WEIGHTED INDUCTIVE LIMITS OF SPACES OF ENTIRE FUNCTIONS

KLAUS D. BIERSTEDT, JOSÉ BONET AND JARI TASKINEN

Abstract Methods developed by Lusky in connection with the isomorphic classification of weighted Banach spaces of entire functions are used to show that projective description holds for weighted inductive limits of spaces of entire functions for radial weights of a certain type. It is shown by an example that our results are not covered by the general theorem of Bierstedt, Meise and Summers from 1982.

Keywords: topological subspace, projective description

2000 *Mathematics subject classification:* Primary 46E10; secondary 30H05; 46A13; 46M40

1. Introduction

Motivated by applications to, among other things, linear partial differential equations, weighted inductive limits of spaces of holomorphic functions, $\mathcal{V}H$ and $\mathcal{V}_0H$, have been the object of quite a lot of research at least since the 1960s. It is important that, in good cases, a basis of the continuous seminorms of the inductive limit topology can again be given by weighted sup-seminorms. In the seminal paper [7], *associated projective hulls* $H\overline{V}$ and $H\overline{V}_0$ were introduced, and it was asked if *projective description* holds; i.e., roughly spoken, if $\mathcal{V}H = H\overline{V}$ and $\mathcal{V}_0H = H\overline{V}_0$ algebraically and topologically. The main theorem of [7], which was, in fact, the first general result of this type, asserted that $\mathcal{V}_0H = \mathcal{V}H = H\overline{V} = H\overline{V}_0$ is true if $\mathcal{V}H$ is a (DFS)-space; that is, if the linking maps between the generating Banach spaces are compact. But the methods used showed that $\mathcal{V}H = H\overline{V}$ is actually true whenever $\mathcal{V}H$ is a (semi-)Montel space; that is, whenever all the bounded subsets are relatively compact. – Let us remark that the projective description problem often is equivalent to asking if the weighted inductive limit of spaces of holomorphic functions is a topological subspace of the weighted inductive limit of the corresponding spaces of continuous functions.

Later on, a different result was obtained in [14]; here the authors worked with radial weights on the unit disc. No conditions on the topology or on the linking maps of the inductive limit were imposed, but all the weights had to be of a certain type. Along these lines, the best projective description results for spaces of functions on the unit disc can be found in [3]; they are obtained using ideas and methods of Lusky [12] for weighted Banach spaces on the unit disc. The weights in the inductive limit must satisfy Lusky's two conditions in a uniform way.

In Lusky's recent article [13], his results were carried over to weighted Banach spaces of entire functions, and we are now profiting from Lusky's methods to get projective description for weighted inductive limits of spaces of entire functions. Here the radial weights must be of a certain rather special form and must again satisfy Lusky's condition in a uniform way. In the case of non-radial weights there are results for weighted (LF)-spaces of entire functions by Bonet, Meise and Melikhov [8], [9].

The organization of this article is as follows: In Section 2, after recalling the definitions, we explain the condition on the weights and give some examples. Then we formulate the technical decomposition theorem for polynomials (Proposition 2.3) which will be used in the proof of our main projective description theorems in Sections 3 and 4 (Theorem 3.2, Theorems 4.1 and 4.2). The proof of Proposition 2.3 is only given in Section 5, and it is here that we apply the methods of Lusky [13] in an essential way. Finally, in Section 6 we point out that our projective description results are not covered by the main theorem of [7]. In fact, we show this by proving that $\mathcal{V}H$ contains a closed subspace isomorphic to a Köthe co-echelon space k_∞ which can be taken to be non-Montel.

2. Definitions, preliminaries, the decomposition result

We first establish some notation. In the sequel, $H(\mathbb{C})$ denotes the space of all entire functions. A *weight* v on $\mathbb{C}$ is a strictly positive continuous function on $\mathbb{C}$. For such a weight, the *weighted Banach spaces of entire functions* are defined by

$$Hv := \{f \in H(\mathbb{C});\ ||f||_v = \sup_{z\in\mathbb{C}} v(z)|f(z)| < +\infty\},$$
$$Hv_0 := \{f \in H(\mathbb{C});\ vf \text{ vanishes at } \infty\},$$

endowed with the norm $\|f\|_v := \sup_{z\in\mathbb{C}} v(z)|f(z)|$.

Let $\mathcal{V} := (v_k)_{k=1}^\infty$ be a decreasing sequence of weights on $\mathbb{C}$. Then the *weighted inductive limits of spaces of entire functions* are defined by

$$\mathcal{V}H := \mathrm{ind}_k Hv_k,$$
$$\mathcal{V}_0H := \mathrm{ind}_k H(v_k)_0;$$

that is, $\mathcal{V}H$ is the increasing union of the Banach spaces Hv_k with the strongest locally convex topology for which all the injections $Hv_k \to \mathcal{V}H$ become continuous, $k \in \mathbb{N}$, and similarly for $\mathcal{V}_0H$. These are locally convex inductive limits. It is clear that $\mathcal{V}_0H$ is continuously embedded in $\mathcal{V}H$, and it is not clear a priori if it is even a topological subspace. This is indeed the case in our setting below.

In an effort to describe the inductive limit topologies by a system of weighted sup-seminorms, Bierstedt, Meise and Summers [7] defined the *associated system*

$$\bar{V} = \bar{V}(\mathcal{V}) := \{\bar{v} \text{ weight on } \mathbb{C};\ \forall k:\ \sup_{\mathbb{C}} \frac{\bar{v}}{v_k} < +\infty\};$$

that is, $\bar{V}$ consists of all weights on $\mathbb{C}$ which are dominated by a function of the form $\inf_k C_k v_k$ with constants $C_k > 0$ for each k. Then the *projective hulls of the weighted inductive limits* are the complete locally convex spaces

$$H\bar{V} := \{f \in H(\mathbb{C});\ p_{\bar{v}}(f) = \sup_G \bar{v}|f| < +\infty\ \forall \bar{v} \in \bar{V}\},$$
$$H\bar{V}_0 := \{f \in H(\mathbb{C});\ \bar{v}f \text{ vanishes at } \infty\ \ \forall \bar{v} \in \bar{V}\},$$

endowed with the topology given by the system $\{p_{\bar{v}};\ \bar{v} \in \bar{V}\}$ of seminorms. $H\bar{V}_0$ is a closed topological subspace of $H\bar{V}$. By the very definition we have continuous linear embeddings $\mathcal{V}H \to H\bar{V}$ and $\mathcal{V}_0H \to H\bar{V}_0$. It was proved in [7] that $\mathcal{V}H = H\bar{V}$ always holds algebraically.

The *projective description problem* asks: When do we have $\mathcal{V}H = H\bar{V}$ topologically, and when is $\mathcal{V}_0H$ a topological subspace of $H\bar{V}_0$? The first counterexamples to projective description in the case of spaces of holomorphic functions are due to Bonet and Taskinen [11]; for counterexamples in the context of spaces of entire functions see Bonet and Melikhov [10]. All known counterexamples so far are in the case of O-growth conditions. For more information on projective description see the survey [2].

In corollaries to our main theorems below two important conditions on the sequence $\mathcal{V}$ occur. $\mathcal{V} = (v_k)_k$ is said to be *regularly decreasing* if for each k there exists $l \geq k$ such that, for each subset S of $\mathbb{C}$ on which $\inf_S \frac{v_l}{v_k} > 0$, also $\inf_S \frac{v_m}{v_k} > 0$ for each $m \geq l$. This condition is taken from [7], where it is proved that $\mathcal{V}_0 H = H\bar{V}_0$ holds algebraically if $\mathcal{V}$ is regularly decreasing. On the other hand, $\mathcal{V}$ is said to satisfy *condition (D)* if there exists an increasing sequence $(S_n)_n$ of subsets of $\mathbb{C}$ such that for each n there is l such that $\inf_{S_n} \frac{v_m}{v_l} > 0$ for each $m \geq l$, while for each k, and for each subset S of $\mathbb{C}$ with $S \cap (\mathbb{C} \setminus S_n) \neq \emptyset$ for each n, there is $k' > k$ with $\inf_S \frac{v_{k'}}{v_k} = 0$. This condition was introduced in [6], and it was proved there that it implies the (algebraic and) topological equality $\mathcal{V}C = C\bar{V}$ for the corresponding spaces of continuous functions (i.e., one replaces in the above definitions the space of entire functions by the space of all continuous functions).

Let us now explain the special form of the weights considered here and a condition which has to be imposed on these weights.

Definition 2.1. Given constants $A > 0$ and $a > 0$, we say that a continuous, radial, strictly positive weight function $v : \mathbb{C} \to \mathbb{R}^+$ of the form

$$v(r) := w(r)e^{-ar}, \qquad r \in [0, \infty), \tag{2.1}$$

belongs to the class $(E)_{A,a}$ if $w : \mathbb{R}^+ \to \mathbb{R}^+$ is differentiable, strictly increasing and has the property

$$\sup_{r \in [0,\infty)} \frac{rw'(r)}{aw(r)} \leq A. \tag{2.2}$$

If $a = 1$, we denote the class by $(E)_A$.

There are natural examples of functions $w(r)$ that can be taken to construct weights in the class $(E)_A$.

Given $A > 0$, the function $w(r) := (1 + r)^\alpha$ satisfies (2.2) if $0 \leq \alpha \leq A$, $a = 1$.

Given $A > 0$, the functions $w(r) := (\log(2 + r))^\beta$ and $w(r) = \log(\log(e^2 + r))$ satisfy (2.2) if $0 \leq \beta \leq A$, $a = 1$.

It is easy to see that if the weights $w_1(r)$ and $w_2(r)$ satisfy (2.2) for some $A > 0$, then $w(r) = w_1(r)w_2(r)$ also satisfies (2.2) for the same a and $2A$ instead of $A > 0$.

Remark 2.2. Assume that $(v_k)_{k=1}^\infty$ is a decreasing sequence of radial weights on $\mathbb{C}$ which belong to the class $(E)_{A,a}$. So $v_k(r) := w_k(r)e^{-ar}$, where each w_k satisfies (2.2). If another weight $\bar{w}$ is defined by

$$\bar{w} := \frac{1}{\sum_{k=1}^\infty c_k w_k^{-1}}, \tag{2.3}$$

where the numbers $c_k \geq 0$ (at least one of which is not zero) are so small that the sum in the denominator converges uniformly on the compact sets of $\mathbb{C}$, then also $\bar{w}(r)$ satisfies

(2.2). Indeed,

$$\frac{1}{\bar{w}(r)} r\bar{w}'(r) = \sum_{k=1}^{\infty} c_k w_k^{-1} \frac{(-1)^2}{\Big(\sum_{k=1}^{\infty} c_k w_k(r)^{-1}\Big)^2} \sum_{k=1}^{\infty} \frac{r c_k w_k'(r)}{w_k(r)^2}$$

$$\le \frac{1}{\sum_{k=1}^{\infty} c_k w_k(r)^{-1}} \sum_{k=1}^{\infty} \frac{A a c_k}{w_k(r)} = Aa. \tag{2.4}$$

The following structural result about the weighted norms $\| \cdot \|_v$ is very important for our main results about projective description of weighted inductive limits. It provides us, among other things, with some kind of finite dimensional decomposition of the inductive limit space $\mathcal{V}_0 H$. Notice that inside any finite dimensional block, every weight is a constant times e^{-ar}; the latter factor is independent of the given weight.

Proposition 2.3. *Let $A, a > 0$ be fixed. There exists a sequence $(T_n)_{n=1}^{\infty}$ of finite rank operators on the space of all polynomials with the following properties:*
(a) The operators T_n satisfy $T_n T_m = 0$ if $|n-m| \ge 2$, and $T_n T_{n+1} = T_{n+1} T_n$.
(b) For every polynomial f we have $\sum_n T_n f = f$, and the sum is finite.
(c) There is a constant $D \ge 1$ such that for every $r > 0$ and every polynomial f we have

$$\sup_{|z|=r} |T_n f(z)| \le D \sup_{|z|=r} |f(z)|. \tag{2.5}$$

(d) There exist increasing positive sequences $(\varrho_n)_{n=1}^{\infty}$ and $(\sigma_n)_{n=1}^{\infty}$, $\varrho_n < \sigma_n$, such that for each weight $v \in (E)_{A,a}$, $v(r) = w(r)e^{-ar}$, there exists a constant $C(v) > 0$ such that for every polynomial f,

$$\frac{1}{D} \sup_{n\in\mathbb{N}} \sup_{\varrho_n \le |z| \le \sigma_n} w(\varrho_n) e^{-ar} |T_n f(z)| \le \|f\|_v \le C(v) \sup_{n\in\mathbb{N}} \sup_{\varrho_n \le |z| \le \sigma_n} w(\varrho_n) e^{-ar} |T_n f(z)|;$$

here D is the constant of statement (c), which does not depend on the weight v.

The proof of this proposition is presented in Section 5. We will use it to prove our main results concerning projective description.

3. Main results in the case of o–growth conditions

We assume in the next two sections that $\mathcal{V} := (v_k)_{k=1}^{\infty}$ is a decreasing sequence of weights on $\mathbb{C}$ belonging to the class $(E)_{A,a}$ for some $A, a > 0$. We start with the following technical lemma, which shows how to use Proposition 2.3 to decompose polynomials keeping certain estimates. In the next two sections we will use the notation

$$U_k := \{ f \in H v_k \mid \|f\|_{v_k} \le 1 \},$$

and Γ will indicate the absolutely convex hull of a set.

Lemma 3.1. *Let a decreasing sequence $(\varepsilon_k)_{k=1}^{\infty}$, with $\varepsilon_k > 0$ for all k, be given. Define*

$$\bar{v} := \frac{2}{\sum_{k=1}^{\infty} 4^{-k} \alpha_k v_k^{-1}} = \frac{2e^{-ar}}{\sum_{k=1}^{\infty} 4^{-k} \alpha_k w_k^{-1}} =: \bar{w}(r) e^{-ar}, \tag{3.1}$$

where we assume that the numbers $\alpha_k \ge 0$ are so small that the sum in the denominator converges uniformly on the compact subsets of $\mathbb{C}$, and moreover satisfy the inequality $\alpha_k \le \varepsilon_k (3DC(v_k))^{-1}$ and $\alpha_1 > 0$, $\alpha_k \ge \alpha_{k+1}$ for every k.

Then every polynomial f with $\|f\|_{\bar v} \le 1$ belongs to the set

$$U := \Gamma\left(\cup_{k=1}^{\infty} \varepsilon_k U_k\right) = \{f = \sum_k \lambda_k f_k \in \mathcal{V}_0 H;\ \sum_k |\lambda_k| \le 1\ ,\ \|f\|_k \le \varepsilon_k\} \subset \mathcal{V}_0 H. \tag{3.2}$$

If $\alpha_k = 0$ for $k > m$ for some $m \in \mathbb{N}$, then

$$f \in \Gamma(\cup_{k=1}^{m} \varepsilon_k U_k). \tag{3.3}$$

We remark that for the proof of Theorem 3.2 we need the case $\alpha_k \ne 0$ for every k, whereas for the proof of Theorems 4.1 and 4.2 we use the case $\alpha_k = 0$ for $k > m$ for some $m > 0$.

Proof of Lemma 3.1. For each $n \in \mathbb{N}$ choose $k(n) \in \mathbb{N}$ such that

$$\bar v(\varrho_n) \ge 2^{k(n)} \alpha_{k(n)}^{-1} v_{k(n)}(\varrho_n) \text{ and } \alpha_{k(n)} \ne 0.$$

Such a $k(n)$ must exist, since otherwise

$$\bar v(\varrho_n) \le \inf_{\substack{k\in\mathbb{N}\\ \alpha_k \ne 0}} 2^k \alpha_k^{-1} v_k(\varrho_n) = \frac{1}{\sup\limits_{k\in\mathbb{N}} 2^{-k}\alpha_k v_k(\varrho_n)^{-1}} < \frac{2}{\sum\limits_{k=1}^{\infty} 4^{-k}\alpha_k v_k(\varrho_n)^{-1}} = \bar v(\varrho_n), \tag{3.4}$$

a contradiction. Denote, for every $k \in \mathbb{N}$,

$$\mathbb{N}_k := \{n \in \mathbb{N} \mid k(n) = k\}. \tag{3.5}$$

Some of the sets $\mathbb{N}_k$ may be empty, but nevertheless $\mathbb{N}$ is the disjoint union of the sets $\mathbb{N}_k$. We remark that if $\alpha_k = 0$ for $k > m$, then

$$\mathbb{N}_k = \emptyset \qquad \text{for } k > m. \tag{3.6}$$

Let now f be a polynomial such that $\|f\|_{\bar v} \le 1$. For every k we define

$$f_k := 2^k \sum_{n\in\mathbb{N}_k} T_n f. \tag{3.7}$$

The sum has in fact only finitely many terms, since f is a polynomial. We claim that $\|f_k\|_k \le \varepsilon_k$ for every k. We aim to use (2.3). First notice that by (a) of Proposition 2.3

$$\begin{aligned} T_n f_k &= 2^k \sum_{m\in\mathbb{N}_k} T_n T_m f \\ &= 2^k(\chi(k,n-1)T_nT_{n-1} + \chi(k,n)T_n^2 + \chi(k,n+1)T_nT_{n+1})f \\ &= 2^k(\chi(k,n-1)T_{n-1}T_n + \chi(k,n)T_n^2 + \chi(k,n+1)T_{n+1}T_n)f, \end{aligned} \tag{3.8}$$

where $\chi(k,n) := 1$, if $n \in \mathbb{N}_k$, and $\chi(k,n) := 0$ otherwise. Hence, $d)$ and $c)$ of Proposition 2.3 imply

$$\begin{aligned} \|f_k\|_{v_k} \le C(v_k)2^k \sup_{n\in\mathbb{N}} \sup_{\varrho_n\le|z|\le\sigma_n} w_k(\varrho_n)e^{-ar}|(\chi(k,n-1)T_{n-1}T_n \\ + \chi(k,n)T_n^2 + \chi(k,n+1)T_{n+1}T_n)f(z)| \\ \le C(v_k)2^k \sup_{n\in\mathbb{N}_k} \sup_{\varrho_n\le r\le\sigma_n} w_k(\varrho_n)e^{-ar} \sup_{\theta\in[0,2\pi]} (|(T_{n-1}T_nf)(re^{i\theta})| \end{aligned}$$

$$+ |(T_n^2 f)(re^{i\theta})| + |(T_{n+1}T_n f)(re^{i\theta})|)$$
$$\leq C(v_k)2^k \sup_{n\in\mathbb{N}_k} \sup_{\varrho_n\leq r\leq\sigma_n} w_k(\varrho_n)e^{-ar}3 \sup_{\theta\in[0,2\pi]} |(T_n f)(re^{i\theta})|$$
$$= C(v_k)3\cdot 2^k \sup_{n\in\mathbb{N}_k} \sup_{\varrho_n\leq |z|\leq\sigma_n} w_k(\varrho_n)e^{-ar}|(T_n f)(z)|. \tag{3.9}$$

By the choice of the set $\mathbb{N}_k$ we have $2^k\alpha_k^{-1}w_k(\varrho_n) \leq \bar{w}(\varrho_n)$, hence (3.9) is bounded by

$$3C(v_k)\alpha_k \sup_{n\in\mathbb{N}_k} \sup_{\varrho_n\leq |z|\leq\sigma_n} \bar{w}(\varrho_n)e^{-ar}|(T_n f)(z)|$$
$$\leq 3C(v_k)D\alpha_k\|f\|_{\bar{v}} \leq \varepsilon_k. \tag{3.10}$$

In the second inequality of (3.10) we have used the first inequality of part (d) of Proposition 2.3.

Since $f = \sum_k 2^{-k}f_k$, we have proved $f \in \Gamma(\cup_{k=1}^{\infty}\varepsilon_k U_k)$. Now, the statement (3.3) follows from (3.6). □

Theorem 3.2. *Under our general assumptions on the sequence $\mathcal{V}$ of weights, the space $\mathcal{V}_0 H$ is a topological subspace of its projective hull $H\bar{V}_0$, and hence also of $\mathcal{V}H$.*

Proof. Since the polynomials are dense in $\mathcal{V}_0 H$ by [4], it is enough to show that both spaces induce the same topology on the space of polynomials. An arbitrary 0-neighborhood of the space $\mathcal{V}_0 H$ contains a set of the form $\Gamma(\cup_{k=1}^{\infty}\varepsilon_k U_k)$ for some sequence $(\varepsilon_k)_{k=1}^{\infty}$. We simply apply Lemma 3.1 choosing all the numbers α_k strictly positive. Then the norm $\|\cdot\|_{\bar{v}}$ is continuous on $H\bar{V}_0$, and the result follows from the lemma. □

Corollary 3.3. *If, in addition, the sequence $\mathcal{V}$ is regularly decreasing, then we obtain $\mathcal{V}_0 H = H\bar{V}_0$ algebraically and topologically.*

4. Main results in the case of O–growth conditions

Analogously to the paper [3] we first solve the topological subspace problem for our weight families in the case of spaces defined with O–growth conditions. As we did in section 3, we assume that $\mathcal{V} := (v_k)_{k=1}^{\infty}$ is a decreasing sequence of weights on $\mathbb{C}$ belonging to the class $(E)_{A,a}$ for some $A, a > 0$.

Theorem 4.1. *Let $\mathcal{V} := (v_k)_{k=1}^{\infty}$ be a decreasing sequence of weights as in Section 2. Then $\mathcal{V}H$ is a topological subspace of the space $\mathcal{V}C$.*

Proof. We follow the ideas of the proof of Theorem 4.1 of [3]. Fix a neighborhood of zero $U \subset \mathcal{V}H$ of the form $U = \Gamma(\cup_{k=1}^{\infty}\varepsilon_k U_k)$, where again

$$U_k := \{f \in Hv_k \mid \|f\|_{v_k} \leq 1\} \tag{4.1}$$

and the sequence $(\varepsilon_k)_{k=1}^{\infty}$ is assumed strictly decreasing. Set $\beta_k := \varepsilon_k(3DC(v_k))^{-1}$ for each k, and define for each $m \in \mathbb{N}$,

$$w_m := \frac{2}{\sum_{k=1}^{m} 4^{-k}\beta_k v_k^{-1}}. \tag{4.2}$$

Remark 2.2 implies that every weight w_m also belongs to the class $(E)_{A,a}$. Moreover, the sequence $(w_m)_{m=1}^{\infty}$ is decreasing and hence, denoting

$$W_m := \{f \in Cv_m \mid \|f\|_{w_m} \leq 1\},$$

the set $\cup_m W_m$ is an absolutely convex neighborhood of 0 in $\mathcal{V}C$. Let us prove that $W_0 := W \cap \mathcal{V}H \subset U$.

Let $f \in W_0$. By the definition of W, there exists an m such that $w_m(z)|f(z)| \le 1$ for all $z \in \mathbb{C}$. Using the Cesàro means of (the partial sums of) the Taylor series of f about 0 as in [4, Prop. 1.2 and its proof], one obtains a sequence $(V_n f)_n$ of polynomials, $V_n f \to f$ for the compact open topology of $H(\mathbb{C})$, and

$$\sup w_m(z)|(V_n f)(z)| \le \sup w_m(z)|f(z)| \le 1 \tag{4.3}$$

for all n. Fix n and apply Lemma 3.1 to the polynomial $V_n f$ with the numbers $\alpha_k = \beta_k$ if $k \le m$ and $\alpha_k := 0$ for $k > m$. Compare with (4.2). Lemma 3.1 implies

$$V_n f \in \Gamma(\cup_{k=1}^m \varepsilon_k U_k). \tag{4.4}$$

This set is compact in the compact open topology, by Montel's theorem, and moreover, $V_n f \to f$ in the compact open topology as $n \to \infty$. Since n is arbitrary, (4.4) is true also for f replacing $V_n f$, and we get that $W \cap \mathcal{V}H \subset U$. □

In our next result we use the associated weights $\tilde{v}$ which were defined and studied in [5].

Theorem 4.2. *Let $\mathcal{V} := (v_k)_{k=1}^\infty$ be a decreasing sequence of weights as in Section 2. Then the topological equality $\mathcal{V}H = H\bar{V}$ holds if and only if*

$$\begin{aligned} &\forall (\lambda_k)_{k=1}^\infty,\ \lambda_k > 0,\ \exists \bar{v} \in \bar{V}\ \forall n \in \mathbb{N}\ \forall M > 0\ \exists m > n : \\ &\min\Big(\frac{M}{v_n}, \frac{1}{\bar{v}}\Big)^{\sim} \le \sum_{k=1}^m \frac{\lambda_k}{v_k}. \qquad (+) \end{aligned} \tag{4.5}$$

Proof. From Proposition 1 of [3] we know that the condition (+) is necessary. The converse can be proved using Lemma 3.1 as follows. We again fix an arbitrary neighborhood of 0,

$$U := \Gamma(\cup_{k=1}^\infty \varepsilon_k U_k) \subset \mathcal{V}H, \tag{4.6}$$

where the $\varepsilon_k > 0$ form a decreasing sequence and where we set $\beta_k := \varepsilon_k (3DC(v_k))^{-1}$ for each k. We find a weight $\bar{v}$ as in (+) with the numbers $\lambda_k := 4^{-k}\beta_k$, and we claim that

$$W := \{f \in H\bar{V} \mid \|f\|_{\bar{v}} \le 1\} \subset U. \tag{4.7}$$

Fix $f \in W$. Then there exist $M > 0$ and $n \in \mathbb{N}$ such that $|f| \le \min(M/v_n, 1/\bar{v})$. By the condition (+) this means that $\|f\|_{w_m} \le 1$ for some m, where

$$w_m := \frac{1}{\sum_{k=1}^m 4^{-k}\beta_k v_k^{-1}}. \tag{4.8}$$

As in the proof of Theorem 4.1 we consider the Cesàro sums $V_n f$ instead of f. We apply Lemma 3.1 to w_m and $V_n f$, choosing $\alpha_k := \beta_k$, if $k \le m$, and $\alpha_k := 0$ otherwise to conclude

$$V_n f \in \Gamma(\cup_{k=1}^m \varepsilon_k U_k), \tag{4.9}$$

for every n. Using the compact open topology and Montel's theorem we infer also that (4.9) holds for f instead of $V_n f$. □

Bastin [1] showed that condition (D) of [6] is equivalent to the condition (+) without the associated weight in the left hand side of the inequality. Accordingly, we obtain the following consequence, which could also have been deduced from Theorem 4.1 and from the main theorem of [6].

Corollary 4.3. *Let $\mathcal{V} := (v_k)_{k=1}^{\infty}$ be a decreasing sequence of weights as in Section 2 which satisfies condition (D). Then the topological equality $\mathcal{V}H = H\bar{V}$ holds.*

5. Proof of Proposition 2.3

This section is strongly influenced by Lusky [13]. For the applications already given above some of the estimates have to be repeated carefully in the present case, since it is not enough to show that each individual weight satisfies condition (B) of Lusky [13]. In fact all the weights in the sequence defining the inductive limits and a fundamental system of the associated weights must share certain characteristics. See the proofs in the two sections before.

The following numbers play an important role in the definition of the finite rank operators T_n below; cf. [13]. Here $[x]$ is the largest integer not larger than x.

Definition 5.1. We define inductively the numbers $m_n \in \mathbb{R}^+$, $n \in \mathbb{N}$, by $m_1 = 1$ and $m_{n+1} := m_n + \sqrt{m_n}$.

The operators T_n are defined as in [13] by

$$T_n : f = \sum_{k=0}^{\infty} f_k z^k \mapsto \sum_{m_{n-1}<k\le m_n} \frac{k-[m_{n-1}]}{[m_n]-[m_{n-1}]} f_k z^k + \sum_{m_n<k\le m_{n+1}} \frac{[m_{n+1}]-k}{[m_{n+1}]-[m_n]} f_k z^k. \tag{5.1}$$

In the notation of Section 3 of [13], we have $T_n = V_{m_{n+1},m_n} - V_{m_n,m_{n-1}}$.

Observe first that the operators T_n are multipliers with respect to Taylor series: If $f := \sum_{k=0}^{\infty} f_k z^k$, then

$$T_n : f \mapsto \sum_{m_{n-1}\le k\le m_{n+1}} t_{nk} f_k z^k \tag{5.2}$$

for some numbers t_{nk}, $0 \le t_{nk} \le 1$. Statements *a*) and *b*) of Proposition 2.3 are clear from the definition; see the proof of [13, Proposition 6.4].

To show that *c*) holds we proceed as follows. First of all it is easy to see that it is enough to show the estimate for $r = 1$. The estimate for $r = 1$ follows from the uniform boundedness of the operators T_n with respect to the maximum on the unit circle: by Lemma 3.3 (c) in [13], we have the following estimate of the corresponding operator norm

$$\|T_n\| \le 4\left(\frac{[m_{n+1}]-[m_{n-1}]}{[m_n]-[m_{n-1}]}\right)\left(3+4\frac{[m_{n+1}]-[m_{n-1}]}{[m_{n+1}]-[m_n]}\right).$$

Using the recursive definition of the sequence (m_n) it is easy to see that this sequence is bounded by some constant $D \ge 1$.

We now concentrate on the proof of *d*). For the rest of this section we fix a weight $v = we^{-ar} \in (E)_{A,a}$.

Definition 5.2. Given a number $m > 0$ we denote by r_m a global maximum point of the function

$$r^m v(r). \tag{5.3}$$

Lemma 5.3. *The function w (which satisfies (2.2)) is bounded by a polynomial of degree A, and moreover*

$$w(2r) \leq 2^{Aa} w(r) \qquad \text{for all } r \in \mathbb{R}^+. \tag{5.4}$$

For every m we have

$$\frac{m}{a} \leq r_m \leq \frac{m}{a} + A. \tag{5.5}$$

The proof is postponed a bit.

The following lemma can be directly proved by looking carefully at Proposition 5.2 of [13]. It looks already much like Proposition 2.3. However, the difference is that the numbers r_m etc. in general depend on the weight. For the application to inductive limits this does not yet suffice. Nevertheless, Proposition 2.3 can be derived from Lemma 5.4. We denote $s_n := r_{m_n}$, where $m_{n+1} := m_n + \sqrt{m_n}$.

Lemma 5.4. *We have*

$$c \sup_{n\in\mathbb{N}} \sup_{s_{n-1}\leq|z|\leq s_{n+1}} |T_n f(z)| v(z) \leq \|f\|_v \leq C \sup_{n\in\mathbb{N}} \sup_{s_{n-1}\leq|z|\leq s_{n+1}} |T_n f(z)| v(z).$$

Proof of Proposition 2.3. We define the numbers ϱ_n and σ_n by

$$\varrho_n := \frac{1}{a} m_{n-1}\ , \quad \sigma_n := \frac{1}{a} m_{n+1} + A \tag{5.6}$$

for every n. Clearly (5.4) implies that, for some constant $\tilde{C} > 0$,

$$w(a^{-1} m_{n-1}) \geq \tilde{C} w(a^{-1} m_{n+1} + A) \tag{5.7}$$

for each n. So $v(r)$ can be replaced by $w(\varrho_n)e^{-ar}$ for our $r = |z|$. So, we have obtained

$$\sup_{s_{n-1}\leq|z|\leq s_{n+1}} |T_n f(z)| v(z) \leq C \sup_{\varrho_n\leq|z|\leq\sigma_n} |T_n f(z)| w(\varrho_n) e^{-ar}. \tag{5.8}$$

Moreover, since the degree of the polynomial $T_n f$ is at most $m_{n+1} \leq c m_{n-1}$ (where $c := c(a) > 1$), Lemma 3.1.a) of [13], (5.5) and (5.6) imply for every $\varrho_n \leq r \leq s_{n-1}$,

$$\begin{aligned} \sup_{|z|=r} |T_n f(z)| &\leq \Big(\frac{s_{n-1}}{\varrho_n}\Big)^{c m_{n-1}} \sup_{|z|=s_{n-1}} |T_n f(z)| \\ &\leq \Big(\frac{a^{-1} m_{n-1} + A}{a^{-1} m_{n-1}}\Big)^{c m_{n-1}} \sup_{|z|=s_{n-1}} |T_n f(z)| \\ &\leq e^{cA} \sup_{|z|=s_{n-1}} |T_n f(z)|. \end{aligned} \tag{5.9}$$

Also $e^{-ar} \geq e^{-a s_{n-1}}$ for $\varrho_n \leq r \leq s_{n-1}$.

Similarly we obtain for $s_{n+1} \leq r \leq \sigma_n$,

$$\sup_{|z|=r} |T_n f(z)| \leq e^{cA} \sup_{|z|=s_{n+1}} |T_n f(z)| \tag{5.10}$$

and $e^{-ar} \geq C e^{-a s_{n+1}}$ (see (5.6) for the latter).

Combining (5.9), (5.10) and (5.7) we get

$$\sup_{\varrho_n\leq|z|\leq\sigma_n} |T_n f(z)| w(\varrho_n) e^{-ar} \leq c \sup_{s_{n-1}\leq|z|\leq s_{n+1}} |T_n f(z)| v(z). \tag{5.11}$$

Proposition 2.3 follows. $\square$

Proof of Lemma 5.3. Since w satisfies (2.2), we have $rw'(r) - Aaw(r) \le 0$ for all r. This implies the first statement, since the function $r^{-Aa}w(r)$ becomes decreasing; its derivative is $r^{-Aa-1}(rw'(r) - Aaw(r))$.

This fact also implies (5.4), since

$$(2r)^{-Aa}w(2r) - r^{-Aa}w(r) \le 0, \quad \text{or}$$

$$w(2r) \le \frac{(2r)^{Aa}}{r^{Aa}}w(r) = 2^{Aa}w(r). \tag{5.12}$$

By elementary differential calculus, the number r_m is a zero of the derivative of the function $r^m w(r)e^{-ar}$; i.e.,

$$r^{m-1}e^{-ar}\Big(w(r)(m - ar) + rw'(r)\Big). \tag{5.13}$$

Hence, r_m solves

$$r = \frac{rw'(r)}{aw(r)} + \frac{m}{a} \ge \frac{m}{a}. \tag{5.14}$$

Since the assumption (2.2) holds, we find that $r_m \ge m/a$ cannot be larger than $m/a + A$. □

To prove Lemma 5.4 we need

Lemma 5.5. *There exists an $N \in \mathbb{N}$, independent of the weight v, as follows. There exists $\gamma > 1$ such that*

$$\Big(\frac{s_{n+1}}{s_n}\Big)^{m_{n+1}}\frac{v(s_{n+1})}{v(s_n)} \ge \gamma \quad \text{and} \quad \Big(\frac{s_n}{s_{n+1}}\Big)^{m_n}\frac{v(s_n)}{v(s_{n+1})} \ge \gamma \tag{5.15}$$

for all $n \ge N$.

Proof. We first remark that

$$s_{n+1} = s_n + \frac{1}{a}\sqrt{m_n} + a(n),$$

$$s_n = \frac{1}{a}m_n + b(n), \tag{5.16}$$

where always $|a(n)| \le A$ and $0 \le b(n) \le A$. The second identity follows from Lemma 5.3. As for the first, we have $s_{n+1} = r_{m_{n+1}}$, and by the choice of the numbers m_n and Lemma 5.3,

$$r_{m_{n+1}} \le \frac{1}{a}m_{n+1} + A = \frac{1}{a}m_n + \frac{1}{a}\sqrt{m_n} + A \le s_n + \frac{1}{a}\sqrt{m_n} + A,$$

and on the other hand

$$r_{m_{n+1}} \ge \frac{1}{a}m_{n+1} = \frac{1}{a}m_n + \frac{1}{a}\sqrt{m_n} \ge s_n - A + \frac{1}{a}\sqrt{m_n}.$$

Fix $\delta > 0$ so small that

$$\gamma := \min((1-\delta)e^{1/4}, (1-\delta)^2 e^{1/2}) > 1. \tag{5.17}$$

Let us show that (5.16) implies the lower bound $w(s_n)/w(s_{n+1}) \ge 1 - \delta$ for $n \ge N$, where N can be chosen independently of v. As in (5.12) we get

$$w(s_{n+1}) \le \frac{s_{n+1}^{Aa}}{s_n^{Aa}}w(s_n) \le \Big(\frac{s_n + a^{-1}\sqrt{m_n} + A}{s_n}\Big)^{Aa}w(s_n)$$

$$\leq \Big(\frac{a^{-1}m_n + a^{-1}\sqrt{m_n} + 2A}{a^{-1}m_n}\Big)^{Aa} w(s_n), \tag{5.18}$$

and the statement follows from the fact that $m_n \to \infty$ as $n \to \infty$. Notice that A does not depend on v.

The rest of the proof is based on the estimate

$$e^y e^{-\frac{1}{2}\frac{y^2}{x}} \leq \Big(1 + \frac{y}{x}\Big)^x \leq e^y e^{-\frac{1}{4}\frac{y^2}{x}}, \tag{5.19}$$

valid, say, for all $x > 2y > 1$. To prove this, recall that $\log(1+t) = \sum_{n=1}^{\infty}(-1)^{n+1}t^n/n$ for $0 \leq t < 1$. We get

$$y - \frac{1}{2}\frac{y^2}{x} \leq x\log\Big(1 + \frac{y}{x}\Big) \leq y - \frac{1}{4}\frac{y^2}{x}, \tag{5.20}$$

and (5.19) follows by raising this to the exponent of e.

To prove the second inequality of (5.15) we estimate

$$\begin{aligned}
&\Big(\frac{s_n}{s_{n+1}}\Big)^{m_n}\frac{v(s_n)}{v(s_{n+1})} \\
&= \Big(\frac{a^{-1}m_n + b(n)}{a^{-1}m_n + a^{-1}\sqrt{m_n} + a(n) + b(n)}\Big)^{m_n}\frac{w(s_n)}{w(s_{n+1})}e^{aa^{-1}\sqrt{m_n}+aa(n)} \\
&\geq (1-\delta)\Big(\frac{a^{-1}m_n + a^{-1}\sqrt{m_n} + a(n) + b(n)}{a^{-1}m_n + b(n)}\Big)^{-m_n}e^{\sqrt{m_n}+aa(n)} \\
&\geq (1-\delta)\Big(1 + \frac{a^{-1}\sqrt{m_n} + a(n)}{a^{-1}m_n + b(n)}\Big)^{-m_n}e^{\sqrt{m_n}+aa(n)} \\
&\geq (1-\delta)\Big(1 + \frac{\sqrt{m_n} + aa(n)}{m_n}\Big)^{-m_n}e^{\sqrt{m_n}+aa(n)} \\
&\geq (1-\delta)e^{-\sqrt{m_n}-aa(n)}e^{\frac{1}{4}\frac{m_n+aa(n)}{m_n}}e^{\sqrt{m_n}+aa(n)} \\
&\geq (1-\delta)e^{1/4}.
\end{aligned} \tag{5.21}$$

The third but last row was estimated using (5.19)

As for the first inequality of (5.15), we estimate

$$\begin{aligned}
&\Big(\frac{s_{n+1}}{s_n}\Big)^{m_{n+1}}\frac{v(s_{n+1})}{v(s_n)} \\
&= \Big(\frac{a^{-1}m_n + a^{-1}\sqrt{m_n} + a(n) + b(n)}{a^{-1}m_n + b(n)}\Big)^{m_{n+1}}\frac{w(s_{n+1})}{w(s_n)}e^{-aa^{-1}\sqrt{m_n}-aa(n)} \\
&\geq \Big(\frac{a^{-1}m_n + a^{-1}\sqrt{m_n} + a(n) + b(n)}{a^{-1}m_n + b(n)}\Big)^{m_n+\sqrt{m_n}}e^{-\sqrt{m_n}-aa(n)} \\
&\geq \Big(1 + \frac{a^{-1}\sqrt{m_n} + a(n)}{a^{-1}m_n + b(n)}\Big)^{m_n+ab(n)+\sqrt{m_n}-ab(n)}e^{-\sqrt{m_n}-aa(n)} \\
&\geq \Big(1 + \frac{\sqrt{m_n} + aa(n)}{m_n + ab(n)}\Big)^{m_n+b(n)}\Big(1 + \frac{\sqrt{m_n} + aa(n)}{m_n + ab(n)}\Big)^{\sqrt{m_n}-ab(n)}e^{-\sqrt{m_n}-aa(n)} \\
&\geq e^{\sqrt{m_n}+aa(n)}e^{-\frac{1}{2}\frac{(\sqrt{m_n}+aa(n))^2}{m_n}}\Big(1 + \frac{\sqrt{m_n} + aa(n)}{m_n + ab(n)}\Big)^{\sqrt{m_n}-ab(n)}e^{-\sqrt{m_n}-aa(n)}.
\end{aligned} \tag{5.22}$$

Again we used (5.19) at the end. Since the numbers $|a(n)|$ and $b(n) \geq 0$ have the v–independent bound A, one can find a v–independent number $N' \in \mathbb{N}$ such that, for all $n \geq N'$, both

$$e^{-\frac{1}{2}\frac{(\sqrt{m_n}+a(n))^2}{m_n}} \geq (1-\delta)e^{-\frac{1}{2}} \tag{5.23}$$

and

$$\Big(1 + \frac{\sqrt{m_n} + aa(n)}{m_n + ab(n)}\Big)^{\sqrt{m_n}-ab(n)} \geq (1-\delta)e \tag{5.24}$$

hold. As a consequence, (5.22) has the lower bound $(1-\delta)^2 e^{1/2}$. □

Proof of Lemma 5.4, repeated from Lemma 5.2 of [13]. We define the operators $T_n := V_{m_{n+1},m_n} - V_{m_n,m_{n-1}}$, where

$$V_{p,q}f(z) := \sum_{0 \leq k \leq q} f_k z^k + \sum_{q<k\leq p} \frac{[p]-[k]}{[p]-[q]} f_k z^k \tag{5.25}$$

for $f(z) := \sum_{k=0}^{\infty} f_k z^k$.

The inequality on the left side of (5.4) follows easily from statement (c) which was shown before.

As for the other side, we assume that f is only a polynomial. Write f as $f = \sum_p T_p f$, where $T_p f \in \operatorname{span}\{z^j;\ [m_{p-1}] + 1 \leq j \leq [m_{p+1}]\}$. Consider an arbitrary fixed r with $r \geq m_{N+4}$ (N as in Lemma 5.5), and choose $n > N + 2$ such that $s_{n-1} \leq r \leq s_n$. We estimate using Lemma 3.1 of [13]

$$\begin{aligned}
\sup_{|z|=r} |f(z)|v(z) &\leq \sum_{p=0}^{\infty} \sup_{|z|=r} |T_p f(z)|v(z) \\
&\leq \sum_{p\leq N} \sup_{|z|=r} |T_p f(z)|v(z) + \sum_{N<p\leq n-2} \Big(\frac{r}{s_{p+1}}\Big)^{m_{p+1}} \sup_{|z|=s_{p+1}} |T_p f(z)|v(r) \\
&\quad + \sum_{p=n-1}^{n+1} \sup_{|z|=r} |T_p f(z)|v(z) + 2\sum_{p\geq n+2} \Big(\frac{r}{s_{p-1}}\Big)^{m_{p-1}} \sup_{|z|=s_{p-1}} |T_p f(z)|v(r) \\
&\leq \sum_{p\leq N} \sup_{|z|=r} |T_p f(z)|v(z) + \sum_{N<p\leq n-2} \Big(\frac{r}{s_{p+1}}\Big)^{m_{p+1}} \frac{v(r)}{v(s_{p+1})} \sup_{|z|=s_{p+1}} |T_p f(z)|v(z) \\
&\quad + \sum_{p=n-1}^{n+1} \sup_{|z|=r} |T_p f(z)|v(z) + 2\sum_{p\geq n+2} \Big(\frac{r}{s_{p-1}}\Big)^{m_{p-1}} \frac{v(r)}{v(s_{p-1})} \sup_{|z|=s_{p-1}} |T_p f(z)|v(z). \qquad (5.26)
\end{aligned}$$

Since the functions $T_p f$, $p \leq N$, belong to a fixed $[n_{N+1}]$–dimensional subspace, we can find, by the equivalence of all norms in a finite dimensional space, a positive constant $C := C(v)$ such that

$$\sum_{p\leq N} \sup_{|z|=r} |T_p f(z)|v(z) \leq C \sup_{p\leq N} \sup_{\varrho_p \leq |z| \leq \sigma_p} |T_p f(z)| w(\varrho_n) e^{-ar}. \tag{5.27}$$

In the case $N < p \leq n-2$ we have by Lemma 5.5,

$$\Big(\frac{r}{s_{p+1}}\Big)^{m_{p+1}} \frac{v(r)}{v(s_{p+1})} \leq \Big(\frac{s_{p+2}}{s_{p+1}}\Big)^{m_{p+1}} \frac{v(s_{p+2})}{v(s_{p+1})} \cdot \Big(\frac{s_{p+3}}{s_{p+2}}\Big)^{m_{p+2}} \frac{v(s_{p+3})}{v(s_{p+2})} \cdot$$

$$\dots \cdot \Big(\frac{s_{n-1}}{s_{n-2}}\Big)^{m_{n-2}} \frac{v(s_{n-1})}{v(s_{n-2})} \cdot \Big(\frac{r}{s_{n-1}}\Big)^{m_{n-1}} \frac{v(r)}{v(s_{n-1})}$$

$$\leq \Big(\frac{1}{\gamma}\Big)^{n-p-2}. \tag{5.28}$$

For $p \geq n+2$ we obtain in the same way

$$\Big(\frac{r}{s_{p-1}}\Big)^{m_{p-1}} \frac{v(r)}{v(s_{p-1})} \leq \Big(\frac{r}{s_{n+1}}\Big)^{m_{n+1}} \frac{v(r)}{v(s_{n+1})} \cdot \Big(\frac{s_{n+1}}{s_{n+2}}\Big)^{m_{n+2}} \frac{v(s_{n+1})}{v(s_{n+2})} \cdot$$

$$\dots \cdot \Big(\frac{s_{p-2}}{s_{p-1}}\Big)^{m_{p-1}} \frac{v(s_{p-2})}{v(s_{p-1})}$$

$$\leq \Big(\frac{1}{\gamma}\Big)^{p-1-n}. \tag{5.29}$$

Since $\gamma > 1$, the lemma follows from (5.26)–(5.29). □

6. Our results apply to certain spaces $\mathcal{V}H$ which are not Montel

The next result shows that it is possible to construct spaces of type $\mathcal{V}H$ in our setting which are not (semi-) Montel spaces; hence the projective description does not follow from the general result of Bierstedt, Meise and Summers [7] for Montel spaces $\mathcal{V}H$.

Proposition 6.1. *Given a Köthe coechelon space $k_\infty(\mathcal{A})$ such that the decreasing sequence $\mathcal{A} = (a_k)_{k=1}^\infty$ of weights on $\mathbb{N}$ satisfies, for some constant $B \geq 1$,*

$$1 \leq B^{-1} a_k(n+1) \leq a_k(n) \leq a_k(n+1) \leq n^{10} \tag{6.1}$$

for all k and n, there exists a weight sequence $\mathcal{V}$ as in Section 2 such that $\mathcal{V}H$ contains a closed subspace isomorphic to $k_\infty(\mathcal{A})$.

In the proof below we actually assume that B has some upper bound like e^{100}. It is obvious that the bound n^{10} in (6.1) is artificial. It might be possible to prove the result with an arbitrarily large (fixed) bound instead of n^{10}.

Proof. To define the weights w_k we first denote

$$t_n := e^{100^n} \qquad \text{and} \qquad \tau_n := n^{10} t_n < t_{n+1}. \tag{6.2}$$

For every k we define the function $F_k : [0, \infty[\to [0, 1[$ by

$$F_k(r) := \sum_{n=1}^{\infty} t_n^{-1} \chi_{kn}(r), \tag{6.3}$$

where χ_{kn} is the characteristic function of the interval $[t_n, t_n + t_n(a_k(n) - a_k(n-1))]$. We set

$$w_k(r) := \int_0^r F_k(x) dx. \tag{6.4}$$

Then it follows from the definitions that w_k is increasing (since F_k is positive) and that $w_k(r) = a_k(n)$, if $r \in [t_n + t_n(a_k(n) - a_k(n-1)), t_{n+1}]$, in particular if $r \in [\tau_n, t_{n+1}]$. (Notice that the latter interval does not depend on k.)

We show that every w_k does belong to the class $(E)_A$. Take $k, n \in \mathbb{N}$ and then let $r \in [t_n, t_n + t_n(a_k(n) - a_k(n-1))]$. (Otherwise $w_k'(r) = 0$ and there is nothing to prove.)

We have $w_k'(r) = t_n^{-1}$ and of course $r \le t_n + t_n a_k(n)$. On the other hand $w_k(r) \ge a_k(n-1)$, by the remarks on the properties of w_k above. This suffices, in view of (6.1).

For all $n \in \mathbb{N}$ we now consider the monomials $e_n(z) := z^{s_n}$, where $s_n := [t_{n+1}/3] + 1$.

We claim that for some C,

$$\frac{1}{C} \sup_{n\in\mathbb{N}} |b_n| a_{kn} s_n^{s_n} e^{-s_n} \le \| \sum_{n=1}^{\infty} b_n e_n \|_k \le C \sup_{n\in\mathbb{N}} |b_n| a_{kn} s_n^{s_n} e^{-s_n},$$

where the complex numbers b_n satisfy that the suprema are finite. Note: $s_n \in [\tau_n, t_{n+1}]$ *does not depend on* k, hence (6) implies that the mapping which associates e_n with the nth canonical coordinate vector of $k_\infty(\mathcal{A})$ is the desired isomorphism. Observe that we prove in fact that a diagonal transform of $k_\infty(\mathcal{A})$ is isomorphic to a closed subspace of $\mathcal{V}H$.

Given $m \in \mathbb{N}$, we evaluate the value of

$$\max_{r\in[t_m,t_{m+1}]} e_n(r) v_k(r) = \max_{r\in[t_m,t_{m+1}]} r^{s_n} w_k(r) e^{-r} =: J_{knm}. \tag{6.5}$$

We actually claim that

$$J_{knm} \le 10^{-|n-m|} J_{knn}. \tag{6.6}$$

We remark that the maximum point of the function $r^\alpha e^{-r}$, $r > 0$, is at the point $r = \alpha$, as shown by elementary differentiation, and that

$$\frac{d}{dr} r^\alpha e^{-r} > 0 \ \text{ for } r < \alpha \ \text{ and } \ \frac{d}{dr} r^\alpha e^{-r} < 0 \ \text{ for } r > \alpha. \tag{6.7}$$

If $m = n$, the value of (6.5), or J_{knn}, is thus

$$\begin{aligned} J_{knn} = a_k(n) s_n^{s_n} e^{-s_n} &\ge \left(\frac{1}{3}\right)^{\frac{1}{3} e^{100^{n+1}}} a_k(n) e^{\frac{1}{3} 100^{n+1} e^{100^{n+1}}} e^{-\frac{1}{3} e^{100^{n+1}}} \\ &\ge a_k(n) e^{10 \cdot 100^n e^{100^{n+1}}} \end{aligned} \tag{6.8}$$

since $w_k(s_n)$ happens to be equal to $a_k(n)$, and this is the maximum of w_k on the interval $[t_n, t_{n+1}]$ under consideration.

If $m < n$, then

$$\begin{aligned} J_{knm} &\le a_k(m) t_{m+1}^{s_n} \le a_k(m) e^{\frac{1}{3} 100^{m+1} e^{100^{n+1}}} \\ &\le a_k(n) 10^{-|m-n|} e^{10 \cdot 100^n e^{100^{n+1}}} \le 10^{-|m-n|} J_{knn}. \end{aligned} \tag{6.9}$$

If $m > n$, then, by (6.7),

$$J_{knm} \le a_k(m) t_m^{s_n} e^{-t_m} \le a_k(m) e^{\frac{1}{3} 100^m e^{100^{n+1}}} e^{-e^{100^m}}. \tag{6.10}$$

To see that this is also bounded by $10^{-|m-n|} J_{knn}$, in the case $m = n + 1$, (6.10) has to be compared with the second but last expression of (6.8), and in the case $m > n + 1$ the estimate follows from the bound (see (6.1))

$$a_k(n) B^{m-n} e^{-100^{m-n} e^{100^{m-1}}} \tag{6.11}$$

of (6.10).

Finally, (6) follows from (6.6): First (6.6) implies, with $r = |z|$, that

$$\begin{aligned}\|\sum_{n=1}^{\infty} b_n e_n\|_k &\le \sup_{m\in\mathbb{N}} \sup_{r\in[t_m,t_{m+1}]} |\sum_{n=1}^{\infty} b_n e_n(z)| v_k(r) \\ &\le \sup_{m\in\mathbb{N}} \sum_{n=1}^{\infty} |b_n| J_{knm} \\ &\le C \sup_{m\in\mathbb{N}} \sum_{n=1}^{\infty} 10^{-|n-m|} |b_m| J_{kmm} \\ &\le C \sup_{m\in\mathbb{N}} |b_m| J_{kmm} \le C \sup_{m\in\mathbb{N}} |b_m| a_k(m) s_m^{s_m} e^{-s_m}. \qquad (6.12)\end{aligned}$$

Hence, the second inequality of (6) follows from the first identity of (6.8). Now choose N such that

$$b_N a_k(N) s_N^{s_N} e^{-s_N} \ge \frac{9}{10} \sup_{n\in\mathbb{N}} |b_n| a_k(n) s_n^{s_n} e^{-s_n}. \qquad (6.13)$$

For the first inequality we estimate

$$\begin{aligned}\|\sum_{n=1}^{\infty} b_n e_n\|_k &\ge \sup_{r\in[t_N,t_{N+1}]} |\sum_{n=1}^{\infty} b_n e_n| v_k(r) \\ &\ge |b_N| J_{kNN} - \sum_{n\neq N} |b_n| J_{knN} \\ &\ge |b_N| J_{kNN} - \sum_{n\neq N} 10^{-|n-N|} |b_n| J_{knn} \\ &\ge |b_N| J_{kNN} - \frac{10}{9} \sum_{n\neq N} 10^{-|n-N|} |b_N| J_{kNN} \ge C |b_N| J_{kNN}. \qquad (6.14)\end{aligned}$$

□

Acknowledgements. The research of the first two authors was partially supported by FEDER and MEC Project MTM2004-02262, and the research of J. Taskinen was partially supported by the Academy of Finland project "Functional analysis and applications". The present work was initiated during a stay of Taskinen in Paderborn, it was continued during visits of Taskinen in Valencia in 2004 and Bonet in Helsinki 2006, which were supported by the Finnish Mathematical Society International Visitor Program 2005–2006. The research continued during a stay of Bierstedt in Valencia and of Bonet in Paderborn in late 2006. We are grateful for the kind hospitality of all the institutions involved.

References

[1] F. Bastin, Weighted spaces of continuous functions, *Bull. Soc. R. Sci. Liège* **59** (1990) 1-81.

[2] K.D. Bierstedt, A survey of some results and open problems in weighted inductive limits and projective description for spaces of holomorphic functions, *Bull. Soc. Roy. Sci. Liège* **70** (2001) 167–182.

[3] K.D. Bierstedt, J. Bonet, Projective description of weighted (LF)–spaces of holomorphic functions on the disc. *Proc. Edinburgh Math. Soc.* **46** (2003) 435–450.

[4] K.D. Bierstedt, J. Bonet, A. Galbis, Weighted spaces of holomorphic functions on bounded domains, *Michigan Math. J.* **40** (1993) 271–297.

[5] K.D. Bierstedt, J. Bonet, J. Taskinen, Associated weights and spaces of holomorphic functions, *Studia Math.* **127** (1998) 137–168.

[6] K.D. Bierstedt, R. Meise, Distinguished echelon spaces and the projective description of weighted inductive limits of type $\mathcal{V}_d\mathcal{C}(X)$, pp. 169–226 in *Aspects of Mathematics and its Applications*, North-Holland Math. Library **34**, Amsterdam 1986.

[7] K.D. Bierstedt, R. Meise, W.H. Summers, A projective description of weighted inductive limits, *Trans. Amer. Math. Soc.* **272** (1982) 107–160.

[8] J. Bonet, R. Meise, Ultradistributions of Roumieu type and projective descriptions, *J. Math. Anal. Appl.* **255**(2001) 122–136.

[9] J. Bonet, R. Meise, S.N. Melikhov, The dual of the space of holomorphic functions on locally closed convex sets, *Publ. Mat.* **49** (2005) 487-509.

[10] J. Bonet, S.N. Melikhov, Interpolation of entire functions and projective descriptions, J. Math. Anal. Appl. **205**, No. 2 (1997), 454–460.

[11] J. Bonet, J. Taskinen, The subspace problem for weighted inductive limits of spaces of holomorphic functions, *Mich. Math. J.* **42** (1995) 259–268.

[12] W. Lusky, On weighted spaces of harmonic and holomorphic functions, *J. London Math. Soc.* **51** (1995) 309-320.

[13] W. Lusky, On the isomorphism classes of weighted spaces of harmonic and holomorphic functions. *Studia Math.* **175** (2006) 19-45.

[14] P. Mattila, E. Saksman, J. Taskinen, Weighted spaces of harmonic and holomorphic functions: Sequence space representations and projective descriptions, *Proc. Edinburgh Math. Soc.* **40** (1997), 41–62.

Authors' addresses:

(KDB) Mathematical Institute, University of Paderborn, Warburger Str. 100, D-33098 Paderborn, Germany
e-mail: klausd@upb.de

(JB) E.T.S. Arquitectura and IMPA, Departamento de Matemática Aplicada, Universidad Politécnica de Valencia, E-46071 Valencia, Spain
e-mail: jbonet@mat.upv.es

(JT) Department of Mathematics and Statistics, University of Helsinki, P.O. Box 68, FIN-00014 Helsinki, Finland
e-mail: jari.taskinen@helsinki.fi