# The spectrum of Volterra operators on Korenblum type spaces of analytic functions

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**Abstract**

The continuity, compactness and the spectrum of the Volterra integral operator $V_g$ with symbol an analytic function $g$, when acting on the classical Korenblum space and other related weighted Fréchet or (LB) spaces of analytic functions on the open unit disc, are investigated.

## 1 Introduction, notation and preliminaries

The aim of this note is to investigate the continuity and the spectrum of the Volterra integral operator when it acts on certain weighted Fréchet or (LB) spaces of analytic functions on the open unit disc $\mathbb{D}$ of the complex plane. Given a non-constant analytic function $g \in H(\mathbb{D})$, the *Volterra operator* $V_g$ *with symbol* $g$ is defined on the space $H(\mathbb{D})$ of analytic functions on the unit disc $\mathbb{D}$ by

$$V_g(f)(z) := \int_0^z f(\zeta)g'(\zeta)d\zeta \qquad (z \in \mathbb{D}).$$

We investigate the Volterra operator $V_g$ when acting on spaces which appear as intersections or unions of the growth Banach spaces of analytic functions defined below. The characterizations of the continuity and compactness of $V_g$ in terms of the symbol $g$ in this frame are presented in Proposition 2.2, 2.3, 2.5 and 2.6 in Section 2. The spectrum of $V_g$ is investigated in Section 3. It will be shown that the spectrum can be unbounded or not closed. Here one of our main results is Theorem 3.11. The behavior of the Volterra operator on the Korenblum space is different from the one in the other inductive limit spaces. Compare for example Proposition 2.2 with 2.3, and Proposition 3.9 with 3.10. The behavior of $V_g$ also differs from the known one when it acts between the growth Banach spaces used to define the unions or intersections. This case was thoroughly investigated by Malman [24].

In this article the space $H(\mathbb{D})$ is endowed with the Fréchet topology of uniform convergence on compact sets. When we write a space, we mean a Hausdorff locally convex space. We refer the reader to [25] for results and terminology about functional analysis, and in particular about Fréchet and (LB)-spaces.

The inspiration and motivation for the present research come from two sources. Malman [24] investigated recently the spectrum of Volterra operators on growth Banach spaces of analytic functions, and Albanese, Ricker and the author [2] studied the continuity,

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compactness and spectrum of the Cesàro operator in the same context of unions and intersections of growth spaces considered here. The Cesàro operator is closely related to Volterra operators. Our results and their proofs below require a different approach and new ingredients.

For each $\gamma > 0$, the growth Banach spaces of analytic functions are defined as

$$A^{-\gamma} := \{f \in H(\mathbb{D}) \colon \|f\|_{-\gamma} := \sup_{z\in\mathbb{D}} (1-|z|)^{\gamma}|f(z)| < \infty\}$$

and

$$A_0^{-\gamma} := \{f \in H(\mathbb{D}) \colon \lim_{|z|\to 1^-} (1-|z|)^{\gamma}|f(z)| = 0 \}.$$

The articles [8] and [24] define these spaces using the weight $(1-|z|^2)$ instead of $(1-|z|)$. Since $1-|z| \leq 1-|z|^2 \leq 2(1-|z|)$, the spaces are the same and the norms are equivalent. Both $A^{-\gamma}$ and $A_0^{-\gamma}$ are Banach spaces when endowed with the norm $\|.\|_{-\gamma}$. The space $A_0^{-\gamma}$, which is a closed subspace of $A^{-\gamma}$, coincides with the closure of the polynomials on $A^{-\gamma}$; see e.g. [10]. The space $H^\infty$ of bounded analytic functions on $\mathbb{D}$ is contained in $A_0^{-\gamma}$ for each $\gamma > 0$. These Banach spaces, as well as their intersections and unions, play a relevant and important role in connection with the interpolation and sampling of analytic functions; see [19, Section 4.3].

For each pair $0 < \mu_1 < \mu_2$ we have $A^{-\mu_1} \subset A_0^{-\mu_2}$, with the natural inclusion being continuous. Moreover, for each $\gamma > 0$, $A^{-\gamma}$ is canonically isomorphic to the bidual Banach space $(A_0^{-\gamma})''$ of $A_0^{-\gamma}$, [10, 28].

The weighted spaces of analytic functions we consider in this paper are defined as follows.

$$A_+^{-\gamma} := \cap_{\mu>\gamma} A^{-\mu} = \{f \in H(\mathbb{D}) \colon \sup_{z\in\mathbb{D}} (1-|z|)^{\mu}|f(z)| < \infty \ \forall \mu > \gamma\},$$

in which case also

$$A_+^{-\gamma} = \cap_{\mu>\gamma} A_0^{-\mu}.$$

for each $\gamma \geq 0$. And

$$A_-^{-\gamma} := \cup_{\mu<\gamma} A^{-\mu} = \{f \in H(\mathbb{D}) \colon \sup_{z\in\mathbb{D}} (1-|z|)^{\mu}|f(z)| < \infty \text{ for some } \mu < \gamma\}$$

in which case also

$$A_-^{-\gamma} = \cup_{\mu<\gamma} A_0^{-\mu},$$

for each $0 < \gamma \leq \infty$.

The space $A_+^{-\gamma}$ is a Fréchet space when endowed with the locally convex topology generated by the increasing sequence of norms $|||f|||_k := \sup_{z\in\mathbb{D}} (1-|z|)^{\gamma+\frac{1}{k}}|f(z)|$, for $f \in A_+^{-\gamma}$ and each $k \in \mathbb{N}$. We note, for $0 < \mu_1 < \mu_2$, that the natural inclusion $A^{-\mu_1} \subset A_0^{-\mu_2}$ is actually a *compact operator* between Banach spaces. Consequently, $A_+^{-\gamma}$ is a *Fréchet Schwartz* space, [21, §21.1 Example 1(b)]. In particular, bounded subsets of $A_+^{-\gamma}$ are relatively compact, [25, Remark 24.24]. Moreover, for every $\mu > \gamma > 0$ we have $A^{-\gamma} \subset A_+^{-\gamma} \subset A_0^{-\mu}$ with continuous inclusions.

Each space $A_-^{-\gamma}$ is endowed with the finest locally convex topology such that all the natural inclusion maps $A^{-\mu} \subset A_-^{-\gamma}$, for $\mu < \gamma$, are continuous. In particular, the space $A_-^{-\gamma} = \cup_{k\in\mathbb{N}} A^{-(\gamma-\frac{1}{k})}$ is the complete (DFS)-space

$$A_-^{-\gamma} := \operatorname*{ind}_k A^{-(\gamma-\frac{1}{k})} = \operatorname*{ind}_k A_0^{-(\gamma-\frac{1}{k})},$$

which is necessarily a *Schwartz* space, [25, Proposition 25.20]. Of course, the inductive limit is taken over all $k \in \mathbb{N}$ such that $(\gamma - \frac{1}{k}) > 0$. Furthermore, the (LB)-space $A_-^{-\gamma}$ is *regular*, i.e., every bounded set $B \subset A_-^{-\gamma}$ is contained and bounded in the Banach space $A^{-\mu}$ for some $0 < \mu < \gamma$.

The *Korenblum space* $A_-^{-\infty}$, [22], denoted $A^{-\infty}$, is defined by

$$A^{-\infty} := \cup_{0<\gamma<\infty} A^{-\gamma} = \cup_{n\in\mathbb{N}} A^{-n}$$

and is endowed with the finest locally convex topology such that all the natural inclusion maps $A^{-n} \subset A^{-\infty}$ are continuous, that is, $A^{-\infty} = \operatorname{ind}_n A^{-n}$. It is well-know that $A^{-\infty}$ is an algebra with continuous pointwise multiplication which is closed under derivation and integration. Further information can be found in [19, Section 4.3]. We mention, for all $\gamma > 0$, that $A_-^{-\gamma} \subset A_0^{-\gamma} \subset A^{-\gamma} \subset A_+^{-\gamma}$ with continuous inclusions.

If $g(z) = z, z \in \mathbb{D}$, the Volterra operator $V_g$ coincides with the integration operator, which we denote by $J$. If one takes $g(z) = -\log(1-z), z \in \mathbb{D}$, then $C(f)(z) := (1/z)V_g(f)(z), z \in \mathbb{D}, z \neq 0$, and $C(f)(0) := f(0)$ is the Cesàro operator. This operator when acting on $A_+^{-\gamma}$ and $A_-^{-\gamma}$ was investigated in [2].

The functions $g$ and $g - g(0)$ define the same Volterra operator. This is why we assume sometimes that $g(0) = 0$. Clearly $V_g$ defines a continuous operator on $H(\mathbb{D})$. The Volterra operator for holomorphic functions on the unit disc was introduced by Pommerenke [27] and he proved that $V_g$ is bounded on the Hardy space $H^2$, if and only if $g \in BMOA$. Aleman and Siskakis [6] extended this result for $H^p, 1 \leq p < \infty$, and they considered later in [7] the case of weighted Bergman spaces. Volterra operators on weighted Banach spaces of holomorphic functions on the disc of type $H^\infty$ have been investigated recently in [8], thus extending results in [20] and [30].

Aleman, Constantin, Peláez and Persson [3], [4] and [5] investigated the spectra of Volterra and Cesàro operators on several spaces of holomorphic functions on the disc. See also [1]. The spectra of Volterra operators acting on growth spaces has been recently investigated by Malman [24]. Constantin started in [13] the study of the Volterra operator on spaces of entire functions. She characterized the continuity of $V_g$ on the classical Fock spaces and investigated its spectrum. The investigation of the spectrum of the Volterra operator for weighted spaces of entire functions was continued in [12] and [14].

## 2 Continuous Volterra operators on $A_+^{-\gamma}$ and $A_-^{-\gamma}$

The Volterra operator $V_g, g \in H(\mathbb{D})$, is continuous (respectively compact) on $A^{-\alpha}$ and, equivalently on $A_0^{-\alpha}$, if and only if $g$ belongs to the Bloch space $\mathcal{B}$ (respectively to the little Bloch space $\mathcal{B}_0$). This was proved by Hu [20] and Stević [30]; see also Proposition 3.1 in [24]. Recall that a function $g \in H(\mathbb{D})$ belongs to $\mathcal{B}$ if and only if $\sup_{z\in\mathbb{D}}(1-|z|)|g'(z)| < \infty$, and $g$ belongs to $\mathcal{B}_0$ if and only if $\lim_{|z|\to 1}(1-|z|)|g'(z)| = 0$. We refer the reader to [32] for more information about Bloch spaces. Our next lemma is a consequence of Theorem 1 in [8] and it is stated for further reference below.

**Lemma 2.1** *Let $g \in H(\mathbb{D})$ be an analytic function and let $\alpha > 0$ and $\beta > 0$.*

*(i) The operator $V_g : A^{-\alpha} \to A^{-\beta}$ is continuous if and only if $V_g : A_0^{-\alpha} \to A_0^{-\beta}$ is continuous and if and only if*

$$\sup_{z\in\mathbb{D}}(1-|z|)^{\beta-\alpha+1}|g'(z)| < \infty.$$

*(ii) If $\alpha < \beta$, then $V_g : A^{-\alpha} \to A^{-\beta}$ is continuous if and only if $V_g : A_0^{-\alpha} \to A_0^{-\beta}$ is continuous and if and only if $g \in A^{-(\beta-\alpha)}$.*

**Proof.** (i) It is well known that the weight $V(z) = (1-|z|)^\gamma, \gamma > 0$, is essential in the sense of Section 2 in [8] and [9]. Therefore Theorem 1 in [8] implies that $V_g : A^{-\alpha} \to A^{-\beta}$ is continuous (or equivalently $V_g : A_0^{-\alpha} \to A_0^{-\beta}$ is continuous) if and only if $\sup_{z\in\mathbb{D}}(1-|z|)^{\beta-\alpha+1}|g'(z)| < \infty$.

(ii) If $\alpha < \beta$, part (i) implies that the continuity of $V_g$ is equivalent to $g' \in A^{-(\beta-\alpha+1)}$. Since $\beta - \alpha > 0$, we can apply a classical result of Hardy and Littlewood, see Theorem 5.5 in [16], to get that this is equivalent to $g \in A^{-(\beta-\alpha)}$. □

**Proposition 2.2** *Let $g \in H(\mathbb{D})$ be an analytic function.*

*(1) Let $0 \le \gamma < \infty$. The operator $V_g : A_+^{-\gamma} \to A_+^{-\gamma}$ is continuous if and only if $g \in A_+^{-0}$.*

*(2) Let $0 < \gamma < \infty$. The operator $V_g : A_-^{-\gamma} \to A_-^{-\gamma}$ is continuous if and only if $g \in A_+^{-0}$.*

**Proof.** (1) Since $A_+^{-\gamma} = \cap_{\mu>\gamma} A_0^{-\mu}$, and the polynomials are dense in each $A_0^{-\mu}$, the Fréchet space $A_+^{-\gamma}$ is the reduced projective limit of the Banach spaces $A_0^{-\mu}, \mu > \gamma$. Accordingly, $V_g : A_+^{-\gamma} \to A_+^{-\gamma}$ is continuous if and only if for each $\mu > \gamma$ there is $\gamma < \alpha < \mu$ such that $V_g : A_0^{-\alpha} \to A_0^{-\mu}$ is continuous.

Assume that $g \in A^{-\eta}$ for each $\eta > 0$. Given $\mu > \gamma$, take any $\gamma < \alpha < \mu$. By Lemma 2.1 (ii), $V_g : A_0^{-\alpha} \to A_0^{-\mu}$ is continuous since $g \in A^{-(\mu-\alpha)}$. Therefore $V_g : A_+^{-\gamma} \to A_+^{-\gamma}$ is continuous.

Now suppose that $V_g : A_+^{-\gamma} \to A_+^{-\gamma}$ is continuous. Fix $\eta > 0$. Given $\mu := \gamma + \eta$, there is $\gamma < \alpha < \mu$ such that $V_g : A_0^{-\alpha} \to A_0^{-\mu}$ is continuous. By Lemma 2.1 (ii), $g \in A^{-(\mu-\alpha)}$. Since $\mu - \alpha < \eta$, we have $g \in A^{-\eta}$. Since $\eta > 0$ is arbitrary, $g \in A_+^{-0}$.

(2) It follows from Grothendieck's factorization theorem (see Theorem 24.33 in [25]) that the operator $V_g : A_-^{-\gamma} \to A_-^{-\gamma}$ is continuous if and only if for each $0 < \alpha < \gamma$ there is $\alpha < \mu < \gamma$ such that $V_g : A^{-\alpha} \to A^{-\mu}$ is continuous. The rest of the argument is very similar to the Fréchet case (1). □

The function $g(z) = -\log(1-z), z \in \mathbb{D}$ is unbounded and $V_g$ defines a continuous operator on $A^{-\alpha}$ for each $\alpha > 0$, hence on $A_+^{-\gamma}, \gamma \ge 0$, and on $A_-^{-\gamma}, \gamma > 0$.

Define the weight $v_{log}(z) := (\log(e/(1-|z|))^{-1}$, $z \in \mathbb{D}$. We say that $g \in H(\mathbb{D})$ has *logarithmic mean growth* if $\sup_{z\in\mathbb{D}} v_{log}(z)|g(z)| < \infty$. It is well known that every analytic function $g$ in the Bloch space $\mathcal{B}$ has logarithmic mean growth, that is $g$ satisfies $\sup_{z\in\mathbb{D}} v_{log}(z)|g(z)| < \infty$. See e.g. page 106 in [32]. There are analytic functions $g \notin \mathcal{B}$ which have logarithmic mean growth. See for example Theorem 1.5 in [11] and the results in [17]. It is easy to see that every analytic function with logarithmic mean growth belongs to $A_+^{-0}$. Therefore the Volterra operator $V_g$ for such symbols $g$ is continuous on $A_+^{-\gamma}$ and $A_-^{-\gamma}$ for each $\gamma$, but it is not continuous on each $A^{-\alpha}$ for each $\alpha > 0$.

The space $H_{v_{log}}$ of all analytic functions with logarithmic mean growth is a Banach space for the norm $||g||_{\log} := \sup_{z\in\mathbb{D}} v_{log}(z)|g(z)|$, and it is contained but different from

$A_+^{-0}$. Indeed, if they were equal, the closed graph theorem would imply that the space is simultaneously a Banach space and a Fréchet Schwartz space, hence finite dimensional. A contradiction.

The case of the Korenblum space below should be compared with Proposition 3.2 in [12].

**Proposition 2.3** *Let $g \in H(\mathbb{D})$ be an analytic function. The operator $V_g : A^{-\infty} \to A^{-\infty}$ is continuous if and only if $g \in A^{-\infty}$.*

**Proof.** Grothendieck's factorization theorem can be applied to conclude that $V_g : A^{-\infty} \to A^{-\infty}$ is continuous if and only if for each $n \in \mathbb{N}$ there is $m \in \mathbb{N}, m > n$, such that $V_g : A^{-n} \to A^{-m}$ is continuous. The conclusion follows from Lemma 2.1 (ii). □

The Volterra operator $V_g$ for a symbol $g$ with logarithmic mean growth is continuous on $A^{-\infty}$, but it does not act (continuously) from each step $A^{-n}$ into itself.

An operator $T : X \to X$ on a space $X$ is called *compact* (resp. *bounded*) if there exists a neighbourhood $U$ of 0 such that $T(U)$ is a relatively compact (resp. bounded) subset of $X$. Every bounded operator is continuous. If the bounded subsets of $X$ are relatively compact, as it happens if $X$ is one of the spaces $A_+^{-\gamma}$ or $A_-^{-\gamma}$, then bounded and compact operators $T : X \to X$ coincide. The following lemma is well-known and it follows from the definitions involved.

**Lemma 2.4** (i) *Let $E := \mathrm{proj}_m E_m$ and $F := \mathrm{proj}_n F_n$ be Fréchet spaces which are projective limits of Banach spaces. Assume that $E$ is dense in $E_m$ and that $E_{m+1} \subset E_m$ with a continuous inclusion for each $m \in \mathbb{N}$ (resp. $F_{n+1} \subset F_n$ with a continuous inclusion for each $n \in \mathbb{N}$). Let $T : E \to F$ be a continuous linear operator. Then $T$ is bounded if and only if there exists $m_0 \in \mathbb{N}$ such that, for every $n \in \mathbb{N}$, the operator $T$ has a unique continuous linear extension $T_{m_0,n} : E_{m_0} \to F_n$.*

(ii) *Let $X = \mathrm{ind}_n X_n$ and $Y = \mathrm{ind}_m Y_m$ be two (LB)-spaces which are increasing unions of Banach spaces $X = \cup_{n=1}^{\infty} X_n$ and $Y = \cup_{m=1}^{\infty} Y_m$. Let $T\colon X \to Y$ be a continuous linear map. Assume that $Y$ is a regular (LB)-space. Then $T$ is bounded if and only if there exists $m \in \mathbb{N}$ such that $T(X_n) \subset Y_m$ and $T\colon X_n \to Y_m$ is continuous for all $n \geq m$.*

**Proposition 2.5** *Let $g \in H(\mathbb{D})$ be an analytic function.*

*(1) Let $0 \leq \gamma < \infty$. The operator $V_g : A_+^{-\gamma} \to A_+^{-\gamma}$ is compact if and only if there is $\varepsilon \in ]0,1[$ such that $\sup_{z\in\mathbb{D}}(1-|z|)^{1-\varepsilon}|g'(z)| < \infty$.*

*(2) Let $0 < \gamma < \infty$. The operator $V_g : A_-^{-\gamma} \to A_-^{-\gamma}$ is compact if and only if there is $\varepsilon \in ]0,1[$ such that $\sup_{z\in\mathbb{D}}(1-|z|)^{1-\varepsilon}|g'(z)| < \infty$.*

**Proof.** (1) First suppose that $V_g : A_+^{-\gamma} \to A_+^{-\gamma}$ is bounded (or equivalently compact in this case). By Lemma 2.4 (i) applied to $E = F = A_+^{-\gamma} = \cap_{\mu>\gamma} A_0^{-\mu}$, there is $\mu > \gamma$ such that for each $\gamma < \alpha < \mu$ the operator $V_g : A_0^{-\mu} \to A_0^{-\alpha}$ is continuous. Take any $\alpha$ with $\gamma < \alpha < \mu$ and apply Lemma 2.1 (i) to conclude that

$$\sup_{z\in\mathbb{D}}(1-|z|)^{1+\alpha-\mu}|g'(z)| < \infty.$$

The conclusion follows taking $\varepsilon := \mu - \alpha$.

Now assume that $\sup_{z\in\mathbb{D}}(1-|z|)^{1-\varepsilon}|g'(z)| < \infty$ for some $\varepsilon \in ]0,1[$. Given $\gamma \geq 0$, select $\mu := \gamma + \varepsilon$. For each $\gamma < \alpha < \mu$ we have

$$\sup_{z\in\mathbb{D}}(1-|z|)^{1+\alpha-\mu}|g'(z)| \leq \sup_{z\in\mathbb{D}}(1-|z|)^{1-\varepsilon}|g'(z)| < \infty.$$

This implies by Lemma 2.1 (i) that $V_g : A_0^{-\mu} \to A_0^{-\alpha}$ is continuous for each $\gamma < \alpha < \mu$. We can apply Lemma 2.4 (i) to conclude that $V_g : A_+^{-\gamma} \to A_+^{-\gamma}$ is bounded, hence compact.

(2) By Lemma 2.4 (ii) the operator $V_g : A_-^{-\gamma} \to A_-^{-\gamma}$ is compact, or equivalently bounded, if and only if there is $\mu < \gamma$ such that for each $\mu < \alpha < \gamma$ the operator $V_g : A^{-\alpha} \to A^{-\mu}$ is continuous. By Lemma 2.1 (i) this in turn is equivalent to $\sup_{z\in\mathbb{D}}(1-|z|)^{1+\mu-\alpha}|g'(z)| < \infty$ for each $\mu < \alpha < \gamma$. Observe that now $\mu - \alpha < 0$. The proof now is completed similarly as in part (1). $\square$

**Proposition 2.6** *Let $g \in H(\mathbb{D})$ be an analytic function. The operator $V_g : A^{-\infty} \to A^{-\infty}$ is compact if and only if $g$ is constant, that is $V_g = 0$.*

**Proof.** If $V_g : A^{-\infty} \to A^{-\infty}$ is compact, then it follows from Lemma 2.4 (ii) that there is $m \in \mathbb{N}$ such that for each $n \in \mathbb{N}, n > m$ the operator $V_g : A^{-n} \to A^{-m}$ is continuous. In particular, $V_g : A^{-(m+2)} \to A^{-m}$ is continuous. Now Lemma 2.1 (i) implies $\sup_{z\in\mathbb{D}}(1-|z|)^{-1}|g'(z)| < \infty$. That is, there is $M > 0$ such that $|g'(z)| \leq M(1-|z|)$ for each $z \in \mathbb{D}$. Cauchy inequalities (or the maximum modulus principle) imply that $g'(z) = 0$ for each $z \in \mathbb{D}$ and $g$ is constant. $\square$

If $g$ is a non-constant polynomial with $g(0) = 0$, then the Volterra operator is compact on $A_-^{-\gamma}$ for each $0 < \gamma < \infty$ but it is continuous and not compact on $A^{-\infty}$.

# 3 Spectra of Volterra operators on $A_+^{-\gamma}$ and $A_-^{-\gamma}$

Let $T : X \to X$ be a continuous operator on a space $X$. We write $T \in \mathcal{L}(X)$. The *resolvent set* $\rho(T)$ of $T$ consists of all $\lambda \in \mathbb{C}$ such that $R(\lambda, T) := (\lambda I - T)^{-1}$ is a continuous linear operator, that is $T - \lambda I : X \to X$ is bijective and has a continuous inverse. Here $I$ stands for the identity operator on $X$. The set $\sigma(T) := \mathbb{C} \setminus \rho(T)$ is called the *spectrum* of $T$. The *point spectrum* $\sigma_{pt}(T)$ of $T$ consists of all $\lambda \in \mathbb{C}$ such that $(\lambda I - T)$ is not injective. If we need to stress the space $X$, then we write $\sigma(T; X)$, $\sigma_{pt}(T; X)$ and $\rho(T; X)$. Unlike for Banach spaces $X$, it may happen that $\rho(T) = \emptyset$ or that $\rho(T)$ is not open. This is why some authors (see e.g. [31]) prefer to consider $\rho^*(T)$ instead of $\rho(T)$ consisting of all $\lambda \in \mathbb{C}$ for which there exists $\delta > 0$ such that $B(\lambda, \delta) := \{z \in \mathbb{C} \colon |z - \lambda| < \delta\} \subset \rho(T)$ and $\{R(\mu, T) \colon \mu \in B(\lambda, \delta)\}$ is equicontinuous in $\mathcal{L}(X)$. Define the *Waelbroeck spectrum* $\sigma^*(T) := \mathbb{C} \setminus \rho^*(T)$, which is a closed set containing $\sigma(T)$. If $T \in \mathcal{L}(X)$ with $X$ a Banach space, then $\sigma(T) = \sigma^*(T)$.

We recall a few preliminary results about the spectrum of the Volterra operator on $H(\mathbb{D})$ to be found in [3] and [4]; see also [12] for detailed proofs in the case of entire functions.

For each non-constant analytic function $g \in H(\mathbb{D})$ with $g(0) = 0$ the Volterra operator $V_g$ associated with $g$ acts continuously on $H(\mathbb{D})$.

**Proposition 3.1** *Let $g \in H(\mathbb{D})$ be a non-constant analytic function such that $g(0) = 0$.*

*(1) The operator $V_g - \lambda I : H(\mathbb{D}) \to H(\mathbb{D})$ is injective for each $\lambda \in \mathbb{C}$. In particular $\sigma_{pt}(V_g, H(\mathbb{D})) = \emptyset$. Moreover, $0 \in \sigma(V_g, H(\mathbb{D}))$.*

*(2) Given $\lambda \in \mathbb{C}$, $\lambda \neq 0$, and $h \in H(\mathbb{D})$, the equation $f - (1/\lambda)V_g f = h$ has a unique solution given by*

$$f(z) = R_{\lambda,g}h(z) = h(0)e^{\frac{g(z)}{\lambda}} + e^{\frac{g(z)}{\lambda}} \int_0^z e^{-\frac{g(\zeta)}{\lambda}} h'(\zeta)d\zeta, \qquad z \in \mathbb{D}.$$

*(3) The operator $V_g$ satisfies $\sigma(V_g, H(\mathbb{D})) = \{0\}$ and $\sigma_{pt}(V_g, H(\mathbb{D})) = \emptyset$.*

**Lemma 3.2** *Let $X \subset H(\mathbb{D})$ be a locally convex space that contains the constants and such that the inclusion $X \subset H(\mathbb{D})$ is continuous. Assume that $V_g : X \to X$ is continuous for some non-constant analytic function $g$ such that $g(0) = 0$. Then*

$$\{0\} \cup \{\lambda \in \mathbb{C} \setminus \{0\} \mid e^{\frac{g}{\lambda}} \notin X\} \subset \sigma(V_g, X).$$

**Lemma 3.3** *Let $X \subset H(\mathbb{D})$ be a locally convex space that contains the constants and such that the inclusion $X \to H(\mathbb{D})$ is continuous. Assume that $V_g : X \to X$ is continuous for some non-constant entire function $g$ such that $g(0) = 0$. The following conditions are equivalent:*

*(i) $\lambda \in \rho(V_g, X)$.*

*(ii) $R_{\lambda,g} : X \to X$ is continuous.*

*(iii) (a) $e^{\frac{g}{\lambda}} \in X$, and*

*(b) $S_{\lambda,g} : X_0 \to X_0$, $S_{\lambda,g}h(z) := e^{\frac{g(z)}{\lambda}} \int_0^z h'(\zeta)e^{-\frac{g(\zeta)}{\lambda}} d\zeta$, $z \in \mathbb{C}$, is continuous on the subspace $X_0$ of $X$ of all the functions $h \in X$ with $h(0) = 0$.*

The following two abstract lemmas about the spectrum of operators between Fréchet or (LB)-spaces will be useful later. We refer the reader to the Appendix of [2].

**Lemma 3.4** *Let $X = \cap_{n\in\mathbb{N}} X_n$ be a Fréchet space which is the intersection of a sequence of Banach spaces $((X_n, \|\cdot\|_n))_{n\in\mathbb{N}}$ satisfying $X_{n+1} \subset X_n$ with $\|x\|_n \leq \|x\|_{n+1}$ for each $n \in \mathbb{N}$ and $x \in X_{n+1}$. Let $T \in \mathcal{L}(X)$ satisfy the following condition:*

(A) *For each $n \in \mathbb{N}$ there exists $T_n \in \mathcal{L}(X_n)$ such that the restriction of $T_n$ to $X$ (resp. of $T_n$ to $X_{n+1}$) coincides with $T$ (resp. with $T_{n+1}$).*

*Then $\sigma(T;X) \subset \cup_{n\in\mathbb{N}}\sigma(T_n;X_n)$ and $R(\lambda,T)$ coincides with the restriction of $R(\lambda,T_n)$ to $X$ for each $n \in \mathbb{N}$ and each $\lambda \in \cap_{n\in\mathbb{N}}\rho(T_n;X_n)$.*

*Moreover, if $\cup_{n\in\mathbb{N}}\sigma(T_n;X_n) \subset \overline{\sigma(T;X)}$, then $\sigma^*(T;X) = \overline{\sigma(T;X)}$.*

**Lemma 3.5** *Let $E = \operatorname{ind}_n E_n$ be a Hausdorff inductive limit of Banach spaces. Let $T \in \mathcal{L}(E)$ satisfy the following condition:*

*(A) For each $n \in \mathbb{N}$ the restriction $T_n$ of $T$ to $E_n$ maps $E_n$ into itself and $T_n \in \mathcal{L}(E_n)$.*

*Then the following properties are satisfied.*

(i) $\sigma_{pt}(T;E) = \cup_{n\in\mathbb{N}}\sigma_{pt}(T_n;E_n)$.

(ii) $\sigma(T;E) \subset \cap_{m\in\mathbb{N}} \left(\cup_{n=m}^{\infty}\sigma(T_n;E_n)\right)$.

(iii) *If $\cup_{n=m}^{\infty}\sigma(T_n;E_n) \subset \overline{\sigma(T;E)}$ for some $m \in \mathbb{N}$, then*

$$\sigma^*(T;E) = \overline{\sigma(T;E)}.$$

**Proposition 3.6** *Let $g \in H^\infty$ be a bounded analytic function with $g(0) = 0$.*

*1) If $\gamma \geq 0$, then $\sigma(V_g, A_+^{-\gamma}) = \sigma^*(V_g, A_+^{-\gamma}) = \{0\}$.*

*(2) If $0 < \gamma \leq \infty$, then $\sigma(V_g, A_-^{-\gamma}) = \sigma^*(V_g, A_-^{-\gamma}) = \{0\}$.*

*The same statements hold if $g \in \mathcal{B}_0$.*

**Proof.** By Lemma 3.2, $\{0\}$ is contained in the spectrum of $V_g$ in $A_+^{-\gamma}$ and in $A_-^{-\gamma}$. If $g \in H^\infty$, the argument in the proof of Corollary 5.5 in [24] shows that $\sigma(V_g, A^{-\alpha}) = \{0\}$ for each $\alpha > 0$. The conclusion in part (1) follows from Lemma 3.4 and in part (2) from Lemma 3.5 (ii) and (iii).

If $g \in \mathcal{B}_0$, then $V_g$ is compact on each growth Banach space $A^{-\alpha}$ and its spectrum reduces to $\{0\}$; see [24]. $\square$

The following Lemma is a particular case of Theorem 5.6 in Malman [24].

**Lemma 3.7** *If $g(z) = c\log(1/(1 - \overline{w}z)), z \in \mathbb{D}$ with $c, w \in \mathbb{C}, c \neq 0, |w| = 1$, then $V_g : A^{-\alpha} \to A^{-\alpha}, \alpha > 0$, is continuous and $\sigma(V_g, A^{-\alpha}) = \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\frac{c}{\lambda}) \geq \alpha\}$.*

In the notation of Lemma 3.7, we understand $0 \in \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\frac{c}{\lambda}) \geq \alpha\}$. With this in mind, this set coincides with the disc $\{\lambda \in \mathbb{C} \mid |\lambda - \frac{c}{2\alpha}| \leq \frac{|c|}{2\alpha}\}$. Our next two results should be compared with Theorems 1.2 and 1.3 in [2].

**Proposition 3.8** *Let $g(z) = c\log(1/(1 - \overline{w}z)), z \in \mathbb{D}$ with $c, w \in \mathbb{C}, c \neq 0, |w| = 1$, For each $\gamma \geq 0$ the operator $V_g : A_+^{-\gamma} \to A_+^{-\gamma}$ is continuous. Moreover $\sigma(V_g, A_+^{-\gamma}) = \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\frac{c}{\lambda}) > \gamma\}$ and $\sigma^*(V_g, A_+^{-\gamma}) = \overline{\sigma(V_g, A_+^{-\gamma})}$.*

**Proof.** The operator $V_g : A_+^{-\gamma} \to A_+^{-\gamma}$ is continuous by Proposition 2.2 (1).

In order to determine the spectrum of $V_g$, observe that the analytic function $g$ satisfies

$$\left|e^{g(z)/\lambda}\right| \simeq \frac{1}{|1 - \overline{w}z|^{\mathrm{Re}(c/\lambda)}}.$$

See (1) in page 26 of [24].

Now take $\lambda \in \mathbb{C}$ such that $\gamma < \mathrm{Re}(c/\lambda)$ and select $\gamma < \beta < \mathrm{Re}(c/\lambda)$. Then $e^{g/\lambda} \notin A^{-\beta}$, hence $e^{g/\lambda} \notin A_+^{-\gamma}$. This implies $\lambda \in \sigma(V_g, A_+^{-\gamma})$ by Lemma 3.2. We have shown

$$\{\lambda \in \mathbb{C} \mid \mathrm{Re}(\frac{c}{\lambda}) > \gamma\} \subset \sigma(V_g, A_+^{-\gamma}).$$

On the other hand, we can apply Lemmas 3.4 and 3.7 to conclude

$$\sigma(V_g, A_+^{-\gamma}) \subset \bigcup_{k \in \mathbb{N}} \sigma(V_g, A^{-(\gamma + (1/k))}) =$$

$$= \bigcup_{k \in \mathbb{N}} \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\frac{c}{\lambda}) \geq \gamma + \frac{1}{k}\} = \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\frac{c}{\lambda}) > \gamma\}.$$

Thus $\sigma(V_g, A_+^{-\gamma}) = \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\frac{c}{\lambda}) > \gamma\}$. Moreover, we have

$$\bigcup_{k \in \mathbb{N}} \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\frac{c}{\lambda}) \geq \gamma + \frac{1}{k}\} \subset \overline{\sigma(V_g, A_+^{-\gamma})},$$

and we can apply Lemma 3.4 to conclude $\sigma^*(V_g, A_+^{-\gamma}) = \overline{\sigma(V_g, A_+^{-\gamma})}$. $\square$

If $\gamma = 0$ in Proposition 3.8, then $\sigma(V_g, A_+^{-0}) = \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\frac{c}{\lambda}) > 0\}$ is the union of $\{0\}$ and an open half plane of $\mathbb{C}$ with 0 at the boundary, which depends on $c$. Therefore it is unbounded. If $\gamma > 0$, then $\sigma(V_g, A_+^{-0})$ is bounded but not closed.

**Proposition 3.9** *Let $g(z) = c\log(1/(1-\overline{w}z)), z \in \mathbb{D}$ with $c, w \in \mathbb{C}, c \neq 0, |w| = 1$, For each $\gamma \in ]0,\infty[$ the operator $V_g : A_-^{-\gamma} \to A_-^{-\gamma}$ is continuous, and $\sigma(V_g, A_-^{-\gamma}) = \sigma^*(V_g, A_-^{-\gamma}) = \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\frac{c}{\lambda}) \geq \gamma\}$.*

**Proof.** It follows from Proposition 2.2 (2) that $V_g : A_-^{-\gamma} \to A_-^{-\gamma}$ is continuous.

Fix $\lambda \in \mathbb{C}$ such that $\gamma \leq \mathrm{Re}(c/\lambda)$. In particular $\alpha < \mathrm{Re}(c/\lambda)$ for each $\alpha < \gamma$. We apply the estimate (10) in [24] mentioned in the proof of Proposition 3.8 to conclude $e^{g/\lambda} \notin A^{-\alpha}$ for each $\alpha < \gamma$. This implies $e^{g/\lambda} \notin A_-^{-\gamma}$. Hence $\lambda \in \sigma(V_g, A_-^{-\gamma})$ by Lemma 3.2. We have shown

$$\{\lambda \in \mathbb{C} \mid \mathrm{Re}(\frac{c}{\lambda}) \geq \gamma\} \subset \sigma(V_g, A_-^{-\gamma}).$$

On the other hand, Lemmas 3.5 and 3.7 yield

$$\sigma(V_g, A_-^{-\gamma}) \subset \bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} \sigma(V_g, A^{-(\gamma-(1/n))}) = \bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\frac{c}{\lambda}) \geq \gamma - \frac{1}{n}\} =$$

$$= \bigcap_{m\in\mathbb{N}} \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\frac{c}{\lambda}) > \gamma - \frac{1}{m}\} = \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\frac{c}{\lambda}) \geq \gamma\}.$$

It remains to prove the statement about the Waelbroeck spectrum. We adapt an argument in [2, Proposition 2.9]. If $\lambda \notin \sigma(V_g, A_-^{-\gamma})$ there are $r > 0$ and $n(0) \in \mathbb{N}$ such that, for all $n \geq n(0)$,

$$\overline{B(\lambda, r)} \cap \{\lambda \mid \mathrm{Re}(c/\lambda) \geq \gamma - \frac{1}{n}\} = \emptyset.$$

Therefore $\overline{B(\lambda, r)} \subset \rho(V_g, A_-^{-\gamma})$, as well as $\overline{B(\lambda, r)} \subset \rho(V_g, A^{-(\gamma-(1/n))})$ for each $n \geq n(0)$. In particular, $\{R(\mu, V_g) \colon \mu \in \overline{B(\lambda, r)}\}$ is equicontinuous, that is, operator norm bounded in $\mathcal{L}(A^{-(\gamma-\frac{1}{n})})$ for every $n \geq n(0)$. We claim that the set $\{R(\mu, V_g) \colon \mu \in \overline{B(\lambda, r)}\}$ is equicontinuous in $\mathcal{L}(A_-^{-\gamma})$. This would verify that $\lambda \notin \sigma^*(V_g, A_-^{-\gamma})$. To establish the claim it suffices to show that $\{R(\mu, V_g)f \colon \mu \in \overline{B(\lambda, r)}\}$ is a bounded set in $A_-^{-\gamma}$ for all $f \in A_-^{-\gamma}$ by the Banach-Steinhaus principle, which is applicable since $A_-^{-\gamma}$ is a (DFS)-space. Fix $f \in A_-^{-\gamma}$. Then $f \in A^{-(\gamma-\frac{1}{n})}$ for some $n \geq n_0$. So, $\{R(\mu, V_g)f \colon \mu \in \overline{B(\lambda, r)}\}$ is a bounded set in $A_0^{-(\gamma-\frac{1}{n})}$ and hence, also in $A_-^{-\gamma}$. This completes the proof. □

For $A^{-\infty}$ we have the following complement of Propositions 3.6 (2) and 3.9 for symbols with logarithmic mean growth. Observe that in this case we cannot use in general Lemma 3.5, because the Volterra operator need not act from each step into itself.

**Proposition 3.10** *If $g \in H(\mathbb{D})$ is an analytic function with $g(0) = 0$ such that*

$$\sup_{z\in\mathbb{D}} v_{log}(z)|g(z)| < \infty, \text{ with } v_{log}(z) := (\log(e/(1-|z|))^{-1}, \ z \in \mathbb{D},$$

*then $V_g : A^{-\infty} \to A^{-\infty}$ is continuous and $\sigma(V_g, A^{-\infty}) = \sigma^*(V_g, A^{-\infty}) = \{0\}$.*

**Proof.** Proposition 2.3 implies that $V_g : A^{-\infty} \to A^{-\infty}$ is continuous. By Lemma 3.2, $\{0\}$ is contained in the spectrum of $V_g$ in $A^{-\infty}$. Fix $\lambda \neq 0$. Since $\sup_{z\in\mathbb{D}} v_{log}(z)|g(z)| \leq M$ for some $M > 0$, we have

$$|e^{g(z)/\lambda}| \leq \left(\frac{e}{(1-|z|}\right)^{M/|\lambda|}$$

for each $z \in \mathbb{D}$. Thus $e^{g/\lambda} \in A^{-\infty}$. The space $A^{-\infty}$ is and algebra with continuous multiplication which is closed under differentiation and under the integration operator. Therefore both conditions (a) and (b) in Lemma 3.3 (iii) are satisfied. Hence $\lambda \in \rho(V_g, A^{-\infty})$ and $\sigma(V_g, A^{-\infty})$ is $\{0\}$.

It remains to show that every $\lambda \neq 0$ belongs to $\rho^*(V_g, A^{-\infty})$. To see this, given $\lambda \neq 0$, select $\varepsilon > 0$ and $n(0) \in \mathbb{N}$ such that $|\mu| > 1/n(0)$ if $|\mu - \lambda| < \varepsilon$. We must prove that the set $\{R(\mu, V_g) \mid |\mu - \lambda| < \varepsilon\}$ is equicontinuous in $\mathcal{L}(A^{-\infty})$. Since $A^{-\infty} = \text{ind}_n A^{-n}$, it is enough to show that for each $n \in \mathbb{N}$ there are $m > n$ and $K > 0$ such that for each $h \in A^{-n}$

$$\sup_{z\in\mathbb{D}}(1-|z|)^m |R(\mu, V_g)h(z)| \leq K \sup_{z\in\mathbb{D}}(1-|z|)^n |h(z)|.$$

By assumption $\sup_{z\in\mathbb{D}} v_{log}(z)|g(z)| \leq M$ for some $M > 0$. Therefore, if $|\mu - \lambda| < \varepsilon$, since $|\mu| > 1/n(0)$, we get

$$|e^{\frac{g(z)}{\mu}}| \leq \left(\frac{e}{(1-|z|)}\right)^{n(0)M}.$$

Now, given $n \in \mathbb{N}$ we select $m \in \mathbb{N}$ with $m > n + 2n(0)M + 1$. By Proposition 3.1 (2) we get, for each $z \in \mathbb{D}$,

$$R(\mu, V_g)h(z) = (1/\mu)h(0)e^{\frac{g(z)}{\mu}} + (1/\mu)e^{\frac{g(z)}{\mu}} \int_0^z e^{-\frac{g(\zeta)}{\mu}} h'(\zeta)d\zeta.$$

For $z \in \mathbb{D}$ and $h \in A^{-n}$ we have

$$|R(\mu, V_g)h(z)| \leq n(0)|h(0)| \left(\frac{e}{(1-|z|)}\right)^{n(0)M} + n(0) \left(\frac{e}{(1-|z|)}\right)^{2n(0)M} |z| \max_{|\zeta|=|z|} |h'(\zeta)|.$$

The differentiation operator is continuous from $A^{-n}$ into $A^{-(n+1)}$ by Theorem 5.5 in [16] and the closed graph theorem, hence there is $C > 0$ such that $\sup_{z\in\mathbb{D}}(1-|z|)^{n+1}|h'(z)| \leq C \sup_{z\in\mathbb{D}}(1-|z|)^n |h(z)|$ for each $h \in A^{-n}$. This implies, for $h \in A^{-n}$,

$$\sup_{z\in\mathbb{D}}(1-|z|)^m |R(\mu, V_g)h(z)| \leq (n(0)e^{n(0)M} + Cn(0)e^{2n(0)M}) \sup_{z\in\mathbb{D}}(1-|z|)^n |h(z)|.$$

This completes the proof. □

It follows from Proposition 3.10 that $\sigma(V_g, A_+^{-\infty}) = \sigma^*(V_g, A_+^{-\infty}) = \{0\}$ for $g(z) = c\log(1/(1-\overline{w}z)), z \in \mathbb{D}$ with $c, w \in \mathbb{C}, c \neq 0, |w| = 1$. Compare with Proposition 3.9. Now we investigate the spectrum of $V_g$ on the Korenblum space $A_+^{-\infty}$ for symbols that are not of logarithmic mean growth.

**Theorem 3.11** *Let $g \in A^{-\infty}$ satisfy $g(0) = 0$. The following conditions are equivalent.*

*(1) The function $g$ is of logarithmic mean growth.*

*(2) $\sigma^*(V_g, A_+^{-\infty}) = \{0\}$*

*(4) $\sigma^*(V_g, A_+^{-\infty})$ is bounded.*

**Proof.** Condition (1) implies (2) by proposition 3.10. Clearly (2) implies (3). We show that (3) implies (1). Assume that $\sigma^*(V_g, A_+^{-\infty})$ is bounded. There is $\delta > 0$ such that $\lambda \in \rho^*(V_g, A_+^{-\infty})$ for each $\lambda \in \mathbb{C}$ with $|\lambda| = \delta$. The definition of the complement of Waelbroeck spectrum and a compactness argument imply that the set $\{R(\lambda, V_g) \mid |\lambda| = \delta\}$ is equicontinuous in $\mathcal{L}(A_+^{-\infty})$. Hence the set $\{R(\lambda, V_g)1 \mid |\lambda| = \delta\}$ is bounded in $A^{-\infty}$. Since

$A^{-\infty} = \text{ind}_n A^{-n}$ is a regular inductive limit, there is $N \in \mathbb{N}$ such that $\{R(\lambda, V_g)1 \mid |\lambda| = \delta\}$ is bounded in $A^{-N}$. Now we use $R(\lambda, V_g)1 = (1/\lambda)e^{g/\lambda}$ to conclude that there is $M > 0$ such that, for each $z \in \mathbb{D}$ and $|\lambda| = \delta$,

$$(1 - |z|)^N |(1/\lambda)e^{g(z)/\lambda}| \leq M.$$

That is, $(1 - |z|)^N |e^{g(z)/\lambda}| \leq M\delta$ for each $z \in \mathbb{D}$ and $|\lambda| = \delta$. For each $z \in \mathbb{D}$ there is $|\theta(z)| = 1$ such that $g(z) = |g(z)|\theta(z)$. Then

$$(1 - |z|)^N e^{|g(z)|/\delta} = (1 - |z|)^N e^{g(z)/(\overline{\theta(z)}\delta)} \leq M\delta.$$

Therefore, for each $z \in \mathbb{D}$,

$$\frac{|g(z)|}{\delta} \leq \log(M\delta) - N \log(1 - |z|).$$

There is $r(0) \in ]0, 1[$ such that, if $r(0) < |z| < 1$, then $\frac{|g(z)|}{\delta} \leq -2N \log(1 - |z|)$, hence $|g(z)| \leq -2N\delta \log(1 - |z|)$, and $g$ is of logarithmic mean growth. □

**Lemma 3.12** *If $g \in A^{-\infty}$ satisfies that $e^{g/\lambda}$ and $e^{-g/\lambda}$ belong to $A^{-\infty}$, then $\mu\lambda \in \rho(V_g, A^{-\infty})$ for each $\mu > 0$.*

**Proof.** It is easy to see that if $e^{g/\lambda} \in A^{-\infty}$, then $e^{g/\mu\lambda} \in A^{-\infty}$ for each $\mu > 0$. Accordingly, it is enough to prove the result for $\mu = 1$.

To show that $\lambda \in \rho(V_g, A^{-\infty})$, we check that the two conditions in Lemma 3.3 (iii) are satisfied. Condition (a), i.e. $e^{g/\lambda} \in A^{-\infty}$, holds by assumption. On the other hand, the operator

$$S_{\lambda,g}h(z) := e^{\frac{g(z)}{\lambda}} \int_0^z h'(\zeta)e^{-\frac{g(\zeta)}{\lambda}} d\zeta, \ z \in \mathbb{C},$$

is continuous on $A^{-\infty}$ because the operators of differentiation, integration and of multiplication by a function in $A^{-\infty}$ are all continuous. Therefore condition (b) is also satisfied and the proof is complete. □

**Proposition 3.13** *If $g \in A^{-\infty}$ with $g(0) = 0$ satisfies that $\sigma(V_g, A_+^{-\infty})$ is bounded, then $\sigma(V_g, A_+^{-\infty}) = \{0\}$.*

**Proof.** If $\sigma(V_g, A_+^{-\infty})$ is bounded, then there is $\delta > 0$ such that $\lambda \in \rho(V_g, A_+^{-\infty})$ for each $\lambda \in \mathbb{C}$ with $|\lambda| = \delta$. Hence, by Lemma 3.3, both $e^{g/\lambda}$ and $e^{-g/\lambda}$ belong to $A^{-\infty}$ for each $|\lambda| = \delta$. We apply Lemma 3.12 to conclude that $\rho(V_g, A^{-\infty}) = \mathbb{C} \setminus \{0\}$. □

**Proposition 3.14** *If $g \in A^{-\infty}$ with $g(0) = 0$ satisfies $e^g \notin A^{-\infty}$, then $[0, +\infty[ \subset \sigma(V_g, A_+^{-\infty})$.*

*This holds in particular for the functions $g_s(z) := -1 + 1/(1 - z)^s, s > 0$.*

**Proof.** Since $e^g \notin A^{-\infty}$, it follows that $e^{g/\mu} \notin A^{-\infty}$ for each $\mu > 0$, as it is easy to see proceeding by contradiction. The conclusion follows from Lemma 3.2.

The function $h_s(z) = 1/(1 - z)^s = g_s(z) + 1, s > 0$ belongs to $A^{-\infty}$, but $e^{h_s} \notin A^{-\infty}$, since otherwise there would be $N \in \mathbb{N}$ such that $\sup_k k^{-N} \exp(k^s) < \infty$, which is not the case. □

We conclude the paper with the following concrete example.

**Proposition 3.15** *The function $g(z) = z/(1-z)$, which belongs to $A^{-\infty}$ satisfies*

*(1)* $\{\lambda \in \mathbb{C} \mid \lambda \text{ not a negative real number}\} \subset \sigma(V_g, A_+^{-\infty})$*, and*

*(2)* $\sigma^*(V_g, A_+^{-\infty}) = \mathbb{C}$.

**Proof.** Set $h(z) = 1/(1-z) = g(z) + 1$. We show that $e^{h/\lambda} \notin A^{-\infty}$ for each $\lambda \in \mathbb{C}$ which is not a negative real number. Then Lemma 3.2 implies statement (1). Statement (2) follows from (1) since $\overline{\sigma(V_g, A_+^{-\infty})} \subset \sigma^*(V_g, A_+^{-\infty})$ holds in general.

Assume first that $e^{h/\mu} \in A^{-\infty}$ for some $\mu > 0$. There are $n \in \mathbb{N}$ and $M > 0$ such that $(1-|z|)^n |e^{h(z)/\mu}| \leq M$ for each $z \in \mathbb{D}$. Evaluating at the points $z_k = 1 - (1/k), k \in \mathbb{N}$, we get $e^{k/\mu} \leq M k^n$ for each $n \in \mathbb{N}$; a contradiction.

Now let $\lambda \in \mathbb{C}, \lambda \neq 0$, satisfy that its imaginary part is not 0, and write $1/\lambda = \alpha + i\beta, \beta \neq 0$. Select $\gamma > 0$ such that $\gamma > \beta^2/\alpha$.

If $z = a + ib \in \mathbb{D}$, then

$$\mathrm{Re}\left(\frac{h(z)}{\lambda}\right) = \mathrm{Re}\left(\frac{\alpha + i\beta}{(1-a) - ib}\right) = \frac{\alpha(1-a) - b\beta}{(1-a)^2 + b^2}.$$

For $k > (\gamma^2 + \beta^2)/\gamma$, we define $a_k := 1 - (\gamma/k)$, $b_k := -\beta/k$ and $z_k := a_k + ib_k$. Then $|z_k|^2 = 1 - \frac{2\gamma}{k} + \frac{\gamma^2+\beta^2}{k^2} < 1$ and $z_k \in \mathbb{D}$. Moreover,

$$\mathrm{Re}\left(\frac{h(z_k)}{\lambda}\right) = \frac{\alpha\gamma + \beta^2}{\gamma^2 + \beta^2} k.$$

Hence

$$\left|e^{\frac{h(z_k)}{\lambda}}\right| = \exp\left(\frac{\alpha\gamma + \beta^2}{\gamma^2 + \beta^2} k\right).$$

Assume that $e^{\frac{h(z)}{\lambda}} \in A^{-\infty}$. Then there are $s \in \mathbb{N}$ and $M > 0$ such that for each $k > (\gamma^2 + \beta^2)/\gamma$ we have

$$(1 - |z_k|)^s \left|e^{\frac{h(z_k)}{\lambda}}\right| \leq M.$$

Since $\frac{\gamma}{k} - \frac{\gamma^2+\beta^2}{k^2} > 0$, we have

$$1 - |z_k| \geq \frac{1 - |z_k|^2}{2} = \frac{1}{2}\left(\frac{2\gamma}{k} - \frac{\gamma^2 + \beta^2}{k^2}\right) \geq \frac{1}{2}\frac{\gamma}{k}.$$

This yields, for all $k > (\gamma^2 + \beta^2)/\gamma$,

$$\left(\frac{\gamma}{2k}\right)^s \exp\left(\frac{\alpha\gamma + \beta^2}{\gamma^2 + \beta^2} k\right) \leq M.$$

This contradiction completes the proof. □

It is easy to see that $f(z) := e^{1/(z-1)} \in H^\infty \subset A_+^{-\infty}$. However $1/f(z) = e^{1/(1-z)} \notin A_+^{-\infty}$. Compare with the assumption in Lemma 3.12. Incidentally, this observation implies the known fact that the algebra $A_+^{-\infty}$ is not inverse closed.

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