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ON THE BOUNDEDNESS OF TOEPLITZ OPERATORS WITH RADIAL SYMBOLS OVER WEIGHTED SUP-NORM SPACES OF HOLOMORPHIC FUNCTIONS

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Abstract. We prove sufficient conditions for the boundedness and compactness of Toeplitz operators T_a in weighted sup-normed Banach spaces H_v^∞ of holomorphic functions defined on the open unit disc $\mathbb{D}$ of the complex plane; both the weights v and symbols a are assumed to be radial functions on $\mathbb{D}$. In an earlier work by the authors it was shown that there exists a bounded, harmonic (thus non-radial) symbol a such that T_a is not bounded in any space H_v^∞ with an admissible weight v. Here, we show that a mild additional assumption on the logarithmic decay rate of a radial symbol a at the boundary of $\mathbb{D}$ guarantees the boundedness of T_a.

The sufficient conditions for the boundedness and compactness of T_a, in a number of variations, are derived from the general, abstract necessary and sufficient condition recently found by the authors. The results apply for a large class of weights satisfying the so called condition (B), which includes in addition to standard weight classes also many rapidly decreasing weights.

1. Introduction and main results.

In the article [2] we studied Toeplitz operators T_a with radial symbols a on the analytic function spaces H_v^∞ on the unit disc $\mathbb{D} \subset \mathbb{C}$, endowed with weighted sup-norms for a large class of radial weights v satisfying the so called condition (B); this excludes the unweighted or constant weight case. In particular, in Theorem 3.6 of the citation (repeated in this paper in Theorem 2.1) we obtained a general sufficient and necessary condition for the boundedness and compactness of $T_a : H_v^\infty \to H_v^\infty$. Also, we observed that the boundedness of a non-radial symbol does not necessarily imply the boundedness of the Toeplitz operator. In fact, Theorem 2.3 of [2] contains an example of a bounded harmonic symbol a such that $T_a : H_v^\infty \to H_v^\infty$ is not bounded for any weight v under consideration.

The criterion for the boundedness of the Toeplitz operator in Theorem 3.6 of [2] is quite abstract, and it may not be easy to verify it for concrete weights and symbols. Some examples were presented in the citation under quite special assumptions either on the symbol or on the weight. Here, our aim is to use Theorem 3.6 of [2] to prove concrete sufficient conditions for the boundedness and compactness of $T_a : H_v^\infty \to H_v^\infty$. These conditions are much more general than in the examples of the citation, and the sufficient conditions for the symbol are easy to formulate and control. In all of our results we assume that the weight v satisfies condition (B) of [10], see also Definition 1.1 below, and a mild technical condition (1.1). These assumptions hold for example for the important classes of standard, normal and exponential weights (Proposition 1.3). Then, in the first main result, Theorem 1.2, we show that for the

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boundedness of $T_a : H_v^\infty \to H_v^\infty$ it suffices that the symbol a is differentiable near the unit circle and $\limsup a'$ or $-\limsup a'$ is bounded from above, and $a \to 0$ at a slow, logarithmic speed as $r \to 1$. In Theorem 1.4 the decay requirements for a are replaced by decay conditions on a'. In the case of normal weights the smoothness requirements of the symbol can be relaxed, see Theorem 1.6. Finally, in Theorem 1.7 we find a stronger decay condition for a which guarantees the boundedness of T_a in the case of exponential weights without a smoothness assumption on a.

All of these theorems also contain the analogous statements on the compactness of the Toeplitz operator. The proofs of Theorems 1.2 and 1.6 will be presented in Section 3 and that of Theorem 1.7 in Section 4.

We refer to the papers [3], [4], [5], [7], [8], [11], [12], [13], [14], [15], [16], [17], [19], [20], [21], [22], [23] for classical and recent results on the boundedness and compactness of Toeplitz operators on Bergman spaces

Let us turn to the exact definitions and formulation of the main results. By a weight v on the unit disc $\mathbb{D}$ we mean a continuous function with $v(z) = v(|z|)$ for all $z \in \mathbb{D}$, $\lim_{|z|\to 1} v(z) = 0$ and $v(r) \geq v(s)$ if $1 > s > r > 0$. Put

$$H_v^\infty = \big\{h : \mathbb{D} \to \mathbb{C} : h \text{ holomorphic},\ \|h\|_v := \sup_{z\in\mathbb{D}} |h(z)|v(|z|) < \infty\big\},$$
$$L_v^\infty = \big\{h : \mathbb{D} \to \mathbb{C} : h \text{ measurable },\ \|h\|_v := \operatorname*{ess\,sup}_{z\in\mathbb{D}} |h(z)|v(|z|) < \infty\big\}.$$

Let μ be the Lebesgue area measure on $\mathbb{D}$ endowed with v as density, i.e. $d\mu(re^{i\varphi}) = v(r)rdrd\varphi$ and denote the weighted L^p- and Bergman spaces by

$$L_v^p = \left\{g : \mathbb{D} \to \mathbb{C} : \|g\|_{p,v}^p := \int_{\mathbb{D}} |f|^p d\mu < \infty\right\} \text{ and } A_v^p = \{h \in L_v^p : h \text{ holomorphic}\},$$

where $1 \leq p < \infty$. In the unweighted case v is omitted in the notation.

Now let $a \in L^1$. We define the Toeplitz operator T_a with symbol a on H_v^∞ by $T_a h = P_v(a \cdot h)$ for $h \in H_v^\infty$, where $P_v : L_v^2 \to H_v^2$ is the orthogonal projection. Then $T_a h$ is a holomorphic function, at least if $a \cdot h \in L_v^2$. The definition of the Toeplitz operator in the present setting is discussed in detail in Section 1 of [2] and we do not wish to repeat the details here. However, we emphasize that even if $T_a h$ is a well a well defined analytic function, is not necessarily an element of H_v^∞ and T_a need not be a bounded operator.

In the following we consider radial symbols $a \in L^1$, i.e. functions with $a(z) = a(|z|)$ for almost all $z \in \mathbb{D}$. As for general notation, $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and c, C, C' denote generic positive constants, the exact value of which may change from place to place, but does not depend on the variables, indices or functions in the given expressions, unless otherwise indicated. By 1_A we denote the characteristic function of a set A, i.e. a function which equals 1 on A and 0 outside A; the domain of 1_A will be clear from the context. For other general terminology and definitions, see [6] and [22].

We will need the following definition. Let v be a weight on $\mathbb{D}$. Consider $m > 0$ and let r_m be a point where the function $r^m v(r)$ attains its absolute maximum on $[0, 1]$. It is easy to see that $r_n \geq r_m$ if $n \geq m$ and $\lim_{m\to\infty} r_m = 1$.

Definition 1.1. (i) The weight v satisfies condition (B), if

$$\forall b_1 > 1\ \exists b_2 > 1\ \exists c > 0\ \forall m, n > 0$$

$$\left(\frac{r_m}{r_n}\right)^m \frac{v(r_m)}{v(r_n)} \le b_1 \quad \text{and} \quad m, n, |m-n| \ge c \quad \Rightarrow \left(\frac{r_n}{r_m}\right)^n \frac{v(r_n)}{v(r_m)} \le b_2,$$

(ii) v is called *normal* if

$$\sup_{n\in\mathbb{N}} \frac{v(1-2^{-n})}{v(1-2^{-n-1})} < \infty \quad \text{and} \quad \inf_k \limsup_{n\to\infty} \frac{v(1-2^{-n-k})}{v(1-2^{-n})} < 1,$$

(iii) v is called an *exponential weight of type* (α, β) for some constants $\alpha > 0$ and $\beta > 0$ if $v(r) = \exp(-\alpha/(1-r)^\beta)$ for all r.

Note that the numbers m and n in (i) need not be integers. For example all normal weights as well as all exponential weights satisfy (B) (see [10]). Standard weights $(1-r)^\alpha$ and $(1-r^2)^\alpha$ are normal for all $\alpha > 0$, but no exponential weight is normal: the first condition in (ii) is not satisfied. Neither is the weight $v(r) = 1/(1-\log(1-r))$ is normal, since it decays too slowly to 0 in order to satisfy the second condition in (i).

We show

Theorem 1.2. *Let v satisfy (B) and assume that there is some $\epsilon > 0$ with*

$$\sup_{n=1,2,\dots} \frac{\int_0^1 r^{n-n^\epsilon} v(r)dr}{\int_0^1 r^n v(r)dr} < \infty. \tag{1.1}$$

Let $a \in L^1$ be real valued and radial such that the restriction $a|_{[\delta,1[}$ is differentiable for some $\delta \in]0,1[$ with

$$\limsup_{r\to 1} a'(r) < \infty \qquad \text{or} \qquad \liminf_{r\to 1} a'(r) > -\infty. \tag{1.2}$$

If

$$\limsup_{r\to 1} |a(r)\log(1-r)| < \infty \tag{1.3}$$

then T_a is a bounded operator $H_v^\infty \to H_v^\infty$.

If

$$\limsup_{r\to 1} |a(r)\log(1-r)| = 0 \tag{1.4}$$

then T_a is compact on H_v^∞.

Of course, Theorem 1.2 can be applied to complex valued symbols a as well. Here Re a and Im a have to satisfy the assumptions of the theorem.

We prove Theorem 1.2 in Section 3. Condition (B) and (1.1) are satisfied for many weights, in particular we have

Proposition 1.3. *All normal and exponential weights (see Definition 1.1) satisfy (B) and condition* (1.1)*.*

Indeed, it was proven in [10] that normal and exponential weights satisfy (B). Condition (1.1) with $\epsilon = 1/2$ follows for normal weights from Lemma 4.5. of [3]. The remaining claim in Proposition 1.3 about condition (1.1) for exponential weights will be proved in Section 4; see the remark after the proof of Corollary 4.2.

Examples. Assume that v is a weight satisfying (B). The symbol $a(r) = 1/(1-\log(1-r))$ satisfies the second condition (1.2) and, of course, (1.3) so that T_a :

$H_v^\infty \to H_v^\infty$ is bounded. The same is true for $a(r) = (1-r)^\delta$ with any $\delta > 0$, and this symbol even satisfies (1.4) so that T_a is compact on H_v^∞. (1.3). Moreover,

$$a(r) = \begin{cases} \log 2\,, & \text{if } 0 \le r \le 1/2, \\ -\log r\,, & \text{if } 1/2 < r < 1, \end{cases}$$

satisfies (1.2), (1.4) as well.

Next we present a reformulation of Theorem 1.2.

Theorem 1.4. *Let v satisfy (B) and (1.1). Let $a \in L^1$ be a radial symbol, and assume that $a|_{[\delta,1[}$ is differentiable for some $\delta \in]0,1[$, a' satisfies (1.2) and, for some constant $C > 0$, there holds the bound*

$$|a'(r)| \le \frac{C}{(1-r)\big(\log(1-r)\big)^2} \quad \text{for } r \in]\delta, 1[. \tag{1.5}$$

Then, T_a is a bounded operator $H_v^\infty \to H_v^\infty$. Moreover, if

$$\lim_{r\to 1} |a'(r)|(1-r)\big(\log(1-r)\big)^2 = 0 \tag{1.6}$$

holds, then T_a is compact, if and only if $\lim_{r\to 1} a(r) = 0$.

Proof. We can assume that a is real-valued (otherwise consider Re a and Im a separately). Assume (1.5) holds. For all $r \in]\delta, 1[$ we get by the change of the integration variable $\log(1-s) =: x$ and $dx/ds = -1/(1-s)$ that

$$\int_r^1 |a'(s)| ds \le C \int_r^1 \frac{1}{(1-s)\big(\log(1-s)\big)^2} ds = C \int_{-\infty}^{\log(1-r)} \frac{1}{x^2} dx = \frac{C}{|\log(1-r)|}. \tag{1.7}$$

Thus, we can extend a as a continuous function to $]\delta, 1]$ by defining

$$a(1) = \int_\delta^1 a'(s) ds + a(\delta) \ \big(= \lim_{r\to 1} a(r) \ \big),$$

and by (1.7) we obtain for all $r \in]\delta, 1[$

$$|a(r) - a(1)| = \Big| \int_r^1 a'(s) ds \Big| \le \frac{C}{|\log(1-r)|}. \tag{1.8}$$

This means, the function $a - a(1)$ satisfies (1.3) so that the Toeplitz operator $T_{a-a(1)}$ is bounded. Since $T_{a(1)}$ is a multiple of the identity, $T_a = T_{a-a(1)} + T_{a(1)}$ is bounded.

If (1.6) holds, then we can repeat the calculation (1.7)–(1.8) so that the constant C is replaced by a positive function $C(r)$ with $C(r) \to 0$ as $r \to 1$. Then, we see from the analogue of (1.8) that the function $a - a(1)$ even satisfies (1.4); hence the operator $T_{a-a(1)}$ is compact, and if in addition $a(1) = 0$ then also T_a is compact. If $\lim_{r\to 1} a(r) = a(1) \ne 0$, then T_a is a compact perturbation of a non-zero multiple of the identity which is not compact. $\Box$

All examples presented after Proposition 1.3 also satisfy the assumptions of Theorem 1.4.

The sufficient condition for the boundedness can be put into the following, very simple form. This should be compared with corresponding results for non-radial symbols in [3]: we proved that for holomorphic f on $\mathbb{D}$, the operator T_f is bounded, if and only f is bounded while there are bounded harmonic g on $\mathbb{D}$ where T_g is unbounded on H_v^∞.

Corollary 1.5. *If the symbol a is radial and continuously differentiable on $[0,1]$, then $T_a : H_v^\infty \to H_v^\infty$ is bounded.*

For normal weights we can relax the assumptions on a of Theorem 1.2 considerably.

Theorem 1.6. *Let v be a normal weight. If $a \in L^1$ is radial and satisfies (1.3) then T_a is a bounded operator $H_v^\infty \to H_v^\infty$.*

If a satisfies (1.4) then T_a is compact on H_v^∞.

We prove Theorem 1.6 in Section 3. There is a variant of Theorem 1.2 for exponential weights, too, without the restrictive smoothness requirements on a.

Theorem 1.7. *Let v be an exponential weight of type (α, β). Assume that $a \in L^1$ is radial and*

$$\limsup_{r\to 1} |a(r)|(1-r)^{-1/2-\beta/4} < \infty. \tag{1.9}$$

Then, T_a is a bounded operator $H_v^\infty \to H_v^\infty$.

If

$$\limsup_{r\to 1} |a(r)|(1-r)^{-1/2-\beta/4} = 0 \tag{1.10}$$

then T_a is compact on H_v^∞.

We prove Theorem 1.7 in Section 4.

2. Preliminaries.

To prove the theorems of Section 1 we need to recall some results of [10] and [2]. We refer to these papers for a more detailed exposition.

Let v be a weight on $\mathbb{D}$. Fix $b > 2$. We define by induction the indices $0 \le m_1 < m_2 < \dots$ such that

$$b = \min\left(\left(\frac{r_{m_n}}{r_{m_{n+1}}}\right)^{m_n} \frac{v(r_{m_n})}{v(r_{m_{n+1}})}, \left(\frac{r_{m_{n+1}}}{r_{m_n}}\right)^{m_{n+1}} \frac{v(r_{m_{n+1}})}{v(r_{m_n})}\right).$$

This is always possible according to Lemma 5.1. of [10]. (Actually it suffices to choose the indices such that the preceding minimum lies between b and some constant $b_1 > b$.) Formula (6.1) of [10] implies that

$$\sup_n \frac{m_{n+1} - m_n}{m_n - m_{n-1}} < \infty \tag{2.1}$$

so that we also have $\sup_n m_{n+1}/m_n < \infty$ and $\sup_n (m_{n+1} - m_{n-1})/m_{n-1} < \infty$.

Now let $h(\varphi) = \sum_{k\in\mathbb{Z}} b_k e^{ik\varphi}$ be a formal series with some numbers $b_k \in \mathbb{C}$ and $\varphi \in [0, 2\pi]$. Take the preceding numbers m_k and define for every $n \in \mathbb{N}$ the operator

$$\begin{aligned}(W_n h)(\varphi) \;&=\; \sum_{m_{n-1}<|k|\le m_n} \frac{|k| - [m_{n-1}]}{[m_n] - [m_{n-1}]} b_k e^{ik\varphi} + \sum_{m_n<|k|\le m_{n+1}} \frac{[m_{n+1}] - |k|}{[m_{n+1}] - [m_n]} b_k e^{ik\varphi} \\ &=: \sum_{k\in\mathbb{Z}} \beta_k b_k e^{ik\varphi} \qquad (2.2)\end{aligned}$$

with coefficients $\beta_k = \beta_k(n)$ satisfying $|\beta_k| \leq 1$ for all k and n. Here $[r]$ is the largest integer not larger than r. Obviously, $W_n h$ is always a continuous function $[0, 2\pi] \to \mathbb{C}$. The following is Theorem 3.6. of [2].

Theorem 2.1. *Let the weight v satisfy (B). If $a \in L^1$ is radial then T_a is bounded as operator $H_v^\infty \to H_v^\infty$ if and only if*

$$\sup_n \int_0^{2\pi} |(W_n f_a)(\varphi)| d\varphi < \infty$$

and T_a is a compact operator $H_v^\infty \to H_v^\infty$, if and only if

$$\lim_{n\to\infty} \int_0^{2\pi} |(W_n f_a)(\varphi)| d\varphi = 0.$$

Here, $f_a(\varphi)$ is for $\varphi \in [0, 2\pi]$ the formal series

$$f_a(\varphi) = \sum_{j=0}^{\infty} \gamma_j e^{ij\varphi} \qquad \textit{with} \quad \gamma_n = \frac{\int_0^1 r^{2n+1} v(r) a(r) dr}{\int_0^1 r^{2n+1} v(r) dr}.$$

We recall that for radial symbols the Toeplitz operators reduces into a Taylor coefficient multiplier: if $h(z) = \sum_{n=0}^{\infty} h_n z^n$, then $T_a(z) = \sum_{n=0}^{\infty} \gamma_n h_n z^n$.

Examples. If v is normal, then one can take $m_n = 2^{kn}$ for suitable fixed $k > 0$ (see [10], Example 2.4, and [9]).

For $v(r) = \exp(-\alpha/(1-r)^\beta)$ one can take $m_n = \beta^2(\beta/\alpha)^{1/\beta} n^{2+2/\beta} - \beta^2 n^2$, and $r_{m_n} = 1 - (\alpha/(\beta n^2))^{1/\beta}$. This follows from (3.15), (3.16) and (3.30) of [1]. (There is a misprint in Theorem 3.1. of [1], two times the exponent 2 is missing in the description of m_n.)

Corollary 2.2. *Let the weight satisfy (B) and assume that $a \in L^1$ is radial and satisfies $a|_{[s,1]} = 0$ for some $s \in]0, 1[$. Then $T_a : H_v^\infty \to H_v^\infty$ is compact.*

Proof. We have

$$\left| \frac{\int_0^1 a(r) r^k v(r) dr}{\int_0^1 r^k v(r) dr} \right| \leq \frac{\int_0^s |a(r)| r^k v(r) dr}{\int_{(1+s)/2}^1 r^k v(r) dr}$$
$$\leq \left(\frac{2s}{1+s} \right)^k \frac{\int_0^s |a(r)| v(r) dr}{\int_{(1+s)/2}^1 v(r) dr}.$$

Hence, with f_a as in Theorem 2.1,

$$\int_0^{2\pi} |(W_n f_a)(\varphi)| d\varphi \leq c_1(m_{n+1} - m_{n-1}) \left(\frac{2s}{1+s} \right)^{m_{n-1}} \leq c_2 m_{n-1} \left(\frac{2s}{1+s} \right)^{m_{n-1}}$$

for universal constants c_1, c_2. Here we used (2.1). The right-hand side goes to 0 as n goes to ∞. Hence Theorem 2.1 finishes the proof. $\square$

For $r > 0$ and an integrable function f on $r \cdot \partial\mathbb{D}$ we put

$$M_1(f, r) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\varphi})| d\varphi.$$

It is well-known that $M_1(f, r)$ is increasing with respect to r if f is a harmonic function.

Let R be the Riesz projection, $R : \sum_{k=-\infty}^{\infty} a_k r^{|k|} e^{ik\varphi} \mapsto \sum_0^\infty a_k r^{|k|} e^{ik\varphi}$. In the following we consider the Poisson kernel p,

$$p(re^{i\varphi}) = \sum_{k=-\infty}^{\infty} r^{|k|} e^{ik\varphi}, \quad \text{where } re^{i\varphi} \in \mathbb{D}.$$

It is well-known that $p \geq 0$ and that $M_1(p, r) = 1$ for all $r \in [0, 1[$. The following lemma will be needed later.

Lemma 2.3. *Let v satisfy condition (B) and consider the preceding numbers m_n and operators W_n. Then we have*

$$\sup_n \sup_{0 \leq r < 1} M_1(RW_n p, r) < \infty.$$

Proof. According to Lemma 3.3 of [10] we have

$$\begin{aligned} M_1(RW_n p, r) \leq 4\Big(\frac{[m_{n+1}] - [m_{n-1}]}{[m_n] - [m_{n-1}]}\Big)\Big(3 + 4\frac{[m_{n+1}] - [m_{n-1}]}{[m_{n+1}] - [m_n]}\Big) \\ \cdot\Big(1 + \frac{[m_{n+1}] - [m_{n-1}]}{[m_{n-1}]}\Big) M_1(p, r). \end{aligned}$$

Since $M_1(p, r) = 1$, the lemma follows in view of (2.1). □

3. Estimates for $\int_0^{2\pi} |(W_n f_a)(\varphi)| d\varphi$.

For the proofs of the theorems of Section 1 we will need the following estimate.

Proposition 3.1. *Let v be a weight which satisfies (B) and let m_n be the numbers defined above Theorem 2.1. Assume that $a \in L^1$ is radial. Then there is a universal constant $c > 0$ with*

$$\begin{aligned} \int_0^{2\pi} |(W_n f_a)(\varphi)| d\varphi \leq c \log(m_n) \cdot \Bigg(\Bigg| \frac{\int_0^1 a(r) r^{2[m_{n-1}]+1} v(r) dr}{\int_0^1 r^{2[m_{n-1}]+1} v(r) dr} \Bigg| \\ + \sum_{[m_{n-1}] < k \leq [m_{n+1}]} \Bigg| \frac{\int_0^1 a(r) r^{2k+1} v(r) dr}{\int_0^1 r^{2k+1} v(r) dr} - \frac{\int_0^1 a(r) r^{2k-1} v(r) dr}{\int_0^1 r^{2k-1} v(r) dr} \Bigg| \Bigg) \end{aligned} \tag{3.1}$$

and

$$\int_0^{2\pi} |(W_n f_a)(\varphi)| d\varphi \leq c \log(m_n) \frac{\int_0^1 |a(r)| r^{2[m_{n-1}]+1} v(r) dr}{\int_0^1 r^{2[m_{n+1}]+1} v(r) dr} \tag{3.2}$$

for all n large enough.

To prove Proposition 3.1 we need a lemma. Given $m \in \mathbb{N}$, let Q_m be the following projection acting on formal series (cf. (2.2)),

$$Q_m\Big(\sum_{l=0}^{\infty} b_l e^{il\varphi}\Big) = \sum_{l=0}^{m} b_l e^{il\varphi}.$$

It is well-known (see for example (2.7) in Ch. I, [18]) that

$$\int_0^{2\pi} \Big| Q_m\Big(\sum_{l=0}^{\infty} b_l e^{il\varphi}\Big)\Big| d\varphi \leq d \log m \int_0^{2\pi} \Big| \sum_{l=0}^{\infty} b_l e^{il\varphi}\Big| d\varphi \tag{3.3}$$

where the coefficients b_l for example form an ℓ^2-sequence so that the sum and the integral on the right converge and $d > 0$ is a constant independent of m.

Lemma 3.2. *Let $f(e^{i\varphi}) = \sum_{k=0}^n b_k e^{ik\varphi}$ for some $b_k \in \mathbb{C}$ and $n \in \mathbb{N}$, and let $g(e^{i\varphi}) = \sum_{k=0}^n \alpha_k b_k e^{ik\varphi}$ for some coefficients $\alpha_k \in \mathbb{C}$. Then,*

$$\int_0^{2\pi} |g(e^{i\varphi})| d\varphi \leq c \log n \Big(|\alpha_0| + \sum_{k=1}^n |\alpha_k - \alpha_{k-1}| \Big) \int_0^{2\pi} |f(e^{i\varphi})| d\varphi \tag{3.4}$$

where $c > 0$ is a constant independent of n and f.

Proof. We obtain, with $\beta_j = \alpha_j - \alpha_{j-1}$ for $j = 1, \dots, n$ and $\beta_0 = \alpha_0$,

$$g(e^{i\varphi}) = \sum_{j=0}^n \beta_j \sum_{k=j}^n b_k e^{ik\varphi}.$$

Hence

$$\int_0^{2\pi} |g(e^{i\varphi})| d\varphi \leq \left(|\alpha_0| + \sum_{k=1}^n |\alpha_k - \alpha_{k-1}| \right) \sup_j \int_0^{2\pi} \Big| \big((\mathrm{id} - Q_{j-1}) f \big)(e^{i\varphi}) \Big| d\varphi$$

from which we infer (3.4), in view of (3.3). $\square$

Proof of Proposition 3.1. Let f_a be again as in Theorem 2.1. We have

$$(W_n f_a)(e^{i\varphi}) = \sum_{k=[m_{n-1}]}^{[m_{n+1}]} \frac{\int_0^1 a(r) r^{2k+1} v(r) dr}{\int_0^1 r^{2k+1} v(r) dr} \cdot \beta_k e^{ik\varphi}$$

for certain β_k with $|\beta_k| \leq 1$ (where $\beta_{[m_{n-1}]} = \beta_{[m_{n+1}]} = 0$; see (2.2)). Now put $h(re^{i\varphi}) = \sum_{j=0}^{[m_{n+1}]-[m_{n-1}]} \beta_{j+m_{n-1}} r^j e^{ij\varphi}$ so that h is a polynomial, hence a holomorphic function. We obtain

$$M_1(h, r) \leq M_1(h, 1) = M_1(W_n Rp, 1) \quad \text{for } r \leq 1,$$

where p is the Poisson kernel. Since $W_n Rp$ is a polynomial we clearly find a radius $r(n) \in [0, 1[$ such that

$$M_1(W_n Rp, 1) \leq 2 M_1(W_n Rp, r(n)) \quad \text{for all } n. \tag{3.5}$$

We use Lemma 3.2 with $f(e^{i\varphi}) = h(re^{i\varphi})$ for fixed r, $b_j = \beta_{j+[m_{n-1}]}$ and

$$\alpha_j = \frac{\int_0^1 a(s) s^{2(j+[m_{n-1}])+1} v(s) ds}{\int_0^1 s^{2(j+[m_{n-1}])+1} v(s) ds}$$

and obtain

$$\int_0^{2\pi} |W_n f_a(e^{i\varphi})| d\varphi$$

$$\leq c \log([m_{n+1}] - [m_{n-1}]) \left(|\alpha_0| + \sum_{k=1}^{[m_{n+1}]-[m_{n-1}]} |\alpha_k - \alpha_{k-1}| \right) M_1(W_n Rp, 1).$$

Then Lemma 2.3 proves (3.1).

To show (3.2) we see that

$$\int_0^{2\pi} |W_n f_a(e^{i\varphi})| d\varphi$$

$$\leq \int_0^1 \int_0^{2\pi} \Bigg| \sum_{k=[m_{n-1}]}^{[m_{n+1}]} \frac{r^{2k+1} a(r) v(r)}{\int_0^1 s^{2k+1} v(s) ds} \cdot \beta_k e^{ik\varphi} \Bigg| d\varphi dr$$

$$= \int_0^1 |a(r)| r^{2[m_{n-1}]+1} v(r)$$
$$\cdot \int_0^{2\pi} \Big| \sum_{k=[m_{n-1}]}^{[m_{n+1}]} \frac{1}{\int_0^1 s^{2k+1} v(s) ds} \cdot r^{2(k-[m_{n-1}])} \beta_k e^{i(k-[m_{n-1}])\varphi} \Big| d\varphi dr.$$

Now put

$$\tilde{h}(re^{i\varphi}) = \sum_{j=0}^{[m_{n+1}]-[m_{n-1}]} \beta_{j+[m_{n-1}]} r^j e^{ij\varphi}.$$

Again we obtain

$$M_1(\tilde{h}, r) \leq M_1(\tilde{h}, 1) = M_1(W_n Rp, 1) \quad \text{for } r \leq 1.$$

We use Lemma 3.2 with $f(e^{i\varphi}) = \tilde{h}(re^{i\varphi})$ for fixed r, $b_j = \beta_{j+[m_{n-1}]} r^j$ and

$$\alpha_j = \Big(\int_0^1 s^{2(j+[m_{n-1}])+1} v(s) ds \Big)^{-1}.$$

Then α_j is increasing and

$$|\alpha_0| + \sum_{k=1}^{[m_{n+1}]-[m_{n-1}]} |\alpha_k - \alpha_{k-1}| = |\alpha_{[m_{n+1}]-[m_{n-1}]}|$$

The preceding estimate and (2.1), (3.4), (3.5) yield constants $c_1, c_2 > 0$ with

$$\int_0^{2\pi} |W_n f_a(e^{i\varphi})| d\varphi$$
$$\leq 2\pi c_1 \log([m_{n+1}] - [m_{n-1}]) \int_0^1 \frac{|a(r)| r^{2[m_{n-1}]+1}}{\int_0^1 s^{2[m_{n+1}]+1} v(s) ds} \cdot v(r) M_1(\tilde{h}, r) dr$$
$$\leq 2\pi c_2 \log([m_n]) \int_0^1 \frac{|a(r)| r^{2[m_{n-1}]+1}}{\int_0^1 s^{2[m_{n+1}]+1} v(s) ds} \cdot v(r) M_1(\tilde{h}, 1) dr.$$

Now, Lemma 2.3 also shows (3.2). □

We recall the following

Lemma 3.3. *Let v be normal. Then there is a universal constant $c > 0$ such that, for any k, m with $0 < k \leq m \leq 2k$, we have*

$$\frac{\int_0^1 r^k v(r) dr}{\int_0^1 r^m v(r) dr} \leq c.$$

Proof. This is Lemma 4.5. of [2].

Lemma 3.4. *For a function $a : [0,1] \to \mathbb{C}$, $\epsilon > 0$ and $\delta \in [0, 1[$ there are constants $c_1, c_2 > 0$ with*

$$(a) \qquad c_1 \sup_{n \geq 1/(1-\delta)} \sup_{\delta \leq r \leq 1} |a(r)| r^n \log n \leq \sup_{\delta \leq r < 1} |a(r) \log(1-r)|$$
$$\leq c_2 \sup_{n \geq 1/(1-\delta)} \sup_{\delta \leq r \leq 1} |a(r)| r^n \log n$$

(b)
$$c_1 \sup_{n\geq 1/(1-\delta)} \sup_{\delta\leq r\leq 1} |a(r)|r^n n^\epsilon \leq \sup_{\delta\leq r<1} |a(r)|/(1-r)^\epsilon \leq c_2 \sup_{n\geq 1/(1-\delta)} \sup_{\delta\leq r\leq 1} |a(r)|r^n n^\epsilon$$

Proof. Put $r = 1-1/n$, $n \geq 1/(1-\delta)$, and observe that $1/(1-1/n)^n$ is bounded. □

Proof of Theorem 1.6. The inequalities (1.3) or (1.4) and Lemma 3.4 imply that there is $\delta \in [0,1[$ such that $\sup_{\delta\leq r<1} |a(r)r^n| \leq c_0/\log n$ for all $n > 1$ and some constant c_0. Without loss of generality, we may assume that $\delta = 0$, otherwise we take $a_1 = a\cdot 1_{[\delta,1]}$ instead of a and use the fact that $a = a_1 + a_2$ where $a_2 = a\cdot 1_{[0,\delta]}$ yields the compact operator T_{a_2}. We apply Proposition 3.1. At first, we get

$$\frac{\int_0^1 |a(r)|r^{2m_{n-1}+1}v(r)dr}{\int_0^1 r^{2m_{n+1}+1}v(r)dr} \leq \Big(\sup_{0\leq r<1} |a(r)|r^{m_{n-1}}\Big)\frac{\int_0^1 r^{m_{n-1}+1}v(r)dr}{\int_0^1 r^{2m_{n+1}+1}v(r)dr}$$

According to (2.1) we have $\sup_k (m_{k+1}-m_k)/(m_k - m_{k-1}) < \infty$. This implies that $m_{n-1}+1 \leq m_{n+1}+1 \leq 2^q(m_{n-1}+1)$ for some $q \in \mathbb{N}$ which is independent of n. If we apply Lemma 3.3 q times we see that

$$\frac{\int_0^1 s^{m_{n-1}+1}v(s)ds}{\int_0^1 s^{2m_{n+1}+1}v(s)ds} \leq c^q$$

where c is the constant of Lemma 3.3. By (3.2) this shows, for some constant c_1

$$\int_0^{2\pi} |(W_n f_a)(\varphi)|d\varphi \leq c_1 \log(m_n)\left(\sup_{0\leq r<1} |a(r)|r^{m_{n-1}}\right)c^q \leq \frac{\log(m_n)}{\log(m_{n-1})}c^q$$

where we used (1.3) and Lemma 3.4(a). With (2.1) we see that $\sup_n \int_0^{2\pi} |(W_n f_a)(\varphi)|d\varphi < \infty$. If (1.4) holds then the same estimate yields

$$\lim_{n\to\infty} \int_0^{2\pi} |(W_n f_a)(\varphi)|d\varphi = 0.$$

So Theorem 1.6 follows from Theorem 2.1. □

In order to prove Theorem 1.2 we need

Lemma 3.5. *Let v be a weight on $\mathbb{D}$ and let $a : [0,1] \to \mathbb{R}_+$ be continuous and non-increasing. Then*

$$\frac{\int_0^1 a(r)r^k v(r)dr}{\int_0^1 r^k v(r)dr} \geq \frac{\int_0^1 a(r)r^{k+1}v(r)dr}{\int_0^1 r^{k+1}v(r)dr} \qquad \textit{for all} \quad k = 1,2,\ldots.$$

Proof. For $t \in [0,1]$ put

$$F(t) = \left(\int_0^t a(r)r^k v(r)dr\right)\left(\int_0^t r^{k+1}v(r)dr\right),$$
$$G(t) = \left(\int_0^t a(r)r^{k+1}v(r)dr\right)\left(\int_0^t r^k v(r)dr\right).$$

Then F and G are differentiable and the mean value theorem yields $s \in]0,1[$ with

$$\frac{F(1)-F(0)}{G(1)-G(0)} = \frac{F'(s)}{G'(s)}$$

(Here we can assume that a is not the zero function.) Hence

$$\begin{aligned}&\frac{\left(\int_0^1 a(r)r^k v(r)dr\right)\left(\int_0^1 r^{k+1}v(r)dr\right)}{\left(\int_0^1 a(r)r^{k+1}v(r)dr\right)\left(\int_0^1 r^k v(r)dr\right)}\\ &= \frac{\left(\int_0^s a(r)r^k v(r)dr\right)s^{k+1}v(s) + a(s)s^k v(s)\left(\int_0^s r^{k+1}v(r)dr\right)}{\left(\int_0^s a(r)r^{k+1}v(r)dr\right)s^k v(s) + a(s)s^{k+1}v(s)\left(\int_0^s r^k v(r)dr\right)}.\end{aligned}$$

Since a is non-increasing we have

$$s^k v(s)\int_0^s \big(a(s)-a(r)\big)r^k(s-r)v(r)dr \le 0$$

which implies

$$\begin{aligned}&\Big(\int_0^s a(r)r^k v(r)dr\Big)s^{k+1}v(s) + a(s)s^k v(s)\Big(\int_0^s r^{k+1}v(r)dr\Big)\\ &\ge \Big(\int_0^s a(r)r^{k+1}v(r)dr\Big)s^k v(s) + a(s)s^{k+1}v(s)\Big(\int_0^s r^k v(r)dr\Big).\end{aligned}$$

Since a is non-negative we obtain

$$\frac{F(1)-F(0)}{G(1)-G(0)} \ge 1$$

and hence

$$\frac{\int_0^1 a(r)r^k v(r)dr}{\int_0^1 r^k v(r)dr} \ge \frac{\int_0^1 a(r)r^{k+1}v(r)dr}{\int_0^1 r^{k+1}v(r)dr}. \qquad \square$$

Proof of Theorem 1.2. Let the symbol a satisfy the assumptions of Theorem 1.2. Let us assume that

$$\limsup_{r\to 1} a'(r) < \infty, \tag{3.6}$$

otherwise we can consider $-a$. We may even assume that a is differentiable on $[0,1[$. Indeed put

$$a_1(r) = \begin{cases} a(r) & r \in [\delta,1] \\ a(\delta) - a'(\delta)(\delta - r) & r \in [0,\delta] \end{cases}$$

Then a_1 is differentiable on $[0,1[$. Let $a_2 = a \cdot 1_{[0,\delta]}$ and $a_3 = a_1 \cdot 1_{[0,\delta]}$. According to Corollary 2.2, T_{a_2} and T_{a_3} are compact. Since $a = a_1 + a_2 - a_3$ it suffices to assume (by perhaps taking a_1 instead of a) that a is differentiable on $[0,1[$.

Moreover it suffices to assume that a is decreasing. Indeed otherwise consider

$$\tilde{a}(r) = a(r) + d(1-r) \tag{3.7}$$

instead of a where $d > 0$ is a constant so large that $\tilde{a}' < 0$ which exists in view of (3.6). The symbol $\tilde{a}$ satisfies (1.3) or (1.4), too. If we have proved Theorem 1.2 for $\tilde{a}$ then it is also correct for the symbol $1-r$. (Here take $a = 0$ in (3.7)). So let us assume that a is differentiable everywhere, satisfies (1.3) or (1.4) and is decreasing. Since $\lim_{r\to\infty} a(r) = 0$ we obtain $a(r) \ge 0$ for all r.

We use again the terminology of Theorem 2.1. In view of Lemma 3.5 with Proposition 3.1 we see that

$$\int_0^{2\pi} |(W_n f_a)(\varphi)| d\varphi$$
$$\leq c \log(m_n) \cdot \left(\frac{\int_0^1 a(r) r^{2[m_{n-1}]+1} v(r) dr}{\int_0^1 r^{2[m_{n-1}]+1} v(r) dr} + \frac{\int_0^1 a(r) r^{2[m_{n+1}]+1} v(r) dr}{\int_0^1 r^{2[m_{n+1}]+1} v(r) dr} \right) \tag{3.8}$$

With (1.1) we obtain, for $k = m_{n-1}$ and $k = m_{n+1}$,

$$\frac{\int_0^1 a(r) r^{2k+1} v(r) dr}{\int_0^1 r^{2k+1} v(r) dr} \leq c_1 \sup_r a(r) r^{k^\epsilon}$$

for a universal constant c_1. According to Lemma 3.4(a) and (1.3) we obtain a constant $c_2 > 0$ with

$$\sup_r a(r) r^{k^\epsilon} \leq c_2 (\epsilon \log k)^{-1}.$$

If we insert the last estimates in (3.8) we obtain $\sup_n \int_0^{2\pi} |(W_n f_a)(\varphi)| d\varphi < \infty$ and we can apply Theorem 2.1. If we even have (1.4), then the same estimates yield $\lim_{n\to\infty} \int_0^{2\pi} |(W_n f_a)(\varphi)| d\varphi = 0$ and again Theorem 2.1 finishes the proof. □

4. Exponential weights.

We now turn to the proof of Theorem 1.7. Let us fix an exponential weight v of type (α, β), i.e. $v(r) = \exp(-\alpha/(1-r)^\beta)$ for all r. By analyzing the function $j \mapsto j^{\beta+1} - j^\beta$ we see that for every $k > 0$ there is exactly one $j = j(k) > 1$ with

$$k = \alpha\beta(j^{\beta+1} - j^\beta). \tag{4.1}$$

With this notation we see that $r_k := 1 - 1/j$ is the unique maximum point of the function $f(r) = r^k v(r)$. Hence f is increasing for $0 \leq r \leq r_k$ and decreasing for $r_k \leq r < 1$. Moreover, in view of (4.1), there are constants $c_1, c_2 > 0$ with

$$c_1 k^{1/(\beta+1)} \leq j \leq c_2 k^{1/(\beta+1)} \quad \text{for all} \quad k \geq 1. \tag{4.2}$$

Proposition 4.1. *Let $k \geq 1$ and the number $j = j(k) > 1$ be as chosen above and let $0 < \delta < 1$. Then there is a constant $d > 0$, independent of k, such that*

$$\int_0^1 r^k \exp\Big(-\frac{\alpha}{(1-r)^\beta}\Big) dr \leq d \int_{1-1/(\delta j)}^1 r^k \exp\Big(-\frac{\alpha}{(1-r)^\beta}\Big) dr.$$

Proof. Fix $1 > \gamma > \delta$ and put $x = 1 - 1/(\delta j)$, $y = 1 - 1/(\gamma j)$. We may only consider large enough j (and hence k) such that $0 < x$ and $\delta j > 1$. Then we have $0 < x < y < 1$. We use

$$\exp\Big(-\frac{k}{t-1}\Big) \leq \Big(1 - \frac{1}{t}\Big)^k \leq \exp\Big(-\frac{k}{t}\Big)$$

whenever $1 < t$. This implies

$$\begin{aligned}\int_0^x r^k \exp\Big(-\frac{\alpha}{(1-r)^\beta}\Big) dr &\leq x^{k+1} \exp\Big(-\frac{\alpha}{(1-x)^\beta}\Big) \\ &\leq x^k \exp\Big(-\frac{\alpha}{(1-x)^\beta}\Big) \\ &\leq \exp\Big(-\alpha\Big(\frac{\beta}{\delta} + \delta^\beta\Big) j^\beta + \frac{\alpha\beta}{\delta} j^{\beta-1}\Big) =: u_1.\end{aligned}$$

Moreover we have

$$\int_y^{r_k} r^k \exp\Big(-\frac{\alpha}{(1-r)^\beta}\Big)dr \geq y^k \exp\Big(-\frac{\alpha}{(1-y)^\beta}\Big)\Big(1-\frac{1}{j}-y\Big)$$

$$\geq \exp\Big(-\alpha\beta\frac{j^{\beta+1}}{\gamma j-1}+\alpha\beta\frac{j^\beta}{\gamma j-1}-\alpha\gamma^\beta j^\beta\Big)\frac{1-\gamma}{\gamma j}$$

$$= \exp\Big(-\alpha\Big(\frac{\beta}{\gamma}+\gamma^\beta\Big)j^\beta-\frac{\alpha\beta}{\gamma}j^{\beta-1}\Big(\frac{1}{\gamma-1/j}\Big)$$

$$+\alpha\beta\frac{j^{\beta-1}}{\gamma-1/j}-\log\Big(\frac{\gamma j}{1-\gamma}\Big)\Big) =: u_2.$$

Put $g(t)=\beta/t+t^\beta$ for $t>0$. We easily see that g is decreasing for $0<t<1$. Moreover

$$\frac{u_1}{u_2} \leq \exp\Big(-\alpha\big(g(\delta)-g(\gamma)\big)j^\beta+d_1 j^{\beta-1}+d_1'\log j\Big)$$

for some universal constants d_1, d_1'. Since $g(\delta)-g(\gamma)>0$ this implies $\limsup_{r\to\infty} u_1/u_2<\infty$. Hence $u_1\leq d_2u_2$ for some universal constant d_2. We obtain

$$\int_0^1 r^k \exp\Big(-\frac{\alpha}{(1-r)^\beta}\Big)dr$$

$$= \int_0^x r^k \exp\Big(-\frac{\alpha}{(1-r)^\beta}\Big)dr + \int_x^1 r^k \exp\Big(-\frac{\alpha}{(1-r)^\beta}\Big)dr$$

$$\leq d_2\int_y^{r_k} r^k \exp\Big(-\frac{\alpha}{(1-r)^\beta}\Big)dr + \int_x^1 r^k \exp\Big(-\frac{\alpha}{(1-r)^\beta}\Big)dr$$

$$\leq (1+d_2)\int_x^1 r^k \exp\Big(-\frac{\alpha}{(1-r)^\beta}\Big)dr.$$

We finally put $d=1+d_2$. □

Corollary 4.2. *There is a constant $c>0$ such that*

$$\int_0^1 r^{k-k^{1/(\beta+1)}} \exp\Big(-\frac{\alpha}{(1-r)^\beta}\Big)dr \leq c\int_0^1 r^k \exp\Big(-\frac{\alpha}{(1-r)^\beta}\Big)dr$$

whenever $k\geq 1$.

Proof. It is enough to consider sufficiently large k. Let $l=l(k)$ be such that

$$k-k^{1/(\beta+1)}=\alpha\beta(l^{\beta+1}-l^\beta).$$

For $k\geq k_0$, k_0 sufficiently large, there is a constant $c_0>0$ such that

$$(k-k^{1/(\beta+1)})^{1/(\beta+1)}\geq c_0k^{1/(\beta+1)}.$$

Taking into account (4.2) for $k-k^{1/(\beta+1)}$ instead of k we find a constant $c_1>0$ with

$$l\geq c_1k^{1/(\beta+1)}. \tag{4.3}$$

Let $\delta=1/2$ and apply Proposition 4.1 for $k-k^{1/(\beta+1)}$ instead of k. Together with (4.3) this yields

$$\int_0^1 r^{k-k^{1/(\beta+1)}} \exp\Big(-\frac{\alpha}{(1-r)^\beta}\Big)dr$$

$$\leq d\int_{1-2/l}^1 r^{k-k^{1/(\beta+1)}} \exp\Big(-\frac{\alpha}{(1-r)^\beta}\Big)dr$$

$$\leq \frac{d}{(1-2/l)^{k^{1/(\beta+1)}}} \int_0^1 r^k \exp\Big(-\frac{\alpha}{(1-r)^\beta}\Big)dr$$
$$\leq \frac{d}{\big(1-2c_1k^{-1/(\beta+1)}\big)^{k^{1/(\beta+1)}}} \int_0^1 r^k \exp\Big(-\frac{\alpha}{(1-r)^\beta}\Big)dr.$$

In order to complete the proof it is enough to take c such that

$$\frac{d}{\big(1-2c_1k^{-1/(\beta+1)}\big)^{k^{1/(\beta+1)}}} \leq c. \quad \square$$

Corollary 4.2 proves (1.1) (with $\epsilon = 1/(\beta+1)$) for exponential weights and thus completes the proof of Proposition 1.3 .

Proof of Theorem 1.7. By possibly taking $a \cdot 1_{[\delta,1]}$ instead of a for suitable δ we can assume without loss of generality, in view of Lemma 3.4, that $\sup_{0\leq r\leq 1}|a(r)|r^k \leq c_0/k^{1/2+\beta/4}$ for all k and some constant c_0. To obtain the indices m_n of Theorem 2.1 we use (4.1) with $j = (\beta n^2/\alpha)^{1/\beta}$ (see (3.30), (3.15) and (3.16) of [1]). Hence

$$m_n = \frac{\beta^{2+1/\beta}}{\alpha^{1/\beta}} n^{2+2/\beta} - \beta^2 n^2. \tag{4.4}$$

Let f_a be again as in Theorem 2.1. We need to show that

$$\sup_n \int_0^1 |W_n f_a(e^{i\varphi})|d\varphi < \infty.$$

We have

$$(W_n f_a)(e^{i\varphi}) = \sum_{k=[m_{n-1}]}^{[m_{n+1}]} \frac{\int_0^1 a(r)r^{2k+1}v(r)dr}{\int_0^1 r^{2k+1}v(r)dr} \cdot \delta_k e^{ik\varphi}$$

for certain δ_k with $|\delta_k| \leq 1$.

The equality (4.4) implies that there is a constant $c_1 > 0$ such that

$$(2k+1)^{1/(\beta+1)} \geq c_1 n^{(2+2/\beta)/(\beta+1)} = c_1 n^{2/\beta} \tag{4.5}$$

for all $k \geq m_{n-1}$. Moreover, by an application of the mean value theorem to the function $n \mapsto m_n$ of (4.4) we may assume that

$$2(m_{n+1} - m_{n-1}) + 1 \leq c_2 n^{1+2/\beta}. \tag{4.6}$$

The remark in the beginning of the proof, Corollary 4.2, (4.5) and the assumption (1.9) on a yield a constant $c_3 > 0$ such that, for $m_{n-1} \leq k < m_{n+1}$, we have

$$\left|\frac{\int_0^1 a(r)r^{2k+1}v(r)dr}{\int_0^1 r^{2k+1}v(r)dr}\right|$$
$$\leq \Big(\sup_r |a(r)|r^{(2k+1)^{1/(\beta+1)}}\Big) \cdot \frac{\int_0^1 r^{2k+1-(2k+1)^{1/(\beta+1)}}v(r)dr}{\int_0^1 r^{2k+1}v(r)dr}$$
$$\leq c_3 \frac{1}{(n^{2/\beta})^{1/2+\beta/4}} = c_3 \frac{1}{n^{1/\beta+1/2}}.$$

This implies by (4.6)

$$\int_0^1 |W_n f_a(e^{i\varphi})|d\varphi \leq \Big(\int_0^1 |W_n f_a(e^{i\varphi})|^2 d\varphi\Big)^{1/2}$$

$$= \left(\sum_{k=m_{n-1}}^{m_{n+1}} \left| \frac{\int_0^1 a(r) r^{2k+1} v(r) dr}{\int_0^1 r^{2k+1} v(r) dr} \right|^2 |\delta_k|^2 \right)^{1/2}$$
$$\leq \left((2(m_{n+1} - m_{n-1}) + 1) \frac{c_3^2}{n^{1+2/\beta}} \right)^{1/2}$$
$$\leq c_2^{1/2} c_3.$$

Hence, $\sup_n \int_0^{2\pi} |(W_n f_a)(\varphi)| d\varphi < \infty$. So Theorem 2.1 concludes the first part of Theorem 1.7. If (1.10) holds then the same estimates as above show that $\lim_{n\to\infty} \int_0^{2\pi} |W_n f_a(e^{i\varphi})| d\varphi = 0$. Again, with Theorem 2.1 we see that then T_a is compact. □

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