OPERATORS ACTING IN THE DUAL SPACES OF DISCRETE CESÀRO SPACES

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Abstract. The discrete Cesàro (Banach) sequence spaces $\mathrm{ces}(r), 1 < r < \infty$, have been thoroughly investigated for over 45 years. Not so for their dual spaces $d(s) \cong (\mathrm{ces}(r))'$, with $\frac{1}{s} + \frac{1}{r} = 1$, which are somewhat unwieldy. Our aim is to undertake a further study of the spaces $d(s)$ and of various operators acting between these spaces. It is shown that $d(s) \subseteq d(t)$ whenever $s \le t$, with the inclusion being compact if $s < t$. The classical Cesàro operator C is continuous from $d(s)$ into $d(t)$ precisely when $s \le t$ and compact precisely when $s < t$. Moreover, C even maps the *larger* space $\mathrm{ces}(s)$ continuously into $d(s)$. This is a consequence of the Hardy-Littlewood maximal theorem and the remarkable property, for each $1 < s < \infty$, that $x \in \mathbb{C}^{\mathbb{N}}$ satisfies $C(C(|x|)) \in d(s)$ if and only if $C(|x|) \in d(s)$. These results are used to analyze the spectrum and to determine the norm and the mean ergodicity of C acting in $d(s)$. Similar properties for multiplier operators are also treated.

1. Introduction

Given an element $x = (x_n)_n = (x_1, x_2, \ldots)$ of $\mathbb{C}^{\mathbb{N}}$ set $|x| := (|x_n|)_n \in \mathbb{C}^{\mathbb{N}}$ and define $x \ge 0$ if $|x| = x$. The Cesàro operator $C : \mathbb{C}^{\mathbb{N}} \longrightarrow \mathbb{C}^{\mathbb{N}}$ is given by

$$C(x) := \left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)_n, \quad x \in \mathbb{C}^{\mathbb{N}};$$

it satisfies $|C(x)| \le C(|x|)$ for each $x \in \mathbb{C}^{\mathbb{N}}$ and is a vector space isomorphism of $\mathbb{C}^{\mathbb{N}}$ onto itself. Define

$$\mathrm{ces}(p) := \{x \in \mathbb{C}^{\mathbb{N}} : \|x\|_{\mathrm{ces}(p)} := \|C(|x|)\|_p < \infty\}, \quad 1 < p < \infty, \tag{1.1}$$

with $\|\cdot\|_p$ denoting the standard norm in ℓ_p. A detailed investigation of the Banach spaces $\mathrm{ces}(p), 1 < p < \infty$, also called *discrete Cesàro spaces*, was undertaken in [6]; see also the references therein. They are reflexive, p-concave, complex Banach lattices for the coordinatewise order. In recent years there has been a keen interest in these spaces and in various linear operators acting in them (e.g., the Cesàro operator, multiplier operators, inclusion maps, convolution operators); see, for instance, [1], [3], [4], [6], [8], [10], [11], [12], [13], [14], [18], [21], [24], [27], [30], [32], [33] and the references therein.

The dual Banach spaces $(\mathrm{ces}(p))'$ of $\mathrm{ces}(p), 1 < p < \infty$, are rather unwieldy. A description of these dual spaces was first presented in [21] as the solution to a problem posed by the Dutch Mathematical Society in 1968. An alternate and more tractable *isomorphic* identification of $(\mathrm{ces}(p))'$ is presented in [6, Corollary 12.17]. To describe this identification consider

$$d(s) := \left\{x \in \ell_\infty : \hat{x} := \left(\sup_{k\ge n} |x_k|\right)_n \in \ell_s\right\}, \quad 1 < s < \infty, \tag{1.2}$$

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which is a Banach space for the norm $\|x\|_{d(s)} := \|\hat{x}\|_s$. The sequence $\hat{x}$ is called the *least decreasing majorant* of x. Then $(\mathrm{ces}(p))'$ is isomorphic to $d(p')$, where $\frac{1}{p} + \frac{1}{p'} = 1$, with the duality specified by

$$\langle u, x\rangle := \textstyle\sum_{n=1}^{\infty} u_n x_n, \quad u \in \mathrm{ces}(p), \quad x \in d(p');$$

see Section 2. The dual spaces $(\mathrm{ces}(p))' \cong d(p')$ are less prominent than the discrete Cesàro spaces $\mathrm{ces}(p)$, although they have received some attention; see, for example, [6], [10], [11], [18], [21], [24], [30] and the references therein. The aim of this paper is to further study the spaces $d(s), 1 < s < \infty$, in detail as well as the properties of certain linear operators acting in them. Let us be more precise.

In Section 2 the Banach spaces $d(s), 1 < s < \infty$, are investigated; they are reflexive and the standard canonical vectors $e_n := (\delta_{kn})_k$, for $n \in \mathbb{N}$, form an unconditional basis. It is shown that $d(s)$ is lattice isomorphic to the dual Banach lattice $(\mathrm{ces}(s'))'$ of $\mathrm{ces}(s'), 1 < s < \infty$, and hence, $d(s)$ is s-convex. Whenever $1 < r \leq s$ it turns out that $d(r) \subsetneq d(s)$ (cf. Proposition 2.7(i)) with a compact inclusion map $d(r) \subseteq d(s)$ if $r < s$; see Remark 4.2. Whenever $r \neq s$, it is shown in Proposition 2.7(ii) that $d(r)$ and $d(s)$ are not isomorphic as Banach spaces. Although $d(s) \subsetneq \ell_s$ is a proper containment for every $1 < s < \infty$, it is clear that every eventually decreasing, non-negative sequence $x \in \ell_s$ does lie in $d(s)$ because x and $\hat{x}$ coincide except for at most finitely many coordinates. The Banach spaces $\mathrm{ces}(p), 1 < p < \infty$, have the following remarkable property. Let $1 < p < \infty$ and $x \in \mathbb{C}^{\mathbb{N}}$. Then

$$C(|x|) \in \mathrm{ces}(p) \text{ if and only if } x \in \mathrm{ces}(p), \tag{1.3}$$

[6, Theorem 20.31]. In view of (1.1) this can be equivalently formulated as

$$CC(|x|) \in \ell_p \text{ if and only if } C(|x|) \in \ell_p; \tag{1.4}$$

see Section 3. The relationship (1.4) between the spaces ℓ_p and the operator C acting in these spaces has the following analogue for $d(s)$; see Proposition 3.7. Namely, for $1 < s < \infty$ and $x \in \mathbb{C}^{\mathbb{N}}$ it turns out that

$$CC(|x|) \in d(s) \text{ if and only if } C(|x|) \in d(s).$$

A crucial property of $d(s)$ is that it is *solid* in $\mathbb{C}^{\mathbb{N}}$, that is, if $x \in d(s)$ and $y \in \mathbb{C}^{\mathbb{N}}$ satisfy $|y| \leq |x|$ then also $y \in d(s)$.

Since $d(s) \cong (\mathrm{ces}(s'))'$, it is clear from (1.1) that the Cesàro operator C is closely connected to the spaces $d(s), 1 < s < \infty$; clarifying this connection is the purpose of Section 3. It turns out that $C : d(s) \longrightarrow d(s)$, denoted by $C_{d(s)}$, is continuous with operator norm $\|C_{d(s)}\|_{\mathrm{op}} = s'$ (cf. Proposition 3.2). Moreover, the spectrum of $C_{d(s)}$, denoted by $\sigma(C_{d(s)})$, is given by

$$\sigma(C_{d(s)}) = \{\lambda \in \mathbb{C} : \left|\lambda - \tfrac{s'}{2}\right| \leq \tfrac{s'}{2}\}; \tag{1.5}$$

see Corollary 3.5. Consequently, $C_{d(s)}$ fails to be mean ergodic. Moreover, the dual operator $C'_{d(s)} : (d(s))' \longrightarrow (d(s))'$ of $C_{d(s)}$ has a large point spectrum which implies that $C_{d(s)}$ cannot be supercyclic. Clearly $C_{d(s)}$ is a positive operator (i.e., $C_{d(s)}(x) \geq 0$ for every $x \in d(s)$ satisfying $x \geq 0$) and hence, it is also a *regular operator* in the complex Banach lattice $d(s)$. It is shown that the *order spectrum* of $C_{d(s)}$, that is, its spectrum relative to the Banach algebra of all regular operators in $d(s)$, coincides with its spectrum $\sigma(C_{d(s)})$ as given in (1.5); see Proposition 3.6. An interesting and important fact (cf. Proposition 3.4) which will often be used is that C maps ℓ_s into the smaller space $d(s)$ for every $1 < s < \infty$. Perhaps more surprising is that C even maps $\mathrm{ces}(s)$, which is genuinely larger that ℓ_s, into $d(s)$; see Corollary 3.8.

Given a pair $1 < s, t < \infty$, an element $a \in \mathbb{C}^{\mathbb{N}}$ is called a $(d(s), d(t))$-*multiplier* if it multiplies $d(s)$ into $d(t)$, that is, if $ax \in d(t)$ for every $x \in d(s)$, where $ax := (a_n x_n)_n$ is the coordinatewise product. The closed graph theorem implies that the associated linear $(d(s), d(t))$-*multiplier operator* $M^a_{d(s),d(t)} : x \longmapsto ax$ is then continuous from $d(s)$ into $d(t)$. We denote $M^a_{d(s),d(s)}$ simply by $M^a_{d(s)}$; it is the operator acting in $d(s)$ via the diagonal matrix having the scalars $\{a_n : n \in \mathbb{N}\}$ in its diagonal. The vector space $\mathcal{M}_{d(s),d(t)}$ of all $(d(s), d(t))$-multipliers (briefly $\mathcal{M}_{d(s)}$ if $s = t$) has been identified by G. Bennett, [6, pp.69-70].

In Section 4 we investigate certain features of the operators $M^a_{d(s),d(t)}$ for all $a \in \mathcal{M}_{d(s),d(t)}$ and all pairs $1 < s, t < \infty$. For example, those multipliers $a \in \mathcal{M}_{d(s),d(t)}$ for which $M^a_{d(s),d(t)} : d(s) \longrightarrow d(t)$ is a *compact operator* are characterized in Propositions 4.1 and 4.4. Moreover, if $a \in \mathcal{M}_{d(s)} = \ell_\infty$, then it is shown that the spectrum of $M^a_{d(s)} : d(s) \longrightarrow d(s)$ is given by

$$\sigma(M^a_{d(s)}) = \overline{a(\mathbb{N})}, \quad 1 < s < \infty,$$

where $a(\mathbb{N}) := \{a_n : n \in \mathbb{N}\} \subseteq \mathbb{C}$, and that $\|M^a_{d(s)}\|_{\mathrm{op}} = \|a\|_\infty$; see Proposition 4.5. In addition, those $a \in \mathcal{M}_{d(s)}$ for which the operator $M^a_{d(s)}$ is *mean ergodic* are identified, as well as those for which $M^a_{d(s)}$ is *uniformly mean ergodic*; see Propositions 4.7 and 4.9, respectively.

In the final Section 5 it is shown that C maps $d(s)$ into $d(t)$, necessarily continuously, precisely when $1 < s \le t < \infty$. Moreover, *all pairs* $1 < s, t < \infty$ are identified for which C maps $d(s)$ into ℓ_t and for which C maps ℓ_s into $d(t)$ (cf. Proposition 5.3), as well as the subclasses of these continuous operators which are actually *compact*; see Proposition 5.4. This will require a knowledge of the continuity and compactness properties of various *inclusion maps* between pairs of spaces coming from the class $\{d(s), \ell_q, \mathrm{ces}(p) : 1 < p, q, s < \infty\}$; see Propositions 5.1 and 5.2.

2. The spaces $d(s)$ for $1 < s < \infty$

That the canonical vectors $\{e_n : n \in \mathbb{N}\}$, with $e_n := (\delta_{nk})_k$ for $n \in \mathbb{N}$, form a Schauder basis in $d(s)$, $1 < s < \infty$, is known, [1, Lemma 2.3]. The following result is a strengthening of this fact.

Proposition 2.1. *For each $1 < s < \infty$ the vectors $\{e_n : n \in \mathbb{N}\}$ form an unconditional Schauder basis for the Banach space $d(s)$. Moreover, $\|e_n\|_{d(s)} = n^{1/s}$ for each $n \in \mathbb{N}$.*

Proof. We need to verify the following criterion: Given $x \in d(s)$ and $\epsilon > 0$ there exists $M_\epsilon \in \mathbb{N}$ such that for all finite sets $\sigma = \{\sigma_1 < \sigma_2 < \ldots \sigma_k\} \subseteq \mathbb{N}$ with $\sigma_1 > M_\epsilon$ we have $\| \Sigma_{n\in\sigma} x_n e_n \|_{d(s)} < \epsilon$; see [26, Vol. I, Proposition 1.c.1]. To verify this condition it suffices to show (as $\{e_n : n \in \mathbb{N}\}$ is a Schauder basis for $d(s)$), that for every σ as stated, we have

$$\| \textstyle\sum_{n\in\sigma} x_n e_n \|_{d(s)} \leq \|x - x^{(N)}\|_{d(s)}, \quad \forall\, N > M_\epsilon, \tag{2.1}$$

where $x^{(N)} := \Sigma_{k>N} x_k e_k$. From the definition of $\hat{u}$ for $u \in \mathbb{C}^{\mathbb{N}}$ it is routine to verify that

$$(\textstyle\sum_{n\in\sigma} x_n e_n)\hat{} \leq (x - x^{(N)})\hat{}, \quad \forall\, N > M_\epsilon.$$

This inequality and (1.2) clearly imply (2.1).

For each $n \in \mathbb{N}$ we have $\hat{e}_n = (1, \ldots, 1, 0, 0, \ldots)$ with 1 occurring n times. It is then immediate that $\|e_n\|_{d(s)} = n^{1/s}$. □

It follows immediately from Proposition 2.1 that $x = \sum_{n=1}^\infty x_n e_n$ for each $x \in d(s)$ and hence, that $0 = \lim_{n\to\infty} \|x_n e_n\|_{d(s)} = \lim_{n\to\infty} |x_n| n^{1/s}$. This is Proposition 1 of [30].

It was already stated that $d(s)$ is *solid* in $\mathbb{C}^{\mathbb{N}}$; this is immediate from (1.2) and the fact that if $x \in d(s)$ and $y \in \mathbb{C}^{\mathbb{N}}$ satisfy $|y| \leq |x|$, then $\hat{y} \leq \hat{x}$ and so $\|\hat{y}\|_s \leq \|\hat{x}\|_s < \infty$. The *Köthe dual* of a solid Banach sequence space $(X, \|\cdot\|_X)$ with $c_{00} \subseteq X \subseteq \mathbb{C}^{\mathbb{N}}$, is the Banach space

$$X^\times := \left\{ x \in \mathbb{C}^{\mathbb{N}} : \textstyle\sum_{n=1}^\infty |x_n y_n| < \infty, \quad \forall\, y \in X \right\}$$

when equipped with the norm

$$\|x\|_{X^\times} := \sup\{\textstyle\sum_{n=1}^\infty |x_n y_n| : \|y\|_X \leq 1\}, \quad x \in X^\times$$

[22, Section 30]. Here $c_{00} \subseteq \mathbb{C}^{\mathbb{N}}$ is the linear space of all vectors having only finitely many non-zero coordinates. The duality for the pair of Banach spaces X and $X^\times$ is given by

$$[u, x] := \textstyle\sum_{n=1}^\infty u_n x_n, \quad x = (x_n)_n \in X^\times, \quad u = (u_n)_n \in X.$$

Then $X^\times$ is a closed subspace of the dual Banach space X' of X and $\|v\|_{X^\times} = \|v\|_{X'}$ for all $v \in X^\times$. For the following result we refer to [6, p.61 & Corollary 12.17], where $\langle \cdot, \cdot \rangle$ denotes the duality between ces(p) and its dual Banach space (ces(p))$'$.

Lemma 2.2. *Let* $1 < p < \infty$ *and* $\frac{1}{p} + \frac{1}{p'} = 1$. *Then* $(\mathrm{ces(p)})^\times = d(p')$ *and* $(d(p'))^\times = \mathrm{ces(p)}$ *with equivalent norms. Moreover, the map* $\Phi_p : (\mathrm{ces(p)})' \longrightarrow d(p')$ *defined by*

$$\Phi_p(f) := (\langle e_n, f \rangle)_n, \quad f \in (\mathrm{ces(p)})' \tag{2.2}$$

is a linear isomorphism of the dual Banach space $(\mathrm{ces(p)})'$ *onto the Banach space* $d(p')$ *and satisfies*

$$\frac{1}{p'} \|\Phi_p(f)\|_{d(p')} \leq \|f\|_{(\mathrm{ces(p)})'} \leq (p-1)^{1/p} \|\Phi_p(f)\|_{d(p')}, \quad f \in (\mathrm{ces(p)})',$$

and

$$\langle x, f \rangle = \langle \textstyle\sum_{n=1}^\infty x_n e_n, f \rangle = \sum_{n=1}^\infty x_n \langle e_n, f \rangle = [x, \Phi_p(f)], \quad x \in \mathrm{ces(p)},\ f \in (\mathrm{ces(p)})'.$$

Remark 2.3. Since the Banach spaces ces(p), $1 < p < \infty$, are known to be reflexive, [6, p.61], the dual Banach spaces (ces(p))$'$, $1 < p < \infty$, are also reflexive. In view of Lemma 2.2, the Banach spaces $d(s)$, $1 < s < \infty$, are necessarily reflexive. □

Let $E_{\mathbb{R}}$ be a real Banach lattice and $E = E_{\mathbb{R}} \oplus iE_{\mathbb{R}}$ be the complex Banach lattice that it generates. The modulus $|z|$ of an element $z = x + iy$ in E is given by

$$|z| := \sup_{0\le\theta<2\pi} \mathrm{Re}(e^{i\theta}z) = \sup_{0\le\theta<2\pi} (\cos(\theta)x + \sin(\theta)y)$$

and satisfies $|z| \in E^+$ (defined to be the positive cone $E^+_{\mathbb{R}} := \{x \in E_{\mathbb{R}} : x \ge 0\}$ of $E_{\mathbb{R}}$). The norm in E is defined by $\|z\| := \|\,|z|\,\|_{E_{\mathbb{R}}}$, where $\|\cdot\|_{E_{\mathbb{R}}}$ is the norm in $E_{\mathbb{R}}$. The positive cone of the real dual Banach space $(E_{\mathbb{R}})'$ is specified by

$$x' \le y' \text{ if and only if } \langle x, x'\rangle \le \langle x, y'\rangle, \quad \forall x \in E^+_{\mathbb{R}}.$$

Then $(E_{\mathbb{R}})'$ is also a real Banach lattice, called the *dual lattice* of $E_{\mathbb{R}}$. Moreover, the dual Banach space E' is then the complexification $(E_{\mathbb{R}})' \oplus i(E_{\mathbb{R}})'$ of $(E_{\mathbb{R}})'$; [28, p.71] and [34, p.235, Corollary 3]. Standard references for complex Banach lattices are [28], [34]. The relevant spaces in this paper are the real Banach lattices $E_{\mathbb{R}} = d_{\mathbb{R}}(s) := \{x \in \mathbb{R}^{\mathbb{N}} : \|\hat{x}\|_s < \infty\}$ for $1 < s < \infty$. The norm in $d_{\mathbb{R}}(s)$ is again given by (1.2) and its lattice order is defined to be the coordinatewise one, that is, $x \le y$ if and only if $x_n \le y_n$ for all $n \in \mathbb{N}$. This is precisely the lattice ordering induced in $d_{\mathbb{R}}(s)$ via the unconditional Schauder basis $\{e_n : n \in \mathbb{N}\}$ when it is normalized; see [26, Vol.II, p.2]. Since $\|\cdot\|_s$ is a lattice norm in ℓ_s, it follows from (2.4) below that $\|\cdot\|_{d_{\mathbb{R}}(s)}$ is a *lattice norm* in $d_{\mathbb{R}}(s)$, that is, if $x, y \in d_{\mathbb{R}}(s)$ satisfy $|x| \le |y|$, then $\|x\|_{d_{\mathbb{R}}(s)} \le \|y\|_{d_{\mathbb{R}}(s)}$. The complex Banach lattice $d(s) = d_{\mathbb{R}}(s) \oplus id_{\mathbb{R}}(s)$ is then the complexification of $d_{\mathbb{R}}(s)$. Moreover, $(\mathrm{ces}(p))' = (\mathrm{ces}_{\mathbb{R}}(p))' \oplus i(\mathrm{ces}_{\mathbb{R}}(p))'$, for $1 < p < \infty$, is the dual Banach lattice of $\mathrm{ces}(p) = \mathrm{ces}_{\mathbb{R}}(p) \oplus i\ \mathrm{ces}_{\mathbb{R}}(p)$, where $\mathrm{ces}_{\mathbb{R}}(p) := \{x \in \mathbb{R}^{\mathbb{N}} : \|C(|x|)\|_p < \infty\}$.

Let $E = E_{\mathbb{R}} \oplus iE_{\mathbb{R}}$ and $F = F_{\mathbb{R}} \oplus iF_{\mathbb{R}}$ be complex Banach lattices. For any continuous linear operator $T : E \longrightarrow F$ there exist canonical continuous, $\mathbb{R}$-linear operators $\mathrm{Re}\, T$ and $\mathrm{Im}\, T$ from $E_{\mathbb{R}}$ into $F_{\mathbb{R}}$ such that $T(x) = (\mathrm{Re}\, T)(x) + i(\mathrm{Im}\, T)(x)$ for all $x \in E_{\mathbb{R}}$. A continuous $\mathbb{R}$-linear operator $S : E_{\mathbb{R}} \longrightarrow F_{\mathbb{R}}$ is called a *(Banach) lattice homomorphism* if $S(x \wedge y) = S(x) \wedge S(y)$ and $S(x \vee y) = S(x) \vee S(y)$ for all $x, y \in E_{\mathbb{R}}$. This is equivalent to the condition that $|S(x)| = S(|x|)$ for all $x \in E_{\mathbb{R}}$, [28, Proposition 1.3.11]. In particular, every lattice homomorphism S is necessarily a positive operator (denoted by $S \ge 0$), i.e., $S(E^+_{\mathbb{R}}) \subseteq F^+_{\mathbb{R}}$. Generally, whenever $S \ge 0$ we have $|S(x)| \le S(|x|)$ for all $x \in E_{\mathbb{R}}$, [34, p.58]. For complex Banach lattices E, F a continuous linear operator $T : E \longrightarrow F$ is called a lattice homomorphism if $|T(z)| = T(|z|)$ for $z \in E$, [34, p.136], i.e., if both $\mathrm{Re}\, T$ and $\mathrm{Im}\, T$ are lattice homomorphisms. If, in addition, T is a bijection, then it is called a *lattice isomorphism*.

The following result is a strengthening of Lemma 2.2.

Proposition 2.4. *Let* $1 < p < \infty$ *and* $\frac{1}{p} + \frac{1}{p'} = 1$. *Then the linear Banach space isomorphism* $\Phi_p : (\mathrm{ces(p)})' \longrightarrow d(p')$ *as given by* (2.2) *is actually a Banach lattice isomorphism.*

Proof. In view of the above discussion and the fact that $(\mathrm{ces}(p))'$, resp. $d(p')$, is the complexification of $(\mathrm{ces}_{\mathbb{R}}(p))'$, resp. of $d_{\mathbb{R}}(p')$, it suffices to show that the restriction $\Phi_p : (\mathrm{ces}_{\mathbb{R}}(p))' \longrightarrow d_{\mathbb{R}}(p')$ of (2.2) is a Banach lattice isomorphism.

Since $\{e_n : n \in \mathbb{N}\} \subseteq (\mathrm{ces}_{\mathbb{R}}(p))^+$, it is clear from (2.2) and the definition of the order in $d(p')$ that $\Phi_p \geq 0$. Accordingly,

$$|\Phi_p(f)| \leq \Phi_p(|f|), \quad f \in (\mathrm{ces}_{\mathbb{R}}(p))'. \tag{2.3}$$

By Lemma 2.2 the linear operator Φ_p is a Banach space isomorphism and so, in particular, the inverse operator $\Phi_p^{-1} : d_{\mathbb{R}}(p') \longrightarrow (\mathrm{ces}_{\mathbb{R}}(p))'$ exists and is linear and continuous. If $y = (y_n)_n \in d_{\mathbb{R}}(p')$, then $\Phi_p^{-1}(y) \in (\mathrm{ces}_{\mathbb{R}}(p))'$ is given by

$$\langle x, \Phi_p^{-1}(y)\rangle := \sum_{n=1}^{\infty} x_n y_n, \quad x = (x_n)_n \in \mathrm{ces}_{\mathbb{R}}(p),$$

from which it is clear that $\Phi_p^{-1} \geq 0$. Accordingly,

$$|\Phi_p^{-1}(y)| \leq \Phi_p^{-1}(|y|), \quad y \in d_{\mathbb{R}}(p').$$

Fix $f \in (\mathrm{ces}_{\mathbb{R}}(p))'$ and set $y := \Phi_p(f)$. By the previous inequality

$$|\Phi_p^{-1}(\Phi_p(f))| \leq \Phi_p^{-1}(|\Phi_p(f)|)$$

and so

$$0 \leq |f| = |\Phi_p^{-1}(\Phi_p(f))| \leq \Phi_p^{-1}(|\Phi_p(f)|), \quad f \in (\mathrm{ces}_{\mathbb{R}}(p))'.$$

Apply $\Phi_p \geq 0$ to this inequality yields

$$\Phi_p(|f|) \leq |\Phi_p(f)|, \quad f \in (\mathrm{ces}_{\mathbb{R}}(p))'.$$

Combining this inequality with (2.3) it follows that $\Phi_p(|f|) = \Phi_p(f)|$ for all $f \in (\mathrm{ces}_{\mathbb{R}}(p))'$. Hence, Φ_p is a Banach lattice homomorphism and, being a bijection, it is a Banach lattice isomorphism. □

Corollary 2.5. *For each* $1 < s < \infty$, *the Banach lattice* $d(s)$ *is* s*-convex.*

Proof. Set $p := s'$. According to Proposition 2.1(iii) of [14] the Banach lattice $\mathrm{ces}(p)$ is p-concave. Hence, its dual Banach lattice $(\mathrm{ces}(p))'$ is $s = p'$ convex; combine Proposition 1.d.4 of [26, Vol.II], which is formulated for real Banach lattices, with [29, Lemma 2.49]. Since $(\mathrm{ces}(p))'$ is Banach lattice isomorphic to $d(p') = d(s)$ by Proposition 2.4, it follows that $d(s)$ is s-convex. □

Remark 2.6. Since the Banach lattice $d(s), 1 < s < \infty$, is reflexive (cf. Remark 2.3), it is known that $d(s)$ has *order continuous norm* (i.e., if $0 \leq x^{(N)} \in d(s)$, for $N \in \mathbb{N}$, is a decreasing sequence with $x^{(N)} \downarrow 0$ in the order of $d(s)$, then $\lim_{N\to\infty} \|x^{(N)}\|_{d(s)} = 0$). Moreover, $d(s)$ has the property that whenever a sequence $\{u^{(N)}\}_{N=1}^{\infty} \subseteq (d(s))^+$ is increasing and satisfies $\sup_N \|u^{(N)}\|_{d(s)} < \infty$, then there exists $u \in (d(s))^+$ such that $u^{(N)} \uparrow u$ and $\lim_{N\to\infty} u^{(N)} = u$ in $d(s)$, [36, Theorem 114.8]. The Banach lattice $d(s)$ is also *Dedekind complete*, [36, Theorem 113.4]. □

We now exhibit some further properties of the class of Banach spaces $d(s), 1 < s < \infty$.

Proposition 2.7. (i) *For each pair* $1 < s \leq t < \infty$, *we have* $d(s) \subseteq d(t)$ *with*

$$\|x\|_{d(t)} \leq \|x\|_{d(s)}, \quad x \in d(s).$$

(ii) *For each pair* $1 < s, t < \infty$ *with* $s \neq t$, *the Banach spaces* $d(s)$ *and* $d(t)$ *are not isomorphic. In particular, for* $s < t$ *the containment in part* (i) *is proper.*

(iii) *For each* $1 < s < \infty$ *we have* $d(s) \subseteq \ell_s$*, with a proper containment, and*

$$\|x\|_s \le \|x\|_{d(s)}, \quad x \in d(s).$$

In particular, the natural inclusion map $d(s) \subseteq \ell_s$ *is continuous.*

(iv) *For each pair* $1 < s, t < \infty$*, the Banach spaces* ℓ_s *and* $d(t)$ *are not isomorphic.*

(v) *For each pair* $1 < s \le t < \infty$ *we have* $d(s) \subseteq \ell_t$ *with the continuous inclusion map satisfying*

$$\|x\|_t \le \|x\|_{d(s)}, \quad x \in d(s).$$

Moreover, the containment $d(s) \subseteq \ell_t$ *is then proper.*

(vi) *Let* $1 < s < \infty$*. For each* $x = (x_n)_n \in d(s)$ *we have*

$$|x_n| \le \|x\|_{d(s)}, \quad n \in \mathbb{N}.$$

Hence, convergence of a sequence in $d(s)$ *implies its coordinatewise convergence.*

Proof. (i) Let $x \in d(s)$. Then $\hat{x} \in \ell_s \subseteq \ell_t$ and $\|\hat{x}\|_t \le \|\hat{x}\|_s$. It follows from (1.2) that $\|x\|_{d(t)} \le \|x\|_{d(s)}$.

(ii) If $d(s)$ and $d(t)$ are isomorphic as Banach spaces, then also their dual spaces $(d(s))'$ and $(d(t))'$ are isomorphic, that is, $\mathrm{ces}(s')$ and $\mathrm{ces}(t')$ are isomorphic (cf. Lemma 2.2). Since $s' \ne t'$, this is *not* the case, [1, Proposition 3.3].
Suppose that $s < t$ and $d(s) = d(t)$. By part (i) and the open mapping theorem the identity map from $d(s)$ into $d(t)$ would be an isomorphism, which is a contradiction.

(iii) It is clear from the definition of $\hat{x}$ that

$$0 \le |x| \le \hat{x}, \quad \forall\, x \in \ell_\infty. \tag{2.4}$$

Fix $x \in d(s)$. By definition $\hat{x} \in \ell_s$ and so (2.4) yields that also $x \in \ell_s$. Hence, $d(s) \subseteq \ell_s$. Since $\|\cdot\|_s$ is a lattice norm it follows from (2.4) that

$$\|x\|_s = \|\,|x|\,\|_s \le \|\hat{x}\|_s = \|x\|_{d(s)}.$$

Set $m_k = [k^s]$ for $k \in \mathbb{N}$, where $[\cdot]$ denotes *integer part*, and define

$$x = (1, 0, \dots, 0, \tfrac{1}{2}, 0, \dots, 0, \tfrac{1}{3}, 0, \dots, 0, \tfrac{1}{4}, 0 \dots)$$

with 0 occuring m_k times between $\frac{1}{k-1}$ and $\frac{1}{k}$, for $k \ge 2$. Clearly $x \in \ell_s$ as $\sum_{j=1}^\infty \frac{1}{j^s} < \infty$. On the other hand

$$\hat{x} = (1, \tfrac{1}{2}, \dots, \tfrac{1}{2}, \tfrac{1}{3}, \dots, \tfrac{1}{3}, \tfrac{1}{4}, \dots, \tfrac{1}{4}, \dots),$$

where $\frac{1}{k}$ occurs $1 + m_k$ times, for each $k \ge 2$. Since $\frac{1+m_k}{k^s} \ge 1$ for each $k \in \mathbb{N}$, it follows that $\hat{x} \notin \ell_s$ as $\sum_{k=1}^\infty (\hat{x})_k^s = \sum_{k=1}^\infty \frac{1+m_k}{k^s} = \infty$. Hence, $x \notin d(s)$.

(iv) Assume that ℓ_s and $d(t)$ are isomorphic. Then their dual Banach spaces $(\ell_s)'$ and $(d(t))'$) are also isomorphic, that is, $\ell_{s'}$ and $\mathrm{ces}(t')$ are isomorphic. This contradicts [6, Proposition 15.13].

(v) Observe that $d(s) \subseteq \ell_s$ (by part (iii)) and so $d(s) \subseteq \ell_t$. Fix $x \in d(s)$. Then $x \in \ell_t$ and so $\|x\|_t \le \|x\|_s$. Part (iii) implies that $\|x\|_t \le \|x\|_{d(s)}$.

Suppose that $d(s) = \ell_t$. Then the identity map from $d(s)$ into ℓ_t is surjective and hence, is an isomorphism. This contradicts part (iv).

(vi) Let $x = (x_n)_n \in d(s)$. Fix $n \in \mathbb{N}$. Then (2.4) implies that $|x_n e_n| \le |x| \le \hat{x}$. Since $\|\cdot\|_s$ is a lattice norm, it follows that

$$|x_n| = \|x_n e_n\|_s \le \|\hat{x}\|_s = \|x\|_{d(s)}. \qquad \square$$

Remark 2.8. (i) The example given in the proof of Proposition 2.7(iii) can be modified to show, for all choices of $1 < s, t < \infty$, that $\ell_s \nsubseteq d(t)$. To see this fix $1 < t < \infty$. Choose $m \in \mathbb{N}$ such that $m > t$ and define

$$x = \left(1, 0, \tfrac{1}{2}, 0, \dots, 0, \tfrac{1}{2^2}, 0, \dots, 0, \tfrac{1}{2^3}, 0, \dots\dots, \tfrac{1}{2^{n-1}}, 0, \dots, 0, \tfrac{1}{2^n}, 0, \dots\right),$$

where 0 occurs 2^{mn} times between $\frac{1}{2^{n-1}}$ and $\frac{1}{2^n}$ for each $n \geq 2$. Clearly $x \in \ell_1$ with $\|x\|_1 = \sum_{j=0}^{\infty} \frac{1}{2^j} = 2$. Observe that

$$\hat{x} = \left(1, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2^2}, \dots, \tfrac{1}{2^2}, \tfrac{1}{2^3}, \dots, \tfrac{1}{2^3}, \dots, \tfrac{1}{2^3}, \dots\dots, \tfrac{1}{2^n}, \dots, \tfrac{1}{2^n}, \dots\right),$$

where $\frac{1}{2^n}$ occurs $2^{nm} + 1$ times for each $n \geq 2$. Accordingly,

$$\|\hat{x}\|_t^t = 1 + 2\tfrac{1}{2^t} + \textstyle\sum_{n=2}^{\infty}(1 + 2^{nm})\tfrac{1}{2^{nt}} > 1 + 2\tfrac{1}{2^t} + \sum_{n=2}^{\infty} \tfrac{2^{nm}}{2^{nt}} = \infty$$

because $2^{n(m-t)} > 1$ for all $n \geq 2$. So, $\hat{x} \notin \ell_t$. Hence, $x \in \ell_1 \subseteq \ell_s$ for all $1 < s < \infty$ but, $x \notin d(t)$.

(ii) Since $\ell_s \subseteq \mathrm{ces}(s)$ for $1 < s < \infty$, the same x from part (i) shows that $\mathrm{ces}(s) \nsubseteq d(t)$ for all choices of $1 < s, t < \infty$.

(iii) The proof of Proposition 2.7(iv) uses Proposition 15.13 of [6], in whose proof the following property of the spaces $d(s)$ is established.

(2.5) *Let $1 < s < \infty$. Then there exists an unconditionally convergent series $\sum_{k=1}^{\infty} x^{(k)}$ in $d(s)$ such that $\sum_{k=1}^{\infty} \|x^{(k)}\|_{d(s)}^r = \infty$ for every $1 \leq r < \infty$.*

By the classical Orlicz theorem, [16, Ch.IV, §1,(5)a], such a sequence cannot exist in any space ℓ_s, for $1 < s < \infty$. □

For our next result we will require an equivalent norm for the Cesàro spaces $\mathrm{ces}(p)$, for $1 < p < \infty$, [18, Theorem 4.1]. Namely, $x \in \mathbb{C}^{\mathbb{N}}$ belongs to $\mathrm{ces}(p)$ if and only if

$$\|x\|_{[p]} := \left(\textstyle\sum_{j=0}^{\infty} 2^{j(1-p)} \left(\sum_{k=2^j}^{2^{j+1}-1} |x_k|\right)^p\right)^{1/p} < \infty. \tag{2.6}$$

Proposition 2.9. (i) *Let $1 < s < \infty$. Then $d(s)$ contains a complemented subspace which is isomorphic to ℓ_s.*

(ii) *For any pair $1 < s, t < \infty$ with $s \neq t$, the Banach spaces $d(s)$ and ces(t) are not isomorphic.*

Proof. (i) Set $p := s'$. According to (2.6) the sectional (hence, complemented) subspace

$$Y := \{x \in \mathrm{ces}(p) : x_k = 0 \ \text{ unless } \ k = 2^j \ \text{ for some } j = 0, 1, 2, \dots\}$$

is isomorphic to a weighted ℓ_p-space (as $\|x\|_{[p]} = (\sum_{j=0}^{\infty} 2^{j(1-p)} |x_{2^j}|^p)^{1/p}$ for $x \in Y$) and hence, Y is also isomorphic to ℓ_p. That is, $\mathrm{ces}(p) = Y \oplus Z$ with Y isomorphic to ℓ_p. Accordingly, $(\mathrm{ces}(p))' \cong Y' \oplus Z'$, that is, $d(s) \simeq \ell_s \oplus Z'$.

(ii) Assume that $d(s)$ and $\mathrm{ces}(t)$ are isomorphic. By part (i) it follows that ℓ_s is isomorphic to a closed subspace of $\mathrm{ces}(t)$. The same argument as in the proof of Proposition 3.3 in [1], replacing there p with s and q with t, shows that this is impossible. □

It would be interesting to know whether or not $d(s)$ isomorphic to $\mathrm{ces}(s)$.

3. The Cesàro operator acting in $d(s)$

The aim of this section is to make a detailed analysis of the Cesàro operator when it acts in the spaces $d(s), 1 < s < \infty$. We begin by collecting some useful elementary inequalities. Recall that $C : \mathbb{C}^{\mathbb{N}} \longrightarrow \mathbb{C}^{\mathbb{N}}$ is a positive operator.

Lemma 3.1. (i) *Let $x \in (\mathbb{R}^{\mathbb{N}})^+$ be a decreasing sequence (i.e. $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$). Then also $C(x)$ is a decreasing sequence.*
(ii) $0 \leq |C(x)| \leq C(|x|) \leq C(\hat{x}), \quad \forall x \in \ell_\infty$.
(iii) $0 \leq (C(x))\hat{} \leq C(\hat{x}), \quad x \in \ell_\infty$.

Proof. (i) Direct calculation yields

$$(C(x))_n - (C(x))_{n+1} = \frac{1}{n}\sum_{j=1}^{n} x_j - \frac{1}{(n+1)}\sum_{j=1}^{n+1} x_j = \frac{(x_1 + \ldots + x_n) - nx_{n+1}}{n(n+1)},$$

for each $n \in \mathbb{N}$. Since $x \geq 0$ is decreasing, it is clear that $(C(x))_n - (C(x))_{n+1} \geq 0$ for $n \in \mathbb{N}$ and hence, $C(x)$ is a decreasing sequence.

(ii) Fix $x \in \ell_\infty$. Then

$$|C(x)| = \Big(|\tfrac{1}{n}\sum_{j=1}^{n} x_j|\Big) \leq \Big(\tfrac{1}{n}\sum_{j=1}^{n} |x_j|\Big)_n = C(|x|). \tag{3.1}$$

Since $0 \leq |x| \leq \hat{x}$ (see (2.4)) and C is a positive operator we have $C(|x|) \leq C(\hat{x})$. Combining this with (3.1) yields the desired inequalities.

(iii) It follows from part (ii) and the definition of $\hat{u}$ for $u \in \ell_\infty$ that

$$(C(x))\hat{} = (|C(x)|)\hat{} \leq (C(\hat{x}))\hat{}, \quad x \in \ell_\infty. \tag{3.2}$$

Since $\hat{x} \in \ell_\infty^+$ is decreasing, it follows from part (i) that $C(\hat{x}) \geq 0$ is also decreasing. Hence, $(C(\hat{x}))\hat{} = C(\hat{x})$ and the required inequality follows from (3.2). □

We require some notation. Consider a pair $1 < s, t < \infty$. Denote by $C_{d(s),d(t)}$ (resp. $C_{d(s),t}; C_{s,d(t)}; C_{s,t}$) the Cesàro operator C when it acts from $d(s)$ into $d(t)$ (resp. from $d(s)$ into ℓ_t; from ℓ_s into $d(t)$; from ℓ_s into ℓ_t), whenever this operator exists. The closed graph theorem then ensures that this operator is continuous. With $c(p)$ as an abbreviation for the space $\mathrm{ces}(p), 1 < p < \infty$, it is clear which Cesàro operator is meant by the notation $C_{d(s),c(p)}; C_{c(p),d(s)}; C_{c(p),c(q)}; C_{s,c(p)}; C_{c(p),s}$ for $1 < s, q < \infty$. For $w \in \{d(s), c(p), t\}$ the operator $C_{w,w}$ is denoted simply by C_w. It is known that $\|C_s\|_{\mathrm{op}} = s'$ and $\|C_{c(p)}\|_{\mathrm{op}} = p'$ for all $1 < p, s < \infty$; see [20, Theorem 326] and [14, Theorem 5.1], respectively.

Given Banach spaces X, Y we denote the space of all continuous linear operators from X into Y by $L(X, Y)$. If $X = Y$, we simply write $L(X)$ for $L(X, X)$. Let $T \in L(X)$. Then

$$\rho(T) := \{\lambda \in \mathbb{C} : (T - \lambda I)^{-1} \text{ exists in } L(X)\}$$

is called the *resolvent set* of T; its complement $\sigma(T) := \mathbb{C} \setminus \rho(T)$ is the *spectrum* of T. The quantity $r(T) := \sup\{|\lambda| : \lambda \in \sigma(T)\}$ is the *spectral radius* of T and satisfies

$$r(T) \leq \|T\|_{\mathrm{op}}, \tag{3.3}$$

[17, Ch.VII, Lemma 3.4]. A complex number λ for which there exists $x \in X\setminus\{0\}$ such that $T(x) = \lambda x$ is called an *eigenvalue* of T and x is called an *eigenvector* of T corresponding to λ. The set of all eigenvalues of T is denoted by $\sigma_{pt}(T)$ and satisfies $\sigma_{pt}(T) \subseteq \sigma(T)$.

Proposition 3.2. *Let* $1 < s < \infty$.

(i) *The Cesàro operator* $C_{d(s)} : d(s) \longrightarrow d(s)$ *is continuous.*
(ii) *The point spectrum* $\sigma_{pt}(C_{d(s)}) = \emptyset$.
(iii) $\{\lambda \in \mathbb{C} : |\lambda - \frac{s'}{2}| \le \frac{s'}{2}\} \subseteq \sigma(C_{d(s)})$.
(iv) $\|C_{d(s)}\|_{\text{op}} = s'$.

Proof. (i) Let $x \in d(s) \subseteq \ell_\infty$. By Lemma 3.1(iii) we have $0 \le (C(x))\hat{} \le C(\hat{x})$. Since $\hat{x} \in \ell_s$ and $C_s : \ell_s \longrightarrow \ell_s$ is continuous, we can conclude that $C(\hat{x}) = C_s(\hat{x}) \in \ell_s$ and hence, by the previous inequality, that $(C(x))\hat{} \in \ell_s$ with $\|(C(x))\hat{}\|_s \le \|C(\hat{x})\|_s$. Accordingly, $C(x) \in d(s)$ and

$$\|C(s)\|_{d(s)} = \|(C(x))\hat{}\|_s \le \|C(\hat{x})\|_s = \|C_s(\hat{x})\|_s. \tag{3.4}$$

Moreover,

$$\|C_s(\hat{x})\|_s \le \|C_s\|_{\text{op}} \, \|\hat{x}\,\|_s = s'\|x\,\|_{d(s)}. \tag{3.5}$$

It follows from (3.4) and (3.5) that $C_{d(s)} : d(s) \longrightarrow d(s)$ is continuous with $\|C_{d(s)}\|_{\text{op}} \le s'$.

(ii) Suppose that there exists a point $\lambda \in \sigma_{pt}(C_{d(s)})$. Choose $x \in d(s)\setminus\{0\}$ such that $C(x) = \lambda x$. Since $d(s) \subseteq \ell_s$ (cf. Proposition 2.7(iii)) we can conclude that $\lambda \in \sigma_{pt}(C_s)$, that is, $\sigma_{pt}(C_s) \ne \emptyset$. This is known *not* to be the case, [23, Theorem 1(a)].

(iii) It is known for the dual operator $C'_s : (\ell_s)' \longrightarrow (\ell_s)'$ of C_s that

$$\{\lambda \in \mathbb{C} : |\lambda - \tfrac{s'}{2}| < \tfrac{s'}{2}\} \subseteq \sigma_{pt}(C'_s), \tag{3.6}$$

[23, Theorem 1(b)]. Let λ satisfy $|\lambda - \frac{s'}{2}| < \frac{s'}{2}$ and choose $u \in (\ell_s)'\setminus\{0\}$ such that $C'_s(u) = \lambda u$. Proposition 2.7(iii) implies that $(\ell_s)' \subseteq (d(s))'$ and so $u \in (d(s))'\setminus\{0\}$, that is, $\lambda \in \sigma_{pt}(C'_{d(s)})$. This shows that

$$\{\lambda \in \mathbb{C} : |\lambda - \tfrac{s'}{2}| < \tfrac{s'}{2}\} \subseteq \sigma_{pt}(C'_{d(s)}) \subseteq \sigma(C'_{d(s)}) = \sigma(C_{d(s)}), \tag{3.7}$$

[17, Ch.VII, Lemma 3.7] Since $\sigma(C_{d(s)})$ is a closed subset of $\mathbb{C}$, it follows that the closure of the set in the left-side of (3.7) is also contained in $\sigma(C_{d(s)})$.

(iv) It follows from part (iii) that the spectral radius $r(C_{d(s)}) \ge s'$ and hence, via (3.3), that $\|C_{d(s)}\|_{op} \ge s'$. In the proof of part (i) the reverse inequality $\|C_{d(s)}\|_{\text{op}} \le s'$ was established. Hence, $\|C_{d(s)}\|_{\text{op}} = s'$. □

Remark 3.3. Let $1 < s < \infty$. For each $\lambda \in \mathbb{C}$ satisfying $|\lambda - \frac{s'}{2}| < \frac{s'}{2}$ it follows from (3.6) that the range of the operator $(C_s - \lambda I)$ is *not dense* in $d(s)$. □

In order to show that equality holds in Proposition 3.2(iii) we require the following result concerning the Cesàro operator $C_{s,d(s)}$ which was stated in [6, p.3] without a proof.

Given $x \in c_0$ the Hardy-Littlewood maximal sequence $M(x) = ((M(x))_k)_k$ of x is defined by

$$(M(x))_k := \sup\left\{\frac{1}{n-m+1}\sum_{j=m}^{n} |x_j| : 1 \le m \le k, \quad n \ge k\right\}, \quad k \in \mathbb{N}. \tag{3.8}$$

The operator $M : x \longmapsto M(x)$ is called the *Hardy-Littlewood maximal operator*.

Proposition 3.4. *For each* $1 < s < \infty$, *the Cesàro operator* C *maps* ℓ_s *into* $d(s)$, *that is,* $C_{s,d(s)}$ *exists and is continuous.*

Proof. Fix $x \in c_0$. For each $k \in \mathbb{N}$ it follows from (3.8), for the choice $m = 1$, that

$$(M(x))_k \geq \sup_{n\geq k} \tfrac{1}{n} \textstyle\sum_{j=1}^{n} |x_j| = \sup_{n\geq k}(C(|x|))_n = ((C(|x|))\hat{})_k,$$

that is,

$$(C(|x|))\hat{} \leq M(x), \quad x \in c_0. \tag{3.9}$$

By the Corollary on p.527 of [7] and the comment following it, we have the well known fact that $M(\ell_s) \subseteq \ell_s$. It follows from (3.9) that $(C(|x|))\hat{} \in \ell_s$ whenever $x \in \ell_s \subseteq c_0$, that is, $C(|x|) \in d(s)$ for every $x \in \ell_s$. Accordingly, C maps ℓ_s into $d(s)$. As noted earlier, the continuity of $C_{s,d(s)}$ is then a consequence of the closed graph theorem. □

Corollary 3.5. *For each* $1 < s < \infty$, *we have*

$$\sigma(C_{d(s)}) = \{\lambda \in \mathbb{C} : |\lambda - \tfrac{s'}{2}| \leq \tfrac{s'}{2}\}. \tag{3.10}$$

Proof. Fix $\lambda \in \mathbb{C}$ such that $|\lambda - \frac{s'}{2}| > \frac{s'}{2}$. Since $\sigma(C_s)$ equals the right-side of (3.10), [23, Theorem 2], the operator $T := (C_s - \lambda I)^{-1} \in L(\ell_s)$ exists. Set $S := (C_s - \lambda I) \in L(\ell_s)$ so that $TS = I = ST$ in $L(\ell_s)$.

Step 1. *T maps $d(s)$ into $d(s)$.*

Fix $x \in \ell_s$. Then $T(x) \in \ell_s$ and so $CT(x) \in d(s)$; see Proposition 3.4. This shows that CT maps ℓ_s into $d(s)$. It is routine to verify that $CT = TC$ in $L(\ell_s)$ and so also TC maps ℓ_s into $d(s)$. In particular, TC maps $d(s)$ into $d(s)$. Note that $\lambda \neq 0$ and that $\lambda T = I + TC$ as an identity in $L(\ell_s)$. This identity implies that λT maps $d(s)$ into $d(s)$ and hence, so does T.

Step 2. *The restriction $\tilde{T} := T|_{d(s)}$ belongs to $L(d(s))$.*

Suppose that the sequence $\{x^{(n)}\}_{n=1}^{\infty} \subseteq d(s)$ satisfies $x^{(n)} \longrightarrow 0$ in $d(s)$ and that $\tilde{T}(x^{(n)}) \longrightarrow y$ in $d(s)$ for $n \longrightarrow \infty$. Since the inclusion $d(s) \subseteq \ell_s$ is continuous (cf. Proposition 2.7(iii)), we have that $x^{(n)} \longrightarrow 0$ in ℓ_s and $T(x^{(n)}) = \tilde{T}(x^{(n)}) \longrightarrow y$ in ℓ_s for $n \longrightarrow \infty$. But, $T \in L(\ell_s)$ and so $y = 0$. Accordingly, $\tilde{T}$ is a closed operator in $d(s)$ and hence, is continuous.

Step 3. *The restriction $\tilde{S} := S|_{d(s)} = (C_s - \lambda I)|_{d(s)}$ belongs to $L(d(s))$.*

The continuity of $\tilde{S}$ in $d(s)$ follows from Proposition 3.2(i) and the continuous inclusion $d(s) \subseteq \ell_s$.

To complete the proof of the Corollary let $x \in d(s)$. Then $d(s) \subseteq \ell_s$ implies that

$$\tilde{T}\tilde{S}(x) = \tilde{T}(S(x)) = T(S(x)) = x.$$

Similarly, $\tilde{S}\tilde{T}(x) = x$. Hence, $\tilde{S}\tilde{T} = I = \tilde{T}\tilde{S}$ as an identity in $L(d(s))$. That is, the operator $\tilde{S} = (C - \lambda I)|_{d(s)}$ is invertible in $L(d(s))$ with inverse $\tilde{T} = (C_s - \lambda I)^{-1}|_{d(s)}$. Accordingly, $\lambda \in \rho(C_{d(s)})$. Hence, we have shown that $\{\lambda \in \mathbb{C} : |\lambda - \frac{s'}{2}| > \frac{s'}{2}\} \subseteq \rho(C_{d(s)})$, that is, $\sigma(C_{d(s)}) \subseteq \{\lambda \in \mathbb{C} : |\lambda - \frac{s'}{2}| \leq \frac{s'}{2}\}$. The reverse containment is Proposition 3.2(iii) and so the equality (3.10) is established. □

Let E be a complex Banach lattice. An operator $T \in L(E)$ is called *regular* if it is a finite linear combination of positive operators. The complex vector space of all regular operators is denoted by $L^r(E)$; it is a unital Banach algebra for the norm

$$\|T\|_r := \inf\{\|S\|_{\mathrm{op}} : 0 \leq S \in L(E),\ |T(z)| \leq S(|z|)\ \forall z \in E\}, \quad T \in L^r(E).$$

Then $\|T\|_{\mathrm{op}} \leq \|T\|_r$ for $T \in L^r(E)$ with equality whenever $T \geq 0$. The spectrum of $T \in L^r(E)$, considered as an element of the Banach algebra $L^r(E)$, is denoted by $\sigma_{\mathrm{o}}(T)$

and is called its *order spectrum*. Then $\rho_{\mathrm{o}}(T) := \mathbb{C} \setminus \sigma_{\mathrm{o}}(T)$ is the order resolvent set of T. Clearly

$$\sigma(T) \subseteq \sigma_{\mathrm{o}}(T), \quad T \in L^r(E). \tag{3.11}$$

Standard references for the above concepts and facts are [2], [34], [35].

Since C is a positive operator in $\mathbb{C}^{\mathbb{N}}$ it is a regular operator in the Banach lattices $\ell_s, d(s)$ and $\mathrm{ces}(s)$, for $1 < s < \infty$. It is known that

$$\sigma_{\mathrm{o}}(C_s) = \sigma_{\mathrm{o}}(C_{\mathrm{ces}(s)}) = \sigma(C_s) = \sigma(C_{\mathrm{ces}(s)}) = \{\lambda \in \mathbb{C} : |\lambda - \tfrac{s'}{2} \le \tfrac{s'}{2}\},$$

[8]. We now show that the same identities hold for $C_{d(s)}$. First we require some notation.

Let $\Sigma_0 := \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$. The formula for the inverse operator $(C - \lambda I)^{-1} : \mathbb{C}^{\mathbb{N}} \longrightarrow \mathbb{C}^{\mathbb{N}}$ whenever $\lambda \in \mathbb{C} \setminus \Sigma_0$ is known, [31, p.266]. Namely, for $n \in \mathbb{N}$ the n-th row of the lower triangular matrix determining $(C - \lambda I)^{-1}$ has the entries

$$\frac{-1}{n\lambda^2 \prod_{k=m}^{n}(1 - \frac{1}{k\lambda})}, \quad 1 \le m < n, \quad \text{and } \frac{n}{1 - n\lambda} = \frac{1}{(\frac{1}{n} - \lambda)}, \quad m = n, \tag{3.12}$$

with all other entries in row n being 0. We write

$$(C - \lambda I)^{-1} = D_\lambda - \tfrac{1}{\lambda^2} E_\lambda, \tag{3.13}$$

where the diagonal matrix $D_\lambda = (d_{nm}(\lambda))_{n,m=1}^{\infty}$ is given by

$$d_{nn}(\lambda) := \frac{1}{(\frac{1}{n} - \lambda)} \quad \text{and} \quad d_{nm}(\lambda) := 0 \quad \text{if } n \ne m. \tag{3.14}$$

Moreover, $E_\lambda = (e_{nm}(\lambda))_{n,m=1}^{\infty}$ is the lower triangular matrix given by $e_{1m}(\lambda) = 0$, for $m \in \mathbb{N}$, and for all $n \ge 2$ by

$$e_{nm}(\lambda) := \begin{cases} \frac{1}{n \prod_{k=m}^{n}(1 - \frac{1}{k\lambda})} & \text{if } 1 \le m < n \\ 0 & \text{if } m \ge n. \end{cases} \tag{3.15}$$

Proposition 3.6. *For each $1 < s < \infty$ the order spectrum of $C_{d(s)}$ satisfies*

$$\sigma_{\mathrm{o}}(C_{d(s)}) = \sigma(C_{d(s)}) = \{\lambda \in \mathbb{C} : |\lambda - \tfrac{s'}{2}| \le \tfrac{s'}{2}\}.$$

Proof. In view of (3.11), with $T := C_{d(s)}$, it suffices to verify that $\rho(C_{d(s)}) \subseteq \rho_{\mathrm{o}}(C_{d(s)})$. We decompose $\rho(C_{d(s)}) = \left\{\lambda \in \mathbb{C} : |\lambda - \frac{s'}{2}| > \frac{s'}{2}\right\}$ (see (3.10)) into two disjoint parts, namely

$$\rho_1 := \{\lambda \in \mathbb{C} \setminus \{0\} : \mathrm{Re}(\tfrac{1}{\lambda}) \le 0\} = \{u \in \mathbb{C} \setminus \{0\} : \mathrm{Re}(u) \le 0\}$$

and its complement $\rho_2 := \rho(C_{d(s)}) \setminus \rho_1$.

First fix $\lambda \in \rho_1$ in which case $(C_{d(s)} - \lambda I)^{-1} \in L(d(s))$. Then $\lambda \notin \Sigma_0$ and so we may consider D_λ and E_λ as specified by (3.14) and (3.15), respectively. It is shown on p.72 of [14] that

$$|e_{nm}(\lambda)| \le \tfrac{1}{n}, \quad 1 \le m < n, \quad n \in \mathbb{N}. \tag{3.16}$$

Warning. In [14] the set $\mathbb{N} = \{0, 1, 2, \ldots\}$ is used rather than $\mathbb{N} = \{1, 2, \ldots\}$ which is used here and so the inequalities from [14] are slightly different when they are stated here. Back to our proof, it is clear from the definition of C that the matrix $A = (a_{nm})_{n,m=1}^{\infty}$ of C

is lower triangular with its n-th row, for each $n \in \mathbb{N}$, given by $a_{nm} := \frac{1}{n}$ for $1 \leq m \leq n$ and $a_{nm} = 0$ for $m > n$. Let $B = (b_{nm})_{n,m=1}^{\infty}$ be the matrix of E_λ. It is clear from (3.16) that

$$|b_{nm}| \leq a_{nm}, \quad m, n \in \mathbb{N}.$$

Since the space $X := d(s)$ has the property that $A = C_{d(s)}$ maps X into X continuously, it follows via Lemma 3.1 and Corollary 3.2 from [8] that $E_\lambda : d(s) \longrightarrow d(s)$ is continuous and regular. Also the diagonal operator D_λ is continuous and regular in $d(s)$, [8, Lemma 3.3]. It follows from (3.13) that $(C_{d(s)} - \lambda I)^{-1} \in L^r(d(s))$.

Concerning ρ_2 it is known that $\rho_2 = \bigcup_{0<\alpha<1/s'} \Gamma_\alpha$, where

$$\Gamma_\alpha := \{z \in \mathbb{C}\backslash\{0\} : \mathrm{Re}(\tfrac{1}{z}) = \alpha\} = \{z \in \mathbb{C}\backslash\{0\} : |z - \tfrac{1}{2\alpha}| = \tfrac{1}{2\alpha}\};$$

see the proof of Proposition 5.1 in [8]. Fix $\lambda \in \rho_2$. Then there exists a unique $\alpha \in (0, \frac{1}{s'})$ such that $\lambda \in \Gamma_\alpha$ namely, $\alpha := \mathrm{Re}(\frac{1}{\lambda})$. Then, for $1 \leq m < n$ and $n \geq 2$ we have from (3.15) that

$$(3.17) \qquad \begin{aligned} |e_{nm}(\lambda)| &= \frac{1}{n|\Pi_{k=m}^{n}(1-\frac{1}{k\lambda})|} \\ &\leq \frac{1}{n\Pi_{k=m}^{n}(1-\frac{\alpha}{k})} = e_{nm}(\tfrac{1}{\alpha}), \end{aligned}$$

where we have used that $\frac{1}{|z|} \leq \frac{1}{|\mathrm{Re}(z)|}$ whenever $\mathrm{Re}(z) \neq 0$ and $(1 - \frac{\alpha}{k}) > 0$ for all $k \in \mathbb{N}$. Moreover, $\frac{1}{\alpha} > s'$ implies that $\frac{1}{\alpha} \in \rho(C_{d(s)})$, that is , $(C_{d(s)} - \frac{1}{\alpha}I)^{-1} \in L(d(s))$. Since $D_{1/\alpha} \in L(d(s))$, it follows that

$$E_{1/\alpha} = \alpha^2(D_{1/\alpha} - (C_{d(s)} - \tfrac{1}{\alpha}I)^{-1}) \in L(d(s)).$$

Accordingly, (3.17) and Corollary 3.2 of [8] imply that $E_\lambda \in L^r(d(s))$. It then follows from [8, Lemma 3.3] and (3.13) that $(C_{d(s)} - \lambda I)^{-1} \in L^r(d(s))$. □

A remarkable property of the spaces $\mathrm{ces}(p), 1 < p < \infty$, as stated in (1.3), is due to Bennett, [6, Theorem 20.31]. In view of (1.1), which implies (via Lemma 3.1(ii)) that $C : \mathrm{ces}(p) \longrightarrow \ell_p$, this property can be equivalently formulated as follows. Let $1 < p < \infty$ and $x \in \mathbb{C}^{\mathbb{N}}$. Then

$$(3.18) \qquad C^2(|x|) \in \ell_p \textit{ if and only if } C(|x|) \in \ell_p.$$

Clearly (3.18) is a relationship between the class of spaces $\ell_p, 1 < p < \infty$, and the action of a particular operator, namely C, in these spaces. It is known that (3.18) remains valid if ℓ_p is replaced with c_0 or ℓ_∞, [14]. The same is true for the class of spaces $N^p, 1 < p < \infty$, arising in harmonic analysis and introduced in [5]; see [15, Theorem 2.4]. Namely, $N^p := \{x \in \mathbb{C}^{\mathbb{N}} : |x| \leq \mathcal{F}(f)$ for some $f \in L^p(\mathbb{T})\}$, where $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ is the circle group and $\mathcal{F}(f)$ is the Fourier transform of f. Another class of such spaces is given by the following result.

Proposition 3.7. *Let* $1 < s < \infty$ *and* $x \in \mathbb{C}^{\mathbb{N}}$. *Then*

$$(3.19) \qquad C^2(|x|) \in d(s) \textit{ if and only if } C(|x|) \in d(s).$$

Proof. Fix $1 < s < \infty$. Define $\rho : \mathbb{C}^{\mathbb{N}} \longrightarrow [0, \infty]$ via

$$\rho(x) := \left(\textstyle\sum_{n=1}^{\infty}((\hat{x})_n)^s\right)^{1/s}, \quad x \in \mathbb{C}^{\mathbb{N}}.$$

Then $\mathbb{X} := \{x \in \mathbb{C}^{\mathbb{N}} : \rho(x) < \infty\}$ is precisely the space $d(s)$ and so we know that C maps $\mathbb{X}$ into $\mathbb{X}$ continuously. Furthermore,

$$\mathrm{ces}(\mathbb{X}) := \{x \in \mathbb{C}^{\mathbb{N}} : C(|x|) \in \mathbb{X}\}$$

is the space $\{x \in \mathbb{C}^{\mathbb{N}} : C(|x|) \in d(s)\}$. Since C is a positive operator in $\mathbb{C}^{\mathbb{N}}$ we have $C(|C(|x|)|) = C^2(|x|)$ for each $x \in \mathbb{C}^{\mathbb{N}}$. So, (3.19) will follow from Theorem 3 of [13] and the discussion after that result once we establish the following property of $\mathbb{X} = d(s)$:

$$x \in \mathbb{C}^{\mathbb{N}} \text{ belongs to } d(s) \text{ whenever } [x] \in d(s). \tag{3.20}$$

Here, given $x = (x_n)_n \in \mathbb{C}^{\mathbb{N}}$ the element $[x] \in \mathbb{C}^{\mathbb{N}}$ is defined by $[x] := (x_1, x_1, x_2, x_2, x_3, x_3, \ldots)$. So, suppose that $[x] \in d(s)$. It is routine to verify that

$$[x]\hat{} = ((\hat{x})_1, (\hat{x})_1, (\hat{x})_2, (\hat{x})_2, (\hat{x})_3, (\hat{x})_3, \ldots) = [\hat{x}].$$

Since $[x] \in d(s)$ implies that $[x]\hat{} \in \ell_s$, the previous identity shows that $[\hat{x}] \in \ell_s$. Accordingly, $2\sum_{j=1}^{\infty}((\hat{x})_j)^s < \infty$ which clearly implies that $\hat{x} \in \ell_s$, that is, $x \in d(s)$. This establishes (3.20) and thereby completes the proof of the proposition. □

Since the containment $\ell_s \subseteq \mathrm{ces}(s)$ is strict for each $1 < s < \infty$, [14, Remark 2.2(ii)], the following result is an improvement on Proposition 3.4.

Corollary 3.8. *For each* $1 < s < \infty$, *the Cesàro operator* C *maps* $\mathrm{ces}(s)$ *into* $d(s)$, *that is,* $C_{\mathrm{ces}(s),d(s)}$ *exists and is continuous.*

Proof. If $x \in \mathrm{ces}(s)$, then Lemma 3.1(ii) implies that

$$\|C(x)\|_s = \|\,|C(x)|\,\|_s \le \|C(|x|)\|_s = \|x\|_{\mathrm{ces}(s)},$$

which shows that $C : \mathrm{ces}(s) \longrightarrow \ell_s$ is continuous. Since also $C : \ell_s \longrightarrow d(s)$ is continuous (cf. Proposition 3.4), it follows that the composition C^2 maps $\mathrm{ces}(s)$ continuously into $d(s)$. That is,

$$C^2(x) \in d(s), \quad \forall\, x \in \mathrm{ces}(s). \tag{3.21}$$

Let $0 \le x \in \mathrm{ces}(s)$. It follows from (3.21) that $C^2(x) = C^2(|x|) \in d(s)$ and hence, by Proposition 3.7, that $C(x) = C(|x|) \in d(s)$. This shows that C maps the positive cone $(\mathrm{ces}(s))^+$ of $\mathrm{ces}(s)$ into $d(s)$. Since each $x \in \mathrm{ces}(s)$ can be written as $(\mathrm{Re}(x))^+ - (\mathrm{Re}(x))^- + i((\mathrm{Im}(x))^+ - (\mathrm{Im}(x))^-)$ with $\{(\mathrm{Re}(x))^+, (\mathrm{Re}(x))^-, (\mathrm{Im}(x))^+, (\mathrm{Im}(x))^-\} \subseteq (\mathrm{ces}(s))^+$, it follows that C maps $\mathrm{ces}(s)$ into $d(s)$. The continuity of $C_{\mathrm{ces}(s),d(s)}$ is a consequence of the closed graph theorem. □

Since $C : d(s) \longrightarrow d(s)$ is a positive operator, we may ask if there exist $d(s)$-*valued* extensions of C to larger, *solid* Banach lattices in $\mathbb{C}^{\mathbb{N}}$. Proposition 3.4 and Corollary 3.8 show that this is indeed possible. The *largest* of those solid Banach lattices for which such a continuous $d(s)$-valued extension is possible is denoted by $[C, d(s)]_s$. With obvious notation, it is known that $[C, \ell_s]_s = \mathrm{ces}(s)$ and that $[C, \mathrm{ces}(s)]_s = \mathrm{ces}(s)$; see p.62 and Theorem 2.5 of [14], respectively.

Corollary 3.9. *For each* $1 < s < \infty$ *it is the case that*

$$[C, d(s)]_s = \mathrm{ces}(s).$$

Proof. Clearly $[C, d(s)]_s \subseteq [C, \ell_s]_s$. Via the above discussion we see that $[C, d(s)]_s \subseteq$ ces(s). On the other hand, Corollary 3.8 shows that ces$(s) \subseteq [C, d(s)]_s$. □

Given any Banach space operator $T \in L(X)$ define its sequence of Cesàro averages by $T_{[n]} := \frac{1}{n}\sum_{m=1}^{n} T^m$ for $n \in \mathbb{N}$. Setting $T_{[0]} := I$ it is routine to verify that

$$\frac{1}{n}T^n = T_{[n]} - \frac{(n-1)}{n}T_{[n-1]}, \quad n \in \mathbb{N}. \tag{3.22}$$

If the sequence $\{T_{[n]}\}_{n=1}^{\infty}$ converges in the strong operator topology (briefly, s.o.t.), that is, $\lim_{n\to\infty} T_{[n]}(x)$ exists in X for each $x \in X$, then T is called *mean ergodic*, [17, Ch. VIII]. We note that $\sigma(T) \subseteq \overline{\mathbb{D}}$ whenever an operator $T \in L(X)$ satisfies $\lim_{n\to\infty} \frac{1}{n}T^n = 0$ in the s.o.t., [17, p.709, Lemma 1]. Here $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. It is clear from (3.22) that necessarily $\lim_{n\to\infty} \frac{1}{n}T^n = 0$ in the s.o.t. whenever T is mean ergodic. An operator $T \in L(X)$ is called *uniformly mean ergodic* if $\{T_{[n]}\}_{n=1}^{\infty}$ is a convergent sequence in $L(X)$ for the operator norm.

Proposition 3.10. *For each* $1 < s < \infty$, *the Cesàro operator* $C_{d(s)} : d(s) \longrightarrow d(s)$ *fails to be mean ergodic.*

Proof. If $C_{d(s)}$ was mean ergodic, then the discussion prior to the proposition shows that necessarily $\sigma(C_{d(s)}) \subseteq \overline{\mathbb{D}}$. But, this is *not* the case by Corollary 3.5. □

Let X be a separable Banach space and $T \in L(X)$. If, for some $x \in X$, the projective orbit $\{\lambda T^n(x) : \lambda \in \mathbb{C},\ n = 0, 1, 2, \ldots\}$ is dense in X, then T is called *supercyclic*. For the theory of such operators see [19], for example. Since the projective orbit coincides with $\bigcup_{n=0}^{\infty} T^n(\mathrm{span}\{x\})$, we see that being supercyclic is the same as being 1-supercyclic in the sense of [9].

Proposition 3.11. *For each* $1 < s < \infty$, *the Cesàro operator* $C_{d(s)} : d(s) \longrightarrow d(s)$ *fails to be supercyclic.*

Proof. It follows from (3.7) that the dual operator $C'_{d(s)}$ of $C_{d(s)}$ has at least two linearly independent eigenvectors. Hence, $C_{d(s)}$ is *not* supercyclic, [9, Theorem 2.1]. □

4. Multiplier operators from $d(s)$ into $d(t)$

Let $a = (a_n)_n \in \mathbb{C}^{\mathbb{N}}$. According to table 6 on p.69 of [6] we have (in the notation of Section 1) that:

(4.1) *Let* $1 < s \le t < \infty$ *and* $a \in \mathbb{C}^{\mathbb{N}}$. *Then* $a \in \mathcal{M}_{d(s),d(t)}$ *if and only if* $(a_n n^{\frac{1}{t}-\frac{1}{s}})_n \in \ell_\infty$.

Note that $(\frac{1}{t} - \frac{1}{s}) \le 0$. In particular, $\ell_\infty \subseteq \mathcal{M}_{d(s),d(t)}$. If $s = t$, then $\mathcal{M}_{d(s)} = \ell_\infty$. For fixed $a \in \ell_\infty$ it is routine to verify that

$$(ax)\hat{}\le \|a\|_\infty \hat{x}, \quad x \in \ell_\infty. \tag{4.2}$$

For each $1 < s < \infty$, it follows from (4.2) that

$$\|M^a_{d(s)}(x)\|_{d(s)} = \|(ax)\hat{}\|_s \le \|a\|_\infty \|\hat{x}\|_s = \|a\|_\infty \|x\|_{d(s)}, \quad x \in d(s).$$

Given $n \in \mathbb{N}$, the vector $x := n^{-1/s} e_n$ satisfies $\|x\|_{d(s)} = 1$. Moreover,

$$(ax)\hat{} = n^{-1/s}|a_n|(1, 1, \ldots 1, 0, 0, \ldots)$$

with 1 occuring n-times and so $\|(ax)\hat{}\|_s = |a_n|$, that is, $\|M^a_{d(s)}(x)\|_{d(s)} = |a_n|$. Accordingly, $M^a_{d(s)} : d(s) \longrightarrow d(s)$ is continuous and satisfies

$$\|M^a_{d(s)}\|_{\mathrm{op}} = \|a\|_\infty, \quad a \in \ell_\infty, \quad 1 < s < \infty. \tag{4.3}$$

The following fact is from table 22 on p.70 of [6]:

(4.4) *For* $1 < t < s < \infty$, *we have* $\mathcal{M}_{d(s),d(t)} = d(r)$ *with* $\frac{1}{r} = \frac{1}{t} - \frac{1}{s}$.

Let $1 < s, t < \infty$ and $a \in \mathcal{M}_{d(s),d(t)}$, that is, $M^a_{d(s),d(t)} \in L(d(s), d(t))$. The dual Banach spaces are given by $(d(s))' = \mathrm{ces}(s')$ and $(d(t))' = \mathrm{ces}(t')$; see Lemma 2.2. Moreover, $s \le t$ if and only if $t' \le s'$. Accordingly, the identities

$$\langle M^a_{d(s),d(t)}(x), y\rangle = \langle ax, y\rangle = \langle x, ay\rangle, \quad x \in d(s), \quad y \in \mathrm{ces}(t'),$$

show that the dual operator $(M^a_{d(s),d(t)})' \in L(ces(t'), ces(s'))$ of $M^a_{d(s),d(t)}$ is precisely the multiplier operator $M^a_{c(t'),c(s')} : y \longmapsto ay$ from $\mathrm{ces}(t')$ to $\mathrm{ces}(s')$. Here we have used the fact (as the Copson space $\mathrm{cop}(p) = \mathrm{ces}(p)$ for $1 < p < \infty$, [6, p.47]) that the multiplier spaces $\mathcal{M}_{c(t'),c(s')}$ as given in tables 16 and 32 on pp.69–70 of [6] are consistent with the dual map $T \longmapsto T'$ from $L(d(s), d(t))$ to $L((d(t))', (d(s))') = L(\mathrm{ces}(t'), \mathrm{ces}(s'))$. That is, $a \in \mathcal{M}_{d(s),d(t)}$ is a multiplier from $d(s)$ to $d(t)$ if and only if $a \in \mathcal{M}_{c(t'),c(s')}$ is a multiplier from $\mathrm{ces}(t')$ into $\mathrm{ces}(s')$. This will allow us to use duality arguments to deduce information about $M^a_{d(s),d(t)}$ via known facts about $M^a_{c(t'),c(s')}$.

Proposition 4.1. *Let* $1 < s \le t < \infty$ *and* $a \in \mathcal{M}_{d(s),d(t)}$. *Then the continuous multiplier operator* $M^a_{d(s),d(t)} : d(s) \longrightarrow d(t)$ *is compact if and only if* $(a_n n^{\frac{1}{t}-\frac{1}{s}})_n \in c_0$.

Proof. According to Schauder's theorem, [17, Ch. VI, Theorem 5.2], $M^a_{d(s),d(t)}$ is a compact operator if and only if $(M^a_{d(s),d(t)})' = M^a_{c(t'),c(s')}$ is compact. Since $t' \le s'$, it follows from [1, Proposition 2.2] that $M^a_{c(t'),c(s')}$ is compact if and only if $(a_n n^{\frac{1}{s'}-\frac{1}{t'}})_n \in c_0$. But, $(\frac{1}{s'} - \frac{1}{t'}) = (\frac{1}{t} - \frac{1}{s})$ and so $M^a_{d(s),d(t)}$ is compact if and only if $(a_n n^{\frac{1}{t}-\frac{1}{s}})_n \in c_0$. □

Remark 4.2. (i) Let $1 < s < t < \infty$. For the constant sequence $a = (1, 1, \ldots) \in \ell_\infty \subseteq \mathcal{M}_{d(s),d(t)}$ it follows from Proposition 4.1 that the natural inclusion map $i_{d(s),d(t)} = M^a_{d(s),d(t)}$ from $d(s)$ into $d(t)$ is *compact* because $(a_n n^{\frac{1}{t}-\frac{1}{s}})_n = (n^{\frac{1}{t}-\frac{1}{s}})_n \in c_0$.

(ii) For the case when $s = t$ and $a \in \mathcal{M}_{d(s)} = \ell_\infty$, Proposition 4.1 implies that the multiplier operator $M^a_{d(s)} : d(s) \longrightarrow d(s)$ is compact if and only if $a \in c_0$. □

Lemma 4.3. *Let* $1 < t < s < \infty$ *and* r *satisfy* $\frac{1}{r} = \frac{1}{t} - \frac{1}{s}$. *Then there exists a constant* $B_{s,t} > 0$ *such that*

$$\|M^a_{d(s),d(t)}\|_{\mathrm{op}} \le B_{s,t}\, \|a\|_{d(r)}, \quad \forall a \in \mathcal{M}_{d(s),d(t)} = d(r).$$

Proof. That $\mathcal{M}_{d(s),d(t)} = d(r)$ is precisely (4.4). Since the norm $\|M^a_{d(s),d(t)}\|_{\mathrm{op}}$ of $M^a_{d(s),d(t)}$ coincides with the norm $\|(M^a_{d(s),d(t)})'\|_{\mathrm{op}}$ of the dual operator $(M^a_{d(s),d(t)})' = M^a_{c(t'),c(s')}$ and $1 < s' < t' < \infty$, the stated inequality follows from Lemma 2.4 of [1]. □

The following result, for $1 < t < s < \infty$, shows that *every* multiplier operator $M^a_{d(s),d(t)}$ for $a \in \mathcal{M}_{d(s),d(t)}$ is compact.

Proposition 4.4. *Let* $1 < t < s < \infty$. *For* $a \in \mathbb{C}^{\mathbb{N}}$ *the following assertions are equivalent.*

(i) *$a \in M_{d(s),d(t)}$, that is $M^a_{d(s),d(t)} : d(s) \longrightarrow d(t)$ is continuous.*
(ii) *$M^a_{d(s),d(t)} : d(s) \longrightarrow d(t)$ is compact.*
(iii) *$a \in d(r)$, where $\frac{1}{r} = \frac{1}{t} - \frac{1}{s}$.*

Proof. (i) ⇔ (iii) is (4.4) and (ii) ⇒ (i) is clear.

(iii) ⇒ (ii). Using Lemma 4.3 above in place of Lemma 2.4 in [1] it is clear that the proof of (iii) ⇒ (ii) in Proposition 2.5 of [1] can be adapted to apply here; one needs to replace $M^a_{p,q}$ ($= M^a_{c(p),c(q)}$) in the notation from there in [1] with $M^a_{d(s),d(t)}$ here. □

In view of (4.3), the following fact can be verified by modifying suitably the proofs of Lemma 2.6 and Proposition 2.7 in [1].

Proposition 4.5. *Let $1 < s < \infty$. For each $a \in \ell_\infty = \mathcal{M}_{d(s)}$ we have that*

$$\|M^a_{d(s)}\|_{\mathrm{op}} = \|a\|_\infty \tag{4.5}$$

and that

$$\sigma(M^a_{d(s)}) = \overline{a(\mathbb{N})} = \overline{\{a_n : n \in \mathbb{N}\}}. \tag{4.6}$$

Remark 4.6. The Banach lattices $d(s)$, $1 < s < \infty$, are also *Banach function spaces* over the σ-finite measure space $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$, where μ is counting measure, [36, p.252 and §112]. Since $\mathcal{M}_{d(s)} = \ell_\infty$, it follows from [8, Proposition 2.1(iii)] and Proposition 4.5 that the order spectrum

$$\sigma_{\mathrm{o}}(M^a_{d(s)}) = \sigma(M^a_{d(s)}) = \overline{a(\mathbb{N})}, \quad \forall\, a \in \mathcal{M}_{d(s)}, \quad 1 < s < \infty.$$ □

A Banach space operator $T \in L(X)$ is called *power bounded* if $\sup_{n\in\mathbb{N}} \|T^n\|_{\mathrm{op}} < \infty$. In this case $\lim_{n\to\infty} \frac{1}{n}\|T^n\|_{\mathrm{op}} = 0$. The following result characterizes which multiplier operators are mean ergodic.

Proposition 4.7. *Let $1 < s < \infty$ and $a \in \mathcal{M}_{d(s)} = \ell_\infty$. The following statements are equivalent.*

(i) $\|a\|_\infty \le 1$.
(ii) *The operator $M^a_{d(s)} \in L(d(s))$ is power bounded.*
(iii) *The operator $M^a_{d(s)} \in L(d(s))$ is mean ergodic.*
(iv) *The spectrum $\sigma(M^a_{d(s)}) \subseteq \overline{\mathbb{D}}$.*
(v) $\lim_{n\to\infty} \frac{1}{n}(M^a_{d(s)})^n = 0$ *for the s.o.t. in $L(d(s))$.*

By using the identities (4.5) and (4.6) it is routine to establish Proposition 4.7 by suitably adapting the proof of Proposition 2.8 in [1].

In view of Proposition 4.7 we can suppose that $\|a\|_\infty \le 1$ and that $M^a_{d(s)}$ is power bounded whenever it is mean ergodic. Then $\lim_{n\to\infty} \frac{1}{n}\|(M^a_{d(s)})^n\|_{\mathrm{op}} = 0$ and so, by a known result, [25], the *uniform* mean ergodicity of $M^a_{d(s)}$ is equivalent to the range $(I - M^a_{d(s)})(d(s)) = (M^{\mathbf{1}-a}_{d(s)})(d(s))$ of $I - M^a_{d(s)}$ being a *closed* subspace of $d(s)$, where $\mathbf{1}$ denotes the constant sequence with all coordinates equal to 1.

Given $w \in \mathbb{C}^{\mathbb{N}}$ define its *support* by $S(w) := \{n \in \mathbb{N} : w_n \neq 0\}$. Then $w\chi_{S(w)} = w$ as elements of $\mathbb{C}^{\mathbb{N}}$. If $w \in \ell_\infty$, then the range

$$M^w_{d(s)}(d(s)) := \{wx : x \in d(s)\} = \{wx\chi_{S(w)} : x \in d(s)\}, \tag{4.7}$$

for each $1 < s < \infty$. We also require the *closed* subspace of $d(s)$ which is the range of the continuous projection operator $M_{d(s)}^{\chi_{S(w)}}$, that is,

$$X_{w,d(s)} := \{x\chi_{S(w)} : x \in d(s)\} = M_{d(s)}^{\chi_{S(w)}}(d(s)). \tag{4.8}$$

It is routine to check that $X_{w,d(s)}$ is $M_{d(s)}^{w}$-invariant. Let $\tilde{M}_{d(s)}^{w} : X_{w,d(s)} \longrightarrow X_{w,d(s)}$ be the restriction of $M_{d(s)}^{w}$ to $X_{w,d(s)}$ so that $\tilde{M}_{d(s)}^{w} \in L(X_{w,d(s)})$. Since $w_n \neq 0$ for each $n \in S(w)$, it follows that $\tilde{M}_{d(s)}^{w}$ is *injective*. Hence, $\tilde{M}_{d(s)}^{w}$ is a vector space isomorphism of $X_{w,d(s)}$ onto its range $\tilde{M}_{d(s)}^{w}(X_{w,d(s)})$ in $X_{w,d(s)}$. It follows from (4.7) and (4.8) that $\tilde{M}_{d(s)}^{w}(X_{w,d(s)}) = M_{d(s)}^{w}(d(s))$ *provided* that $M_{d(s)}^{w}(d(s))$ is *closed* in $d(s)$.

The facts recorded in the previous paragraph can be used to establish the following result by adequately adapting the proof of Lemma 2.9 in [1].

Lemma 4.8. *Let $w \in \ell_\infty$ and $1 < s < \infty$. If the range $M_{d(s)}^{w}(d(s))$ is closed in $d(s)$, then $0 \notin \overline{(w\chi_{S(w)})(\mathbb{N})}$.*

Equipped with Lemma 4.8 and the identities (4.7) and (4.8) the following result can be established along the lines of the proof of Proposition 2.10 in [1].

Proposition 4.9. *Let $1 < s < \infty$ and $a \in \mathcal{M}_{d(s)} = \ell_\infty$. The following assertions are equivalent.*

(i) *$M_{d(s)}^{a}$ is uniformly mean ergodic.*
(ii) *$\|a\|_\infty \leq 1$ and $1 \notin \overline{a(\mathbb{N})\setminus\{1\}}$.*

Example 4.10. Let $1 < s < \infty$ and set $a := (1 - \frac{1}{n})_n$. Then $a \in \ell_\infty$ and $\|a\|_\infty = 1$. Proposition 4.7 ensures that $M_{d(s)}^{a}$ is mean ergodic. However, as $1 \in \overline{a(\mathbb{N})\setminus\{1\}}$ it follows from Proposition 4.9 that $M_{d(s)}^{a}$ is *not* uniformly mean ergodic. □

Proposition 4.11. *Let $1 < s < \infty$ and $a \in \mathcal{M}_{d(s)} = \ell_\infty$. The multiplier operator $M_{d(s)}^{a} \in L(d(s))$ fails to be supercyclic.*

Proof. Since $d(s)$ is dense in the Frèchet space $\mathbb{C}^{\mathbb{N}}$ (equipped with its topology of coordinatewise convergence) and the natural inclusion $d(s) \subseteq \mathbb{C}^{\mathbb{N}}$ is continuous (cf. Proposition 2.7(vi)), the supercyclicity of $M_{d(s)}^{a}$ would imply the supercyclicity of the multiplication operator $M^a : \mathbb{C}^{\mathbb{N}} \longrightarrow \mathbb{C}^{\mathbb{N}}$ defined by $x \longmapsto ax$ for $x \in \mathbb{C}^{\mathbb{N}}$. But, it is known that this is *not* the case, [1, Lemma 2.11]. So, $M_{d(s)}^{a}$ is not supercyclic. □

5. Cesàro and inclusion operators

Let $X, Y \in \{d(s), \mathrm{ces}(p), \ell_t : 1 < p, s, t < \infty\}$. The notation when considering the Cesàro operator $C : X \longrightarrow Y$ was introduced in Section 3. We use the analogous notation for the natural inclusion maps $i : X \longrightarrow Y$, whenever they exist. As a sample, $i_{d(s),t}$ (resp. $i_{d(s),c(p)}$) denotes the inclusion map from $X = d(s)$ into $Y = \ell_t$ (resp. from $X = d(s)$ into $Y = \mathrm{ces}(p)$). The main aim of this section is to identify those pairs of spaces X, Y such that $C : X \longrightarrow Y$ and $i : X \longrightarrow Y$ *do exist* (in which case continuity follows from the closed graph theorem) and, for such a pair X, Y, to determine whether or not the operator is *compact*. Recalling the notation $C_p := C_{p,p}$ and $C_{c(p)} := C_{c(p),c(p)}$ and $C_{d(p)} := C_{d(p),d(p)}$, it was already noted earlier that

$$\sigma(C_p) = \sigma(C_{c(p)}) = \sigma(C_{d(p)}) = \{\lambda \in \mathbb{C} : |\lambda - \tfrac{p'}{2}| \leq \tfrac{p'}{2}\}, \quad 1 < p < \infty. \tag{5.1}$$

Accordingly, all such Cesàro operators *fail* to be compact as their spectrum is uncountable. Concerning inclusion maps, it is routine to check that $i : X \longrightarrow Y$ exists if and only if its dual operator $i' : Y' \longrightarrow X'$ exists. Of course, the dual operator i' is the natural inclusion map $i : Y' \longrightarrow X'$. According to Schauder's theorem, $i : X \longrightarrow Y$ is compact if and only if $i : Y' \longrightarrow X'$ is compact. The situation when both spaces X, Y belong to $\{\text{ces}(p), \ell_q : 1 < p, q < \infty\}$ is completely answered in Section 3 of [1]. So, we only need to consider the remaining cases, that is, when at least one of X or Y is $d(s)$ for some $1 < s < \infty$.

Let us begin with various inclusion operators.

Proposition 5.1. *Let* $1 < s, t < \infty$ *be an arbitrary pair.*

(i) *The inclusion map* $i_{d(s),c(t)} : d(s) \longrightarrow c(t)$ *exists if and only if* $s \leq t$.
(ii) *The inclusion map* $i_{d(s),t} : d(s) \longrightarrow \ell_t$ *exists if and only if* $s \leq t$.
(iii) *The inclusion map* $i_{d(s),d(t)} : d(s) \longrightarrow d(t)$ *exists if and only if* $s \leq t$.
(iv) $\ell_s \nsubseteq d(t)$ *and* $\text{ces}(s) \nsubseteq d(t)$.

Proof. (i) Let $s \leq t$. Proposition 2.7(iii) implies that $i_{d(s),s} : d(s) \longrightarrow \ell_s$ exists and, by Proposition 3.2(ii) of [1], also $i_{s,c(t)} : \ell_s \longrightarrow \text{ces}(t)$ exists. Accordingly, the composition $i_{d(s),c(t)} = i_{s,c(t)} \circ i_{d(s),s}$ exists .

Let $s > t$. Suppose that $i_{d(s),c(t)}$ exists. Since the operator $C_{c(s),d(s)} : \text{ces}(s) \longrightarrow d(s)$ exists (cf. Corollary 3.8), it follows that the operator $C_{c(s),c(t)} = i_{d(s),c(t)} \circ C_{c(s),d(s)}$ also exists. But, this is *not* the case, [1, Proposition 3.5(iii)]. So, $i_{d(s),c(t)}$ does not exist.

(ii) Note that $s \leq t$ if and only if the conjugate indices satisfy $t' \leq s'$. Moreover, the dual operator $(i_{d(s),t})'$ is the natural inclusion map $i_{t',c(s')} : \ell_{t'} \longrightarrow \text{ces}(s')$, interpreted to mean that whenever one of the operators exists then so does the other one. The stated claim then follows by setting $p := t'$ and $q := s'$ in Proposition 3.2(ii) of [1].

(iii) Let $s \leq t$. Then the inclusion map $i_{d(s),d(t)} : d(s) \longrightarrow d(t)$ exists by Proposition 2.7(i). On the other hand, Proposition 2.7(ii) shows that $d(t) \subsetneq d(s)$ whenever $s > t$, that is, the inclusion $i_{d(s),d(t)}$ does not exist.

(iv) Remark 2.8(i) implies that $\ell_s \nsubseteq d(t)$. Remark 2.8(ii) shows that $\text{ces}(s) \nsubseteq d(t)$. □

The following result characterizes, for those inclusion maps which exist, precisely when they are compact.

Proposition 5.2. *Let* $1 < s \leq t < \infty$ *be an arbitrary pair.*

(i) *The inclusion map* $i_{d(s),c(t)} : d(s) \longrightarrow c(t)$ *is compact if and only if* $s < t$.
(ii) *The inclusion map* $i_{d(s),t} : d(s) \longrightarrow \ell_t$ *is compact if and only if* $s < t$.
(iii) *The inclusion map* $i_{d(s),d(t)} : d(s) \longrightarrow d(t)$ *is compact if and only if* $s < t$.

Proof. (i) Suppose that $s < t$. By Remark 4.2(i) the inclusion map $i_{d(s),d(t)}$ is compact and by Proposition 5.1(i) the inclusion $i_{d(t),c(t)}$ is continuous. So, $i_{d(s),c(t)} = i_{d(t),c(t)} \circ i_{d(s),d(t)}$ is compact.

For the case when $s = t$, suppose that the continuous inclusion map $i_{d(s),c(s)}$ (cf. Proposition 5.1(i)) is actually compact. Since the Cesàro operator $C_{c(s),d(s)} : \mathrm{ces}(s) \longrightarrow d(s)$ is continuous (cf. Corollary 3.7), it follows that $C_{d(s)} = C_{c(s),d(s)} \circ i_{d(s),c(s)}$ is compact, which is *not* the case by (5.1). Hence, $i_{d(s),c(s)}$ fails to be compact.

(ii) Note that $s \leq t$ if and only if $t' \leq s'$ and that the dual operator $(i_{d(s),t})'$ is the natural inclusion map $i_{t',c(s')} : \ell_{t'} \longrightarrow \mathrm{ces}(s')$. So, by Schauder's theorem, the stated claim follows by setting $p := t'$ and $q := s'$ in Proposition 3.4(iii) of [1].

(iii) If $s < t$, then $i_{d(s),d(t)}$ is compact by Remark 4.2(i). For the case of $s = t$ we see that $i_{d(s),d(s)}$ is the identity operator on $d(s)$ and hence, is surely not compact. □

We now turn our attention to various Cesàro operators. First we need to determine which of these operators actually exist. By the closed graph theorem they are then continuous.

Proposition 5.3. *Let* $1 < s, t < \infty$ *be an arbitrary pair.*

(i) *The Cesàro operator* $C_{d(s),c(t)} : d(s) \longrightarrow \mathrm{ces}(t)$ *exists if and only if* $s \leq t$.
(ii) *The Cesàro operator* $C_{d(s),t} : d(s) \longrightarrow \ell_t$ *exists if and only if* $s \leq t$.
(iii) *The Cesàro operator* $C_{d(s),d(t)} : d(s) \longrightarrow d(t)$ *exists if and only if* $s \leq t$.
(iv) *The Cesàro operator* $C_{s,d(t)} : \ell_s \longrightarrow d(t)$ *exists if and only if* $s \leq t$.
(v) *The Cesàro operator* $C_{c(s),d(t)} : \mathrm{ces}(s) \longrightarrow d(t)$ *exists if and only if* $s \leq t$.

Proof. (i) Suppose that $s \leq t$. Then the inclusion map $i_{d(s),s}$ exists (cf. Proposition 5.1(ii)) as does $C_{s,c(t)}$, [1, Proposition 3.5(ii)]. Hence, also $C_{d(s),c(t)} = C_{s,c(t)} \circ i_{d(s),s}$ exists.

Let $s > t$ and assume that $C_{d(s,),c(t)}$ *does* exist. Since $C_{c(t),d(t)}$ is continuous by Corollary 3.8 and $i_{d(t),d(s)}$ is compact by Proposition 5.2(iii), it follows that the operator

$$(C_{d(s)})^2 = i_{d(t),d(s)} \circ C_{c(t),d(t)} \circ C_{d(s),c(t)}$$

is compact. But, this is impossible because (5.1) and the spectral mapping theorem, [17, VII Theorem 3.11], imply that the set $\sigma((C_{d(s)})^2) = \{\lambda^2 : \lambda \in \sigma(C_{d(s)})\}$ is uncountable (it contains the real interval $[0, (s')^2]$, for example). Hence, $C_{d(s),c(t)}$ *does not exist.*

(ii) Assume that $s \leq t$. Proposition 5.1(ii) implies the existence of $i_{d(s),s}$ and [1, Proposition 3.5(i)] ensures the existence of $C_{s,t}$. The existence of $C_{d(s),t} = C_{s,t} \circ i_{d(s),s}$ is then clear.

Let $s > t$. Suppose that $C_{d(s),t}$ *does* exist. The continuity of $C_{t,d(t)}$ is known (cf. Proposition 3.4) and $i_{d(t),d(s)}$ is compact by Proposition 5.2(iii). Accordingly,

$$(C_{d(s)})^2 = i_{d(t),d(s)} \circ C_{t,d(t)} \circ C_{d(s),t}$$

is compact. The proof of part (i) shows that this is not the case and so $C_{d(s),t}$ *does not exist.*

(iii) Let $s \leq t$. Proposition 3.2(i) shows that $C_{d(t)}$ exists and the map $i_{d(s),d(t)}$ exists by Proposition 2.7(i). So, the existence of $C_{d(s),d(t)} = C_{d(t)} \circ i_{d(s),d(t)}$ is clear.

Let $s > t$ and suppose that $C_{d(s),d(t)}$ exists. Since $i_{d(t),d(s)}$ is compact (cf. Proposition 5.2(iii)), it follows that $C_{d(s)} = i_{d(t),d(s)} \circ C_{d(s),d(t)}$ is also compact which contradicts (5.1). So, $C_{d(s),d(t)}$ *does not exist.*

(iv) Suppose that $s \leq t$. It is well known that $i_{s,t}$ exists and, by Proposition 3.4, also $C_{t,d(t)}$ exists. Accordingly, $C_{s,d(t)} = C_{t,d(t)} \circ i_{s,t}$ surely exists.

Let $s > t$. Note that $i_{d(s),s}$ exists by Proposition 5.1(i) and that $i_{d(t),d(s)}$ is compact by Proposition 5.2(iii). *If* $C_{s,d(t)}$ exists, then $C_{d(s)} = i_{d(t),d(s)} \circ C_{s,d(t)} \circ i_{d(s),s}$ is a compact operator, which is impossible by (5.1). Hence, $C_{s,d(t)}$ *does not exist.*

(v) Let $s \le t$. Then $C_{c(t),d(t)}$ exists by Corollary 3.7 and the map $i_{c(s),c(t)}$ exists by [1, Proposition 3.2(iii)]. Hence, $C_{c(s),d(t)} = C_{c(t),d(t)} \circ i_{c(s),c(t)}$ also exists.

Assume that $s > t$. By Proposition 5.2(i) we see that $i_{d(t),c(s)}$ is compact. So, *if* $C_{c(s),d(t)}$ exists, then $C_{c(s)} = i_{d(t),c(s)} \circ C_{c(s),d(t)}$ is a compact operator, which is impossible by (5.1). Accordingly, $C_{c(s),d(t)}$ *does not exist.* □

Our final result determines precisely which Cesàro operators, when they exist (see Proposition 5.3), are compact.

Proposition 5.4. *Let* $1 < s \le t < \infty$ *be arbitrary.*

(i) *The Cesàro operator* $C_{d(s),c(t)}$ *is compact if and only if* $s < t$.
(ii) *The Cesàro operator* $C_{d(s),t}$ *is compact if and only if* $s < t$.
(iii) *The Cesàro operator* $C_{d(s),d(t)}$ *is compact if and only if* $s < t$.
(iv) *The Cesàro operator* $C_{s,d(t)}$ *is compact if and only if* $s < t$.
(v) *The Cesàro operator* $C_{c(s),d(t)}$ *is compact if and only if* $s < t$.

Proof. (i) Let $s < t$. The inclusion map $i_{d(s),d(t)}$ is compact (cf. Proposition 5.2(iii)) and $i_{d(t),t}$ is continuous (cf. Proposition 5.1(ii)). Since $C_{t,c(t)}$ is also continuous, [1, Proposition 3.5(ii)], it follows that $C_{d(s),c(t)} = C_{t,c(t)} \circ i_{d(t),t} \circ i_{d(s),d(t)}$ is a compact operator.

Suppose that $s = t$. Both $C_{d(s),c(s)}$ and $C_{c(s),d(s)}$ are continuous; see parts (i) and (v) of Proposition 5.3, respectively. *If* $C_{d(s),c(s)}$ were also compact, then $(C_{d(s)})^2 = C_{c(s),d(s)} \circ C_{d(s),c(s)}$ would be compact, which is not the case; see the proof of part (i) of Proposition 5.3(i). So, $C_{d(s),c(s)}$ *fails* to be compact.

(ii) Let $s < t$. By Proposition 5.2(iii) the inclusion $i_{d(s),d(t)}$ is compact and $C_{d(t),t}$ is continuous by Proposition 5.2(ii). Hence, $C_{d(s),t} = C_{d(t),t} \circ i_{d(s),d(t)}$ is compact.

Assume that $s = t$. Both $C_{d(s),s}$ and $C_{s,d(s)}$ are continuous; see parts (ii) and (iv) of Proposition 5.3, respectively. *If* $C_{d(s),s}$ were also compact, then $(C_{d(s)})^2 = C_{s,d(s)} \circ C_{d(s),s}$ would be compact which is not so. Hence, $C_{d(s),s}$ is *not* compact.

(iii) Let $s < t$. By Proposition 5.2(iii) the inclusion $i_{d(s),d(t)}$ is compact. Moreover, $C_{d(t)}$ is continuous by Proposition 5.3(iii). So, $C_{d(s),d(t)} = C_{d(t)} \circ i_{d(s),d(t)}$ is necessarily compact.

For $s = t$ we note that $C_{d(s),d(s)}$ $(= C_{d(s)})$ is *not* compact; see (5.1) and the discussion following it.

(iv) Let $s < t$. Then $i_{d(s),d(t)}$ is compact by Proposition 5.2(iii). Moreover, $C_{s,d(s)}$ is continuous by Proposition 5.3(iv). Hence, $C_{s,d(t)} = i_{d(s),d(t)} \circ C_{s,d(s)}$ is a compact operator.

Suppose that $s = t$. Both $C_{s,d(s)}$ and $i_{d(s),s}$ are continuous by Proposition 5.3(iv) and Proposition 5.1(ii), respectively. *If* $C_{s,d(s)}$ were also compact, then $C_s = i_{d(s),s} \circ C_{s,d(s)}$ would be compact which is not the case. Hence, $C_{s,d(s)}$ *fails* to be compact.

(v) Let $s < t$. Then $i_{d(s),d(t)}$ is compact by Proposition 5.2(iii) and $C_{c(s),d(s)}$ is continuous by Proposition 5.3(v). Accordingly, $C_{c(s),d(t)} = i_{d(s),d(t)} \circ C_{c(s),d(s)}$ is compact.

Assume that $s = t$. Both $C_{c(s),d(s)}$ and $C_{d(s),c(s)}$ are continuous; see parts (v) and (i) of Proposition 5.3, respectively. *If* $C_{c(s),d(s)}$ were also compact, then $(C_{c(s)})^2 = C_{d(s),c(s)} \circ C_{c(s),d(s)}$ would be compact. But, (5.1) and the spectral mapping theorem imply that this is not the case. So, $C_{c(s),d(s)}$ is *not* compact. □

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References

[1] A.A. Albanese, J. Bonet, W.J. Ricker, Multiplier and averaging operators in the Banach spaces ces(p), $1 < p < \infty$, *Positivity*, **23** (2019), 177–193.
[2] W. Arendt, On the o-spectrum of regular operators and the spectrum of measures, *Math. Z.*, **178** (1981), 271–287.
[3] S.V. Astashkin, L. Maligranda, Structure of Cesàro function spaces, *Indag. Math. (N.S.)*, **20** (2009), 329–379.
[4] S.V. Astashkin, L. Maligranda, Structure of Cesàro function spaces: a survey, *Function Spaces X*, pp. 13–40. Banach Center Publ. **102**, Polish Acad. Sci. Inst. Math., Warsaw, 2014.
[5] G.F. Bachelis, On the upper and lower majorant properties in $L^p(G)$, *Quarterly J. Math. Oxford II Ser.*, **24** (1973), 119–128.
[6] G. Bennett, Factorizing the classical inequalities, *Mem. Amer. Math. Soc.*, **120** (Nr. 576), viii + 130 pp. (1996).
[7] G. Bennett, K.-G. Grosse-Erdmann, Weighted Hardy inequalities for decreasing sequences and functions, *Math. Ann.*, **334** (2006), 489–531.
[8] J. Bonet, W.J. Ricker, Order spectrum of the Cesàro operator in Banach lattice sequence spaces, *Positivity*, https://doi.org/10.1007/s11117-019-00699-9
[9] P.S. Bourdon, N.S. Feldman, J.H. Shapiro, Some properties of N-supercyclic operators, *Studia Math.*, **165** (2004), 135–157.
[10] J. Bueno-Contreras, *The Cesàro Spaces of Dirichlet Series*, Ph.D. Thesis, Instituto de Matemàticas Univ. Sevilla IMUS, Universidad de Sevilla, Spain, 2018.
[11] J. Bueno-Contreras, G.P. Curbera, O. Delgado, The Cesàro space of Dirichlet series and its multiplier algebra, *J. Math. Anal. Appl.*, **475** (2019), 1448–1471.
[12] G.P. Curbera, W.J. Ricker, Solid extensions of the Cesàro operator on the Hardy space $H^2(\mathbb{D})$, *J. Math. Anal. Appl.*, **407** (2013), 387–397.
[13] G.P. Curbera, W.J. Ricker, A feature of averaging, *Integral Equ. Oper. Theory*, **76** (2013), 447–449.
[14] G.P. Curbera, W.J. Ricker, Solid extensions of the Cesàro operator on ℓ^p and c_0, *Integral Equ. Oper. Theory*, **80** (2014), 61–77.
[15] G.P. Curbera, W.J. Ricker, The Cesàro operator and unconditional Taylor series in Hardy spaces, *Integral Equ. Oper. Theory*, **83** (2015), 179–195.
[16] M.M. Day, *Normed Linear Spaces*, 3rd Ed., Springer-Verlag, Berlin Heidelberg, 1973.
[17] N. Dunford, J.T. Schwartz, *Linear Operators I*: *General Theory*, Wiley Interscience Publ. Inc., New York, 1964.
[18] K.-G. Grosse-Erdmann, *The Blocking Technique*, *Weighted Mean Operators and Hardy's inequality*, Lecture Notes in Mathematics, **1679**, Springer Verlag, Berlin Heidelberg, 1998.
[19] K.-G. Grosse-Erdmann, A. Peris Manguillot, *Linear Chaos*, Springer, London, 2011.
[20] G.H. Hardy, J.E. Littlewood, G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1934.
[21] A.A. Jagers, A note on Cesàro sequence spaces, *Nieuw Arch. Wisk.*, **22** (1974), 113–124.
[22] G. Köthe, *Topological Vector Spaces I*, Springer, Berlin Heidelberg, 1983.
[23] G. Leibowitz, Spectra of discrete Cesàro operators, *Tamkang J. Math.*, **3** (1972), 123–132.
[24] K. Leśnik, L. Maligranda, Abstract Cesàro spaces. Duality, *J. Math. Anal. Appl.*, **424** (2015), 932–951.
[25] M. Lin, On the uniform ergodic theorem, *Proc. Amer. Math. Soc.*, **43** (1974), 337–340.
[26] J. Lindenstrauss, L. Tzafriri, *Classical Banach Spaces I and II*, Springer Verlag, Berlin Heidelberg New York, 1996.

[27] L. Maligranda, N. Petrot, S. Suantai, On the James constant and B-convexity of Cesàro and Cesàro-Orlicz sequence spaces, *J. Math. Anal. Appl.* **326** (2007), 312–331.
[28] P. Meyer-Nieberg, *Banach Lattices*, Springer, Berlin Heidelberg New York, 1991.
[29] S. Okada, W.J. Ricker, E.A. Sánchez-Pérez, *Optimal Domain and Integral Extension of Operators Acting in Function Spaces*, Operator Theory Advances and Applications, Vol. **180**, Birkhäuser, Berlin, 2008.
[30] J.C. Ramos-Fernández, M.A. Rivera-Sariento, M. Salas-Brown, On the essential norm of multiplication operators acting on Cesàro sequence spaces, *J. Funct. Spaces* 2019, Art. Id. 5069610, 5 pp.
[31] J.B. Reade, On the spectrum of the Cesàro operator, *Bull London Math. Soc.*, **117** (1985), 263–267.
[32] W.J. Ricker, Convolution operators in discrete Cesàro spaces, *Arch. Math.*, **112** (2019), 71–82.
[33] W.J. Ricker, The order spectrum of convolution operators in discrete Cesàro spaces, *Indag. Math. (N.S).*, https://doi.org/10.1016/j.indag.2019.01.008
[34] H.H. Schaefer, *Banach Lattices and Positive Operators*, Springer, Berlin Heidelberg New York, 1974.
[35] H.H. Schaefer, On the o-spectrum of order bounded operators, *Math. Z.*, **154** (1977), 79–84.
[36] A.C. Zaanen, *Riesz Spaces II*, North Holland, Amsterdam, 1983.

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