SOLID CORES AND SOLID HULLS OF WEIGHTED BERGMAN SPACES.

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ABSTRACT. We determine the solid hull for $2 < p < \infty$ and the solid core for $1 < p < 2$ of weighted Bergman spaces A^p_μ, $1 < p < \infty$, of analytic functions functions on the disc and on the whole complex plane, for a very general class of non-atomic positive bounded Borel measures μ. New examples are presented. Moreover we show that the space A^p_μ, $1 < p < \infty$, is solid if and only if the monomials are an unconditional basis of this space.

1. Introduction and preliminaries

Consider $R = 1$ or $R = \infty$ and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. We study holomorphic functions $f : R\cdot\mathbb{D} \to \mathbb{C}$ where $R \cdot \mathbb{D} = \mathbb{D}$ if $R = 1$ and $R \cdot \mathbb{D} = \mathbb{C}$ if $R = \infty$. Let $\hat{f}(k)$ be the Taylor coeffients of f, i.e. $f(z) = \sum_{k=0}^{\infty} \hat{f}(k)z^k$. We take a non-atomic positive bounded Borel measure μ on $[0, R[$ such that $\mu([r, R[) > 0$ for every $r > 0$ and $\int_0^R r^n d\mu(r) < \infty$ for all $n > 0$. Put, for $1 \le p < \infty$,

$$||f||_p = \left(\frac{1}{2\pi}\int_0^R \int_0^{2\pi} |f(re^{i\varphi})|^p d\varphi d\mu(r)\right)^{1/p}$$

and let

$$A^p_\mu = \{f : R \cdot \mathbb{D} \to \mathbb{C} : f \text{ holomorphic with } ||f||_p < \infty\}.$$

A^p_μ is called a *weighted Bergman space*.

Let $H(R \cdot \mathbb{D})$ be the space of all holomorphic functions on $R \cdot \mathbb{D}$ and let $A \subset H(R \cdot \mathbb{D})$ be a subspace containing the polynomials. We want to study the *solid core*

$$s(A) = \{f \in A : g \in A \text{ for all holomorphic } g \text{ with } |\hat{g}(k)| \le |\hat{f}(k)| \text{ for all } k\}$$

and the *solid hull*

$$S(A) = \{g : R\cdot\mathbb{D} \to \mathbb{C} : g \text{ holomorphic, there is } f \in A \text{ with } |\hat{g}(k)| \le |\hat{f}(k)| \text{ for all } k\}.$$

A is called *solid* if $A = S(A)$.

In the first four sections we consider $A = A^p_\mu$ while in section 5 we include the case where A consists of weighted sup-norm spaces of holomorphic functions.

The solid hull and core of spaces of analytic functions has been investigated by many authors. We refer the reader to the recent books [6] and [13] and the many

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references therein. For example in [6] the characterisation of the solid hulls and cores of A^p_μ can be found where $d\mu(r) = (1-r)^\alpha dr$ for some $\alpha > 0$ and $R = 1$.

Originally, our main interest was to replace the "standard weights" $(1-r)^\alpha$ by weights of the form $v_{a,b}(r) = \exp(-a/(1-r)^b)$ for some $a > 0$ and $b > 0$, which are of a completely different nature and require different methods, and hence to consider $d\mu(r) = v_{a,b}(r)dr$. We wanted to extend to weighted Bergman spaces the results of [3], a paper which was entirely devoted to this class of weights $v_{a,b}$ in connection with weighted sup-norms. In the present article we give a characterization of solid hulls of A^p_μ if $2 < p < \infty$ and solid cores if $1 < p < 2$ in our main Theorem 2.1 for much more general μ which, under some mild additional assumptions (Corollary 3.2), resulted in the explicit computation of many examples including $v(r) = \exp(-a/(1-r)^b)$ for $R = 1$ and $v(r) = \exp(-r)$ for $R = \infty$; see Corollaries 3.4 and 3.5. The final sections 4 and 5 are dedicated to Bergman spaces A^p_μ and weighted sup-norm spaces H^∞_v which themselves are solid. We give examples for this situation in connection with holomorphic functions over the complex plane and show that this can never happen for holomorphic functions over the unit disc. The main results are Theorem 4.1 which states that A^p_μ is solid if and only if the monomials $(z^n)_{n=0}^\infty$ are an unconditional basis of A^p_μ, and Theorem 5.2 which ensures that H^∞_v is solid if and only if $(z^n)_{n=0}^\infty$ is a Schauder basis of the closure H^0_v of the polynomials in H^∞_v.

For a holomorphic g and $0 < r$ we define

$$M_p(g,r) = \left(\frac{1}{2\pi}\int_0^{2\pi} |g(re^{i\varphi})|^p d\varphi\right)^{1/p}$$

and $P_n g(z) = \sum_{k=0}^n \hat{g}(k)z^k$. It is well-known that, for $1 < p < \infty$, there are universal constants $c_p > 0$ with $M_p(P_n g, r) \le c_p M_p(g,r)$ where c_p does not depend on g, n or r. Moreover we have $\lim_{n\to\infty} M_p(g - P_n g, r) = 0$. Hence we obtain

$$||P_n f||_p \le c_p ||f||_p \text{ for all } f \in A^p_\mu \text{ and all } n \text{ and } \lim_{n\to\infty} ||f - P_n f||_p = 0.$$

In particular we see that the monomials $z \mapsto z^n$, $n = 0, 1, 2, \ldots$ form a Schauder basis of A^p_μ if $1 < p < \infty$. Details can be seen in [4] and [14].

In the rest of the article $[r]$ denotes the largest integer smaller or equal than $r > 0$.

2. Main general result.

The main result of this section is the following Theorem 2.1 below. There are relevant earlier related works. For example Theorem 4.1 in Pavlović [12] established a useful norm in blocks for certain weighted Bergman spaces. See also earlier work by Mateljević and Pavlović [11].

Theorem 2.1. *Assume that there are constants $d_1, d_2 > 0$, and $\omega_n > 0, n = 1, 2, \ldots$, numbers $0 \le l_1 < l_2 < \ldots$ and radii $s_1 < s_2 < \ldots$ such that, for every $f \in A^p_\mu$,*

$$d_1||f||_p \le \left(\sum_{n=1}^\infty \omega_n^p M_p^p\left((P_{[l_{n+1}]} - P_{[l_n]})f, s_n\right)\right)^{1/p} \le d_2||f||_p. \tag{2.1}$$

(a) If $2<p<\infty$*, then*

$$S(A^p_\mu)=$$

$$\left\{g:R\cdot\mathbb{D}\to\mathbb{C}: g \text{ holomorphic with } \sum_{n=1}^{\infty}\omega_n^p\left(\sum_{k=[l_n]+1}^{[l_{n+1}]}|\hat{g}(k)|^2 s_n^{2k}\right)^{p/2}<\infty\right\}.$$

(b) If $1<p<2$*, then*

$$s(A^p_\mu)=$$

$$\left\{g:R\cdot\mathbb{D}\to\mathbb{C}: g \text{ holomorphic with } \sum_{n=1}^{\infty}\omega_n^p\left(\sum_{k=[l_n]+1}^{[l_{n+1}]}|\hat{g}(k)|^2 s_n^{2k}\right)^{p/2}<\infty\right\}.$$

Theorem 2.1 is proved below. Before presenting the proof we point out that condition (2.1) can be realized for any given μ. Indeed, fix $\beta>16\cdot 3^{p-1}(1+2^p)c_p^p+2$ and use induction to obtain $0=l_1<l_2<l_3\ldots$ and $0\le s_1<s_2\ldots<R$ with

$$\int_0^{s_n} r^{l_np}d\mu=\beta\int_{s_n}^{R}r^{l_np}d\mu \quad\text{and}\quad \int_0^{s_n}r^{l_{n+1}p}d\mu=\frac{1}{\beta}\int_{s_n}^{R}r^{l_{n+1}p}d\mu. \tag{2.2}$$

Instead of starting with $n=1$ we can as well start the induction e.g. with $n=n_0$ for some $n_0\ge 0$ (with $l_1=0$ and arbitrary s_1) and restrict the preceding relations to all $n\ge n_0$. Moreover put

$$\omega_n=\left(\int_0^{s_n}\left(\frac{r}{s_n}\right)^{l_np}d\mu+\int_{s_n}^{R}\left(\frac{r}{s_n}\right)^{l_{n+1}p}d\mu\right)^{1/p}.$$

Then there are constants $d_1,d_2>0$ such that, for every $f\in A^p_\mu$,

$$d_1||f||_p\le\left(\sum_{n=1}^{\infty}\omega_n^pM_p^p\left((P_{[l_{n+1}]}-P_{[l_n]})f,s_n\right)\right)^{1/p}\le d_2||f||_p.$$

This was shown in [5] for $p=1$ and in [10] for $1<p<\infty$ and $R=1$, but with some slight modifications the proofs carry over to the case $R=\infty$.

Example 2.2. (i) Let $d\mu(r)=dr$ where $R=1$. Then we obtain

$$l_n=\frac{1}{p}(a^{n-1}-1)\text{ and } s_n=\left(\frac{\beta}{\beta+1}\right)^{a^{1-n}}\quad\text{where } a=\frac{\log(\beta+1)}{\log(1+\beta)-\log(\beta)}.$$

This can be easily verified using the definition (starting with $n=0$) and induction.

(ii) Let $d\mu(r)=r^\alpha dr$ for some $\alpha>0$ and $R=1$. With example (i) and $l_np+\alpha=(a^{n-1}-1)$, where a is the number in (i), we obtain

$$l_n=\frac{1}{p}(a^{n-1}-1)-\frac{\alpha}{p}\quad\text{and}\quad s_n=\left(\frac{\beta}{\beta+1}\right)^{a^{1-n}}$$

for $n\ge 2$ with $l_1=0$ and $s_1=\beta/(\beta+1)$.

Now we turn to the proof of Theorem 2.1. Let $f : R\cdot\mathbb{D} \to \mathbb{C}$ be holomorphic. Recall that $\hat{f}(n)r^n = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\varphi})e^{-in\varphi}d\varphi$ for each $0 < r < R$ and each $n = 0, 1, 2, \dots$. For $g(re^{i\varphi}) = r^{n(p-1)}e^{-in\varphi}/(\int_0^R r^{np}d\mu)^{1-1/p}$ we have

$$|\hat{f}(n)|\left(\int_0^R r^{np}d\mu\right)^{1/p} = \frac{1}{2\pi}\left|\int_0^R\int_0^{2\pi} f(re^{i\varphi})g(re^{i\varphi})d\varphi d\mu\right| \le ||f||_p.$$

In the following we make use of the Khintchine inequality ([7], 2.b.3.), i.e. for arbitrary b_k and n we have

$$A_p\left(\sum_{k=1}^n |b_k|^2\right)^{1/2} \le \left(\frac{1}{2^n}\sum_{\theta_k=\pm1}\left|\sum_{k=1}^n b_k\theta_k\right|^p\right)^{1/p} \le B_p\left(\sum_{k=1}^n |b_k|^2\right)^{1/2}$$

where A_p, B_p are universal constants not depending on n. (The summation in the central expression runs over the 2^n different possibilities of the change of signs.)

Conclusion of the proof of Theorem 2.1. For a holomorphic function g put

$$\alpha(g) = \left(\sum_{n=1}^\infty \omega_n^p M_p^p\left((P_{[l_{n+1}]} - P_{[l_n]})f, s_n\right)\right)^{1/p}.$$

As assumed, $\alpha(\cdot)$ is equivalent to $||\cdot||_p$. Moreover let

$$\gamma(g) = \left(\sum_{n=1}^\infty \omega_n^p\left(\sum_{k=[l_n]+1}^{[l_{n+1}]} |\hat{g}(k)|^2 s_n^{2k}\right)^{p/2}\right)^{1/p}$$

and $V = \{g : R\cdot\mathbb{D} \to \mathbb{C} : g \text{ holomorphic with } \gamma(g) < \infty\}$. Recall that Parseval's identity implies

$$M_2^2\left((P_{[l_{n+1}]} - P_{[l_n]})f, s_n\right) = \sum_{k=[l_n]+1}^{[l_{n+1}]} |\hat{g}(k)|^2 s_n^{2k}.$$

Proof of (a). Let $g \in S(A_\mu^p)$. Then there is $f \in A_\mu^p$ with $|\hat{g}(k)| \le |\hat{f}(k)|$ for all k. If $2 < p < \infty$ then

$$\gamma(g) \le \gamma(f) \le \alpha(f) \le d_2||f||_p < \infty.$$

Hence $g \in V$.

Now let $g \in V$. Put $\Delta_n = \{+1, -1\}^{[l_{n+1}]-[l_n]}$. For $\Theta_n = (\theta_{[l_n]+1}, \dots, \theta_{[l_{n+1}]}) \in \Delta_n$ put

$$g_{\Theta_n}(\varphi) = \sum_{k=[l_n]+1}^{[l_{n+1}]} \theta_k\hat{g}(k)s_n^k e^{ik\varphi} \text{ and } g_n(\varphi) = \sum_{k=[l_n]+1}^{[l_{n+1}]} \hat{g}(k)s_n^k e^{ik\varphi}.$$

Let $\tilde{\Theta}_n$ be such that

$$M_p(g_{\tilde{\Theta}_n}, s_n) \le \left(\frac{1}{2^{[l_{n+1}]-[l_n]}}\sum_{\Theta_n\in\Delta_n} M_p^p(g_{\Theta_n}, s_n)\right)^{1/p}.$$

The Khintchine inequality yields

$$M_p(g_{\tilde{\Theta}_n}, s_n) \le B_p M_2(g_n, s_n).$$

Put $h = \sum_n g_{\tilde{\Theta}_n}$. Then, by the preceding estimates,

$$d_1 ||h||_p \le \alpha(h) \le B_p \gamma(g) < \infty.$$

Hence $h \in A^p_\mu$. Since by definition $|\hat{h}(k)| = |\hat{g}(k)|$ for all k we obtain $g \in S(A^p_\mu)$.

Proof of (b). We retain the preceding notation. Let $g \in V$ and let $f : R{\cdot}\mathbb{D} \to \mathbb{C}$ be holomorphic with $|\hat{f}(k)| \le |\hat{g}(k)|$ for all k. Then

$$d_1 ||f||_p \le \alpha(f) \le \gamma(f) \le \gamma(g) < \infty.$$

This implies $f \in A^p_\mu$ and hence $g \in s(A^p_\mu)$.

Now let $g \in s(A^p_\mu)$. Let $\tilde{\tilde{\Theta}}_n \in \Delta_n$ be such that

$$\left(\frac{1}{2^{[l_{n+1}]-[l_n]}} \sum_{\Theta_n \in \Delta_n} M_p^p(g_{\Theta_n}, s_n) \right)^{1/p} \le M_p(g_{\tilde{\tilde{\Theta}}_n}, s_n).$$

Put $h = \sum_n g_{\tilde{\tilde{\Theta}}_n}$. Then we obtain $|\hat{h}(k)| = |\hat{g}(k)|$ for all k. Hence $h \in A^p_\mu$. The Khintchine inequality together with the choice of $\tilde{\tilde{\Theta}}_n$ yields

$$\gamma(g) = \gamma(h) \le A_p^{-1} \alpha(h) \le d_2 A_p^{-1} ||h||_p < \infty.$$

We conclude $g \in V$. $\square$

3. **Main examples.**

Quite often it is very difficult to compute the parameters l_n and s_n in (2.2). Therefore it is worthwhile to consider special cases which yield an equivalent representation of the norm $||\cdot||_p$ satisfying (2.1) and which are easier to compute and cover many examples. To this end let $v : [0, R[\to]0, \infty[$ be a weight function, i.e. let v be continuous, decreasing and satisfy

$$\lim_{r \to R} v(r) = 0 \quad \text{ and } \quad \sup_r r^n v(r) < \infty \ \text{ for all } \ n > 0.$$

Moreover, let ν be a non-atomic positive Borel measure on $[0, R[$ such that $\nu([r, R[) > 0$ for every $r > 0$, and $\int_0^R r^n v(r) d\nu(r) < \infty$ for every $n \ge 0$. Put, for $1 \le p < \infty$,

$$||f||_p = \left(\int_0^R M_p^p(f, r) v(r) d\nu(r) \right)^{1/p}$$

Here we consider A^p_μ with $d\mu(r) = v(r) d\nu(r)$. Actually one can relax a bit the conditions on v. It suffices to require that v be decreasing on $[r_0, R[$ for some $r_0 \in]0, R[$. This follows from the fact that, for $d\tilde{\mu} = 1_{[r_0,R[} d\mu$, the L_p-norms with respect to μ and $\tilde{\mu}$ are equivalent. Indeed, using the fact that $M_p(f, r)$ is increasing with respect to r for holomorphic functions f we see that

$$\int_{r_0}^R M_p^p(f, r) d\mu(r) \le \int_0^R M_p^p(f, r) d\mu(r) \le \left(1 + \frac{\mu([r_0, R[)}{\mu([0, R[)} \right) \int_{r_0}^R M_p^p(f, r) d\mu(r).$$

For any $n > 0$ let $r_n \in [0, R[$ be a point where the function $r \mapsto r^n v(r)$ attains its global maximum. It is easily seen that $r_m < r_n$ if $m < n$. In the following we assume that

$$r_n \text{ is the unique global maximum of } r^n v(r) \text{ for all } n$$
$$\text{and there are no further local maxima.} \tag{3.1}$$

For example this is the case if v is differentiable and v'/v is injective. The assumption (3.1) implies that $r^n v(r)$ is decreasing for $r \geq r_n$. Moreover we assume that v satisfies

Condition (b_0)**:** There are numbers $1 < b < K$ and $m_1 < m_2 < \ldots$ with $\lim_{n\to\infty} m_n = \infty$ such that

$$b \leq \left(\frac{r_{m_n}}{r_{m_{n+1}}}\right)^{m_n} \frac{v(r_{m_n})}{v(r_{m_{n+1}})}, \left(\frac{r_{m_{n+1}}}{r_{m_n}}\right)^{m_{n+1}} \frac{v(r_{m_{n+1}})}{v(r_{m_n})} \leq K.$$

Condition (b_0) is exactly the same as condition (b) in [3], except that the treatment of weighted Banach spaces of analytic functions with sup-norms requires $2 < b < K$. We refer the reader to [3] and [9] for more information and examples related to these conditions.

We take the parameters of condition (b_0) and we put

$$I_n = \nu([r_{m_n}, r_{m_{n+1}}])$$

and assume

$$I_n < \infty \quad \text{for all } n \quad \text{and} \quad \limsup_{n\to\infty} \frac{I_n}{\min(I_{n-1}, I_{n+1})} < b. \tag{3.2}$$

Theorem 3.1. *Let $1 < p < \infty$. Assume that v satisfies (b_0) with (3.1), (3.2). Then there are constants $d_1, d_2 > 0$ with*

$$d_1 ||f||_p \leq \left(\sum_{n=1}^{\infty} M_p^p((P_{[m_{n+1}/p]} - P_{[m_n/p]})f, r_{m_n}) v(r_{m_n}) I_n\right)^{1/p} \leq d_2 ||f||_p. \tag{3.3}$$

for all $f \in A^p_\mu$.

In view of (2.1) we can apply Theorem 2.1 with the preceding $l_n = m_n/p$, $\omega_n^p = v(r_{m_n}) I_n$ and $s_n = r_{m_n}$.

Corollary 3.2. *Let $d\mu = v d\nu$.*
(a) If $2 < p < \infty$, then

$$S(A^p_\mu) = \Bigg\{ g : R \cdot \mathbb{D} \to \mathbb{C} : g \text{ holomorphic with}$$
$$\sum_{n=1}^{\infty} v(r_{m_n}) I_n \left(\sum_{k=[m_n/p]+1}^{[m_{n+1}/p]} |\hat{g}(k)|^2 r_{m_n}^{2k}\right)^{p/2} < \infty \Bigg\}.$$

(b) *If* $1 < p < 2$, *then*

$$s(A^p_\mu) = \Bigg\{ g : R\cdot\mathbb{D} \to \mathbb{C} : g \textit{ holomorphic with}$$
$$\sum_{n=1}^{\infty} v(r_{m_n}) I_n \left(\sum_{k=[m_n/p]+1}^{[m_{n+1}/p]} |\hat{g}(k)|^2 r_{m_n}^{2k} \right)^{p/2} < \infty \Bigg\}.$$

Before we prove Theorem 3.1 we present the following examples. They are concrete cases to which Corollary 3.2 applies, thus permitting us to calculate explicitly all the parameters which appear in the solid hull and solid core.

Example 3.3. (i) $R = 1$ and $d\mu(r) = \exp(-\alpha/(1-r)^\beta)dr$ for some $\alpha, \beta > 0$. We take $v(r) = \exp(-\alpha/(1-r)^\beta)$ and $d\nu(r) = dr$. v satisfies condition (b_0) with

$$m_n = \beta \left(\frac{\beta}{\alpha}\right)^{1/\beta} n^{2+2/\beta} - \beta n^2 \quad \text{and} \quad r_{m_n} = 1 - \left(\frac{\alpha}{\beta}\right)^{1/\beta} \frac{1}{n^{2/\beta}}$$

and $b = e^1$ (see [3], Theorem 3.1.) Here $I_n = (\alpha/\beta)^{1/\beta}(n^{-2/\beta} - (n+1)^{-2/\beta})$. Hence

$$\lim_{n\to\infty} \frac{I_n}{\min(I_{n-1}, I_{n+1})} = 1.$$

This shows that (3.2) is satisfied. (3.1) holds, too, according to [3]. So we can apply Corollary 3.2.

(ii) $R = 1$ and $d\mu(r) = (1 - \log(1-r))^{-1}dr$. Here we take

$$v(r) = 1 - r \quad \text{and} \quad d\nu(r) = \frac{dr}{(1-r)(1-\log(1-r))}.$$

$r_m = 1 - 1/(m+1)$ is the only zero of the derivative of $r^m v(r)$. Hence (3.1) is satisfied. If we take $m_n = 9^n$ and hence $r_{m_n} = 1 - 1/(9^n + 1)$ then a simple calculation reveals that v satisfies (b_0) with $b = 3$. We obtain

$$I_n = \int_{r_{m_n}}^{r_{m_{n+1}}} d\nu = \log\left(\frac{1 + \log(9^{n+1}+1)}{1 + \log(9^n + 1)}\right)$$

from which we infer $\lim_{n\to\infty} I_n/\min(I_{n-1}, I_{n+1}) = 1$. This implies (3.2).

(iii) $R = \infty$ and $d\mu(r) = e^{-r}dr$. Here we take $v(r) = e^{-r}$, $d\nu(r) = dr$. $r_m = m$ is the unique zero of the derivative of $r^m v(r)$. Hence (3.1) is satisfied. Put

$$m_1 = 1 \quad \text{and} \quad m_{n+1} = m_n + 2\sqrt{m_n},\ n = 1, 2\ldots, \quad \text{and } r_{m_n} = m_n.$$

A simple calculation yields, with

$$-x - \frac{1}{2}\left(\frac{x}{1-x}\right)^2 \le \log(1-x) \le -x \quad \text{if } \ 0 < x < 1,$$

$$\exp\left(\frac{4\sqrt{m}}{\sqrt{m}+2} - 2\right) \le \left(\frac{r_{m_n}}{r_{m_{n+1}}}\right)^{m_n} \frac{v(r_{m_n})}{v(r_{m_{n+1}})} =$$
$$\exp\left(m \log\left(1 - \frac{2}{\sqrt{m}+2}\right) + 2\sqrt{m}\right) \le \exp\left(\frac{4\sqrt{m}}{\sqrt{m}+2}\right).$$

Similarly, with

$$x - \frac{x^2}{2} \le \log(1+x) \le x \quad \text{for} \ \ 0 < x < 1,$$

$$\exp\left(4 - 2(1 + \frac{2}{\sqrt{m}})\right) \le \exp\left((m + 2\sqrt{m})\log\left(1 + \frac{2}{\sqrt{m}}\right) - 2\sqrt{m}\right)$$
$$= \left(\frac{r_{m_{n+1}}}{r_{m_n}}\right)^{m_{n+1}} \frac{v(r_{m_{n+1}})}{v(r_{m_n})} \le e^4.$$

This shows that condition (b_0) holds. Moreover we easily obtain

$$I_n = 2\sqrt{m_n} \quad \text{and} \quad \lim_{n\to\infty} \frac{I_n}{\min(I_{n-1}, I_{n+1})} = 1$$

which yields (3.2). Observe that in this case we can take $m_n = n^2$; see Theorem 3.1 in [1]. This fact is not surprising, since one can easily prove by induction that our selection of m_n above satisfies $(n-1)^2 \le m_n \le n^2$ for each n.

Corollary 3.4. *Let $R = 1$ and $d\mu(r) = \exp(-1/(1-r))dr$.*
(a) If $2 < p < \infty$, then

$$S(A^p_\mu) = \left\{ g \in H(\mathbb{D}) : \sum_{n=1}^{\infty} e^{-n^2} \left(\frac{1}{n^3}\right) \left(\sum_{k=[n^4/p]+1}^{[(n+1)^4/p]} |\hat{g}(k)|^2 \left(1 - \frac{1}{n^2}\right)^{2k} \right)^{p/2} < \infty \right\}.$$

(b) If $1 < p < 2$, then

$$s(A^p_\mu) = \left\{ g \in H(\mathbb{D}) : \sum_{n=1}^{\infty} e^{-n^2} \left(\frac{1}{n^3}\right) \left(\sum_{k=[n^4/p]+1}^{[(n+1)^4/p]} |\hat{g}(k)|^2 \left(1 - \frac{1}{n^2}\right)^{2k} \right)^{p/2} < \infty \right\}.$$

Proof. Example 3.3 (i) and Corollary 3.2 yield, with $\alpha = \beta = 1$ and $m_n = n^4 - n^2$,

$$S(A^p_\mu) = \Bigg\{ g \in H(\mathbb{D}) :$$
$$\sum_{n=1}^{\infty} e^{-n^2} \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) \left(\sum_{k=[(n^4-n^2)/p]+1}^{[((n+1)^4-(n+1)^2)/p]} |\hat{g}(k)|^2 \left(1 - \frac{1}{n^2}\right)^{2k} \right)^{p/2} < \infty \Bigg\}$$

if $2 < p < \infty$ and

$$s(A^p_\mu) = \Bigg\{ g \in H(\mathbb{D}) :$$
$$\sum_{n=1}^{\infty} e^{-n^2} \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) \left(\sum_{k=[(n^4-n^2)/p]+1}^{[((n+1)^4-(n+1)^2)/p]} |\hat{g}(k)|^2 \left(1 - \frac{1}{n^2}\right)^{2k} \right)^{p/2} < \infty \Bigg\}$$

if $1 < p < 2$. If we let k run, in the preceding summations, from $[n^4/p]+1$ to $[(n+1)^4/p]$ instead then we obtain conditions which are equivalent to the preceding ones and hence characterize again $S(A^p_\mu)$ and $s(A^p_\mu)$. This follows from

$$n^4 - n^2 \leq n^4 \leq (n+1)^4 - (n+1)^2 \quad \text{for all } n.$$

(Compare this with Lemma 3.2. and Example 3.3 (i) in [3].) Then, finally, Corollary 3.4 follows from

$$\left(\frac{1}{2}\right)\frac{1}{n^3} \leq \frac{1}{n^2} - \frac{1}{(n+1)^2} \leq \frac{2}{n^3} \qquad \text{for all } n.$$

□

Corollary 3.5. *Let $R = \infty$ and $d\mu(r) = e^{-r}dr$.*
(a) If $2 < p < \infty$, then

$$S(A^p_\mu) = \left\{ g \in H(\mathbb{C}) : \sum_{n=1}^{\infty} e^{-n^2} 2n \left(\sum_{k=[n^2/p]+1}^{[(n+1)^2/p]} |\hat{g}(k)|^2 n^{2k} \right)^{p/2} < \infty \right\}.$$

(b) If $1 < p < 2$, then

$$s(A^p_\mu) = \left\{ g \in H(\mathbb{C}) : \sum_{n=1}^{\infty} e^{-n^2} 2n \left(\sum_{k=[n^2/p]+1}^{[(n+1)^2/p]} |\hat{g}(k)|^2 n^{2k} \right)^{p/2} < \infty \right\}.$$

Proof. It is a consequence of Example 3.3 (iii) and Corollary 3.2. □

Lemma 3.6. *Let $1 \leq p < \infty$, $0 < r < s$ and $f(z) = \sum_{m \leq j \leq n} \alpha_j z^j$ for some α_j and $0 \leq m < n$. Then we have*

$$\text{(i)} \qquad M_p(f,r) \leq \left(\frac{r}{s}\right)^m M_p(f,s)$$

and

$$\text{(ii)} \qquad M_p(f,s) \leq \left(\frac{s}{r}\right)^n M_p(f,r).$$

Proof. Part (i) follows from the fact that, for holomorphic f, the function $M_p(f,\cdot)$ is increasing in r while (ii) is Lemma 3.1. (i) of [8]. □

Now consider $1 < p < \infty$ and let m_n, I_n satisfy (b_0) and (3.1), (3.2).

Lemma 3.7. *Fix k, n and $r_{m_k} \leq r \leq r_{m_{k+1}}$. Then we have*

$$\text{(i)} \quad \left(\frac{r}{r_{m_n}}\right)^{m_n} \frac{v(r)}{v(r_{m_n})} \leq \left(\frac{1}{b}\right)^{n-k-1} \qquad \text{if } \ k < n$$

and

$$\text{(ii)} \quad \left(\frac{r}{r_{m_n}}\right)^{m_{n+1}} \frac{v(r)}{v(r_{m_n})} \leq K \left(\frac{1}{b}\right)^{k-n-1} \qquad \text{if } \ k \geq n.$$

Proof. If $k < n$ we have

$$\begin{aligned}&\left(\frac{r}{r_{m_n}}\right)^{m_n}\frac{v(r)}{v(r_{m_n})}=\\&\quad\left(\frac{r}{r_{m_{k+1}}}\right)^{m_n}\frac{v(r)}{v(r_{m_{k+1}})}\left(\frac{r_{m_{k+1}}}{r_{m_{k+2}}}\right)^{m_n}\frac{v(r_{m_{k+1}})}{v(r_{m_{k+2}})}\cdots\left(\frac{r_{m_{n-1}}}{r_{m_n}}\right)^{m_n}\frac{v(r_{m_{n-1}})}{v(r_{m_n})}\le\\&\quad\left(\frac{r}{r_{m_{k+1}}}\right)^{m_{k+1}}\frac{v(r)}{v(r_{m_{k+1}})}\left(\frac{r_{m_{k+1}}}{r_{m_{k+2}}}\right)^{m_{k+2}}\frac{v(r_{m_{k+1}})}{v(r_{m_{k+2}})}\cdots\left(\frac{r_{m_{n-1}}}{r_{m_n}}\right)^{m_n}\frac{v(r_{m_{n-1}})}{v(r_{m_n})}\\&\quad\le\left(\frac{1}{b}\right)^{n-k-1}\end{aligned}$$

If $k \geq n+1$ we have

$$\begin{aligned}&\left(\frac{r}{r_{m_n}}\right)^{m_{n+1}}\frac{v(r)}{v(r_{m_n})}=\\&\quad\left(\frac{r}{r_{m_k}}\right)^{m_{n+1}}\frac{v(r)}{v(r_{m_k})}\left(\frac{r_{m_k}}{r_{m_{k-1}}}\right)^{m_{n+1}}\frac{v(r_{m_k})}{v(r_{m_{k-1}})}\cdots\left(\frac{r_{m_{n+1}}}{r_{m_n}}\right)^{m_{n+1}}\frac{v(r_{m_{n+1}})}{v(r_{m_n})}\le\\&\quad\left(\frac{r}{r_{m_k}}\right)^{m_k}\frac{v(r)}{v(r_{m_k})}\left(\frac{r_{m_k}}{r_{m_{k-1}}}\right)^{m_{k-1}}\frac{v(r_{m_k})}{v(r_{m_{k-1}})}\cdots\left(\frac{r_{m_{n+2}}}{r_{m_{n+1}}}\right)^{m_{n+1}}\frac{v(r_{m_{n-1}})}{v(r_{m_n})}K\\&\quad\le K\left(\frac{1}{b}\right)^{k-n-1}\end{aligned}$$

Similarly, for $k = n$,

$$\left(\frac{r}{r_{m_n}}\right)^{m_{n+1}}\frac{v(r)}{v(r_{m_n})}\le\left(\frac{r_{m_{n+1}}}{r_{m_n}}\right)^{m_{n+1}}\frac{v(r_{m_{n+1}})}{v(r_{m_n})}\le K.$$

□

Now fix $k_0 > 0$ and $0 < \rho < b$ such that

$$\frac{I_n}{\min(I_{n-1}, I_{n+1})} \le \rho \quad \text{if} \;\; k \geq k_0. \tag{3.4}$$

Corollary 3.8. *Let $f_n(z) = \sum_{m_n/p \le j < m_{n+1}/p} \alpha_j z^j$ where $n \geq k_0$. Then, for any $k \geq k_0$ we have*

$$\int_{r_{m_k}}^{r_{m_{k+1}}} M_p^p(f_n, r)v(r)d\nu(r) \le c\left(\frac{\rho}{b}\right)^{|n-k|} M_p^p(f_n, r_{m_n})v(r_{m_n})I_n. \tag{3.5}$$

Here $c > 0$ is a universal constant independent of k, n, f_n.

Proof. First let $k < n$. Then Lemma 3.6 (i) and Lemma 3.7 (i) imply

$$\begin{aligned}
&\int_{r_{m_k}}^{r_{m_{k+1}}} M_p^p(f_n, r)v(r)d\nu(r) \\
&\qquad \le M_p^p(f_n, r_{m_n})v(r_{m_n}) \int_{r_{m_k}}^{r_{m_{k+1}}} \left(\frac{r}{r_{m_n}}\right)^{m_n} \frac{v(r)}{v(r_{m_n})} d\nu(r) \\
&\qquad \le c_0 M_p^p(f_n, r_{m_n})v(r_{m_n}) I_n \left(\prod_{j=k}^{n-1} \frac{I_j}{I_{j+1}}\right) \left(\frac{1}{b}\right)^{|n-k|} \\
&\qquad \le c_1 \left(\frac{\rho}{b}\right)^{|n-k|} M_p^p(f_n, r_{m_n})v(r_{m_n}) I_n,
\end{aligned}$$

where c_0, c_1 are universal constants. If $k \ge n$ then we use Lemma 3.6 (ii) and Lemma 3.7 (ii) to get

$$\begin{aligned}
&\int_{r_{m_k}}^{r_{m_{k+1}}} M_p^p(f_n, r)v(r)d\nu(r) \\
&\qquad \le M_p^p(f_n, r_{m_n})v(r_{m_n}) \int_{r_{m_k}}^{r_{m_{k+1}}} \left(\frac{r}{r_{m_n}}\right)^{m_{n+1}} \frac{v(r)}{v(r_{m_n})} d\nu(r) \\
&\qquad \le K b M_p^p(f_n, r_{m_n})v(r_{m_n}) I_n \left(\prod_{j=n}^{k-1} \frac{I_{j+1}}{I_j}\right) \left(\frac{1}{b}\right)^{|n-k|} \\
&\qquad \le c_2 \left(\frac{\rho}{b}\right)^{|n-k|} M_p^p(f_n, r_{m_n})v(r_{m_n}) I_n,
\end{aligned}$$

where c_2 is a universal constant. □

Conclusion of the proof of Theorem 3.1. Let $f \in A^p_\mu$, say $f = \sum_n f_n$ where f_n is as in Corollary 3.8. We can assume that $f_n = 0$ for $n \le k_0$ with k_0 as in (3.4).

To prove the right-hand inequality in Theorem 3.1 we use that $M_p(f_n, r) \le cM_p(f, r)$ for a universal constant independent of r, as well as that, in view of (3.1), $r^{m_n}v(r)$ is decreasing for $r \ge r_{m_n}$. We have

$$\begin{aligned}
&\sum_n M_p^p(f_n, r_{m_n})v(r_{m_n}) I_n \\
&\qquad \le \sum_n \int_{r_{m_n}}^{r_{m_{n+1}}} \left(\frac{r_{m_n}}{r}\right)^{m_n} \frac{v(r_{m_n})}{v(r)} M_p^p(f_n, r)v(r)d\nu(r) \\
&\qquad \le \sum_n \int_{r_{m_n}}^{r_{m_{n+1}}} \left(\frac{r_{m_n}}{r_{m_{n+1}}}\right)^{m_n} \frac{v(r_{m_n})}{v(r_{m_{n+1}})} M_p^p(f_n, r)v(r)d\nu(r) \\
&\qquad \le K \sum_n \int_{r_{m_n}}^{r_{m_{n+1}}} M_p^p(f_n, r)v(r)d\nu(r) \\
&\qquad \le c^p K \sum_n \int_{r_{m_n}}^{r_{m_{n+1}}} M_p^p(f, r)v(r)d\nu(r) \\
&\qquad \le c^p K ||f||_p^p.
\end{aligned}$$

This in particular implies that $\sum_n M_p^p(f_n, r_{m_n})v(r_{m_n})I_n < \infty$.

Now we show the left-hand inequality of Theorem 3.1. Using the Minkowski inequality in the first estimate and Corollary 3.8 in the second one, we obtain

$$\begin{aligned}
||f||_p^p &= \sum_k \int_{r_{m_k}}^{r_{m_{k+1}}} M_p^p(f,r)v(r)d\nu(r) \\
&\leq \sum_k \left(\sum_n \left(\int_{r_{m_k}}^{r_{m_{k+1}}} M_p^p(f_n,r)v(r)d\nu(r) \right)^{1/p} \right)^p \\
&\leq c_1 \sum_k \left(\sum_n \left(\frac{\rho}{b}\right)^{|n-k|/p} \big(M_p^p(f_n, r_{m_n})v(r_{m_n})I_n\big)^{1/p} \right)^p \\
&\leq c_2 \sum_k \sum_n \left(\frac{\rho}{b}\right)^{|n-k|/p} M_p^p(f_n, r_{m_n})v(r_{m_n})I_n \\
&\leq c_3 \sum_n M_p^p(f_n, r_{m_n})v(r_{m_n})I_n.
\end{aligned}$$

Here c_1, c_2, c_3 are universal constants. In the second last inequality we used the Hölder inequality in the following way: Put $a_n = \big(M_p^p(f_n, r_{m_n})v(r_{m_n})I_n\big)^{1/p}$. Then

$$\sum_n \left(\frac{\rho}{b}\right)^{|n-k|/p} a_n \leq \left(\sum_n \left(\frac{\rho}{b}\right)^{|n-k|/p} a_n^p \right)^{1/p} \cdot \left(\sum_n \left(\frac{\rho}{b}\right)^{|n-k|/p} \right)^{1/q},$$

with $1/p + 1/q = 1$. In the last inequality we interchanged the summation over k and n and utilized $\sup_k \sum_n (\rho/b)^{|n-k|/p} = \sup_n \sum_k (\rho/b)^{|n-k|/p} < \infty$. $\square$

4. **Solid Bergman spaces.**

Recall, a Bergman space A_μ^p is solid if $S(A_\mu^p) = A_\mu^p$.

Theorem 4.1. *Let $1 < p < \infty, p \neq 2$. Then the following are equivalent*

(i) A_μ^p is solid

(ii) $s(A_\mu^p) = A_\mu^p$

(iii) The monomials $(z^n)_{n=0}^\infty$ are an unconditional basis of A_μ^p

(iv) The normalized monomials $(z^n/||z^n||_p)_{n=0}^\infty$ are equivalent to the unit vector basis of l^p

(v) $\sup_n(l_{n+1} - l_n) < \infty$ *for the numbers l_n in (2.1)*

Remark 4.2. If $p = 2$ then the normalized monomials are an orthonormal basis for A_μ^2 and all conditions (i)-(iv) are satisfied.

The following example is relevant in connection with Theorem 4.1.

Example 4.3. Consider $R = \infty$ and $v(r) = \exp(-(\log r)^2)$, $d\nu(r) = dr$. (This is included in Example 2.2 of [9].) v is decreasing on $[1, \infty[$ which suffices in view of the remarks in the beginning of section 3. We easily see that $r_m = \exp(m/2)$

is the only zero of the derivative of $r^m v(r)$. Hence (3.1) is satisfied. We get for any $n > 0$ and $m > 0$

$$\left(\frac{r_m}{r_n}\right)^m \frac{v(r_m)}{v(r_n)} = \left(\frac{r_n}{r_m}\right)^n \frac{v(r_n)}{v(r_m)} = \exp\left(\frac{(n-m)^2}{4}\right).$$

So, if we take $m_n = 4n$ then condition (b_0) is satisfied with $b = e^4$. Moreover we have $I_n = \exp(2n+2) - \exp(2n)$. An easy calculation shows that (3.2) holds. Hence we can consider (2.1) with $l_n = m_n/p$. Therefore $\sup_n(l_{n+1} - l_n) = 4/p < \infty$. This means, for $d\mu(r) = v(r)dr$, the Bergman space A^p_μ is solid.

For the preceding example it is essential that $R = \infty$. Indeed, we have

Corollary 4.4. *Let $1 < p < \infty$, $p \neq 2$, and $R = 1$. Then no Bergman space A^p_μ is solid.*

We prove Corollary 4.4 at the end of this section. For the proof of Theorem 4.1 we need the following

Lemma 4.5. *Let (e_n) be a Schauder basis of a Banach space X with basis projections P_n. For $M \subset \mathbb{N}$, let T_M be the linear (not necessarily continuous) operator defined in the linear span of (e_n) by $T_M e_k = e_k$ if $k \in M$ and $T_M e_k = 0$ otherwise.*

If the basis (e_n) is not unconditional, then there is $N \subset \mathbb{N}$ such that, for any n, there exists m_n and $0 \neq y \in P_{m_n}X$ with $n||y|| \leq ||T_N y||$.

Proof. If (e_n) is a conditional basis then there exists an operator of the form T_N which is unbounded on X. Hence there is a sequence $x_k \in X$ with $||x_k|| = 1$ and $\lim_{k\to\infty} ||T_N x_k|| = \infty$ and we find k_n with $n = n||x_{k_n}|| < ||T_N x_{k_n}||$ for all n. Using $T_N P_l = P_l T_N$ for all l we find m_n such that

$$0 < n||P_{m_n} x_{k_n}|| \leq ||P_{m_n} T_N x_{k_n}|| = ||T_N P_{m_n} x_{k_n}|| \quad \text{ for all } n.$$

□

In the following we retain the definition of T_N with respect to the monomials (z^n).

Lemma 4.6. *Let $1 < p < \infty$, $p \neq 2$ and assume that there are constants $c_n > 0$, $d_n > 0$ with $\sup_n d_n/c_n < \infty$, integers $0 < a_n < b_n < a_{n+1}$ and radii s_n such that, for any $f_n \in A^p_\mu$ with $f_n(z) = \sum_{a_n \leq j \leq b_n} \alpha_j z^j$ we have*

$$c_n M_p(f_n, s_n) \leq ||f_n||_p \leq d_n M_p(f_n, s_n).$$

If $\sup_n(b_n - a_n) = \infty$ then the monomials are not unconditional in A^p_μ.

Proof. It is well known that the monomials are a conditional basis sequence with respect to the norm $M_p(\cdot, 1)$. So we find $N \subset \mathbb{N}$ and $y_n \in Y_n := \text{span}\,\{z^j : 0 \leq j \leq m_n\}$ with $M_p(y_n, 1) = 1$ and $n \leq M_p(T_N y_n, 1)$. Find k_n with $b_{k_n} - a_{k_n} > m_n$, put $Y_n = \{z^j : a_{k_n} \leq j \leq b_{k_n}\} \subset A^p_\mu$ and define $S_n : X_n \to Y_n$ by

$$(S_n f)(z) = z^{a_{k_n}} f(z/s_n).$$

Then, according to our assumptions we have $||S_n|| \cdot ||S_n^{-1}|| \leq d_n/c_n < c$ for some universal constant c. Put $M_n = \{a_{k_n} + j : j \in N, j \leq m_n\}$. Then $S_n T_N S_n^{-1} = T_{M_n}|_{X_n}$. If we consider $M = \cup_n M_n$ then the preceding shows that

T_M is unbounded on A^p_μ. This proves that the system of monomials is conditional in A^p_μ. □

Conclusion of the proof of Theorem 4.1. $(i) \Leftrightarrow (ii)$ follows from the definition of solid hull while $(ii) \Leftrightarrow (iii)$ follows from the definition of solid core. (Recall, in any case the monomials are a basis of A^p_μ.) Now (iii) and Lemma 4.6 imply (v). Finally, (v) and (2.1) imply (iv) while (iv) trivially implies (iii). □

Proof of Corollary 4.4. Proposition 3.5 of [8] shows that, for $R = 1$, the assumptions of Lemma 4.6 are always satisfied. Hence the system of monomials can never be unconditional. In view of Theorem 4.1 the Bergman space A^p_μ can never be solid. □

5. Solid weighted spaces of entire functions with sup-norms.

In this section we consider weighted Banach spaces of analytic functions with sup-norms. The main result Theorem 5.2 of this section complements Theorem 4.1. This result was announced in Remark 5.6 of [3]. Here, as in section 3, a continuous weight $v : \mathbb{C} \to]0,\infty[$ is a function satisfying

$$v(z) = v(|z|), z \in \mathbb{C}, \quad v(r) \geq v(s) \text{ if } 0 \leq r < s$$
$$\text{and } \lim_{r\to\infty} r^n v(r) = 0 \text{ for all } n \geq 0.$$

We deal with the weighted space H_v^∞ over $\mathbb{C}$, i.e.

$$H_v^\infty = \{f : \mathbb{C} \to \mathbb{C} : f \text{ holomorphic }, ||f||_v := \sup_{z\in\mathbb{C}} |f(z)|v(z) < \infty\}.$$

Let H_v^0 be the closure of the polynomials in H_v^∞.

Similarly to the weighted L_p-norms in section 3 and 4 one sees that it suffices to require only $v(r) \geq v(s)$ for $r_0 \leq r < s$ and some $r_0 > 0$ since $||f||_v$ and $\sup_{r_0\leq|z|<\infty} |f(z)|v(z)$ are equivalent for holomorphic f.

Again, for $n > 0$ let $r_n \in [0,\infty[$ be a point where the function $r \mapsto r^n v(r)$ attains its global maximum. The next lemma can be easily proved with induction (which was done in [9], Lemma 5.1.). The indices m_n are needed in the following.

Lemma 5.1. *For any $b > 2$ there are numbers $0 < m_1 < m_2 < \ldots$ with $\lim_{n\to\infty} m_n = \infty$ and*

$$b = \min\left(\left(\frac{r_{m_n}}{r_{m_{n+1}}}\right)^{m_n} \frac{v(r_{m_n})}{v(r_{m_{n+1}})}, \left(\frac{r_{m_{n+1}}}{r_{m_n}}\right)^{m_{n+1}} \frac{v(r_{m_{n+1}})}{v(r_{m_n})} \right).$$

Actually, one can show that Lemma 5.1 works for all $b > 1$ but we need $b > 2$ in the following proof.

There are examples of weights on $\mathbb{C}$ such that the monomials $(z^n)_{n=0}^\infty$ are a Schauder basis in the Banach space H_v^0. This is the same as saying that the Taylor series of each element in H_v^0 converges with respect to the weighted sup-norm $||\cdot||_v$. In the known examples, in this case, $(z^n/||z^n||_v)_{n=0}^\infty$ is equivalent to the unit vector basis of c_0. Moreover, here H_v^∞ is solid. We show that this is always true provided that $(z^n)_{n=0}^\infty$ is a Schauder basis of H_v^0. We also characterize this situation by a property for the indices m_n of Lemma 5.1. Our arguments are similar to those of [8].

Let $h(z) = \sum_{k=0}^{\infty} b_k z^k$. As before let P_n be the partial sum operators, i.e.

$$(P_n h)(z) = \sum_{k=0}^{n} b_k z^k.$$

If the monomials are a basis of H_v^0 then $\sup_n ||P_n|_{H_v^0}|| = \sup_n ||P_n|_{H_v^\infty}|| < \infty$. For any k we have

$$|b_k| \cdot ||z^k||_v = |b_k| r_k^k v(r_k) = \left| \frac{1}{2\pi} \int_0^{2\pi} h(r_k e^{i\varphi}) e^{-ik\varphi} d\varphi \right| v(r_k) \leq ||h||_v. \tag{5.1}$$

Moreover take the numbers m_n of Lemma 5.1 and put

$$(R_n h)(z) = \sum_{k=0}^{m_{n-1}} b_k z^k + \sum_{m_{n-1}<k\leq m_n} \frac{[m_n] - k}{[m_n] - [m_{n-1}]} b_k z^k.$$

Finally put $M_\infty(h, r) = \sup_{|z|=r} |h(z)|$.

Theorem 5.2. *The following are equivalent*

(i) $\sup_n (m_{n+1} - m_n) < \infty$ *where* m_n *are the indices of Lemma 5.1*

(ii) $(z^n)_{n=0}^{\infty}$ *is a Schauder basis of* H_v^0.

(iii) $(z^n/||z^n||_v)_{n=0}^{\infty}$ *is equivalent to the unit vector basis of* c_0.

(iv) H_v^∞ *is solid.*

(v) H_v^0 *is solid.*

Proof. Put $V_n = R_n - R_{n-1}$. According to Proposition 5.2 in [9], since we assumed $b > 2$ in Lemma 5.1, the norms $||h||_v$ and $\sup_n \sup_{r_{m_{n-1}} \leq r \leq r_{m_{n+1}}} M_\infty(V_n h, r) v(r)$ are equivalent. Since Lemma 3.3 in [9] implies that the operators V_n are uniformly bounded on H_v^∞, we obtain constants $c_1 > 0$ and $c_2 > 0$ with

$$c_1 \sup_n ||V_n h||_v \leq ||h||_v \leq c_2 ||V_n h||_v \quad \text{for all } h \in H_v^\infty. \tag{5.2}$$

(i) $\Rightarrow$ (ii): Observe that, by definition of V_n, dim $V_n(H_v^0) = [m_{n+1}] - [m_{n-1}]$. By (i) we obtain $\sup_n$dim $V_n(H_v^0) < \infty$. With the definition of P_j and (5.1) we see that $\sup_{j,n} ||P_j|_{V_n(H_v^0)}|| \leq \sup_n([m_{n+1}] - [m_{n-1}]) < \infty$. With (5.2) and $P_j V_n = V_n P_j$ for all j and n we conclude that the projections P_j are uniformly bounded. Hence $(z^n)_{n=0}^{\infty}$ is a Schauder basis of H_v^0.

(ii) $\Rightarrow$ (i): Assume that (ii) holds. By definition, $V_n(P_{m_{n+1}} - P_{m_{n-1}}) = V_n$. In view of the uniform boundedness of the V_n and (5.2) we obtain constants $c_1' > 0$ and $c_2' > 0$ with

$$c_1' \sup_n ||(P_{m_{n+1}} - P_{m_n})h||_v \leq ||h||_v \leq c_2' \sup_n ||(P_{m_{n+1}} - P_{m_n})h||_v \tag{5.3}$$

for all $h \in H_v^\infty$. Here the first inequality follows from the uniform boundedness of the P_n in view of (ii) while the second inequality follows from (5.2). Let $t_n \in [0, R[$ be such that

$$t_n = r_{m_n} \quad \text{if} \quad b = \left(\frac{r_{m_{n+1}}}{r_{m_n}}\right)^{m_{n+1}} \frac{v(r_{m_{n+1}})}{v(r_{m_n})}$$

and

$$t_n = r_{m_{n+1}} \quad \text{if} \quad b = \left(\frac{r_{m_n}}{r_{m_{n+1}}}\right)^{m_n} \frac{v(r_{m_n})}{v(r_{m_{n+1}})}$$

in Lemma 5.1 Then Corollary 3.2.(b) of [9] implies

$$||(P_{m_{n+1}} - P_{m_n})h||_v \leq 2bM_\infty((P_{m_{n+1}} - P_{m_n})h, t_n)v(t_n).$$

With (5.3) we obtain

$$\begin{aligned} d_1 \sup_n M_\infty((P_{m_{n+1}} - P_{m_n})h, t_n)v(t_n) \leq ||h||_v \leq \\ d_2 \sup_n M_\infty((P_{m_{n+1}} - P_{m_n})h, t_n)v(t_n) \end{aligned} \tag{5.4}$$

for some contants $d_1 > 0$, $d_2 > 0$ and all $h \in H_v^0$.

It is well-known that there are bounded holomorphic functions whose Taylor series do not converge with respect to $M_\infty(\cdot, 1)$. By going over to suitable Cesaro means if necessary, we see that, for each $n \in \mathbb{N}$, there is a polynomial f of degree N and an index $M \leq N$ such that

$$M_\infty(f, 1) = 1 \quad \text{ but } \quad n \leq M_\infty(P_M f, 1).$$

Proceeding by contradiction, assume that (i) does not hold, that is $\sup_n(m_{n+1} - m_n) = \infty$. Then we find k with dim $(P_{m_{k+1}} - P_{m_k})H_v^0 > N$. Put $h(z) = z^{m_k} f(z)/v(t_k)$. Then, in view of (5.4), we obtain

$$d_1 \leq ||h||_v \leq d_2 \quad \text{ and } \quad \frac{n}{d_2} \leq ||P_{M+m_k}h||_v.$$

This implies that the projections P_j are not uniformly bounded contradicting the assumption (ii). This contradiction implies $\sup_n(m_{n+1} - m_n) < \infty$, and we have checked that (ii) $\Rightarrow$ (i).

Moreover, if $\sup_n(m_{n+1} - m_n) < \infty$ then (5.4) easily implies that the normalized monomials are equivalent to the unit vector basis of c_0. Hence we have (ii) $\Rightarrow$ (iii). (iii) $\Rightarrow$ (ii) is trivial.

(iii) $\Rightarrow$ (iv): By the preceding we know already that (iii) implies (ii) and hence (5.4). If σ_n is the n'th Cesaro mean and $h \in H_v^\infty$ then $\sigma_n h \in H_v^0$. We have $\sigma_n P_j = P_j \sigma_n$ for all n and j. Moreover $||\sigma_n h||_v \leq ||h||_v$ and $\sup_n ||\sigma_n h||_v = ||h||_v$. This implies that (5.4) remains valid for all $h \in H_v^\infty$. This together with the fact that $\sup_n(m_{n+1} - m_n) < \infty$ shows that H_v^∞ is solid.

(iv) $\Rightarrow$ (iii) follows from Theorem 5.2 in [3]. (iv) $\Rightarrow$ (v): If $g \in S(H_v^0)$ then, by definition and (iii),

$$\lim_{n \to \infty} \hat{g}(n)||z^n||_v = 0$$

which implies by (iii) that $g \in H_v^0$.

(v) $\Rightarrow$ (iv): If $g \in S(H_v^\infty)$ then by definition $\sigma_n g \in S(H_v^0) = H_v^0$ for all n. This implies $g \in H_v^\infty$. □

In [9] it was shown that $v(r) = \exp(-(\log r)^2)$, $R = \infty$, satisfies (iv) (and hence all assertions) of Theorem 5.2.

Observe that nowhere in the preceding proof the fact that our functions are defined on $\mathbb{C}$ is used. The arguments work as well for weighted spaces of holomorphic functions over the unit disc $\mathbb{D}$. However in this case $\lim_{n\to\infty} r_n = 1$ and this fact together with

$$4 < b^2 \leq \left(\frac{r_{m_{n+1}}}{r_{m_n}}\right)^{m_{n+1}-m_n}$$

implies $\sup_n(m_{n+1} - m_n) = \infty$ (in view of Lemma 5.1 which remains true over $\mathbb{D}$). This means that in the case of holomorphic functions over $\mathbb{D}$ the preceding theorem is empty. Compare with Corollary 5.3 in [3].

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