FRÉCHET AND (LB) SEQUENCE SPACES INDUCED BY DUAL BANACH SPACES OF DISCRETE CESÀRO SPACES

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Abstract. The Fréchet (resp., (LB)-) sequence spaces $\mathrm{ces}(p+) := \bigcap_{r>p} \mathrm{ces}(r), 1 \leq p < \infty$ (resp. $\mathrm{ces}(p\text{-}) := \bigcup_{1<r<p} \mathrm{ces}(r), 1 < p \leq \infty$), are known to be very different to the classical sequence spaces ℓ_{p+} (resp., ℓ_{p-}). Both of these classes of non-normable spaces $\mathrm{ces}(p+), \mathrm{ces}(p\text{-})$ are defined via the family of reflexive Banach sequence spaces $\mathrm{ces}(p), 1 < p < \infty$. The *dual* Banach spaces $d(q), 1 < q < \infty$, of the discrete Cesàro spaces $\mathrm{ces}(p), 1 < p < \infty$, were studied by G. Bennett, A. Jagers and others. Our aim is to investigate in detail the corresponding sequence spaces $d(p+)$ and $d(p\text{-})$, which have not been considered before. Some of their properties have similarities with those of $\mathrm{ces}(p+), \mathrm{ces}(p\text{-})$ but, they also exhibit differences. For instance, $\mathrm{ces}(p+)$ is isomorphic to a power series Fréchet space of order 1 whereas $d(p+)$ is isomorphic to such a space of infinite order. Every space $\mathrm{ces}(p+), \mathrm{ces}(p\text{-})$ admits an absolute basis but, none of the spaces $d(p+), d(p\text{-})$ have any absolute basis.

1. Introduction

Perhaps the most important class of Banach sequence spaces is $\ell_p, 1 \leq p \leq \infty$, where ℓ_p is equipped with its usual norm $\|\cdot\|_p$. A classical inequality of Hardy, [20, Theorem 326], states for $1 < p < \infty$ and with $\frac{1}{p} + \frac{1}{p'} = 1$ that

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k|\right)^p \leq (p')^p \sum_{n=1}^{\infty} |x_n|^p, \quad x = (x_n)_n \in \ell_p,$$

In terms of the Cesàro operator $C : \mathbb{C}^{\mathbb{N}} \longrightarrow \mathbb{C}^{\mathbb{N}}$, defined by

$$C(x) := (x_1, \tfrac{x_1+x_2}{2}, \ldots, \tfrac{x_1+x_2+\ldots+x_n}{n}, \ldots), \quad x = (x_n)_n \in \mathbb{C}^{\mathbb{N}}. \tag{1.1}$$

Hardy's inequality can be formulated, for $1 < p < \infty$, as

$$\|C(|x|)\|_p \leq p' \|x\|_p, \quad x \in \ell_p, \tag{1.2}$$

where $|x| := (|x_n|)_n$ for $x \in \mathbb{C}^{\mathbb{N}}$. Since $C : \mathbb{C}^{\mathbb{N}} \longrightarrow \mathbb{C}^{\mathbb{N}}$ is a positive operator (i.e., $C(x) \geq 0$, meant in the coordinatewise sense, whenever $x \geq 0$ in $\mathbb{C}^{\mathbb{N}}$) and $|C(x)| \leq C(|x|)$, it follows from (1.2) that $C : \ell_p \longrightarrow \ell_p$ is a continuous linear operator for all $1 < p < \infty$. G. Bennett investigated, in great detail, the closely related spaces

$$\mathrm{ces}(p) := \{x \in \mathbb{C}^{\mathbb{N}} : C(|x|) \in \ell_p\}, \quad 1 < p < \infty, \tag{1.3}$$

which are reflexive Banach spaces relative to the norm

$$\|x\|_{\mathrm{ces}(p)} := \|C(|x|)\|_p, \quad x \in \mathrm{ces}(p), \tag{1.4}$$

and satisfy $\ell_p \subsetneq \mathrm{ces}(p)$; see [9], as well as [7], [8], [13], [15], [19], [25], [27], and the references therein. The Banach spaces $\mathrm{ces}(p)$ have the desirable property (as do the

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spaces ℓ_p) that they are *solid* in $\mathbb{C}^{\mathbb{N}}$, that is, if $x \in \mathrm{ces}(p)$ and $y \in \mathbb{C}^{\mathbb{N}}$ satisfy $|y| \leq |x|$, then $y \in \mathrm{ces}(p)$.

The *dual* Banach spaces $(\mathrm{ces}(p))'$ of $\mathrm{ces}(p), 1 < p < \infty$, are rather complicated, [22]. A more transparent *isomorphic* identification of $(\mathrm{ces}(p))'$ occurs in [9, Corollary 12.17]. Indeed, it is shown there that

$$d(p) := \{x \in \ell_\infty : \hat{x} := (\sup_{k\geq n} |x_k|)_n \in \ell_p\}, \quad 1 < p < \infty, \tag{1.5}$$

is a Banach space for the norm

$$\|x\|_{d(p)} := \|\hat{x}\|_p, \quad x \in d(p), \tag{1.6}$$

which is isomorphic to $(\mathrm{ces}(p'))'$, denoted by $d(p) \simeq (\mathrm{ces}(p'))'$, with the duality given by

$$\langle u, x\rangle := \textstyle\sum_{n=1}^{\infty} u_n x_n, \quad u \in \mathrm{ces}(p'), \quad x \in d(p).$$

It is clear from (1.5) and (1.6) that $d(p)$ is also *solid* in $\mathbb{C}^{\mathbb{N}}$, for $1 < p < \infty$. The Banach spaces $d(p), 1 < p < \infty$, although less prominent than the discrete Cesàro spaces $\mathrm{ces}(p), 1 < p < \infty$, have received some attention; see, e.g., [9], [12], [13], [19], [22], [25]
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Non-normable sequence spaces $X \subseteq \mathbb{C}^{\mathbb{N}}$ are also an important part of functional analysis; see, for example, [10], [11], [18], [23], [?K\"{o}], [28], [33] and the references therein. The classical Fréchet spaces $\ell_{p+} := \bigcap_{p<q} \ell_q$, for $1 \leq p < \infty$, are well understood, [16], [29]. Their analogues $\mathrm{ces}(p+) := \bigcap_{p<q} \mathrm{ces}(q)$ were introduced and investigated in [2]. The (inductive limit) (*LB*)*-space* $\ell_{p-} := \mathrm{ind}_{1<q<p}\ell_q$, for $1 < p \leq \infty$, which is the *strong dual* space of $\ell_{p'+}$, is also well known, [29]; for a recent study of certain aspects of this class of spaces, see [5]. The analogous class of (LB)-spaces $\mathrm{ces}(p\text{-}) := \mathrm{ind}_{1<q<p}\mathrm{ces}(q)$, again an inductive limit, is also analyzed in [5]. The purpose of this note is to introduce and identify properties of the Fréchet spaces $d(p+), 1 \leq p < \infty$, and of the (LB)-spaces $d(p\text{-}), 1 < p \leq \infty$. These spaces have not been considered before.

In the following Section 2 we collect various relevant aspects of the Banach spaces $\ell_p, \mathrm{ces}(p)$ and $d(p)$, for $1 < p < \infty$, which are needed in the sequel. Various inclusions between these spaces are recorded as well as some of the properties of the Cesàro operator when it acts between pairs of such Banach spaces.

Section 3 summarizes some known properties of the non-normable sequence spaces $\ell_{p+}, \mathrm{ces}(p+)$ and $\ell_{p-}, \mathrm{ces}(p\text{-})$. Several new facts (cf. parts (b) of Proposition 3.3(i), (ii), parts (ii), (iv), (v) of Proposition 3.4 and Proposition 3.6) concerning these spaces are also established.

Section 4 is the main one of this note; it exposes properties of the (solid) Fréchet spaces $d(p+), 1 \leq p < \infty$, and of the (solid) (LB)-spaces $d(p\text{-}), 1 < p \leq \infty$, and compares them with those of the corresponding spaces $\ell_{p+}, \mathrm{ces}(p+)$ and $\ell_{p-}, \mathrm{ces}(p\text{-})$.

2. Preliminaries

The non-normable sequence spaces $\ell_{p+}, \mathrm{ces}(p+), d(p+)$ and $\ell_{p-}, \mathrm{ces}(p\text{-}), d(p\text{-})$ are assembled from the Banach spaces $\ell_p, \mathrm{ces}(q), d(s)$ for $1 < p, q, s < \infty$. So, we begin by collecting some facts about various inclusions amongst these Banach spaces. For each $n \in \mathbb{N}$, let $e_n := (\delta_{kn})_k$.

Proposition 2.1. (i) *Each of the Banach spaces* $\ell_p, \mathrm{ces}(p)$ *and* $d(p)$, *for* $1 < p < \infty$, *is separable, reflexive and the canonical vectors* $\{e_n : n \in \mathbb{N}\}$ *form an unconditional9 basis. Moreover,* $(\ell_p)' \simeq \ell_{p'}$ *(isometrically) and* $(\mathrm{ces}(p))' \simeq d(p')$.

(ii) *Let* $1 < p, q < \infty$.

(a) *The inclusion* $\ell_p \subseteq \ell_q$ *is satisfied and continuous if and only if* $p \leq q$. *Moreover, the inclusion is never compact. If* $p < q$, *then* $\ell_p \subsetneq \ell_q$.

(b) *The inclusion* $\mathrm{ces}(p) \subseteq \mathrm{ces}(q)$ *is satisfied and continuous if and only if* $p \leq q$. *Moreover, the inclusion is compact if and only if* $p < q$, *in which case* $\mathrm{ces}(p) \subsetneq \mathrm{ces}(q)$.

(c) *The inclusion* $d(p) \subseteq d(q)$ *is satisfied and continuous if and only if* $p \leq q$. *Moreover, the inclusion is compact if and only if* $p < q$, *in which case* $d(p) \subsetneq d(q)$.

(d) *The inclusion* $\ell_p \subseteq \mathrm{ces}(q)$ *is satisfied and continuous if and only if* $p \leq q$, *in which case* $\ell_p \subsetneq \mathrm{ces}(q)$. *Moreover, the inclusion is compact if and only if* $p < q$.

(e) *The inclusion* $d(p) \subseteq \ell_q$ *is satisfied and continuous if and only if* $p \leq q$, *in which case* $d(p) \subsetneq \ell_q$. *Moreover, the inclusion is compact if and only if* $p < q$.

(f) *The inclusion* $d(p) \subseteq \mathrm{ces}(q)$ *is satisfied and continuous if and only if* $p \leq q$, *in which case* $d(p) \subsetneq \mathrm{ces}(q)$. *Moreover, the inclusion is compact if and only if* $p < q$.

Let us indicate where the various parts of Proposition 2.1 occur in the literature.

(i) For the ℓ_p-spaces the claims are well known. According to [9, p.61], [22, Proposition 2], the spaces $\mathrm{ces}(p)$, $1 < p < \infty$, are reflexive. It was noted in Section 1 that $d(p)$ is isomorphic to $(\mathrm{ces}(p'))'$ and hence, also the spaces $d(p)$, $1 < p < \infty$, are reflexive. For the unconditionality of $\{e_n : n \in \mathbb{N}\}$ as a basis for $\mathrm{ces}(p)$ we refer to [15, Proposition 2.1] and for $d(p)$ to [12, Proposition 2.1]. It is then clear that the spaces $\mathrm{ces}(p), d(p)$, for $1 < p < \infty$, are separable.

(ii) (a) The claims in this setting are all well known.

(b) For the first statement, see [3, Proposition 3.2(iii)], and for the statement about compactness we refer to [3, Proposition 3.4(ii)]. The discussion prior to Proposition 3.3 in [3] shows that $\mathrm{ces}(p) \subsetneq \mathrm{ces}(q)$ whenever $p < q$.

(c) The first statement occurs in [12, Proposition 5.1(iii)] and the statement concerning compactness can be found in [12, Proposition 5.2(iii)]. Proposition 2.7(ii) of [12] reveals that $d(p) \subsetneq d(q)$ whenever $p < q$.

(d) For the first statement we refer to [3, Proposition 3.2(ii)] and [15, Remark 2.2(ii)]. Concerning the compactness statement, see [3, Proposition 3.4(iii)].

(e) For the first statement, see Propositions 2.7(v) and 5.1(ii) in [12], and for the statement concerning compactness we refer to [12, Proposition 5.2(ii)].

(f) Proposition 5.1(i) of [12] shows that $d(p) \subseteq \mathrm{ces}(q)$, with continuity of the inclusion, if and only if $p \leq q$. Moreover, if $p \leq q$, then part (e) yields that $d(p) \subsetneq \ell_q$. Since $\ell_p \subseteq \mathrm{ces}(q)$ by part (d), it follows that also $d(p) \subsetneq \mathrm{ces}(q)$. For the statement about compactness we refer to [12, Proposition 5.2(i)].

It is also relevant to clarify the non-isomorphic nature between the various Banach spaces $\ell_p, \mathrm{ces}(q)$ and $d(s)$, for $1 < p, q, s < \infty$.

Proposition 2.2. *Let* $1 < p, q < \infty$.

(i) *The Banach spaces* ℓ_p *and* ℓ_q *are not isomorphic whenever* $p \neq q$.
(ii) *The Banach spaces* $\mathrm{ces}(p)$ *and* $\mathrm{ces}(q)$ *are not isomorphic whenever* $p \neq q$.
(iii) *The Banach spaces* $d(p)$ *and* $d(q)$ *are not isomorphic whenever* $p \neq q$.
(iv) *The Banach spaces* $d(p)$ *and* $\mathrm{ces}(q)$ *are not isomorphic whenever* $p \neq q$.
(v) *The Banach spaces* $\mathrm{ces}(p)$ *and* ℓ_q *are not isomorphic.*
(vi) *The Banach spaces* $d(p)$ *and* ℓ_q *are not isomorphic.*

For the statements (i) - (vi) in Proposition 2.2 we refer successively to [26, p.54 of Vol.I], to [3, Proposition 3.3], to [12, Proposition 2.7(ii)], to [12, Proposition 2.9(ii)], to [9, Proposition 15.13], and to [12, Proposition 2.7(iv)].

The Cesàro operator $C : \mathbb{C}^{\mathbb{N}} \longrightarrow \mathbb{C}^{\mathbb{N}}$ is a topological isomorphism of the Fréchet space $\mathbb{C}^{\mathbb{N}}$ onto itself, where $\mathbb{C}^{\mathbb{N}}$ is equipped with the coordinatewise convergence topology. It was noted in Section 1 that $C : \ell_p \longrightarrow \ell_p$ is continuous for $1 < p < \infty$. The same is true for $C : d(p) \longrightarrow d(p)$, [12, Proposition 3.2(i)], and for $C : \mathrm{ces}(p) \longrightarrow \mathrm{ces}(p)$; see the proof of Theorem 5.1 in [15]. In relation to the Cesàro operator C, the Banach spaces $\mathrm{ces}(p)$, $1 < p < \infty$, have a remarkable property, [9, Theorem 20.31]. Namely, for $x \in \mathbb{C}^{\mathbb{N}}$,

$$C(|x|) \in \mathrm{ces}(p) \;\textit{ if and only if }\; x \in \mathrm{ces}(p), \tag{2.1}$$

which immediately implies that

$$C^2(|x|) \in \mathrm{ces}(p) \;\textit{ if and only if }\; C(|x|) \in \mathrm{ces}(p). \tag{2.2}$$

In view of (1.4), the criterion (2.1) can be equivalently formulated, for $1 < p < \infty$, as

$$C^2(|x|) \in \ell_p \;\textit{ if and only if }\; C(|x|) \in \ell_p. \tag{2.3}$$

For each $1 < p < \infty$, it is also known, for $x \in \mathbb{C}^{\mathbb{N}}$, that

$$C^2(|x|) \in d(p) \;\textit{ if and only if }\; C(|x|) \in d(p), \tag{2.4}$$

[12, Proposition 3.7]. The criteria (2.2), (2.3) and (2.4) reveal a surprising feature of the Cesàro operator C when it acts on one of the Banach spaces $\ell_p, \mathrm{ces}(p)$ or $d(p)$, for $1 < p < \infty$. For $\mathrm{ces}(p)$ this is perhaps understandable, in view of (1.3) and (1.4). However, the Banach spaces ℓ_p and also $d(p)$, as defined by (1.5) and (1.6), have apriori no connection to C. Actually more is true.

For each $1 < p < \infty$, let respectively $[C, \ell_p]_s, [C, d(p)]_s$ and $[C, \mathrm{ces}(p)]_s$ denote the *largest solid* vector space $X \subseteq \mathbb{C}^{\mathbb{N}}$ such that respectively $C(X) \subseteq \ell_p, C(X) \subseteq d(p)$ and $C(X) \subseteq \mathrm{ces}(p)$. Relevant here is that $C : \mathrm{ces}(p) \longrightarrow \ell_p$ continuously, which is clear from (1.4) and the inequality $|C(x)| \le C(|x|)$, that $C : \ell_p \longrightarrow d(p)$ continuously, [12, Proposition 3.4], and that $C : \mathrm{ces}(p) \longrightarrow d(p)$ continuously, [12, Corollary 3.8], which shows, in turn, that $\mathrm{ces}(p) \subseteq [C, \ell_p]_s$, that $\ell_p \subseteq [C, d(p)]_s$, and that even $\mathrm{ces}(p) \subseteq [C, d(p)]_s$. Of course, also $\mathrm{ces}(p) \subseteq [C, \mathrm{ces}(p)]_s$ as C maps $\mathrm{ces}(p)$ into itself. Actually the following equalities are valid:

$$[C, \ell_p]_s = [C, d(p)]_s = [C, \mathrm{ces}(p)] = \mathrm{ces}(p), \quad 1 < p < \infty. \tag{2.5}$$

Indeed, for the fact that $[C, \ell_p]_s = [C, \mathrm{ces}(p)]_s = \mathrm{ces}(p)$, see p.62 and Theorem 2.5 of [15], and for the equality $[C, d(p)]_s = \mathrm{ces}(p)$ we refer to [12, Corollary 3.9].

We end this section with some inequalities needed later.

Lemma 2.3. *Let* $1 < p < \infty$ *and* $n \in \mathbb{N}$ *be arbitrary.*

(i) $|x_n| \leq n \|x\|_{\mathrm{ces}(p)}, \quad x \in \mathrm{ces}(p)$.

(ii) $|x_n| \leq \|x\|_{d(p)}, \quad x \in d(p)$.

Proof. (i) Given $x \in \mathrm{ces}(p)$ note that

$$|x_n| \leq \textstyle\sum_{k=1}^n |x_k| = n(\frac{1}{n}\sum_{k=1}^n |x_k|) = n(C(|x|))_n \leq n\|C(|x|)\|_p = n\|x\|_{\mathrm{ces}(p)}.$$

(ii) Fix $x \in d(p)$. It is clear from the definition of $\hat{x}$ (cf. (1.4)) that $0 \leq |x_n e_n| \leq |x| \leq |\hat{x}|$, from which it follows that $|x_n| = \|x_n e_n\|_p \leq \|\hat{x}\|_p = \|x\|_{d(p)}$. □

3. The non-normable spaces $\ell_{p+}, \mathrm{ces}(p+)$ and $\ell_{p-}, \mathrm{ces}(p\text{-})$

In this section we collect some facts on certain non-normable sequence spaces in $\mathbb{C}^{\mathbb{N}}$. Given $1 \leq p < \infty$, consider any strictly decreasing sequence

$$\{p_k\}_{k=1}^\infty \subseteq (p, \infty) \quad \text{with} \quad p_k \downarrow p. \tag{3.1}$$

Then $\ell_{p+} := \bigcap_{q>p} \ell_q = \bigcap_{k=1}^\infty \ell_{p_k}$ is a Fréchet space relative to the increasing sequence of norms on ℓ_{p+} given by $\|\cdot\|_{p_k} : x \longmapsto \|x\|_{p_k}$, for $k \in \mathbb{N}$, [18, Ch.1, §2], [?K\"{o}, §30.8]. Since each Banach space ℓ_q, $1 < q < \infty$, is solid, it is clear that ℓ_{p+} is also solid in $\mathbb{C}^{\mathbb{N}}$. Similarly, $\mathrm{ces}(p+) := \bigcap_{q>p} \mathrm{ces}(q) = \bigcap_{k=1}^\infty \mathrm{ces}(p_k)$ is a (solid) Fréchet space relative to the increasing sequence of norms on $\mathrm{ces}(p+)$ given by $\|\cdot\|_{\mathrm{ces}(p_k)} : x \longmapsto \|x\|_{\mathrm{ces}(p_k)}$, for $k \in \mathbb{N}$. Clearly

$$\ell_p \subseteq \ell_{p+} \subseteq \mathbb{C}^{\mathbb{N}}, \quad 1 \leq p < \infty,$$

with continuous inclusions; here Proposition 2.1(ii)(a) is relevant. Also

$$\mathrm{ces}(p) \subseteq \mathrm{ces}(p+) \subseteq \mathbb{C}^{\mathbb{N}}, \quad 1 \leq p < \infty,$$

with continuous inclusions (cf. Proposition 2.1(ii)(b)).

Proposition 3.1. *Let* $1 \leq p < \infty$. *The canonical vectors* $\{e_n : n \in \mathbb{N}\}$ *form an unconditional basis for* ℓ_{p+}. *Moreover,* ℓ_{p+} *is reflexive but not Montel. With continuous inclusions we have*

$$\ell_{p+} \subseteq \ell_{q+}, \quad 1 \leq p \leq q < \infty. \tag{3.2}$$

For every distinct pair $p, q \in [1, \infty)$ *the Fréchet spaces* ℓ_{p+} *and* ℓ_{q+} *are not isomorphic. In particular, the containment* (3.2) *is proper whenever* $p < q$.

Some comments on Proposition 3.1 are in order. That $\{e_n : n \in \mathbb{N}\}$ is an unconditional basis is well known and is a consequence of the fact that this is the case for each Banach space ℓ_q, $1 < q < \infty$; see Proposition 2.1(i). The same is true for the reflexivity of ℓ_{p+} as each space ℓ_q is reflexive for $1 < q < \infty$, [28, Proposition 25.15]. According to [17] the space ℓ_{p+} is *not Montel*. For the continuity of the inclusions in (3.2), see [4, Proposition 26(i)]. According to Proposition 3.3 of [2] the spaces ℓ_{p+} and ℓ_{q+} are not isomorphic whenever $p \neq q$. Finally, if there exist $p < q$ such that $\ell_{p+} = \ell_{q+}$, then the continuity of the inclusion (3.2) and the open mapping theorem for Fréchet spaces, [28, Theorem 24.30], would imply that ℓ_{p+} and ℓ_{q+} are isomorphic, which is not the case.

The properties of the Fréchet spaces $\mathrm{ces}(p+)$, which we now record, are very different to those of ℓ_{p+}. For the remainder of this note we define $\alpha \in \mathbb{C}^{\mathbb{N}}$ by $\alpha := (\log(n))_n$. The definition of a *power series* sequence space of order 2 can be found in [28, IV Section

29], for example. The power series spaces of order p for any $p \in [1, \infty]$ are defined similarly; see e.g., [32].

Proposition 3.2. (i) *Let $1 \leq p < \infty$. The canonical vectors $\{e_n : n \in \mathbb{N}\}$ form an unconditional basis for* ces$(p+)$. *With continuous inclusions we have*

$$\ell_{p+} \subseteq \mathrm{ces}(q+), \quad 1 \leq p \leq q < \infty, \tag{3.3}$$

and these containments are proper. Also, with continuous inclusions we have

$$\mathrm{ces}(p+) \subseteq \mathrm{ces}(q+), \quad 1 \leq p \leq q < \infty, \tag{3.4}$$

and these containments are proper whenever $p < q$.

The space ℓ_{p+} is not isomorphic to ces$(q+)$ *for any $1 \leq q < \infty$.*

(ii) *Each space* ces$(p+)$, *for $1 \leq p < \infty$, is a Fréchet-Schwartz space which is isomorphic to the Köthe echelon space of order one*

$$\Lambda_{-1/p'}(\alpha) := \{x \in \mathbb{C}^{\mathbb{N}} : \textstyle\sum_{n=1}^{\infty} |x_n| n^t < \infty, \quad \forall\, t < (-\frac{1}{p'})\}, \tag{3.5}$$

which is a power series space of finite type $-1/p'$ and order 1. In particular, every space ces$(p+)$ *is isomorphic to the fixed power series space*

$$\Lambda_0^1(\alpha) := \{x \in \mathbb{C}^{\mathbb{N}} : \textstyle\sum_{n=1}^{\infty} |x_n| n^{-1/k} < \infty, \quad \forall k \in \mathbb{N}\} \tag{3.6}$$

and hence, ces$(p+)$ *is not nuclear.*

Concerning Proposition 3.2 consider first part (i). That $\{e_n : n \in \mathbb{N}\}$ is an unconditional basis is established in [2, Proposition 3.5(i)]. For the fact that ℓ_{p+} is not isomorphic to any space ces$(q+)$, for $1 \leq q < \infty$, see [2, Proposition 3.5(iii)]. The continuity of the inclusion $\ell_{p+} \subset$ ces$(q+)$ in (3.3) is established in [4, Proposition 26(ii)]. If $\ell_{p+} =$ ces$(q+)$ for some $1 \leq p \leq q < \infty$, then the continuity of the inclusion (3.3) and the open mapping theorem would imply that ℓ_{p+} and ces$(q+)$ are isomorphic, which is not so. For the continuity of the inclusion in (3.4), see [4, Proposition 26(iii)]. According to [2, Remark 3.4] the inclusion in (3.4) is necessarily proper if $p < q$. Now consider part (ii). That ces$(p+)$ is isomorphic to the space $\Lambda_{-1/p'}(\alpha)$, as given in (3.5), is the statement of Theorem 3.1 in [2] and that each one of these spaces is isomorphic to $\Lambda_0^1(\alpha)$ is precisely Corollary 3.2 of [2]. Consequently, ces$(p+)$ is necessarily a *Fréchet-Schwartz* space but, it is *not* nuclear; see [2, Proposition 3.5(ii)].

The following result summarizes certain properties of the Cesàro operator that will be needed in the sequel.

Proposition 3.3. (i) (a) *For $1 \leq p \leq q < \infty$, the Cesàro operator $C : \ell_{p+} \longrightarrow \ell_{q+}$ is continuous.*

(b) *Let $1 \leq p < \infty$ and $x \in \mathbb{C}^{\mathbb{N}}$. Then*

$$C^2(|x|) \in \ell_{p+} \;\; \textit{if and only if} \;\; C(|x|) \in \ell_{p+}.$$

(c) *For each $1 \leq p < \infty$ we have that $[C, \ell_{p+}]_s =$* ces$(p+)$.

(ii) (a) *For $1 \leq p \leq q < \infty$ the Cesàro operator C :* ces$(p+) \longrightarrow$ ces$(q+)$ *is continuous.*

(b) *Let $1 \leq p < \infty$ and $x \in \mathbb{C}^{\mathbb{N}}$. Then*

$$C^2(|x|) \in \mathrm{ces}(p+) \;\; \textit{if and only if} \;\; C(|x|) \in \mathrm{ces}(p+).$$

(c) *For each* $1 \le p < \infty$ *we have that* $[C, \mathrm{ces}(p+)]_s = \mathrm{ces}(p+)$.

Proof. (i) (a) This is Proposition 28(i) of [4].

(b) Fix $1 \le p < \infty$ and let $x \in \mathbb{C}^{\mathbb{N}}$. If $C(|x|) \in \ell_{p+}$, then part (a) ensures that also $C^2(|x|) \in \ell_{p+}$.

Conversely, suppose that $C^2(|x|) \in \ell_{p+}$. Fix any $q > p$. Since $C^2(|x|) \in \ell_{p+} \subseteq \ell_q$, it follows from (2.3) that $C(|x|) \in \ell_q$. But, $q > p$ is arbitrary, and so $C(|x|) \in \ell_{p+}$.

(c) See [2, Proposition 2.5].

(ii) (a) See Proposition 28(iii) of [4].

(b) The proof of (b) in part (i) can easily be adapted to apply to this case (by using part (a) of (ii) and (2.2) in place of (2.3)).

(c) See [2, Proposition 2.6]. □

We now turn our attention to the (LB)-spaces ℓ_{p-} and ces(p-). Given $1 < p \le \infty$, consider any strictly increasing sequence

$$\{p_k\}_{k=1}^{\infty} \subseteq (1, p) \quad \text{with} \quad p_k \uparrow p. \tag{3.7}$$

Define the linear spaces

$$\ell_{p-} := \bigcup_{1<q<p} \ell_q \quad \text{and} \quad \mathrm{ces}(p\text{-}) := \bigcup_{1<q<p} \mathrm{ces}(q), \tag{3.8}$$

and equip them with the inductive limit topology. In both cases the union is strictly increasing; see parts (a), (b) of Proposition 2.1(ii). Accordingly, $\ell_{p-} = \mathrm{ind}_k \ell_{p_k}$ and $\mathrm{ces}(p\text{-}) = \mathrm{ind}_k \, \mathrm{ces}(p_k)$ are (LB)-spaces, that is, a countable inductive limit of Banach spaces, [10], [11],[28, pp.290–291] . Since the Banach spaces $\ell_q, \mathrm{ces}(q)$, for $1 < q < \infty$, are solid, it is clear from (3.8) that both ℓ_{p-} and ces(p-) are *solid* in $\mathbb{C}^{\mathbb{N}}$.

For the definition of the *strong dual* space X'_β of a locally convex Hausdorff space X we refer to [28, p.269], for example. In the event that X is a Fréchet space, the space X'_β is a complete (DF)-space, [28, Proposition 25.7]. Moreover, a Fréchet space X is reflexive if and only if X'_β is reflexive, [28, Corollary 25.11]. A reflexive Fréchet space X is Montel if and only if X'_β is Montel, [28, Proposition 24.25]. Recall that a locally convex inductive limit is said to be *regular* if each bounded set is contained and bounded in some step.

Proposition 3.4. (i) *For each* $1 < p \le \infty$, *the space* ℓ_{p-} *is a regular (LB)-space which is reflexive but not Montel. It is a (DF)-space which satisfies* $\ell_{p-} \simeq (\ell_{p'+})'_\beta$ *and* $(\ell_{p-})'_\beta \simeq \ell_{p'+}$.

(ii) *For* $1 < p \le q \le \infty$ *the natural inclusions*

$$\ell_{p-} \subseteq \ell_{q-} \tag{3.9}$$

are continuous. If $p < q$, *then* $\ell_{p-} \subsetneq \ell_{q-}$. *Also, the inclusions*

$$\ell_{p-} \subseteq \mathrm{ces}(q\text{-}), \quad 1 < p \le q \le \infty, \tag{3.10}$$

are continuous and proper.

(iii) *For* $1 < p \le q \le \infty$, *both of the Cesàro operators* $C : \ell_{p-} \longrightarrow \ell_{q-}$ *and* $C : \mathrm{ces}(p\text{-}) \longrightarrow \ell_{q-}$ *are continuous.*

(iv) *For* $1 < p \le \infty$ *and* $x \in \mathbb{C}^{\mathbb{N}}$ *it is the case that*

$$C^2(|x|) \in \ell_{p-} \ \text{ if and only if } \ C(|x|) \in \ell_{p-}. \tag{3.11}$$

(v) *The identity* $[C, \ell_{p-}]_s = \mathrm{ces}(p\text{-})$ *is valid for every* $1 < p \le \infty$.

Proof. (i) These facts are essentially known and follow from the general facts stated prior to the proposition; see also [10], [16], [29], for example.

(ii) The continuity of the inclusion (3.9) occurs in [5, Proposition 25(i)]. To see that the containment is proper when $p < q$, choose $r \in (p, q)$ and note that $\ell_{p-} \subseteq \ell_r$. According to part (a) of Proposition 2.1(ii) there exists $x \in \ell_q \setminus \ell_r$. Then $x \in \ell_{q-} \setminus \ell_{p-}$ because $\ell_r \subseteq \ell_{q-}$.

For the continuity of the inclusion (3.10) see [5, Proposition 25(ii)]. Suppose their exist p, q with $1 < p \le q \le \infty$ such that $\ell_{p-} = \mathrm{ces}(q\text{-})$. Since both $\ell_{p-}, \mathrm{ces}(q\text{-})$ have a web and are ultra-bornological, [28, Remark 24.36], and ℓ_{p-} is continuously included in $\mathrm{ces}(q\text{-})$, it follows from the open mapping theorem, [28, Theorem 24.30], that ℓ_{p-} and $\mathrm{ces}(q\text{-})$ are isomorphic. But, this is impossible as ℓ_{p-} is not Montel whereas $\mathrm{ces}(p\text{-})$ is Montel, [5, p.48]. Hence, the inclusion (3.10) is always proper.

(iii) See parts (i) and (iv) of Proposition 27 in [5].

(iv) Fix $p \in (1, \infty]$ and let $x \in \mathbb{C}^{\mathbb{N}}$. If $C(|x|) \in \ell_{p-}$, then also $C^2(|x|) \in \ell_{p-}$ by part (iii). Conversely, suppose that $C^2(|x|) \in \ell_{p-}$. According to (3.8) there exists $q \in (1, p)$ such that $C^2(|x|) \in \ell_q$ and hence, by (2.3), $C(|x|) \in \ell_q \subseteq \ell_{p-}$.

(v) By part (iii) the operator C maps $\mathrm{ces}(p\text{-})$ into ℓ_{p-}, which implies that $\mathrm{ces}(p\text{-}) \subseteq [C, \ell_{p-}]_s$. Conversely, let $X \subseteq \mathbb{C}^{\mathbb{N}}$ be a solid subspace such that $C(X) \subseteq \ell_{p-}$. Given $x \in X$, also $|x| \in X$ and so $C(|x|) \in \ell_{p-}$. Choose $q \in (1, p)$ such that $C(|x|) \in \ell_q \subseteq \mathrm{ces}(q)$; see (3.8). Then (2.1) implies that $x \in \mathrm{ces}(q) \subseteq \mathrm{ces}(p\text{-})$. Accordingly, $X \subseteq \mathrm{ces}(p\text{-})$ which implies that $[C, \ell_{p-}]_s \subseteq \mathrm{ces}(p\text{-})$. □

The (LB)-spaces $\mathrm{ces}(p\text{-})$ are rather different to the (LB)-spaces ℓ_{p-}. This is due to the fact, for $1 < p < q < \infty$, that the natural inclusion map $\mathrm{ces}(p) \subseteq \mathrm{ces}(q)$ is *compact* whereas the inclusion map $\ell_p \subseteq \ell_q$ is *not* compact; see parts (a), (b) of Proposition 2.1(ii). For the definition of a (DFS)-space we refer to [28, p.304 & Proposition 25.20]; it is the strong dual of a Fréchet-Schwartz space. In particular, a (DFS)-space is complete and Montel, [10, pp.61-62].

Proposition 3.5. (i) *With continuous inclusions we have*

$$\mathrm{ces}(p\text{-}) \subseteq \mathrm{ces}(q\text{-}) \subseteq \mathbb{C}^{\mathbb{N}}, \quad 1 < p \le q \le \infty \tag{3.12}$$

and these containments are proper whenever $p < q$.

The space ℓ_{p-} *is not isomorphic to* $\mathrm{ces}(q\text{-})$ *for every pair* $1 < p, q \le \infty$.

(ii) *Each space* $\mathrm{ces}(p\text{-})$, *for* $1 < p \le \infty$, *coincides algebraically and topologically with a countable inductive limit* $k_1(v_p)$ *of weighted* ℓ_1*-spaces.This co-echelon space is isomorphic to the strong dual of the (power series) Fréchet-Schwartz space* $\Lambda^{\infty}_{1/p'}(\alpha)$ *of finite type* $1/p'$ *and infinite order. In particular,* $\mathrm{ces}(p\text{-}) = (\Lambda^{\infty}_{1/p'}(\alpha))'_\beta$ *is a (DFS)-space but, it is not nuclear. Moreover,* $(\mathrm{ces}(p\text{-}))'_\beta \simeq \Lambda^{\infty}_{1/p'}(\alpha)$. *In addition, each space* $\mathrm{ces}(p\text{-})$, *for* $1 < p \le \infty$, *is isomorphic to the fixed (DFS)-space*

$$(\Lambda^{\infty}_{1}(\alpha))'_\beta = \mathrm{ces}(\infty\text{-}). \tag{3.13}$$

(iii) *Whenever* $1 < p \le q \le \infty$, *the Cesàro operator* $C : \mathrm{ces}(p\text{-}) \longrightarrow \mathrm{ces}(q\text{-})$ *is continuous.*

(iv) *For* $1 < p \le \infty$ *and* $x \in \mathbb{C}^{\mathbb{N}}$ *it is the case that*

$$C^2(|x|) \in \mathrm{ces}(p\text{-}) \ \textit{if and only if} \ C(|x|) \in \mathrm{ces}(p\text{-}).$$

(v) *The identity* $[C, \mathrm{ces}(p\text{-})]_s = \mathrm{ces}(p\text{-})$ *is valid for every* $1 < p \leq \infty$.

Some comments relevant to Proposition 3.5 are as follows. For part (i), the continuity of the first inclusion in (3.12) occurs in [5, Proposition 25(iii)]. For the second inclusion in (3.12), recall that the lcH-topology of the Fréchet space $\mathbb{C}^{\mathbb{N}}$ is given by the increasing sequence of seminorms $\{q_m : m \in \mathbb{N}\}$, where

$$q_m(x) := \max_{1 \leq k \leq m} |x_k|, \quad x = (x_n)_n \in \mathbb{C}^{\mathbb{N}}.$$

Given a fixed $m \in \mathbb{N}$ it follows from Lemma 2.3(i), for each $1 < r < \infty$, that

$$q_m(x) \leq m \, \|x\|_{\mathrm{ces}(r)}, \quad x \in \mathrm{ces}(r).$$

Accordingly, the inclusion $\mathrm{ces}(r) \subseteq \mathbb{C}^{\mathbb{N}}$ is continuous for each $r \in (1, \infty)$, which implies that also the inclusion $\mathrm{ces}(q\text{-}) \subseteq \mathbb{C}^{\mathbb{N}}$ is continuous for each $1 < q \leq \infty$, [28, Proposition 24.7]. That the containment (3.12) is *proper* whenever $p < q$ can be argued as in the proof of part (ii) in Proposition 3.4 by replacing the use of Proposition 2.1(ii)(a) there with Proposition 2.1(ii)(b). Corollary 4 of [5] shows that ℓ_{p-} is not isomorphic to $\mathrm{ces}(q\text{-})$ for every $1 < p, q \leq \infty$. All of the assertions in part (ii) are proved on pp. 49-51 of [5]. For part (iii) we refer to [5, Proposition 27(iii)]. The statement in part (iv) follows directly from Proposition 1(i) of [5]. Finally the identity in (v) is Proposition 1(iv) of [5].

The canonical vectors $\{e_n : n \in \mathbb{N}\}$ form an unconditional basis in the Fréchet spaces $\mathrm{ces}(p+)$, $1 \leq p < \infty$ (cf. Proposition 3.2(i)), and a Schauder basis in each (DFS)-space $\mathrm{ces}(p\text{-})$, $1 < p \leq \infty$, [5, Proposition 1]. Actually more is true. We recall the notion of an absolute basis, [23, p.314], [28, p.341]. Given a Schauder basis $\{u_n : n \in \mathbb{N}\}$ of a locally convex Hausdorff space X, for each $x \in X$ there exists a unique element $(x_n)_n \in \mathbb{C}^{\mathbb{N}}$ such that $x = \sum_{n=1}^{\infty} x_n u_n$, with the series converging in X. If, for each continuous seminorm p on X there exist a continuous seminorm q on X and $A > 0$ such that

$$\sum_{n=1}^{\infty} |x_n| p(u_n) \leq A q(x), \quad x \simeq (x_n)_n \in X,$$

then $\{u_n : n \in \mathbb{N}\}$ is called an *absolute basis* of X.

Proposition 3.6. *The canonical vectors* $\{e_n : n \in \mathbb{N}\}$ *form an absolute basis in each space* $\mathrm{ces}(p+)$, $1 \leq p < \infty$, *and in each space* $\mathrm{ces}(p\text{-})$, $1 < p \leq \infty$.

Proof. Fix $1 \leq p < \infty$ and recall (by Proposition 3.2(ii)) that $\mathrm{ces}(p+)$ is isomorphic to the Köthe echelon space $\Lambda_{-1/p'}(\alpha)$. The conclusion then follows from [23, Theorem 14.7.8] or [28, Proposition 27.26].

Now fix $1 < p \leq \infty$. As mentioned in part (ii) of Proposition 3.5, $\mathrm{ces}(p\text{-})$ equals a co-echelon space of order 1 given by $k_1(v_p) = \mathrm{ind}_n \ell_1(\mathrm{v}_n)$ for the decreasing sequence of weights $v_p = \{\mathrm{v}_n : n \in \mathbb{N}\}$, where $\mathrm{v}_n(k) := k^{-1/q_n}$ for $k \in \mathbb{N}$, with $q_n \downarrow p'$; see [5, p.49]. According to [11] the space $k_1(v_p)$ is a Köthe sequence space $K_1(\overline{V})$ for some (uncountable) family of weights $\overline{V}$ associated with v_p. Then Theorem 14.7.8 (and its proof) in [23] show that $\{e_n : n \in \mathbb{N}\}$ is an absolute basis of $\mathrm{ces}(p\text{-})$. □

Remark 3.7. Every absolute basis is an unconditional basis, [23, p.314]. Hence, the Schauder basis $\{e_n : n \in \mathbb{N}\}$ of $\mathrm{ces}(p\text{-})$, $1 < p \leq \infty$, is actually an unconditional basis.

4. The non-normable spaces $d(p+)$ and $d(p\text{-})$

Fix $p \in [1, \infty)$. For any decreasing sequence $\{p_k\}_{k=1}^{\infty}$ satisfying (3.1) the Fréchet space $d(p+) := \bigcap_{q>p} d(q) = \bigcap_{k=1}^{\infty} d(p_k)$ is defined relative to the increasing sequence of norms on $d(p+)$ given by $\|\cdot\|_{d(p_k)} : x \longmapsto \|x\|_{d(p_k)}$, for $k \in \mathbb{N}$; see (1.6). Similarly, let $p \in (1, \infty]$. For any increasing sequence $\{p_k\}_{k=1}^{\infty}$ satisfying (3.7) we define the (LB)-space $d(p\text{-}) := \bigcup_{1<q<p} d(q) = \bigcup_{k=1}^{\infty} d(p_k)$, equipped with the inductive limit topology. Since the Banach spaces $d(q)$, for $1 < q < \infty$, are solid, both $d(p+)$ and $d(p\text{-})$ are *solid* in $\mathbb{C}^{\mathbb{N}}$. The aim of this section is to identify properties of this new class of spaces and to compare them with those of $\ell_{p+}, \mathrm{ces}(p+)$ and $\ell_{p-}, \mathrm{ces}(p\text{-})$ presented in Section 3. It turns out that the spaces $d(p+)$, resp. $d(p\text{-})$, have certain similarities with $\mathrm{ces}(p+)$, resp. $\mathrm{ces}(p\text{-})$, but there are also major differences.

We begin with two lemmas which record a few basic features of the spaces $d(p+)$ and $d(p\text{-})$. Since $d(p) \subseteq \mathbb{C}^{\mathbb{N}}$, with a continuous inclusion, for each $1 < p < \infty$, [12, Proposition 2.7(vi)], it follows that

$$d(p) \subseteq d(p+) \subseteq \mathbb{C}^{\mathbb{N}}, \quad 1 < p < \infty, \tag{4.1}$$

and $d(1+) \subseteq \mathbb{C}^{\mathbb{N}}$, all with continuous inclusions, as well as

$$d(p+) \subseteq d(q+), \quad 1 \le p \le q < \infty, \tag{4.2}$$

where one needs to invoke part (c) of Proposition 2.1(ii) to establish the continuity of the inclusion in (4.2).

Lemma 4.1. *Let* $1 \le p < \infty$.

(i) *The canonical vectors* $\{e_n : n \in \mathbb{N}\}$ *form an unconditional basis in* $d(p+)$.
(ii) *The Fréchet space* $d(p+)$ *is reflexive and separable.*
(iii) *The inclusion in* (4.2) *is proper whenever* $1 \le p < q < \infty$.

Proof. (i) Fix $p \in [1, \infty)$ and recall that $d(p+) = \bigcap_{k=1}^{\infty} d(p_k)$ with $\{e_n : n \in \mathbb{N}\}$ forming an unconditional basis in each Banach space $d(p_k)$, for $k \in \mathbb{N}$; see Proposition 2.1(i). So, given $x \in d(p+)$ and any permutation $\pi : \mathbb{N} \longrightarrow \mathbb{N}$ it is the case that $\lim_{N\to\infty} \|x - \sum_{n=1}^{N} x_{\pi(n)} e_{\pi(n)}\|_{d(p_k)} = 0$ for each $k \in \mathbb{N}$, that is, $\lim_{N\to\infty} \sum_{n=1}^{N} x_{\pi(n)} e_{\pi(n)} = x$ in $d(p+)$. Hence, the series $\sum_{n=1}^{\infty} x_n e_n$ is unconditionally convergent to x in $d(p+)$.

(ii) The reflexivity of the Banach spaces $d(q), 1 < q < \infty$, implies the reflexivity of $d(p+)$. The separability of $d(p+)$ is clear from part (i).

(iii) Suppose that $1 \le p < q < \infty$ satisfy $d(p+) = d(q+)$. For any fixed $r \in (p, q)$ it would follow from the containments $d(p+) \subseteq d(r) \subseteq d(q) \subseteq d(q+)$ that $d(r) = d(q)$. This contradicts part (c) of Proposition 2.1(ii). □

According to parts (d), (e) of Proposition 2.1(ii) it is the case that

$$d(p) \subseteq \ell_q \subseteq \mathrm{ces}(r), \quad 1 < p \le q \le r < \infty, \tag{4.3}$$

with continuous inclusions, which implies that

$$d(p+) \subseteq \ell_{q+} \subseteq \mathrm{ces}(r+), \quad 1 \le p \le q \le r < \infty, \tag{4.4}$$

also with continuous inclusions. A similar argument to the proof of part (iii) of Lemma 4.1 shows that the containments in (4.4) are always *proper*.

Lemma 4.2. (i) *Each (LB)-space $d(p\text{-})$, for $1 < p \leq \infty$, is a (DFS)-space. In particular, it is a Montel space.*

(ii) *With continuous inclusions it is the case that*

$$d(p\text{-}) \subseteq d(q\text{-}) \subseteq \mathbb{C}^{\mathbb{N}}, \quad 1 < p \leq q \leq \infty. \tag{4.5}$$

In addition, if $p < q$, then $d(p\text{-}) \subsetneq d(q\text{-})$.

Proof. (i) Since the natural inclusion $d(q) \subseteq d(r)$ is *compact* whenever $1 < q < r < \infty$ (cf. Proposition 2.1(ii)(c)), the (LB)-space $d(p\text{-})$ is a (DFS)-space, [28, Proposition 25.20].

(ii) An argument similar to that used to establish the continuity of the second inclusion in (3.12) also applies here (by using part (ii) of Lemma 2.3 in place of part (i)) to show that the second inclusion in (4.5) is continuous. Similarly, the continuity of the inclusion $d(s) \subseteq d(t)$ for $1 < s \leq t < \infty$ (cf. Proposition 2.1(ii)(c)), implies that the inclusion $d(p\text{-}) \subseteq d(q\text{-})$ in (4.5) is continuous; see, for example, [5, Lemma 17(i)].

If $1 < p < q \leq \infty$, then $d(p\text{-}) \subsetneq d(q\text{-})$. Indeed, fix $r \in (p, q)$. By Proposition 2.1(ii)(c) there exists $x \in d(r) \backslash d(p)$. Then $x \in d(q\text{-})$ but, $x \notin d(p\text{-})$ as $d(s) \subseteq d(p)$ for all $1 < s < p$. □

The continuity of the inclusions (4.3) imply that

$$d(p\text{-}) \subseteq \ell_{q\text{-}} \subseteq \mathrm{ces}(r\text{-}), \quad 1 < p \leq q \leq r \leq \infty, \tag{4.6}$$

with continuous inclusions; see Lemma 17(i) of [5]. A similar argument to that used in the proof of Lemma 4.2(ii) shows that the containments in (4.6) are always *proper*.

Similarly, the continuity of the inclusion $d(r) \subseteq d(p)$ whenever $1 < r \leq p < \infty$ (cf. Proposition 2.1(ii)(c)), together with [5, Lemma 17(i)] applied when T is the inclusion map from $d(r)$ into $d(p\text{-})$, shows that

$$d(r) \subseteq d(p\text{-}), \quad 1 < r < p \leq \infty, \tag{4.7}$$

with continuous inclusions. An analogous argument to that used in the proof of part (ii) in Lemma 4.2 shows that $d(r) \subsetneq d(p\text{-})$.

The following descriptions of $d(p+)$ and $d(p\text{-})$ exhibit important features of these spaces.

Proposition 4.3. (i) *For $1 < p \leq \infty$ the map $\Phi_p : (\mathrm{ces}(p'+))' \longrightarrow d(p\text{-})$ given by*

$$\Phi_p(f) := (\langle e_n, f \rangle)_n, \quad f \in (\mathrm{ces}(p'+))',$$

is a linear bijection and a topological isomorphism of the (DFS)-space $(\mathrm{ces}(p'+))'_\beta$ onto the (DFS)-space $d(p\text{-}) = \mathrm{ind}_k d(p_k)$, where $\{p_k\}_{k=1}^\infty$ satisfies (3.7).

(ii) For each $1 \leq p < \infty$ the map $\Psi_p : (\mathrm{ces}(p'\text{-}))' \longrightarrow d(p+)$ given by

$$\Psi_p(g) := (\langle e_n, g \rangle)_n, \quad g \in (\mathrm{ces}(p'\text{-}))',$$

is a linear bijection and is a topological isomorphism of the Fréchet-Schwartz space $(\mathrm{ces}(p'\text{-}))'_\beta$ onto the Fréchet-Schwartz space $d(p+)$.

Proof. (i) By Proposition 3.2(ii) the space $\mathrm{ces}(p'+)$ is Fréchet-Schwartz and by Lemma 4.2(i) the space $d(p\text{-})$ is a (DFS)-space. That Φ_p is a bijection and topological isomorphism follows from [2, Proposition 4.6].

(ii) Proposition 3.5(ii) shows that ces(p'-) is a (DFS)-space with compact linking maps (cf. Proposition 2.1(ii)(b)). Using [?K\"{o}, §22.7, Theorem (9)], and [28, Proposition 25.20] it is routine to adapt the proof of Proposition 4.6 in [2] to prove that Ψ_p is a bijection and topological isomorphism. □

Remark 4.4. Since each space $d(p+), 1 \leq p < \infty$, and $d(p\text{-}), 1 < p \leq \infty$, is reflexive, it is isomorphic to its bidual. So, Proposition 4.3 implies that

$$(d(p+))'_\beta \simeq ((\mathrm{ces}(p'\text{-})'_\beta)'_\beta \simeq \mathrm{ces}(p'\text{-}), \quad 1 \leq p < \infty,$$

and that

$$(d(p\text{-}))'_\beta \simeq ((\mathrm{ces}(p'+)'_\beta)'_\beta \simeq \mathrm{ces}(p'+), \quad 1 < p \leq \infty.$$

Corollary 4.5. (i) *Each space $d(p\text{-})$, for $1 < p \leq \infty$, is a non-nuclear, (DFS)-space which is isomorphic to the strong dual $(\Lambda_0^1(\alpha))'_\beta$ of the power series Fréchet space $\Lambda_0^1(\alpha)$ of finite type* 0 *and order* 1.

(ii) *Each space $d(p+)$, for $1 \leq p < \infty$, is a non-nuclear, Fréchet-Schwartz space which is isomorphic to the power series space $\Lambda_0^\infty(\alpha)$ of finite type* 0 *and order infinity.*

Proof. In view of Propositions 3.2(ii), 3.5(ii), and 4.3, only the non-nuclearity of the spaces in parts (i) and (ii) needs to be addressed. According to Proposition 3.2(ii) and Proposition 3.5(ii) the spaces $\mathrm{ces}(q+), 1 \leq q < \infty$, and $\mathrm{ces}(q\text{-}), 1 < q \leq \infty$, are all non-nuclear. Then [30, p.78, Theorem′] implies that the strong dual spaces $d(p\text{-}) \simeq (\mathrm{ces}(p'+))'_\beta$, for $1 < p \leq \infty$, and $d(p+) \cong (\mathrm{ces}(p'\text{-}))'_\beta$, for $1 \leq p < \infty$, are also non-nuclear. □

Proposition 3.6 implies that the unconditional basis $\{e_n : n \in \mathbb{N}\}$ in $\mathrm{ces}(p+), 1 \leq p < \infty$, and in $\mathrm{ces}(p\text{-}), 1 < p \leq \infty$ (cf. Proposition 3.2(i) and Remark 3.7) is actually an absolute basis. This turns out *not* to be the case for the spaces $d(p+)$ and $d(p\text{-})$. For the definition of the operator ideals $\mathcal{L}_p, 1 \leq p \leq \infty$, of all p-factorable operators we refer to [31]; see also [21], where the terminology of an $\mathcal{L}_p$-locally convex space is used.

Theorem 4.6. *The canonical vectors $\{e_n : n \in \mathbb{N}\}$ form an unconditional basis for each Fréchet-Schwartz space $d(p+), 1 \leq p < \infty$, and for each (DFS)-space $d(p\text{-}), 1 < p \leq \infty$. However, none of these spaces have any absolute basis.*

Proof. According to Lemma 4.1(i), $\{e_n : n \in \mathbb{N}\}$ is an unconditional basis in $d(p+)$, for $1 \leq p < \infty$.

Fix $1 < p \leq \infty$ and let $x \in d(p\text{-})$. Then $x \in d(r)$ for some $1 < r < p$. Given any permutation $\pi : \mathbb{N} \longrightarrow \mathbb{N}$, it follows from Proposition 2.1(i) applied to $d(r)$, that $\lim_{N\to\infty} \|x - \sum_{n=1}^N x_{\pi(n)} e_{\pi(n)}\|_{d(r)} = 0$. Since $d(r) \subseteq d(p\text{-})$ continuously (cf. (4.7)), it follows that $\lim_{N\to\infty} \sum_{n=1}^N x_{\pi(n)} e_{\pi(n)} = x$ in $d(p\text{-})$, that is, $x = \sum_{n=1}^\infty x_{\pi(n)} e_{\pi(n)}$ with the series converging in $d(p\text{-})$. But, π is arbitrary and so the series $\sum_{n=1}^\infty x_n e_n$ converges unconditionally to x in $d(p\text{-})$. Hence, $\{e_n : n \in \mathbb{N}\}$ is an unconditional basis for $d(p\text{-})$.

Let $1 \leq p < \infty$ and suppose that $d(p+)$ has some absolute basis. By Proposition 27.26 of [28], $d(p+)$ would be isomorphic to a Köthe echelon space of order 1, that is, $d(p+)$ is an $\mathcal{L}_1$-space. On the other hand, Corollary 4.5(ii) implies that $d(p+)$ is also an

$\mathcal{L}_\infty$-space. According to [31, Theorem 29.7.6] the space $d(p+)$ is then nuclear, which contradicts Corollary 4.5(ii). Hence, $d(p+)$ has *no* absolute basis.

Suppose, for fixed $1 < p \leq \infty$, that $d(p\text{-})$ has an absolute basis. By Theorem 14.7.8 of [23] the complete (DFS)-space $d(p\text{-})$ is isomorphic to a Köthe sequence space of order 1 and hence, is an $\mathcal{L}_1$-space. According to Proposition 4.3(i) and Corollary 4.5(i) the space $d(p\text{-})$ is also an $\mathcal{L}_\infty$-space. Hence, $d(p\text{-})$ is nuclear, [31, Theorem 29.7.6], which contradicts Corollary 4.5(i). So, $d(p\text{-})$ has *no* absolute basis. □

Theorem 4.7. (i) *For each pair* $1 \leq p, q < \infty$, *the Fréchet space* $\mathrm{ces}(p+)$ *is not isomorphic to the Fréchet space* $d(q+)$.

(ii) *For each pair* $1 < p, q \leq \infty$, *the (DFS)-space* $\mathrm{ces}(p\text{-})$ *is not isomorphic to the (DFS)-space* $d(q\text{-})$.

Proof. (i) Fix $1 \leq p, q < \infty$. Since isomorphisms between locally convex Hausdorff spaces map absolute bases to absolute bases, it follows from Proposition 3.6 and Theorem 4.6 that $\mathrm{ces}(p+)$ cannot be isomorphic to $d(q+)$.

(ii) Fix $1 < p, q \leq \infty$. If $\mathrm{ces}(p\text{-})$ and $d(q\text{-})$ are isomorphic, then also their strong dual spaces $d(p'+)$ and $\mathrm{ces}(q'+)$ are isomorphic. This contradicts part (i). □

Remark 4.8. (i) An alternate proof of the fact that no space $d(p+), 1 \leq p < \infty$, has an absolute basis is possible. Since $(d(p+))'_\beta = \mathrm{ces}(p'\text{-})$ *has* an absolute basis (cf. Proposition 3.6), *if* also $d(p+)$ had an absolute basis, then Theorem 21.10.6 of [23] would imply that $d(p+)$ is nuclear; contradiction to Corollary 4.5(ii).

(ii) A further difference between the unconditional basis $\mathcal{E} := \{e_n : n \in \mathbb{N}\}$ when it is considered to belong to one of the spaces $\mathrm{ces}(p+), \mathrm{ces}(p\text{-})$, in contrast to when it is considered in one of the spaces $d(p+), d(p\text{-})$, should be pointed out. For each $q \in (1, \infty)$ it is known that there exist positive constants A_q, B_q such that

$$A_q\, n^{-1/q'} \leq \|e_n\|_{\mathrm{ces}(q)} \leq B_q\, n^{-1/q'}, \quad n \in \mathbb{N}, \tag{4.8}$$

[9, Lemma 4.7]. Fix $p \in [1, \infty)$ and consider $\mathcal{E} \subseteq \mathrm{ces}(p+)$. Given $q > p$ it follows from (4.8) that $\lim_{n\to\infty} e_n = 0$ in $\mathrm{ces}(q)$. Since $\mathrm{ces}(p+) = \bigcap_{k\in\mathbb{N}} \mathrm{ces}(p_k)$ for any $p_k \downarrow p$, it follows that $\lim_{n\to\infty} e_n = 0$ in $\mathrm{ces}(p+)$. Similarly, consider $\mathcal{E} \subseteq \mathrm{ces}(p\text{-})$ for any fixed $1 < p \leq \infty$. Since $\mathrm{ces}(p\text{-}) = \mathrm{ind}_k \mathrm{ces}(p_k)$ for any $1 < p_k \uparrow p$ is equipped with its inductive limit topology, [28, p.280], each inclusion map $\mathrm{ces}(p_k) \subseteq \mathrm{ces}(p\text{-})$ for $k \in \mathbb{N}$ is continuous. Fix any $k_0 \in \mathbb{N}$. It is clear from (4.8) that $\lim_{n\to\infty} e_n = 0$ in the Banach space $\mathrm{ces}(p_{k_0})$. By the previous comment this implies that $\lim_{n\to\infty} e_n = 0$ in $\mathrm{ces}(p\text{-})$.

On the other hand, for each $1 < p < \infty$, it is straight-forward to check that

$$\|e_n\|_{d(p)} = n^{1/p}, \quad n \in \mathbb{N}, \tag{4.9}$$

[4, Lemma 11(ii)]. Since $d(p+) = \bigcap_{k\in\mathbb{N}} d(p_k)$ for $p \in [1, \infty)$ with $p_k \downarrow p$ and $d(p\text{-}) = \mathrm{ind}_k d(p_k)$ for $1 < p \leq \infty$ with $1 < p_k \uparrow p$, it is routine to check using (4.9) and the nature of the bounded subsets in the spaces $d(p+), d(p\text{-})$ that $\mathcal{E}$ is an *unbounded* subset in every such space $d(p+)$ and $d(p\text{-})$. In particular, $\{e_n : n \in \mathbb{N}\}$ *cannot* be a convergent sequence in any of these spaces.

Proposition 4.9. (i) (a) *For each* $1 \leq p \leq q < \infty$ *both of the Cesàro operators* $C : d(p+) \longrightarrow d(q+)$ *and* $C : \mathrm{ces}(p+) \longrightarrow d(q+)$ *are continuous.*

(b) *Let* $1 \leq p < \infty$ *and* $x \in \mathbb{C}^{\mathbb{N}}$. *Then it is the case that*

$$C^2(|x|) \in d(p+) \text{ if and only if } C(|x|) \in d(p+).$$

(c) *For each* $1 \leq p < \infty$ *the identity* $[C, d(p+)]_s = \mathrm{ces}(p+)$ *is valid.*

(ii) (a) *For each* $1 < p \leq q \leq \infty$ *both of the Cesàro operators* $C : d(p\text{-}) \longrightarrow d(q\text{-})$ *and* $C : \mathrm{ces}(p\text{-}) \longrightarrow d(p\text{-})$ *are continuous.*

(b) *Let* $1 < p \leq \infty$ *and* $x \in \mathbb{C}^{\mathbb{N}}$. *Then it is the case that*

$$C^2(|x|) \in d(p\text{-}) \text{ if and only if } C(|x|) \in d(p\text{-}).$$

(c) *For each* $1 < p \leq \infty$ *the identity* $[C, d(p\text{-})]_s = \mathrm{ces}(p\text{-})$ *is valid.*

Proof. (i) (a) The continuity of $C : d(p+) \longrightarrow d(q+)$ follows from Lemma 2.5(i) of [4], the definition of $d(p+)$ and $d(q+)$, and the fact that $C : d(r) \longrightarrow d(s)$ is continuous whenever $1 < r \leq s < \infty$; see [12, Proposition 5.3(iii)]).

A similar argument applies to establish the continuity of $C : \mathrm{ces}(p+) \longrightarrow d(q+)$, where now it is needed that $C : \mathrm{ces}(r) \longrightarrow d(s)$ is continuous whenever $1 < r \leq s < \infty$ (cf. [12, Proposition 5.3(v)]).

(b) The proof of part (b) in Proposition 3.3(i) can easily be adapted to apply to this case (by using part (a) above and (2.4) there in place of (2.3)).

(c) Clearly $\mathrm{ces}(p+) \subseteq [C, d(p+)]_s$ as C maps $\mathrm{ces}(p+)$ into $d(p+)$ by part (a). On the other hand, let $X \subseteq \mathbb{C}^{\mathbb{N}}$ be a solid space such that $C(X) \subseteq d(p+)$. Given $x \in X$ also $|x| \in X$ and hence, $C(|x|) \in d(p+) \subseteq \mathrm{ces}(p+)$. By Proposition 2.2 of [2] we have that $x \in \mathrm{ces}(p+)$. Accordingly, $X \subseteq \mathrm{ces}(p+)$. This implies that $[C, d(p+)]_s \subseteq \mathrm{ces}(p+)$.

(ii) (a) The proof of part (a) in (i) above can be adapted to apply here by replacing Lemma 2.5(i) of [4] used there with Lemma 17(i) of [5].

(b) The proof of part (iv) of Proposition 3.4 can be modified by using part (a) above, the definition $d(p\text{-}) = \bigcup_{1<q<p} d(q)$ and applying (2.4) in place of (2.3).

(c) Using part (a) above, the proof of part (v) of Proposition 3.4 can be adapted to fit the present setting. □

Remark 4.10. (i) Since $C : \ell_r \longrightarrow d(s)$ is continuous whenever $1 < r \leq s < \infty$, [12, Proposition 5.3(iv)], it follows from Lemma 2.5(i) of [4], resp. Lemma 17(i) of [5], that $C : \ell_{p+} \longrightarrow d(q+)$ is continuous whenever $1 \leq p \leq q < \infty$, resp. $C : \ell_{p\text{-}} \longrightarrow d(q\text{-})$ is continuous whenever $1 < p \leq q \leq \infty$.

(ii) The analogue of the stronger version of the "Bennett property" for C acting in $\mathrm{ces}(p)$, $1 < p < \infty$, as it is stated in (2.1), is known to also hold for C acting in $\mathrm{ces}(p+)$, [2, Proposition 2.2], and for C acting in $\mathrm{ces}(p\text{-}), 1 < p \leq \infty$, [5, Proposition 1(i)]. However, it *fails* to hold for C acting in $\ell_{p+}, d(p+)$ and in $\ell_{p\text{-}}, d(p\text{-})$.

Indeed, for ℓ_{p+} with $1 \leq p < \infty$, see [2, Proposition 2.4]. For $\ell_{p\text{-}}$ with $1 < p \leq \infty$, choose $0 \leq x \in \mathrm{ces}(p\text{-}) \setminus \ell_{p\text{-}}$ (possible as the containment (3.10) is proper when $q = p$) and note that $C(|x|) = C(x) \in \ell_{p\text{-}}$ by Proposition 3.4(ii). So, it does *not* follow from $C(|u|) \in \ell_{p\text{-}}$ that necessarily $u \in \ell_{p\text{-}}$.

Concerning $d(p+)$ with $1 \leq p < \infty$, it follows from (4.4) and the ensuing discussion that there exists $0 \leq x \in \ell_{p+} \setminus d(p+)$. Then Proposition 4.9(i)(a) implies that $C(|x|) = C(x) \in d(p+)$. So, $C(|u|) \in d(p+)$ need *not* imply that $u \in d(p+)$. Finally, for $d(p\text{-}), 1 < p \leq \infty$, choose $0 \leq x \in \ell_{p\text{-}} \setminus d(p\text{-})$, which is possible via (4.6) and the ensuing discussion, and note that $C(|x|) = C(x) \in d(p\text{-})$ by part (i) of this remark. That is, it does *not* follow from $C(|u|) \in d(p\text{-})$ that necessarily $u \in d(p\text{-})$.

For a more general version of the *Bennett property* for Banach spaces see [14]. □

We conclude with some comments about the spaces occuring in this paper when they are considered as *locally solid, lc-Riesz spaces*. The standard reference on this topic (for real spaces) is [6]; for complex spaces see [34].

Let $1 \leq p < \infty$. Then $\mathrm{ces}(p+)$, resp. ℓ_{p+}, is the complexification of the corresponding real Riesz space $\mathrm{ces}_{\mathbb{R}}(p+) := \{x \in \mathrm{ces}(p+) : x = (x_n)_n \in \mathbb{R}^{\mathbb{N}}\}$, resp. $(\ell_{p+})_{\mathbb{R}} := \{x \in \ell_{p+} : x = (x_n)_n \in \mathbb{R}^{\mathbb{N}}\}$, where the order in the real spaces is defined coordinatewise. The Fréchet lattices $\mathrm{ces}(p+), \ell_{p+}$ are *Dedekind complete*, that is, every subset of $\mathrm{ces}_{\mathbb{R}}(p+), (\ell_{p+})_{\mathbb{R}}$ which is bounded from above in the order sense has a least upper bound. Moreover, being *reflexive*, each of the (separable) Fréchet lattices $\mathrm{ces}(p+), \ell_{p+}$, for $p \in [1, \infty)$, has a *Lebesgue topology*, that is, if $x^{(\alpha)} \downarrow 0$ is a decreasing net in the order of $\mathrm{ces}(p+), \ell_{p+}$, then $\lim_\alpha x^{(\alpha)} = 0$ in the topology of $\mathrm{ces}(p+), \ell_{p+}$. In addition, the order intervals in $\mathrm{ces}(p+), \ell_{p+}$ are topologically complete. Each Fréchet lattice $\mathrm{ces}(p+)$, for $p \in [1, \infty)$, is *Montel*, which is *not* the case for ℓ_{p+}, for $p \in [1, \infty)$. For these notions and facts (and additional properties) we refer to [2, Section 4]. All of the properties needed for establishing the above facts for $\mathrm{ces}(p+)$ in [2] are also available for $d(p+)$. So, each space $d(p+)$, for $p \in [1, \infty)$, is a (Montel) locally convex Fréchet lattice which is Dedekind complete, has a Lebesgue topology and its order intervals are topologically complete.

Various properties of the associated (LB)-spaces $\ell_{p-}, \mathrm{ces}(p\text{-})$, for $1 < p \leq \infty$, considered as locally solid, lc-Riesz spaces, occur in [5, Section 6]. The (LB)-spaces ℓ_{p-} are reflexive (but, not Montel), Dedekind complete, have a Lebesgue topology and their order intervals are topologically complete. Each space $\mathrm{ces}(p\text{-}), 1 < p \leq \infty$, has the same properties just listed for ℓ_{p-} and, in addition, is Montel. All of the properties needed for establishing the above facts in [5] for $\mathrm{ces}(p\text{-})$ are also available for $d(p\text{-})$. Hence, each space $d(p\text{-}), 1 < p \leq \infty$, is a Montel, locally solid, lc-Riesz space which is Dedekind complete, has a Lebesgue topology and its order intervals are topologically complete.

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