). Applying Lemma 2.1 to the setting \( X := ces(p) \), \( Y := ces(q) \) and the continuous multiplier operator \( T = M^a : \mathbb{C}^N \to \mathbb{C}^N \) (whose restriction to \( X \) is \( M_{p,q}^a \)), it follows that \( \{M_{p,q}^a(f_j)\}_{j=1}^{\infty} \) converges to 0 in \( ces(q) \), that is, \( \lim_{j \to \infty} j^{1/p'}|a_j| \cdot \|e_j\|_{ces(q)} = \lim_{j \to \infty} j^{1/p'}a_j e_j \|_{ces(q)} = 0 \). On the other hand, (1.2) implies that \( A_q \leq j^{1/p'} \| e_j \|_{ces(q)} = B_q \) for \( j \in \mathbb{N} \). It follows that \( \lim_{j \to \infty} j^{1/p'}|a_j| \cdot j^{1/q'} = 0 \). Since \( \frac{1}{p'} - \frac{1}{q'} = \frac{1}{q'} - \frac{1}{p} \) we can conclude that \( (a_n n^{\frac{1}{p} - \frac{1}{q}})_n \) is in \( c_0 \).

For the case when \( p = q \) and \( a \in \mathcal{M}_p = \ell_\infty \), Proposition 2.2 implies that the multiplier operator \( M_{p,q}^a : ces(p) \to ces(p) \) is compact if and only if \( a \in c_0 \).

To treat the cases when \( p > q \) we recall, for each \( r > 1 \), the Banach space

\[
d(r) := \{ x \in \mathbb{C}^N : \| x \|_{d(r)} := \| \hat{x} \|_r < \infty \},
\]

where \( \hat{x} = (\hat{x}_n)_n := (\sup_{k \geq n} |x_k|)_n \) and \( \| \hat{x} \|_r \) is its norm in \( \ell_r \), [3, pp.3-4].

**Lemma 2.3.** Let \( 1 < r < \infty \) and \( x \in d(r) \). Then \( \lim_{N \to \infty} \| x - x^{(N)} \|_{d(r)} = 0 \), where \( x^{(N)} := (x_1, \ldots, x_N, 0, 0, \ldots) \) for each \( N \in \mathbb{N} \).

**Proof.** Given \( N \in \mathbb{N} \) observe that \( x - x^{(N)} = (0, \ldots, 0, x_{N+1}, x_{N+2}, \ldots) \) and hence, \( (x - x^{(N)})^r = (\hat{x}_{N+1}, \ldots, \hat{x}_{N+1}, \hat{x}_{N+2}, \ldots) \) where the first \( (N+1) \)-coordinates are constant \( \hat{x}_{N+1} \). It follows that

\[
\| x - x^{(N)} \|_{d(r)}^r = (N+1)(\hat{x}_{N+1})^r + \sum_{n=N+2}^{\infty} (\hat{x}_n)^r, \quad N \in \mathbb{N}.
\]  

Since \( (\hat{x}_n)^r \) is a decreasing sequence of non-negative terms which belongs to \( \ell_1 \), it is classical that \( \lim_{n \to \infty} n(\hat{x}_n)^r = 0 \), [14, §3.3 Theorem 1]. Let \( \varepsilon > 0 \). Choose \( K \in \mathbb{N} \) such that \( n(\hat{x}_n)^r < \frac{\varepsilon}{2} \) and \( \sum_{n=K}^{\infty} (\hat{x}_n)^r < \frac{\varepsilon}{2} \) for all \( n \geq K \). It follows from (2.4) that \( \| x - x^{(N)} \|_{d(r)}^r \to 0 \) for all \( N \geq K \). The proof is thereby complete. \( \square \)

Let \( 1 < q < p < \infty \) and choose \( r \) according to \( \frac{1}{r} = \frac{1}{q} - \frac{1}{p} \). Then it follows from table 32 on p.70 of [3] that

\[
\mathcal{M}_{p,q} = d(r).
\]  

**Lemma 2.4.** Let \( 1 < q < p < \infty \) and \( r \) satisfy \( \frac{1}{r} = \frac{1}{q} - \frac{1}{p} \). Then there exists a constant \( D_{p,q} > 0 \) such that

\[
\| M_{p,q}^a \|_{op} \leq D_{p,q} \| a \|_{d(r)}, \quad a \in \mathcal{M}_{p,q} = d(r).
\]
Proof. For Banach spaces $X, Y$ let $L(X, Y)$ denote the Banach space of all continuous linear operators from $X$ into $Y$, equipped with the operator norm $\| \cdot \|_{op}$. According to (2.5) the linear map $\Phi: d(r) \rightarrow L(\text{ces}(p), \text{ces}(q))$ specified by $\Phi(a) := M^a_{p,q}$ is well defined. To establish the existence of $D_{p,q}$ it suffices to show that $\Phi$ has closed graph. This is a standard argument after noting that convergence of a sequence in $d(r)$ implies its coordinatewise convergence. □

The following result shows, for $p > q > 1$, that every multiplier operator $M^a_{p,q}$ for $a \in M_{p,q}$ is compact.

**Proposition 2.5.** Let $p > q > 1$. For $a \in \mathbb{C}^N$ the following assertions are equivalent.

(i) $a \in M_{p,q}$, that is, $M^a_{p,q}: \text{ces}(p) \rightarrow \text{ces}(q)$ is continuous.

(ii) $M^a_{p,q}: \text{ces}(p) \rightarrow \text{ces}(q)$ is compact.

(iii) $a \in d(r)$ where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.

**Proof.** (i) $\iff$ (iii) is precisely the characterization (2.5) of Bennett.

(ii) $\implies$ (i) is clear as every compact linear operator is continuous.

(iii) $\implies$ (ii). Let $a(N) := (a_1, \ldots, a_N, 0, 0, \ldots)$ for $N \in \mathbb{N}$. Then $a - a(N) \in d(r)$ for $N \in \mathbb{N}$ and $\lim_{N \to \infty} \|a - a(N)\|_{d(r)} = 0$; see Lemma 2.3. By (2.5) the operators $M^a_{p,q}, M^{a(N)}_{p,q}$ and $M^{a(N)}_{p,q} = M^a_{p,q} - M^{a(N)}_{p,q}$ all belong to $L(\text{ces}(p), \text{ces}(q))$. Lemma 2.4 yields that $\|M^a_{p,q} - M^{a(N)}_{p,q}\|_{op} \leq D_{p,q}\|a - a(N)\|_{d(r)}$, for $N \in \mathbb{N}$. Hence, $M^a_{p,q}$ is compact as each operator $M^{a(N)}_{p,q}$ has finite rank. □

We now consider further properties of multiplier operators for the case when $p = q$. The space $L(\text{ces}(p), \text{ces}(p))$ is simply denoted by $L(\text{ces}(p))$.

**Lemma 2.6.** Let $1 < p < \infty$. Then

$$
\|M^a_{p}\|_{op} = \|a\|_{\infty}, \quad a \in \ell_{\infty} = M_p. \tag{2.6}
$$

**Proof.** Just prior to Proposition 2.2 it was noted that $\|M^a_{p}\|_{op} \leq \|a\|_{\infty}$. On the other hand, since $M^a_{p}(e_j) = a_j e_j$ for $j \in \mathbb{N}$, it is clear that the point spectrum $\sigma_{pt}(M^a_{p})$, consisting of all the eigenvalues of $M^a_{p}$, satisfies

$$
a(N) := \{a_j : j \in \mathbb{N}\} \subseteq \sigma_{pt}(M^a_{p}) \subseteq \sigma(M^a_{p}).
$$

Then the spectral radius inequality for operators, [10, Ch. VII, Lemma 3.4], yields

$$
\|M^a_{p}\|_{op} \geq r(M^a_{p}) := \sup\{\|\lambda\| : \lambda \in \sigma(M^a_{p})\} \geq \sup_{j \in \mathbb{N}} |a_j| = \|a\|_{\infty}.
$$

□

The spectrum of multiplier operators in $L(\text{ces}(p))$ can now be determined.

**Proposition 2.7.** Let $1 < p < \infty$. Then

$$
\sigma(M^a_{p}) = a(\overline{N}) = \{a_j : j \in \mathbb{N}\}, \quad a \in M_p. \tag{2.7}
$$

**Proof.** From the proof of Lemma 2.6 we have $a(\overline{N}) \subseteq \sigma_{pt}(M^a_{p}) \subseteq \sigma(M^a_{p})$. Since $\sigma(M^a_{p})$ is a closed set in $\mathbb{C}$, it follows that $a(\overline{N}) \subseteq \sigma(M^a_{p})$.

Suppose that $\lambda \notin a(\overline{N})$. Then $b = (b_n)_n$ with $b_n := \frac{1}{\lambda - a_n}$ for $n \in \mathbb{N}$ belongs to $\ell_{\infty} = M_p$. Using the formula $\lambda I - M^a_{p} = M^\lambda_{I-a}$ (with $I$ the identity operator
on $ces(p)$ and $1 := (1, 1, \ldots)$ it is routine to check that $(\lambda I - M_p^a)M_p^b = I = M_p^b(\lambda I - M_p^a)$. Hence, $\lambda I - M_p^a$ is invertible in $\mathcal{L}(ces(p))$ and so $\lambda$ lies in the resolvent set of $M_p^a$. This establishes the inclusion $\sigma(M_p^a) \subseteq \sigma(N)$. \hfill $\square$

For a Banach space $X$, an operator $T \in \mathcal{L}(X) := \mathcal{L}(X, X)$ is mean ergodic (resp. uniformly mean ergodic) if its sequence of Cesàro averages

$$T_{[n]} := \frac{1}{n} \sum_{m=1}^{n} T^m, \quad n \in \mathbb{N},$$

converges to some operator $P \in \mathcal{L}(X)$ in the strong operator topology $\tau_s$, i.e.,

$$\lim_{n \to \infty} T_{[n]}(x) = P(x) \text{ for each } x \in X, \quad [10, \text{Ch. VIII}] \text{ (resp. in the operator norm topology } \tau_b).$$

According to [10, Ch. VIII, Corollary 5.2] there then exists the direct sum decomposition

$$X = \text{Ker}(I - T) \oplus (I - T)(X).$$

Moreover, we have the identities

$$(I - T)T_{[n]} = T_{[n]}(I - T) = \frac{1}{n}(T - T^{n+1}), \quad n \in \mathbb{N},$$

and, setting $T_{[0]} := I$, that

$$\frac{1}{n} T^n = T_{[n]} - \frac{(n - 1)}{n} T_{[n-1]}, \quad n \in \mathbb{N}. \quad (2.10)$$

An operator $T \in \mathcal{L}(X)$ is called power bounded if $\sup_{n \in \mathbb{N}} \|T^n\|_{op} < \infty$. In this case it is clear that necessarily $\lim_{n \to \infty} \frac{\|T^n\|_{op}}{n} = 0$. A standard reference for mean ergodic operators is [15]. Finally, define $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.

**Proposition 2.8.** Let $1 < p < \infty$ and $a \in \mathcal{M}_p = \ell_\infty$. The following statements are equivalent.

(i) $\|a\|_\infty \leq 1$.

(ii) The multiplier operator $M_p^a \in \mathcal{L}(ces(p))$ is power bounded.

(iii) The multiplier operator $M_p^a \in \mathcal{L}(ces(p))$ is mean ergodic.

(iv) The spectrum $\sigma(M_p^a) \subseteq \mathbb{D}$.

(v) $\lim_{n \to \infty} \frac{(M_p^a)^n}{n} = 0$ relative to $\tau_s$ in $\mathcal{L}(ces(p))$.

**Proof.** (i) $\implies$ (ii). Since $\mathcal{M}_p$ is an algebra under coordinatewise multiplication in $\mathbb{C}^N$ we have $(M_p^a)^n = M_p^{a^n}$ (where $a^n := (a^n_j)_j$ for $a = (a_j)_j$) and so, via Lemma 2.6,

$$\|(M_p^{a^n})\|_{op} = \|M_p^{a^n}\|_{op} = \|a^n\|_{\infty} \leq 1, \quad n \in \mathbb{N}. \quad (2.7)$$

(ii) $\implies$ (iii). Power bounded operators in reflexive Banach spaces are always mean ergodic, [19].

(i) $\implies$ (iv). Since $\|a\|_{\infty} = \sup\{|\lambda| : \lambda \in a(\mathbb{N})\} \leq 1$, (2.7) implies $\sigma(M_p^a) \subseteq \mathbb{D}$.

(iv) $\implies$ (i). Clear from (2.7).

(iii) $\implies$ (i). Suppose that $\|a\|_{\infty} > 1$. Then there exists $k \in \mathbb{N}$ such that $|a_k| > 1$. Since $(M_p^a)^n(e_k) = a^n_k e_k$ for $n \in \mathbb{N}$, it follows that

$$\frac{\|(M_p^a)^n(e_k)\|_{ces(p)}}{n} = \frac{|a^n_k|^n}{n} \|e_k\|_{ces(p)}, \quad n \in \mathbb{N},$$

with $|a_k| > 1$. Hence, the sequence $\{(M_p^a)^n\}^\infty_{n=1}$ cannot converge to $0 \in \mathcal{L}(ces(p))$ in the topology $\tau_s$, thereby violating a necessary condition for $M_p^a$ to be mean ergodic (see (2.10)); contradiction! So, $\|a\|_{\infty} \leq 1$. 


(iii) \(\iff\) (v). This follows from (2.10).
(v) \(\iff\) (i). See the proof of (iii) \(\iff\) (i).

\[\]
(i) \( M^a_p \) is uniformly mean ergodic.

(ii) \( \|a\|_\infty \leq 1 \) and \( 1 \not\in a(\mathbb{N}) \setminus \{1\} \).

Proof. (i) \( \implies \) (ii). By the discussion immediately after Proposition 2.8 we know that (i) implies \( \|a\|_\infty \leq 1 \) and the range of \( I - M^a_p = M^{1-a}_p \) is closed in \( \text{ces}(p) \). Then \( w := 1 - a \) satisfies the hypothesis of Lemma 2.9. Accordingly, \( 0 \not\in ((1-\alpha)X(1-a))(\mathbb{N}) \) which is equivalent to \( 1 \not\in a(\mathbb{N}) \setminus \{1\} \).

(ii) \( \implies \) (i). The condition \( 1 \not\in a(\mathbb{N}) \setminus \{1\} \) implies that \( u := (1-a)^{-1}X(1-a) \) belongs to \( \ell_{\infty} \). In particular, \( M^a_p \in \mathcal{L}(\text{ces}(p)) \). Moreover, \( w := (1-a) \in \ell_{\infty} \) satisfies (in \( \mathcal{L}(\text{ces}(p)) \)) the identity \( M^w_p = M^{X(w)}_p \). It follows from (2.11) that \( M^w_p(\text{ces}(p)) \subseteq M^{X(w)}_p(\text{ces}(p)) = X_{w,p} \) (see (2.12)). It is routine to verify the reverse inclusion and so actually \( M^w_p = X_{w,p} \). In particular, the range of \( M^{1-a}_p = I - M^a_p \) is closed in \( \text{ces}(p) \). Since \( \|a\|_\infty \leq 1 \) implies that \( M^a_p \) is power bounded (cf. Proposition 2.8), it follows that \( \lim_{n \to \infty} \|M^a_p\|^n_{\text{ces}} = 0 \). Hence, the criterion of Lin can be applied to conclude that \( M^a_p \) is uniformly mean ergodic. \( \square \)

An example of a multiplier operator which is mean ergodic but not uniformly ergodic is \( M^a_p \) with \( a := (1 - \frac{1}{n})_n \).

In (2.9), with \( X := \text{ces}(p) \) and \( T := M^a_p \) (for \( \|a\|_\infty \leq 1 \)), note that

\[ \text{Ker}(I - M^a_p) = \{ x \in \text{ces}(p) : x_n = 0 \text{ for all } n \in \mathbb{N} \text{ with } a_n \neq 1 \}. \]

Concerning the linear dynamics of a continuous linear operator \( T : X \to X \) defined on a separable, locally convex Hausdorff space \( X \), recall that \( T \) is hypercyclic if there exists \( x \in X \) whose orbit \( \{ T^n x : n \in \mathbb{N}_0 := \{0\} \cup \mathbb{N} \} \) is dense in \( X \). If, for some \( x \in X \), the projective orbit \( \{ \lambda T^n x : \lambda \in \mathbb{C}, n \in \mathbb{N}_0 \} \) is dense in \( X \), then \( T \) is called supercyclic. Since this projective orbit coincides with \( \cup_{n=0}^{\infty} T^n(\text{span}\{x\}) \) we see that supercyclic is the same as 1-supercyclic as defined in [4]. Hypercyclicity always implies supercyclicity but not conversely.

Lemma 2.11. Let \( a = (a_n)_n \in \mathbb{C}^\mathbb{N} \) and define the multiplier operator \( M^a : \mathbb{C}^\mathbb{N} \to \mathbb{C}^\mathbb{N} \) by \( M^a(x) := ax \) for \( x \in \mathbb{C}^\mathbb{N} \). Then \( M^a \) is not supercyclic in the Fréchet space \( \mathbb{C}^\mathbb{N} \).

Proof. The continuous dual space \( (\mathbb{C}^\mathbb{N})' \) of \( \mathbb{C}^\mathbb{N} \) is the space \( \varphi \). Clearly \( M^a \) is continuous on \( \mathbb{C}^\mathbb{N} \) and its dual operator \( (M^a)' : \varphi \to \varphi \) is given by \((M^a)'(y) = ay\) for \( y \in \varphi \). Moreover, it follows from \((M^a)'(e_j) = a_j e_j \) for \( j \in \mathbb{N} \) that each canonical basis vector \( e_j \in \varphi \) is an eigenvector of \((M^a)'\). According to Theorem 2.1 of [4] the operator \( M^a \in \mathcal{L}(\mathbb{C}^\mathbb{N}) \) cannot be supercyclic. \( \square \)

Given \( 1 < p < \infty \) and \( a \in \mathbb{C}^\mathbb{N} \) the multiplier operator \( M^a : \mathbb{C}^\mathbb{N} \to \mathbb{C}^\mathbb{N} \) maps \( \ell_p \) into \( \ell_p \) if and only if \( a \in \ell_{\infty} \), [3, 1, p.69]. Denote this restricted operator by \( M^a_{(p)} : \ell_p \to \ell_p \).

Proposition 2.12. Let \( 1 < p < \infty \) and \( a \in \ell_{\infty} \).

(i) The multiplier operator \( M^a_{(p)} \in \mathcal{L}(\ell_p) \) is not supercyclic.

(ii) The multiplier operator \( M^a_{(p)} \in \mathcal{L}(\text{ces}(p)) \) is not supercyclic.

Proof. (i) Since \( \ell_p \) is dense in \( \mathbb{C}^\mathbb{N} \) (as it contains \( \varphi \)) and the natural inclusion \( \ell_p \to \mathbb{C}^\mathbb{N} \) is continuous, the supercyclicity of \( M^{a}_{(p)} \in \mathcal{L}(\ell_p) \) would imply the
supercyclicity of $M^a \in \mathcal{L}(\mathbb{C}^N)$, which is not the case (cf. Lemma 2.11). Hence, $M^a_{\{p\}}$ is not supercyclic.

(ii) Since $ces(p)$ is dense in $\mathbb{C}^N$ and the inclusion $ces(p) \hookrightarrow \mathbb{C}^N$ is continuous, the analogous argument to that of part (i) applies.

\section{The Cesàro operators}

Consider a pair $1 < p, q < \infty$. Denote by $C_{c(p),c(q)}$ (resp. $C_{c(p),q}$; $C_{p,c(q)}$; $C_{p,q}$) the Cesàro operator $C$ when it acts from $ces(p)$ into $ces(q)$ (resp. $ces(p)$ into $\ell_q$; $\ell_p$ into $ces(q)$; $\ell_p$ into $\ell_q$), whenever this operator exists. The closed graph theorem then ensures that this operator is continuous. We use the analogous notation for the natural inclusion maps $i_{c(p),c(q)}$; $i_{c(p),q}$; $i_{p,c(q)}$; $i_{p,q}$ whenever they exist. The main aim of this section is to identify all pairs $p, q$ for which these inclusion operators and Cesàro operators do exist and, for such pairs, to determine whether or not the operator is compact. For each $1 < p < \infty$, the spectrum of $C_{p,p} \in \mathcal{L}(\ell_p)$ is well known, [16, Theorem 2], [20, Theorem 4], and coincides with the spectrum of $C_{c(p),c(p)} \in \mathcal{L}(ces(p))$; see (1.6).

We begin with a preliminary result.

\begin{lemma}
Let $1 < p < \infty$.

(i) The operator $C_{c(p),p} : ces(p) \to \ell_p$ exists and satisfies $\|C_{c(p),p}\|_{op} \leq 1$.

(ii) The largest amongst the class of spaces $\ell_r$, for $1 \leq r < \infty$, which satisfy $\ell_r \subseteq ces(p)$ is the space $\ell_p$.

\end{lemma}

\begin{proof}
(i) Follows from the discussion immediately prior to Proposition 1.1. 

(ii) See Remark 2.2(iii) of [6].
\end{proof}

\begin{proposition}
Let $1 < p, q < \infty$ be an arbitrary pair.

(i) The inclusion map $i_{p,q} : \ell_p \to \ell_q$ exists if and only if $p \leq q$, in which case $\|i_{p,q}\|_{op} = 1$.

(ii) The inclusion map $i_{p,c(q)} : \ell_p \to ces(q)$ exists if and only if $p \leq q$, in which case $\|i_{p,c(q)}\|_{op} \leq q'$.

(iii) The inclusion map $i_{c(p),c(q)} : ces(p) \to ces(q)$ exists if and only if $p \leq q$, in which case $\|i_{c(p),c(q)}\|_{op} \leq 1$.

(iv) $ces(p) \not\subseteq \ell_q$ for all choices of $1 < p, q < \infty$.

\end{proposition}

\begin{proof}
(i) This is well known.

(ii) Lemma 3.1(ii) shows that $\ell_p \not\subseteq ces(q)$ if $p > q$.

Let $p \leq q$. For $x \in \ell_p$ we have $\|i_{p,c(q)}(x)\|_{ces(q)} = \|x\|_{ces(q)}$ with

$$\|x\|_{ces(q)} := \|C(|x|)\|_q \leq \|C_{q,q}\|_{op}\|x\|_q \leq \|C_{q,q}\|_{op}\|x\|_p,$$

where the last inequality follows via part (i). Since $\|C_{q,q}\|_{op} = p'$, [13, Theorem 326], the desired conclusion is clear.

(iii) If $p > q$, then $ces(p) \not\subseteq ces(q)$. Indeed, by Lemma 3.1(ii) there exists $y \in \ell_p$ with $y \not\in ces(q)$. By part (ii), $y \in ces(p)$.

Let $p \leq q$. Fix $x \in ces(p)$. By Lemma 3.1(i) we have $C(|x|) \in \ell_p$ and hence, by part (i), $C(|x|) \in \ell_q$. Accordingly,

$$\|x\|_{ces(q)} := \|C(|x|)\|_q \leq \|C(|x|)\|_p = \|x\|_{ces(p)}.$$

This shows that $i_{c(p),c(q)}$ exists and $\|i_{c(p),c(q)}\|_{op} \leq 1$. 

(iv) For arbitrary $1 < p < \infty$ there exists $x \in ces(p)$ with $x \notin \ell_\infty$, [6, Remark 2.2(ii)]. Then also $x \notin \ell_q$ for every $1 < q < \infty$. \hfill $\square$

If $1 < p < q < \infty$, then the inclusion $ces(p) \subseteq ces(q)$ as guaranteed by Proposition 3.2(ii) is actually proper. Indeed, by Lemma 3.1(ii) there exists $x \in \ell_q$ with $x \notin ces(p)$. Then $y := C(|x|) \in ces(q)$; see Proposition 3.2(ii). But, $x \notin ces(p)$ implies $|x| \notin ces(p)$ and so $y \notin ces(p)$; see Proposition 1.1. That $ces(p) \subsetneq ces(q)$ also follows from the next result.

**Proposition 3.3.** Let $1 < p, q < \infty$ with $p \neq q$. Then $ces(p)$ is not Banach space isomorphic to $ces(q)$.

**Proof.** According to (1.3) the closed (sectional) subspace

$$Y := \{x \in ces(p) : x_k = 0 \text{ unless } k = 2^j \text{ for some } j = 0, 1, 2, \ldots\}$$

is isomorphic to a weighted $\ell_p$-space (as $\|x\|_p = (\sum_{j=0}^{\infty} 2^{j(1-p)}|x_{2^j}|^p)^{1/p}$ for $x \in Y$) and hence, also isomorphic to $\ell_p$. Suppose that $ces(p)$ is isomorphic to $ces(q)$. Then $\ell_p$ is isomorphic to a closed subspace of $ces(q)$. Since $ces(q)$ is isomorphic to a closed subspace of the infinite $\ell_p$-sum $\ell_q(E_n)$ with each $E_n, n \in \mathbb{N}$, a finite dimensional space, [21, Theorem 1], it follows that $\ell_p$ is isomorphic to a closed subspace of $\ell_q(E_n)$. But, $X := \ell_p$ has a shrinking basis (it is reflexive) and so is isomorphic to $\ell_q(D_k)$ with each $D_k, k \in \mathbb{N}$, a finite dimensional space, [18, Theorem 2.d.1]. Since $\ell_q$ is clearly isomorphic to a closed (sectional) subspace of $\ell_q(D_k)$, it follows that $\ell_q$ is isomorphic to a closed subspace of $\ell_p$ with $p \neq q$, which is not the case, [18, p. 54]. So, $ces(p)$ is not isomorphic to $ces(q)$. \hfill $\square$

Via Proposition 3.2 we now determine which inclusion maps are compact.

**Proposition 3.4.** Let $1 < p \leq q < \infty$ be arbitrary.

(i) The inclusion $i_{p,q} : \ell_p \rightarrow \ell_q$ is never compact.

(ii) The inclusion $i_{c(p),c(q)} : ces(p) \rightarrow ces(q)$ is compact if and only if $p < q$.

(iii) The inclusion $i_{p,c(q)} : \ell_p \rightarrow ces(q)$ is compact if and only if $p < q$.

**Proof.** (i) The image under $i_{p,q}$ of the unit basis vectors $\{e_n : n \in \mathbb{N}\} \subseteq \ell_p$ has no Cauchy subsequence (hence, no convergent subsequence) in $\ell_q$ because $\|e_n - e_m\|_q = 2^{q/2}$ for all $n \neq m$.

(ii) Since $i_{c(p),c(p)}$ is the identity operator on $ces(p)$ it is surely not compact.

So, assume that $p < q$. Then the constant element $a := 1$ satisfies $(a_n \frac{1}{n^{1-p}})_n = (\frac{1}{n^{1-p}})_n \in c_0$ and hence, by Proposition 2.2 the multiplier operator $M^1_{p,q} \in \mathcal{L}(ces(p), ces(q))$ is compact. But, $M^1_{p,q}$ is precisely the inclusion operator $i_{c(p),c(q)}$.

(iii) Since $C_{p,p}$ is not compact (by (1.6) its spectrum is an uncountable set) and $C_{p,p} = C_{c(p),c(p)} i_{c(p),c(p)}$, also $i_{p,c(q)}$ fails to be compact. So, assume that $p < q$. Then the factorization $i_{p,c(q)} = i_{c(p),c(q)} i_{p,c(p)}$ together with the compactness of $i_{c(p),c(q)}$ (see part (ii)) shows that $i_{p,c(q)}$ is compact. \hfill $\square$

Now that the continuity and compactness of the various inclusion operators are completely determined we can do the same for the Cesàro operators $C : X \rightarrow Y$ where $X, Y \in \{\ell_p, ces(q) : p, q \in (1, \infty)\}$. We begin with continuity.

**Proposition 3.5.** Let $1 < p, q < \infty$ be an arbitrary pair.

(i) $C_{p,q} : \ell_p \rightarrow \ell_q$ exists if and only if $p \leq q$, in which case $\|C_{p,q}\|_{op} \leq p'$.  

(ii) $C_{p,c(q)} : \ell_p \rightarrow ces(q)$ exists if and only if $p \le q$, in which case $\|C_{p,c(q)}\|_{op} \le p' q'$.

(iii) $C_{c(p),c(q)} : ces(p) \rightarrow ces(q)$ exists if and only if $p \le q$, in which case $\|C_{c(p),c(q)}\|_{op} \le q'$.

(iv) $C_{(p),q} : ces(p) \rightarrow \ell_q$ exists if and only if $p \le q$, in which case $\|C_{(p),q}\|_{op} \le 1$.

Proof. (ii) Let $p > q$. According to Lemma 3.1(ii) there exists $x \in \ell_p \setminus ces(q)$, in which case also $|x| \in \ell_p \setminus ces(q)$. If $C(|x|) \in ces(q)$, then Proposition 1.1 implies that also $|x| \in ces(q)$; contradiction. So, $|x| \in \ell_p$ but $C(|x|) \notin ces(q)$, i.e., 

"$C_{p,c(q)}"$ does not exist.

Suppose then that $p \le q$. Then $C_{p,p} \in \mathcal{L}(\ell_p)$ exists with $\|C_{p,p}\|_{op} = p'$ and $i_{p,c(q)} : \ell_p \rightarrow ces(q)$ exists with $\|i_{p,c(q)}\|_{op} \le q'$ (cf. Proposition 3.2(ii)). Hence, the composition $C_{p,c(q)} = i_{p,c(q)} C_{p,p}$ exists and $\|C_{p,c(q)}\|_{op} \le p' q'$.

(i) Let $p > q$. If $C_{p,q}$ exists, then by Proposition 3.2(ii) $C_{p,c(q)} = i_{p,c(q)} C_{p,q}$ also exists. This contradicts part (ii) which was just proved.

So, assume that $p \le q$. Then $C_{p,p} \in \mathcal{L}(\ell_p)$ exists with $\|C_{p,p}\|_{op} = p'$ and $i_{q,c(q)} \in \ell_q$ exists with $\|i_{q,c(q)}\|_{op} = 1$ (cf. Proposition 3.2(i)). Hence, $C_{p,q} = i_{p,q} C_{p,p}$ exists and $\|C_{p,q}\|_{op} \le p'$.

(ii) Let $p > q$. If $C_{c(p),c(q)}$ exists, then by Proposition 3.2(i) also $C_{p,c(q)} = C_{c(p),c(q)} i_{p,c(p)}$ exists. This contradicts part (ii) above.

So, assume that $p \le q$. Fix $x \in ces(p)$. Then also $|x| \in ces(p)$ and $C(|x|) \in \ell_p \subseteq \ell_q$; see Lemma 3.1(i) and Proposition 3.2(i). Moreover, $|C(x)| \in \ell_q$ as $|C(x)| \le C(|x|)$. Hence,

$$\|C(x)\|_{ces(q)} := \|C(|C(x)|)\|_q \le \|C_{q,q}\|_{op} \|C(x)\|_q \le q' \|C(|x|)\|_q \le q' \|C(x)\|_q \le p' \|x\|_{ces(p)}.$$ 

This shows that $C_{p,c(q)}$ exists and $\|C_{c(p),c(q)}\|_{op} \le q'$.

(iii) Let $p > q$. If $C_{c(p),q}$ exists, then also $C_{p,c(q)} = i_{c(p),c(q)} C_{c(p),q}$ exists (cf. Proposition 3.2(ii)). This contradicts part (iii).

Assume now that $p \le q$. Since $C_{c(p),p}$ exists with $\|C_{c(p),p}\|_{op} \le 1$ (cf. Lemma 3.1(i)) and $i_{p,q}$ exists with $\|i_{p,q}\|_{op} = 1$ (cf. Proposition 3.2(i)), it follows that the composition $C_{c(p),q} = i_{p,q} C_{c(p),p}$ exists and $\|C_{c(p),q}\|_{op} \le 1$.

Concerning the proof of part (iii) of Proposition 3.5 when $p \le q$, it is also clear from $C_{c(p),c(q)} = i_{c(p),c(q)} C_{c(p),c(p)}$ that $C_{c(p),c(q)}$ exists. However, since $\|i_{c(p),c(q)}\|_{op} \le 1$ (cf. Proposition 3.2(iii)) and $\|C_{c(p),c(p)}\|_{op} = p'$, this approach only yields $\|C_{c(p),c(q)}\|_{op} \le p'$ whereas the given proof of (iii) yields $\|C_{c(p),c(q)}\|_{op} \le q'$ which is a better estimate when $p < q$.

We now have all the facts needed to prove the main result of this section.

Proposition 3.6. Let $1 < p \le q < \infty$ be arbitrary.

(i) The Cesàro operator $C_{p,q} : \ell_p \rightarrow \ell_q$ is compact if and only if $p < q$.

(ii) The Cesàro operator $C_{p,c(q)} : \ell_p \rightarrow ces(q)$ is compact if and only if $p < q$.

(iii) The Cesàro operator $C_{c(p),c(q)} : ces(p) \rightarrow ces(q)$ is compact if and only if $p < q$.

(iv) The Cesàro operator $C_{(p),q} : ces(p) \rightarrow \ell_q$ is compact if and only if $p < q$. 

Proof. (i) Since $\sigma(C_{p,p})$ is an uncountable set (see the comments prior to Lemma 3.1), it is clear that $C_{p,p}$ is not compact. So, assume that $p < q$. Since $C_{p,q} = C_{c(q),c(p)}$ with $C_{c(q),q} : ces(q) \to \ell_q$ continuous (cf. Lemma 3.1(i)) and $i_{c(p),c(q)} : \ell_p \to ces(q)$ compact (by Proposition 3.4(iii)), it follows that $C_{p,q}$ is compact.

(ii) For $p = q$ observe that $(C_{(p),(p)})^2 = C_{p,c(p)}C_{c(p),p}$. By (1.6) and the spectral mapping theorem, [10, Ch. VII, Theorem 3.11], we see that

$$\sigma((C_{(p),(p)})^2) = \{ \lambda^2 : |\lambda - \frac{p'}{2}| \leq \frac{p'}{2} \}$$

is an uncountable set and so $(C_{(p),(p)})^2$ is not compact. Hence, also $C_{p,c(p)}$ is not compact.

Assume then that $p < q$. Since the inclusion $i_{c(p),c(q)} : ces(p) \to ces(q)$ is compact (cf. Proposition 3.4(ii)), it is clear from the factorization $C_{p,c(q)} = i_{c(p),c(q)}C_{p,c(p)}$ that also $C_{p,c(q)}$ is compact.

(iii) For $p = q$ it follows from (1.6) that $\sigma(C_{p,c(p)})$ is an uncountable set and $C_{p,c(p)}$ is not compact. Suppose now that $p < q$. Since the inclusion $i_{c(p),c(q)} : ces(p) \to ces(q)$ is compact (by Proposition 3.4(ii)), the factorization $C_{p,c(q)} = i_{c(p),c(q)}C_{p,c(p)}$ shows that $C_{c(p),c(q)}$ is compact.

(iv) For $p = q$ we have $C_{c(p),c(p)} = i_{c(p),c(p)}C_{c(p),p}$. By part (iii) the operator $C_{c(p),c(p)}$ is not compact and hence, also $C_{p,c(p)}$ is not compact.

Assume now that $p < q$. Select any $r$ satisfying $p < r < q$, in which case we have $C_{c(p),q} = C_{c(r),q}i_{c(p),c(r)}$ with $C_{c(r),q}$ continuous (by Proposition 3.5(iv)) and $i_{c(p),c(r)}$ compact (via Proposition 3.4(ii)). Hence, also $C_{c(p),q}$ is compact. \qed

Our final result concerns the mean ergodicity and linear dynamics of Cesàro operators.

**Proposition 3.7.** Let $1 < p < \infty$.

(i) The Cesàro operator $C_{p,p} : \ell_p \to \ell_p$ is not power bounded, not mean ergodic and not supercyclic.

(ii) The Cesàro operator $C_{c(p),c(p)} : ces(p) \to ces(p)$ is not power bounded, not mean ergodic and not supercyclic.

**Proof.** (i) That $C_{p,p}$ is neither power bounded nor mean ergodic is Proposition 4.2 of [1]. It is known that the Cesàro operator $C : \mathbb{C}^N \to \mathbb{C}^N$ is not supercyclic, [2, Proposition 4.3]. Since $\ell_p$ is dense in $\mathbb{C}^N$ and the natural inclusion $\ell_p \subseteq \mathbb{C}^N$ is continuous, the supercyclicity of $C_{p,p}$ in $\ell_p$ would imply that $C : \mathbb{C}^N \to \mathbb{C}^N$ is supercyclic. Hence, $C_{p,p} \in \mathcal{L}(\ell_p)$ is not supercyclic.

(ii) Suppose that $C_{c(p),c(p)}$ is mean ergodic. According to (2.10) we have

$$\lim_{n \to \infty} \frac{(C_{c(p),c(p)})^n}{n} = 0$$

for $\tau_n \in \mathcal{L}(ces(p))$ and hence, $\sigma(C_{c(p),c(p)}) \subseteq \mathbb{B}$, [10, Ch. VIII, Lemma 8.1]. This contradicts (1.6). Hence, $C_{c(p),c(p)}$ cannot be mean ergodic. Since power bounded operators in reflexive Banach spaces are always mean ergodic, [19], it follows that $C_{c(p),c(p)}$ is not power bounded. Arguing as in part (i), since $ces(p)$ is dense in $\mathbb{C}^N$ and the inclusion $ces(p) \subseteq \mathbb{C}^N$ is continuous, it follows that $C_{c(p),c(p)}$ is not supercyclic. \qed

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