The splitting of exact sequences of PLS-spaces and smooth dependence of solutions of linear partial differential equations [1]

José Bonet [a] Paweł Domański [b]

[a]*Departamento de Matemática Aplicada, ETS Arquitectura y Instituto de Matemática Pura y Aplicada, Universidad Politécnica de Valencia, E-46071 Valencia, Spain*

[b]*Faculty of Mathematics and Comp. Sci. A. Mickiewicz University Poznań and Institute of Mathematics Polish Academy of Sciences (Poznań branch) Umultowska 87, 61-614 Poznań, POLAND*

Abstract

We investigate the splitting of short exact sequences of the form

$$0 \longrightarrow X \longrightarrow Y \longrightarrow E \longrightarrow 0,$$

where E is the dual of a Fréchet Schwartz space and X, Y are PLS-spaces, like the spaces of distributions or real analytic functions or their subspaces. In particular, we characterize pairs (E, X) as above such that $\text{Ext}^1(E, X) = 0$ in the category of PLS-spaces and apply this characterization to many natural spaces X and E. In particular, we discover an extension of the $(DN) - (\Omega)$ splitting theorem of Vogt and Wagner. These abstract results are applied to parameter dependence of linear partial differential operators and surjectivity of such operators on spaces of vector valued distributions.

Key words: Splitting of short exact sequences, space of distributions, space of ultradistributions in the sense of Beurling, functor Proj^1, functor Ext^1, PLS-space, locally convex space, Fréchet space, linear partial differential operator, convolution operator, vector valued equation, solvability, analytic dependence on parameters, smooth dependence on parameter
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1 Introduction

The aim of this paper is to study the splitting of short exact sequences of PLS-spaces and its applications to parameter dependence of solutions of linear partial differential equations on spaces of distributions or spaces of real analytic functions, see Section 5, Theorem 5.5.

We study the functor Ext^1 for subspaces of $\mathscr{D}'(\Omega)$ and duals of Fréchet Schwartz spaces. This problem is considered in the framework of the so-called PLS-spaces; this is the smallest class of locally convex spaces containing all duals of Fréchet Schwartz spaces and closed with respect of taking countable products and closed subspaces. This class contains the most important spaces which appear in analytic applications of linear functional analysis, like spaces of (ultra-)distributions, or spaces of real analytic or quasi analytic functions as well as spaces of holomorphic or smooth functions; for more information on PLS-spaces we refer the reader to the survey paper [10]. The crucial result of the present paper (Theorem 3.1) is a characterization of the pairs (F, X), where X is a PLS-space and F is a Fréchet nuclear space such that every short topologically exact sequence of PLS-spaces (all arrows throughout the paper denote linear continuous maps)

$$ 0 \longrightarrow X \stackrel{j}{\longrightarrow} Y \stackrel{q}{\longrightarrow} F' \longrightarrow 0 \tag{1} $$

splits (i.e., q has a linear continuous right inverse) or equivalently, such that $\text{Ext}^1_{PLS}(F', X) = 0$. Topological exactness of (1) means that j is a topological embedding onto the kernel of the continuous and open surjection q. The characterization is given in terms of some inequality preceded by a long sequence of quantifiers, see condition (G) or (G_ε) in Theorem 3.1. The proof is long, technical, complicated and based on the method of the functor Proj^1 for spectra of LB-spaces. The case when both X and F' are substituted by Fréchet spaces (or by duality when all the spaces in the exact sequence are DFS-spaces) was characterized long ago under the assumption that one space is nuclear or one space is a suitable sequence space. In fact, necessity of (G) in the case of Fréchet spaces is due to Vogt [41]; he also introduced a sufficient condition very useful in applications. Sufficiency of an analogue of (G) for both spaces being Fréchet sequence spaces is due to Krone and Vogt [21]. Sufficiency in other cases for Fréchet spaces was an open problem for some time. A breakthrough was made by Frerick [15] who proved the case of all nuclear Fréchet spaces and, finally, Frerick and Wengenroth proved sufficiency in all Fréchet cases in [17]. The condition they all used, called (S^*_2), was slightly

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different from ours - a characterization in the Fréchet case even more similar to ours is given in [46, 5.2.5]. There have been very few splitting results for PLS-spaces so far, see [12], [13], [45], [22], [11, Theorem 2.3], [44], [4], comp. [16] and [46, Sec. 5.3]. However, this is considered as an important problem in the modern theory of locally convex spaces and their analytic applications; see [44].

In [4] we investigated the vanishing of $\text{Ext}^1_{PLS}(F, X)$ for a nuclear Fréchet space F, while in the present paper we attack the same question for the dual F'. This is a different, much more difficult problem. For instance, the reduction to the vanishing of the derived functor Proj^1 for spectra of LB-spaces was standard in [4], but now it requires several new ideas and ingredients, among them a key observation due to Vogt in [44], see Lemma 3.3, proof of Theorem 3.4 (ii)$\Leftrightarrow$(iii). To avoid problems with local splitting we have to dualize the considered short exact sequences and to study sequences of LFS-spaces, see the proof of Theorem 3.4.

Although our condition looks complicated it turns out to be evaluable. Indeed, we characterize in Theorem 4.4 and Corollary 4.5) those PLS-spaces X such that $\text{Ext}^1_{PLS}(\Lambda_r(\alpha)', X) = 0$, where $\Lambda_r(\alpha)$ is a stable power series space, like $H(\mathbb{D}^d), H(\mathbb{C}^d)$ or even $C^\infty(U) \simeq \prod \Lambda_r(\alpha)$. The characterizing condition is of (Ω) type and is called (PA) . On the other hand, it turns out that if X has (PA) and a nuclear Fréchet space F has (Ω) then $\text{Ext}^1_{PLS}(F', X) = 0$ (Theorem 4.1), this is the proper extension of the (DN) -(Ω) splitting theorem [27, 30.1]. That is why the discovery of the condition (PA) as a suitable generalization of the condition (Ω) seems to be one of the main achievements of the paper. It is even more striking if one looks at Proposition 5.4 and compare it with earlier results on the property (Ω) of kernels of hypoelliptic operators (comp. [32], [38], [47, 2.2.6]). We give more examples of natural spaces with property (PA) in Theorem 4.3.

The parameter dependence problem considers whether, for every linear partial differential operator with constant coefficients $P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega), \Omega \subseteq \mathbb{R}^d$ convex open, and every family of distributions $(f_\lambda)_{\lambda \in U} \subseteq \mathscr{D}'(\Omega)$ depending smoothly C^∞ (or holomorphically etc.) on the parameter λ running through an arbitrary C^∞-manifold U (or Stein manifold U etc.), there is an analogous family $(u_\lambda)_{\lambda \in U}$ with the same type of dependence on $\lambda \in U$ such that

$$P(D)u_\lambda = f_\lambda \qquad \forall \; \lambda \in U. \tag{2}$$

Recall that (f_λ) depends holomorphically (smoothly) on $\lambda \in U$ if for every test function φ, $\lambda \mapsto \langle f_\lambda, \varphi \rangle$ is holomorphic (C^∞-smooth). This problem has been extensively studied, even in a much more general setting, for instance, if $P(D)$ depends on λ as well; see [23], [24], [35], [3], [2]. For more historical comments see the introduction of [4]. Using tensor product techniques [20, Ch. 16], the parameter dependence is equivalent to the problem of surjec-

tivity of $P(D)$ on the spaces of vector valued distributions $\mathscr{D}'(\Omega, F)$, where, e.g., $F = C^\infty(U)$ (for smooth dependence) or $F = H(U)$ (for holomorphic dependence). Our splitting results imply that the latter problem has a positive solution for any Fréchet space with property (Ω) (Theorem 5.5), for instance, $F \simeq H(U), C^\infty(U), \Lambda_r(\alpha), C^\infty[0,1]$, etc., see [27, 29.11]. Our method is potentially applicable to arbitrary surjective linear continuous operators $T : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ and even to more general spaces than $\mathscr{D}'$ (like spaces of ultradistributions or real analytic functions).

In these applications of our splitting results, the crucial point is whether $\ker P(D)$ has (PA) , which we prove by means of a trick (see Proposition 5.4 and Theorem 5.1). For more applications of our splitting result for spaces of real analytic functions and Roumieu quasianalytic classes of ultradifferentiable functions see the forthcoming paper [5].

The paper is organized as follows. Section 2 contains preliminaries and notation. In Section 3 we prove the main splitting theorem. In Section 4 we apply it for some natural spaces, especially, sequence spaces, we introduce conditions (PA) and $(P\underline{A})$ and give examples and applications. In Section 5 we apply our theory to the parameter dependence problem.

The authors are very indebted to V. Palamodov for deep remarks on the paper especially related to Theorem 5.1.

2 Preliminaries

In the present section we collect some basic notation which is very similar to the one used in [4].

By an *operator* we mean a linear continuous map. By $L(Z, Y)$ we denote the set of all operators $T : Z \longrightarrow Y$. If $A \subseteq Z$ and $B \subseteq Y$, then $W(A, B) := \{T \in L(Z, Y) : T(A) \subseteq B\}$.

A locally convex space X is a *PLS-space* if it is a projective limit of a sequence of strong duals of Fréchet -Schwartz spaces (i.e., LS-spaces), see the survey paper [10]. If we take strong duals of nuclear Fréchet spaces instead (i.e., LN-spaces), then X is called a *PLN-space*. Every closed subspace and every Hausdorff quotient of a PLS-space is a PLS-space, [12, 1.2 and 1.3]. Every PLS-space is automatically complete and Schwartz, PLN-spaces are even nuclear. Every Fréchet-Schwartz space is a PLS-space and every strongly nuclear Fréchet space is a PLN-space.

Every PLS-space X satisfies $X = \operatorname{proj}_{N \in \mathbb{N}} \operatorname{ind}_{n \in \mathbb{N}} X_{N,n}$, $X_{N,n}$ are Banach

spaces, $X_N := \text{ind}_{n \in \mathbb{N}} \, X_{N,n}$ is a locally convex inductive limit with compact linking maps, and

$\text{proj}_{N \in \mathbb{N}} \, X_N$ denotes the topological projective limit of a sequence $(X_N)_{N \in \mathbb{N}}$. The linking maps will be denoted by $i_N^K : X_K \to X_N$ and $i_N : X \longrightarrow X_N$. If $\overline{i_N X} = X_N$ for each N sufficiently big then we call the spectrum (X_N) *reduced*. We denote the closed unit ball of $X_{N,n}$ by $B_{N,n}$ and its polar in X_N' by $U_{N,n}$. In $E = \text{ind}_{n \in \mathbb{N}} \, E_n$ we always denote by B_n the unit ball of the Banach space $(E_n, ||.||_n)$, by U_n its polar in E_n' and by $j_m^n : E_n \to E_m$ the injective compact linking map. Without loss of generality we assume that for every $M \geq N$, $m \geq n$

$$i_N^M(B_{M,n}) \subseteq B_{N,n}, \quad B_{N,n} \subseteq B_{N,m}, \quad B_n \subseteq B_m.$$

This notation will be kept throughout the paper.

We will use in the category of PLS-spaces the notions of pull-back and push-out as described, for instance, in [46, Def. 5.1.2]. They exist in this category by [12].

Let $A = (a_{N,n}(j))$ be a matrix of non negative elements satisfying the following conditions:

(i) $a_{N,n+1}(j) \leq a_{N,n}(j) \leq a_{N+1,n}(j)$;
(ii) For each j there is N such that for all n $a_{N,n}(j) > 0$;
(iii) $\lim_{j \to \infty} \frac{a_{N,n+1}(j)}{a_{N,n}(j)} = 0$.

We define the Köthe type PLS-sequence spaces $\Lambda^p(A)$ for $1 \leq p < \infty$,

$$\Lambda^p(A) := \{x = (x(j)) : \; \forall \, N \in \mathbb{N} \; \exists \, n \in \mathbb{N} : \quad ||x||_{N,n} < \infty\},$$

where $||x||_{N,n} := \left(\sum_j |x(j)|^p a_{N,n}(j) \right)^{1/p}$. The definition for $p = \infty$ is analogous. Clearly, $\Lambda^p(A) = \text{proj}_{N \in \mathbb{N}} \, \text{ind}_{n \in \mathbb{N}} \, l_p(a_{N,n})$, where $l_p(a_{N,n})$ denotes the weighted l_p-space equipped with the norm $|| \cdot ||_{N,n}$. The condition (iii) implies that $\Lambda^p(A)$ is a PLS-space. Every PLS-sequence space $\Lambda^p(A)$ is isomorphic to a countable product of spaces of the same type for a matrix with strictly positive elements. $\Lambda^p(A)$ is even a PLN-space if instead of (iii), we assume

(iv) $\sum_j \frac{a_{N,n+1}(j)}{a_{N,n}(j)} < \infty$.

If the matrix A does not depend on lower case index n, then we get a Köthe sequence Fréchet space $\lambda_p(A)$ which need not have a continuous norm; if it does not depend on the upper case index N then it becomes a coechelon Köthe sequence LS-space $k_p(A)$. Observe that condition (ii) ensures that in $k_p(A)$ the matrix consists of strictly positive elements.

If $a_{N,n}(j) := \exp(r_N \alpha_j - s_n \beta_j)$ where $\alpha_j, \beta_j > 0$ such that $\alpha_j + \beta_j \to \infty$ and $r_N \nearrow r$, $s_n \nearrow s$ then we call the corresponding Köthe type space $\Lambda(A)$ a

PLS-type power series space and denote it by $\Lambda_{r,s}(\alpha, \beta)$. In fact, it suffices to consider only $r, s = 0, \infty$, comp. [42]. For Fréchet power series spaces $\Lambda_r(\alpha)$ see [27].

The spaces of ultradistributions in the sense of Beurling $\mathscr{D}'_{(\omega)}(\Omega)$ (in particular the space of distributions $\mathscr{D}'(\Omega)$), as well as the spaces of ultradifferentiable functions in the sense of Roumieu $\mathscr{E}_{\{\omega\}}(\Omega)$ (in particular the space of smooth functions $C^\infty(\Omega)$ and space of real analytic functions $\mathscr{A}(\Omega)$) are described in detail in [7], some details are also given in [4].

For further information from functional analysis see [27] ((DN) - (Ω) invariants are explained there) and [20], for the theory of PDE see [18]. For the modern theory of locally convex inductive limits see [1]. More details about notation can be seen in [4].

3   Splitting of short exact sequences

We characterize, under some natural assumptions, when $\mathrm{Ext}^1_{PLS}(E, X) = 0$ whenever X is a PLS-space and E is an LS-space, i.e., the strong dual of a Fréchet Schwartz space.

We consider pairs (E, X) satisfying one of the following standard assumptions:

(a) X is a PLN-space and E is an arbitrary LS-space;
(b) X is a Köthe type PLS-space, $X = \Lambda^\infty(A)$ and E is an arbitrary LS-space;
(c) E is an LN-space and X is an arbitrary PLS-space;
(d) E is a Köthe coechelon LS-space of order 1, $X = k_1(v)$ and X is an arbitrary PLS-space.

These assumptions appear below in the statements of the results in this section. Now, we formulate the main theorem (known for E, X both DFS-spaces see [46, 5.2.5], where the dual version is given):

Theorem 3.1 *Let X be an ultrabornological PLS-space, which is the reduced projective limit*
$X = \mathrm{proj}_{N \in \mathbb{N}} X_N$ of LS-spaces $X_N = \mathrm{ind}_{n \in \mathbb{N}} X_{N,n}$. Let $E = \mathrm{ind}_\nu E_\nu$ be an LS-space. Assume that the pair (E, X) satisfies assumptions (b) or (c) or (d) above, then the following assertions are equivalent:

(1) $\mathrm{Ext}^1_{PLS}(E, X) = 0$;

(2) the pair (E, X) satisfies the condition (G), i.e.,

$$\forall\, N, \nu \; \exists\, M \geq N, \mu \geq \nu \; \forall\, K \geq M, \kappa \geq \mu \; \exists\, n \; \forall\, m \geq n \; \exists\, k \geq m, S$$
$$\forall\, y \in X_N', x \in E_\nu :$$
$$\|y \circ i_N^M\|_{M,m}^* \|j_\mu^\nu x\|_\mu \leq S \left(\|y\|_{N,n}^* \|x\|_\nu + \|y \circ i_N^K\|_{K,k}^* \|j_\kappa^\nu x\|_\kappa \right);$$

(3) the pair (E, X) satisfies the condition (G_ε), i.e.,

$$\forall\, N, \nu \; \exists\, M \geq N, \mu \geq \nu \; \forall\, K \geq M, \kappa \geq \mu \; \exists\, n \; \forall\, m \geq n, \varepsilon > 0 \; \exists\, k \geq m, S$$
$$\forall\, y \in X_N', x \in E_\nu :$$
$$\|y \circ i_N^M\|_{M,m}^* \|j_\mu^\nu x\|_\mu \leq \varepsilon \|y\|_{N,n}^* \|x\|_\nu + S \|y \circ i_N^K\|_{K,k}^* \|j_\kappa^\nu x\|_\kappa;$$

Note that the ultrabornologicity of X follows if the pair (E, X) satisfies (G) and (G_ε) for a non-trivial space E (comp. [46, Cor. 3.3.10]). We conjecture that the above Theorem 3.1 holds also in case (a), i.e., if X is a PLN-space and E an arbitrary LS-space.

First, we recall some tools from the homological theory of locally convex spaces; a nice presentation of the theory is contained in Wengenroth's lecture notes [46], comp. [4]. If (X_N, i_N^K) is a projective spectrum of locally convex spaces, the so-called *fundamental resolution* is defined as an exact sequence:

$$0 \longrightarrow X \longrightarrow \prod_{N \in \mathbb{N}} X_N \overset{\sigma}{\longrightarrow} \prod_{N \in \mathbb{N}} X_N,$$

where X is the projective limit of the spectrum and $\sigma((x_N)) = (i_N^{N+1} x_{N+1} - x_N)$. We define

$$\mathrm{Proj}^1 (X_N) := \prod_{N \in \mathbb{N}} X_N / \mathrm{im}\, \sigma.$$

The value of Proj^1 does not depend on the choice of a reduced spectrum of LS-spaces representing X. Moreover, for PLS-spaces the following conditions are equivalent: (i) $\mathrm{Proj}^1 X = 0$; (ii) X is ultrabornological; (iii) X is barreled; (iv) X is reflexive (see [46, 3.3.10]).

We apply the functor Proj^1 to various spectra of spaces of operators. For example, if $X = \mathrm{proj}_{N \in \mathbb{N}} X_N$, then in the spectrum $L(F, X_N)$ the linking maps are defined by $I_N^K : L(F, X_K) \to L(F, X_N)$, $I_N^K(T) = i_N^K \circ T$ and $I_N : L(F, X) \to L(F, X_N)$, $I_N(T) := i_N \circ T$. For other cases the linking maps are defined analogously.

Lemma 3.2 *If X is a PLS-space such that* $\mathrm{Proj}^1 X = 0$ *and Z is a Banach space, then*

(1) $\mathrm{Proj}^1 L(Z, X_N) = 0$ *if* $X = \Lambda^\infty(A)$;
(2) $\mathrm{Proj}^1 L(X_N', Z) = 0$ *if* $X = \Lambda^\infty(A)$;
(3) $\mathrm{Proj}^1 L(X_N', Z) = 0$ *if* $Z = l_\infty$;

(4) $\mathrm{Proj}^1\, L(X'_N, Z) = 0$ *if X is a PLN-space.*

PROOF. Since $\Lambda^\infty(A)$ is isomorphic to a countable product of spaces of the same type for a strictly positive matrix, we may assume that all the elements in A are strictly positive.

(1): By [46, 3.2.18], $\mathrm{Proj}^1\, X = 0$ implies:

$$\forall\, N\ \exists\, M \geq N\ \forall\, K \geq M\ \exists\, n\ \forall\, m \geq n, \varepsilon > 0\ \exists\, k \geq m, S\ \forall\, i :$$
$$a_{M,m}(i) \geq \min\left(\varepsilon^{-1} a_{N,n}(i), S^{-1} a_{K,k}(i)\right).$$

Since X_N is a coechelon Köthe sequence space $k_\infty(v)$, we may treat elements of $L(Z, X_N)$ as sequences of functionals $(f_i) \subseteq Z'$ and, with this identification,

$$W(B, B_{N,n}) = \{(f_i) : \sup_i \|f_i\| a_{N,n}(i) \leq 1\},$$

where B and $B_{N,n}$ denote as usual the unit balls in Z and $X_{N,n}$ respectively. We will show that

$$W(B, B_{M,m}) \subseteq \varepsilon W(B, B_{N,n}) + SW(B, B_{K,k}).$$

Let $(f_i) \in W(B, B_{M,m})$. We take $g_i := f_i$ if $S^{-1} a_{K,k}(i) \geq \varepsilon^{-1} a_{N,n}(i)$ and 0 otherwise. Then

$$\frac{\|g_i\| a_{N,n}(i)}{\varepsilon} \leq \|g_i\| a_{M,m}(i) \leq 1.$$

Therefore $(g_i) \in \varepsilon W(B, B_{N,n})$ and analogously $(f_i - g_i) \in SW(B, B_{K,k})$. Apply [46, 3.2.14] to conclude.

(2): Treating elements of $L(X'_N, Z)$ as $(f_i) \subseteq Z$ we repeat the proof of (1) for

$$W(B^\circ_{N,n}, B) = \{(f_i) : \sup_i \|f_i\| a_{N,n}(i) \leq 1\}.$$

(3): $L(X'_N, Z) = l_\infty(X_N)$ and the result follows from [46, 3.3.11 and 3.3.16].

(4): This is [2, Lemma 3.5]. $\square$

Next we need a lemma essentially due to Vogt; we give a version we need:

Lemma 3.3 *(see [44, Lemma 3.1]) Let X be a PLS-space and E be an LS-space satisfying one of the assumptions (a) — (d). If $H = E'$ and*

$$0 \longrightarrow H \xrightarrow{\ j\ } F \xrightarrow{\ q\ } G \longrightarrow 0 \tag{3}$$

is a short exact sequence of Fréchet spaces, then we have the following exact sequence:

$$0 \longrightarrow L(X', H) \longrightarrow L(X', F) \longrightarrow L(X', G) \longrightarrow \mathrm{Proj}^1 \, L(X'_N, H_N) \longrightarrow$$
$$\longrightarrow \mathrm{Proj}^1 \, L(X'_N, F_N) \longrightarrow \mathrm{Proj}^1 \, L(X'_N, G_N) \longrightarrow 0.$$

PROOF. This is [46, 3.1.5] applied to spectrum of short exact sequences

$$0 \longrightarrow L(X'_N, H_N) \longrightarrow L(X'_N, F_N) \longrightarrow L(X'_N, G_N) \longrightarrow 0. \qquad \square$$

Now, we are ready to reduce the splitting problem to the vanishing of Proj^1.

Theorem 3.4 *Let X be a PLS-space with $\mathrm{Proj}^1 \, X = 0$ and let E be an LS-space satisfying one of the conditions (a) — (d) then the following assertions are equivalent:*

(i) $\mathrm{Ext}^1_{PLS}(E, X) = 0$;
(ii) $\mathrm{Proj}^1 \, L(X', E'_N) = 0$;
(iii) $\mathrm{Proj}^1 \, L(X'_N, E'_N) = 0$.

PROOF. (i)$\Rightarrow$(ii): Note, that X is ultrabornological, X' a complete LFS-space. For any operator $T : X' \to \prod E'_n$, we get twice the pull-back of the fundamental resolution of E':

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & E' & \longrightarrow & \prod E'_N & \stackrel{\sigma}{\longrightarrow} & \prod E'_N & \longrightarrow & 0 \\
& & \uparrow{\scriptstyle\mathrm{id}} & & \uparrow & & \uparrow{\scriptstyle T} & & \\
0 & \longrightarrow & E' & \stackrel{j}{\longrightarrow} & Y & \stackrel{q}{\longrightarrow} & X' & \longrightarrow & 0 \\
& & \uparrow{\scriptstyle\mathrm{id}} & & \uparrow & & \uparrow{\scriptstyle i'_N} & & \\
0 & \longrightarrow & E' & \stackrel{j_N}{\longrightarrow} & Y_N & \stackrel{q_N}{\longrightarrow} & X'_N & \longrightarrow & 0.
\end{array}
\qquad (4)
$$

We will show in few steps that Y is a complete LFS-space.

Completeness, metrizability and being a Schwartz space are three space properties (see [8, Th. 2.3.3], [34, Th. 3.7]), thus Y is complete and Y_N is a Fréchet Schwartz space. Since $X' = \bigcup X'_N$ also $Y = \bigcup Y_N$ and, by Grothendieck factorization theorem, every bounded set in X' (in Y) is bounded in some X'_N (Y_N, resp.). Since E' is a Fréchet Schwartz space, it is quasinormable. By [27, 26.17], q_N lifts bounded sets and, consequently, also q lifts bounded sets.

We have proved that $Y^u = \text{ind}_{N \in \mathbb{N}} Y_N$ is an LFS-space and it is the ultra-bornological space associated to Y. Then

$$0 \longrightarrow E' \xrightarrow{\ j\ } Y^u \xrightarrow{\ q\ } X' \longrightarrow 0$$

is topologically exact since E' and X' are ultrabornological. By Roelcke's lemma (see [33], [9]), $Y = Y^u$ topologically, so Y is a complete ultrabornological reflexive space by [27, 24.19].

Taking duals:

$$0 \longrightarrow X \xrightarrow{\ q'\ } Y' \xrightarrow{\ j'\ } E \longrightarrow 0$$

is a short topologically exact sequence of PLS-spaces (since q lifts bounded sets), so it splits since $\text{Ext}^1_{PLS}(E, X) = 0$. Thus the original sequence (which is the dual of the previous one, use reflexivity)

$$0 \longrightarrow E' \xrightarrow{\ j\ } Y \xrightarrow{\ q\ } X' \longrightarrow 0$$

also splits and T lifts with respect to σ, see [12, Prop. 1.7]. Thus $\text{Proj}^1 \, L(X', E'_N) = 0$.

(ii)$\Rightarrow$(i): Let us consider the following short topologically exact sequence of PLS-spaces:

$$0 \longrightarrow X \xrightarrow{\ j\ } Y \xrightarrow{\ q\ } E \longrightarrow 0. \tag{5}$$

Since $\text{Proj}^1 \, X = 0$ and $\text{Proj}^1 \, E = 0$, then [46, 3.1.5] implies that $\text{Proj}^1 \, Y = 0$ and X, Y, E are reflexive. By [12, Lemma 1.5], q lifts bounded sets, thus we get by duality and the push-out the following diagram with topologically exact rows (E', Y', X' are LFS-spaces):

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & E'_N & \longrightarrow & Z & \xrightarrow{\ Q\ } & X' & \longrightarrow & 0 \\
 & & \big\uparrow{\scriptstyle i'_N} & & \big\uparrow & & \big\uparrow{\scriptstyle \text{id}} & & \\
0 & \longrightarrow & E' & \xrightarrow{\ q'\ } & Y' & \xrightarrow{\ j'\ } & X' & \longrightarrow & 0.
\end{array}
\tag{6}
$$

If the upper rows splits then i'_N extends to Y' and we obtain the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & E' & \longrightarrow & \prod E'_N & \xrightarrow{\ \sigma\ } & \prod E'_N & \longrightarrow & 0 \\
 & & \big\uparrow{\scriptstyle \text{id}} & & \big\uparrow & & \big\uparrow{\scriptstyle T} & & \\
0 & \longrightarrow & E' & \xrightarrow{\ q'\ } & Y' & \xrightarrow{\ j'\ } & X' & \longrightarrow & 0.
\end{array}
$$

Since $\text{Proj}^1 \, L(X', E'_N) = 0$, T lifts with respect to σ. Therefore the lower row splits [12, 1.7], and, by duality, also (5) splits.

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We prove that the upper row in (6) splits. This is evident if the pair (E, X) satisfies one of the conditions (a), (c) or (d). In case (b) X' is a direct sum of Köthe type LFS-spaces with l_1-type "norms". By [43, Prop. 5.1], every summand is a projective limit of l_1 Banach spaces and splitting of the upper row in (6) follows.

(ii)$\Leftrightarrow$(iii): The proof follows the idea of Vogt [44, Proposition 4.1]. We apply Lemma 3.3 to the canonical resolution of $H = E'$:

$$0 \longrightarrow H \xrightarrow{\ i\ } \prod_{n \in \mathbb{N}} H_n \xrightarrow{\ \sigma\ } \prod_{n \in \mathbb{N}} H_n \longrightarrow 0,$$

where $\sigma((x_n)_{n \in \mathbb{N}}) := (i_n^{n+1} x_{n+1} - x_n)_{n \in \mathbb{N}}$ and $i_n^{n+1} : H_{n+1} \to H_n$ are linking maps. We define

$$\Sigma_1 : \prod_{n \in \mathbb{N}} \prod_{N \in \mathbb{N}} L(X'_N, H_n) \to \prod_{n \in \mathbb{N}} \prod_{N \in \mathbb{N}} L(X'_N, H_n),$$

$$\Sigma_2 : \prod_{N \in \mathbb{N}} L(X'_N, \prod_{n \leq N} H_n) \to \prod_{N \in \mathbb{N}} L(X'_N, \prod_{n \leq N} H_n);$$

$$\Sigma_1((T_{N,n})_{N \in \mathbb{N}, n \in \mathbb{N}}) := (T_{N+1,n} \circ I_{N+1}^N - T_{N,n})_{N \in \mathbb{N}, n \in \mathbb{N}},$$

$$\Sigma_2((T_{N,n})_{N \in \mathbb{N}, n \leq N}) := (T_{N+1,n} \circ I_{N+1}^N - T_{N,n})_{N \in \mathbb{N}, n \leq N}.$$

Here $I_{N+1}^N : X'_N \to X'_{N+1}$ are the natural embeddings. Clearly the following diagram commutes:

$$\begin{array}{ccc}
\prod_{N \in \mathbb{N}} L(X'_N, \prod_{n \leq N} H_n) & \xrightarrow{\ \Sigma_2\ } & \prod_{N \in \mathbb{N}} L(X'_N, \prod_{n \leq N} H_n) \\
A_1 \uparrow & & A_2 \uparrow \\
\prod_{n \in \mathbb{N}} \prod_{N \in \mathbb{N}} L(X'_N, H_n) & \xrightarrow{\ \Sigma_1\ } & \prod_{n \in \mathbb{N}} \prod_{N \in \mathbb{N}} L(X'_N, H_n),
\end{array}$$

where the vertical arrows are the natural projections. Let us observe that A_1 and A_2 are surjective, thus $A_2(\operatorname{im} \Sigma_1) = \operatorname{im} \Sigma_2$. Therefore A_2 induces a surjective map

$$\tilde{A}_2 : \left(\prod_{n \in \mathbb{N}} \prod_{N \in \mathbb{N}} L(X'_N, H_n) \right) / \operatorname{im} \Sigma_1 \to \left(\prod_{N \in \mathbb{N}} L(X'_N, \prod_{n \leq N} H_n) \right) / \operatorname{im} \Sigma_2.$$

Hence

$$\operatorname{Proj}^1 L(X'_N, \prod_{n \leq N} H_n) = \left(\prod_{N \in \mathbb{N}} L(X'_N, \prod_{n \leq N} H_n) \right) / \operatorname{im} \Sigma_2$$

is a surjective image of

$$\left(\prod_{n \in \mathbb{N}} \prod_{N \in \mathbb{N}} L(X'_N, H_n) \right) / \operatorname{im} \Sigma_1.$$

Moreover, im Σ_1 is a product of images of maps:

$$\prod_{N\in\mathbb{N}} L(X'_N, H_n) \to \prod_{N\in\mathbb{N}} L(X'_N, H_n), \quad (T_{N,n})_{N\in\mathbb{N}} \mapsto (T_{N+1,n} \circ I^N_{N+1} - T_{N,n})_{N\in\mathbb{N}},$$

thus

$$\left(\prod_{n\in\mathbb{N}} \prod_{N\in\mathbb{N}} L(X'_N, H_n)\right) / \operatorname{im} \Sigma_1 = \prod_{n\in\mathbb{N}} \operatorname{Proj}^1 L(X'_N, H_n).$$

By Lemma 3.2, $\operatorname{Proj}^1 L(X'_N, H_n) = 0$ and thus $\operatorname{Proj}^1 L(X'_N, \prod_{n\leq N} H_n) = 0$. Therefore, by Lemma 3.3, we have the following exact sequence:

$$0 \longrightarrow L(X', H) \longrightarrow \prod_{n\in\mathbb{N}} L(X', H_n) \xrightarrow{\Sigma_0}$$

$$\xrightarrow{\Sigma_0} \prod_{n\in\mathbb{N}} L(X', H_n) \longrightarrow \operatorname{Proj}^1 L(X'_N, H_N) \longrightarrow 0,$$

where

$$\Sigma_0((T_n)_{n\in\mathbb{N}}) := (i^{n+1}_n T_{n+1} - T_n)_{n\in\mathbb{N}}.$$

Thus

$$\operatorname{Proj}^1 L(X', H_N) \simeq \prod_{n\in\mathbb{N}} L(X', X_n) / \operatorname{im} \Sigma_0 \simeq \operatorname{Proj}^1 L(X'_N, H_N).$$

$\square$

The proof of the next lemma follows from duality and [4, Lemma 4.5].

Lemma 3.5 *(a) Let E be an arbitrary LS-space, $E = \operatorname{ind}_{n\in\mathbb{N}} E_n$. Suppose that $a, c \geq 0$, $b > 0$, $n \leq m \leq k$ and*

$$\forall\, x \in E_n \qquad a\|j^n_m x\|_m \leq b\|x\|_n + c\|j^n_k x\|_k \tag{7}$$

then

$$a(j^n_m)'(B^\circ_m) \subseteq 3bB^\circ_n + 2c(j^n_k)'(B^\circ_k).$$

(b) Let X be an arbitrary PLS-space, $X = \operatorname{proj}_{N\in\mathbb{N}} \operatorname{ind}_{n\in\mathbb{N}} X_{N,n}$, with a reduced spectrum. Suppose that $N \leq M \leq K$, $n \leq m \leq k$, $a, b, c \geq 0$ and

$$\forall\, y \in X'_N \qquad a\|y \circ i^M_N\|^*_{M,m} \leq b\|y\|^*_{N,n} + c\|y \circ i^K_N\|^*_{K,k} \tag{8}$$

then

$$a i^M_N(B_{M,m}) \subseteq 2b B_{N,n} + 2c i^K_N(B_{K,k}).$$

Proof of Theorem 3.1 (1)$\Rightarrow$(2): Let us observe that $L(X'_N, E'_N)$ coincides with $\operatorname{ind}_{n\in\mathbb{N}} L(X'_{N,n}, E'_N)$ algebraically, thus it has a natural LB-space topol-

ogy. Then, by Theorem 3.4 and [46, 3.2.18, 1. implies 3.] (the needed implication does not require LS-topology), we get

$$\forall\, N, \nu\ \exists\, M \geq N, \mu \geq \nu\ \forall\, K \geq M, \kappa \geq \mu\ \exists\, n\ \forall\, m \geq n\ \exists\, k \geq m, S \qquad (9)$$
$$I_{N,\nu}^{M,\mu} W(U_{M,m}, U_\mu) \subseteq S(I_{N,\nu}^{K,\kappa} W(U_{K,k}, U_\kappa) + W(U_{N,n}, U_\nu)),$$

where $I_{N,\nu}^{M,\mu} f := (j_\mu^\nu)' \circ f \circ (i_N^M)'$, $U_\mu = B_\mu^\circ$.

Fix $y \in X_N'$ and $x \in E_\nu$, $x \neq 0$. Since j_μ^ν is injective, $\|j_\mu^\nu x\|_\mu > 0$. There is $\varphi \in U_\mu$:

$$\varphi(j_\mu^\nu x) > (1/2)\|j_\mu^\nu x\|_\mu. \qquad (10)$$

Take an arbitrary element $\xi \in B_{M,m} \subseteq X_M$ and define

$$\xi \otimes \varphi \in W(U_{M,m}, U_\mu) \subseteq L(X_M', E_\mu'), \qquad (\xi \otimes \varphi)(u) := \langle u, \xi\rangle \varphi \quad \text{for } u \in X_M'.$$

By (9),

$$I_{N,\nu}^{M,\mu}(\xi \otimes \varphi) = SI_{N,\nu}^{K,\kappa} P + SQ, \qquad (11)$$

where $P \in W(U_{K,k}, U_\kappa), Q \in W(U_{N,n}, U_\nu)$. For y chosen before we have

$$I_{N,\nu}^{M,\mu}(\xi \otimes \varphi)(y) = \left[(j_\mu^\nu)' \circ (\xi \otimes \varphi) \circ (i_N^M)'\right](y) =$$
$$= (j_\mu^\nu)'((\xi \otimes \varphi)(y \circ i_N^M)) = y(i_N^M \xi)(\varphi \circ j_\mu^\nu)$$
$$SI_{N,\nu}^{K,\kappa} P(y) = S\left[(j_\kappa^\nu)' \circ P \circ (i_N^K)'\right](y) = SP(y \circ i_N^K) \circ j_\kappa^\nu.$$

Evaluating both sides of (11) at fixed $y \in X_N'$ and applying it to fixed $x \in E_\nu$ we obtain

$$y(i_N^M \xi)\varphi(j_\mu^\nu x) = SP(y \circ i_N^K)(j_\kappa^\nu x) + SQ(y)(x).$$

Since $P \in W(U_{K,k}, U_\kappa)$ and $Q \in W(U_{N,n}, U_\nu)$, by (10), we have:

$$(1/2)\|j_\mu^\nu x\|_\mu |y(i_N^M \xi)| \leq S(|P(y \circ i_N^K)(j_\kappa^\nu x)| + |Q(y)(x)|) \leq$$
$$\leq S(\|P(y \circ i_N^K)\|_\kappa^* \|j_\kappa^\nu x\|_\kappa + \|Q(y)\|_\nu^* \|x\|_\nu) \leq$$
$$\leq S(\|y \circ i_N^K\|_{K,k}^* \|j_\kappa^\nu x\|_\kappa + \|y\|_{N,n}^* \|x\|_\nu).$$

Taking supremum over all $\xi \in B_{M,m}$ we get the conclusion for $2S$ instead of S.

$(2)\Rightarrow(3)$: Since E is a reflexive LS-space and E' is quasinormable, we get from [26, Th. 7],

$$\forall\, \tilde{\nu}\ \exists\, \nu \geq \tilde{\nu}\ \forall\, \kappa, \rho > 0\ \exists\, D(\rho)\ \forall\, x \in E \quad \|x\|_\nu \leq \rho\|x\|_{\tilde{\nu}} + D(\rho)\|x\|_\kappa. \quad (12)$$

Moreover, since $\text{Proj}^1 X = 0$ and X is a PLS-space, we can apply [46, 3.2.18]

to get

$$\forall\, N\ \exists\, \tilde{M} \geq N\ \forall\, K\ \exists\, \tilde{n}\ \forall\, m \geq \tilde{n}, \gamma > 0\ \exists\, \tilde{k}, C\ \forall\, y \in X'_N$$

$$\|y \circ i_N^{\tilde{M}}\|^*_{\tilde{M},m} \leq C\|y \circ i_N^K\|^*_{K,\tilde{k}} + \gamma\|y\|^*_{N,\tilde{n}}. \tag{13}$$

Then, by (G) we get:

$$\forall\, \tilde{M}, \nu\ \exists\, M \geq \tilde{M}, \mu \geq \nu\ \forall\, K \geq M, \kappa \geq \mu\ \exists\, n\ \forall\, m \geq n\ \exists\, k \geq m, S$$

$$\forall\, y \in X'_{\tilde{M}}\ \forall\, x \in E_\nu : \tag{14}$$

$$\|y \circ i_{\tilde{M}}^M\|^*_{M,m}\|j_\mu^\nu x\|_\mu \leq S\left(\|y\|^*_{\tilde{M},n}\|x\|_\nu + \|y \circ i_{\tilde{M}}^K\|^*_{K,k}\|j_\kappa^\nu x\|_\kappa\right).$$

We choose quantifiers as follows. For every $\tilde{\nu}$ we find $\nu \geq \tilde{\nu}$ according to (12). Then for arbitrary N we find $\tilde{M} \geq N$ from (13), we apply (14) and find $M \geq \tilde{M}, \mu \geq \nu$. We take arbitrary K, κ, then we find n according to (14) and $\tilde{n} \geq n$ according to (13). We take arbitrary $m \geq \tilde{n}$ and find k, S according to (14). Then we choose $\varepsilon > 0$ arbitrary and γ so small that $S\gamma \leq \varepsilon/2$. Using (13) we find $\tilde{k} \geq k$ and C. Finally, we choose ρ so small that $SC\rho \leq \varepsilon/2$ and $S\rho \leq \varepsilon$. Now, we prove (G_ε). For a given $y \in X'_N$ we consider two cases:

(1) $\|y \circ i_N^{\tilde{M}}\|_{\tilde{M},n} \leq \|y \circ i_N^K\|_{K,\tilde{k}}$; (2) otherwise.

Case (1). By (14) applied to $y \circ i_N^{\tilde{M}} \in X'_{\tilde{M}}$, $x \in E_\nu$, using (12) we get:

$$\|y \circ i_N^M\|^*_{M,m}\|j_\mu^\nu x\|_\mu$$
$$\leq S\left(\|y \circ i_N^{\tilde{M}}\|^*_{\tilde{M},n}\|x\|_\nu + \|y \circ i_N^K\|^*_{K,k}\|j_\kappa^\nu x\|_\kappa\right)$$
$$\leq S\|y \circ i_N^{\tilde{M}}\|^*_{\tilde{M},n}(\rho\|x\|_{\tilde{\nu}} + D(\rho)\|j_\kappa^\nu x\|_\kappa) + S\|y \circ i_N^K\|^*_{K,k}\|j_\kappa^\nu x\|_\kappa$$
$$\leq \varepsilon\|y\|^*_{N,n}\|x\|_{\tilde{\nu}} + S(1 + D(\rho))\|y \circ i_N^K\|^*_{K,\tilde{k}}\|j_\kappa^\nu x\|_\kappa.$$

Case (2). Again, by (14), using first (13) and then (12), we obtain:

$$\|y \circ i_N^M\|^*_{M,m}\|j_\mu^\nu x\|_\mu$$
$$\leq S(\|y \circ i_N^{\tilde{M}}\|^*_{\tilde{M},n}\|x\|_\nu + \|y \circ i_N^K\|^*_{K,k}\|j_\kappa^\nu x\|_\kappa)$$
$$\leq S\gamma\|y\|^*_{N,\tilde{n}}\|x\|_\nu + SC\|y \circ i_N^K\|^*_{K,\tilde{k}}\|x\|_\nu + S\|y \circ i_N^K\|^*_{K,k}\|j_\kappa^\nu x\|_\kappa$$
$$\leq S\gamma\|y\|^*_{N,\tilde{n}}\|x\|_\nu + SC\rho\|y \circ i_N^K\|^*_{K,\tilde{k}}\|x\|_{\tilde{\nu}} + SCD(\rho)\|y \circ i_N^K\|^*_{K,\tilde{k}}\|j_\kappa^\nu x\|_\kappa$$
$$\quad + S\|y \circ i_N^K\|^*_{K,k}\|j_\kappa^\nu x\|_\kappa.$$

Since $\nu \geq \tilde{\nu}$, $\tilde{M} \geq N$, $\tilde{n} \geq n$, we have $\|x\|_\nu \leq \|x\|_{\tilde{\nu}}$, $\|y \circ i_N^{\tilde{M}}\|^*_{\tilde{M},n} \leq \|y\|^*_{N,n} \leq \|y\|^*_{N,\tilde{n}}$ and

$$\|y \circ i_N^M\|^*_{M,m}\|j_\mu^\nu x\|_\mu \leq \varepsilon\|y\|^*_{N,\tilde{n}}\|x\|_{\tilde{\nu}} + (SCD(\rho) + S)\|y \circ i_N^K\|_{K,\tilde{k}}\|j_\kappa^\nu x\|_\kappa.$$

$(3) \Rightarrow (1)$: By Theorem 3.4, it suffices to show that $\operatorname{Proj}^1 L(X'_N, E'_N) = 0$. By [46, 3.2.14], it suffices to show that

$$\forall\, N\ \exists\, M \geq N\ \forall\, K \geq M\ \exists\, n\ \forall\, m \geq n, \varepsilon > 0\ \exists\, k \geq m, S$$
$$I_N^M W(U_{M,m}, U_M) \subseteq S(I_N^K W(U_{K,k}, U_K)) + \varepsilon W(U_{N,n}, U_N), \tag{15}$$

where $I_N^M f := (j_N^M)' \circ f \circ (i_N^M)'$. We will show it separately for the assumptions (b), (c) and (d).

Case (b): $X = \Lambda^\infty(A)$ a Köthe type PLS-space. We assume first that $a_{1,n}(i) > 0$ for each n.

Let e_i be the unit vector in X', then $\|e_i\|_{N,n}^* = 1/a_{N,n}(i)$. Thus, by (G_ε), for $N = \nu$, $K = \kappa$ and M, μ chosen as the maximum of those two and denoted by M and for $x \in E_N$, $y = e_i$:

$$\forall\, N\ \exists\, M \geq N\ \forall\, K \geq M\ \exists\, n\ \forall\, m \geq n, \varepsilon > 0\ \exists\, k \geq m, S\ \forall\, i \in \mathbb{N}\ \forall\, x \in E_N$$
$$\frac{\|j_M^N x\|_M}{a_{M,m}(i)} \leq \varepsilon \frac{\|x\|_N}{a_{N,n}(i)} + S \frac{\|j_K^N x\|_K}{a_{K,k}(i)}.$$

By Lemma 3.5,

$$\frac{1}{a_{M,m}(i)}(j_M^N)'(B_M^\circ) \subseteq \frac{3\varepsilon}{a_{N,n}(i)} B_N^\circ + \frac{2S}{a_{K,k}(i)}(j_K^N)'(B_K^\circ). \tag{16}$$

Now, we identify $W(U_{M,m}, U_M) \subseteq L(X'_M, E'_M)$ a space of vector valued sequences:

$$L(X'_M, E'_M) = \{u = (u(i))_{i \in \mathbb{N}} \subseteq E'_M :\ \exists\, m\ \sup_i a_{M,m}(i)\|u(i)\|_M^* < \infty\}.$$

In particular, $u = (u(i))_{i \in \mathbb{N}} \in W(U_{M,m}, U_M)$ if and only if $u(i) \in (a_{M,m}(i))^{-1} U_M$ for every i. By (16), taking some $v(i) \in (a_{N,n}(i))^{-1} U_N$ and $w(i) \in (a_{K,k}(i))^{-1} U_K$ we have

$$(j_M^N)'(u(i)) = 3\varepsilon v(i) + 2S(j_K^N)'(w(i)) \quad \text{for each } i \in \mathbb{N}.$$

Define $v \in W(U_{N,n}, U_N) \subseteq L(X'_N, E'_N)$ and $w \in W(U_{K,k}, U_K) \subseteq L(X'_K, E'_K)$, by

$$v(x) := (v(i)x)_{i \in \mathbb{N}}, \quad w(z) := (w(i)z)_{i \in \mathbb{N}} \quad \text{for } x \in X'_N, z \in X'_K.$$

Obviously, $I_N^M u = 3\varepsilon v + 2S I_N^K w$ which implies (15) with slightly changed S and ε.

In the general case, $X = \Lambda^\infty(A)$ is a countable product of spaces for which we have proved $\operatorname{Ext}^1_{PLS}(E, X_S) = 0$. This implies (1).

Case (c): E is an LN-space, i.e., a nuclear LS-space.

We assume that E_ν is Hilbert and $j_{\nu+1}^\nu : E_\nu \to E_{\nu+1}$ is nuclear for every $\nu \in \mathbb{N}$. By Lemma 3.5 and (G_ε) applied for $\nu = N + 2$, $\kappa = K + 2 > \nu$ and $M = \mu$ we get:

$$\forall\, N \,\exists\, M \geq N \,\forall\, K \geq M \,\exists\, n \,\forall\, m \geq n, \varepsilon > 0 \,\exists\, k \geq m, S \,\forall\, x \in E_{N+2}$$
$$\|j_M^{N+2}x\|_M i_N^M B_{M,m} \subseteq \varepsilon\|x\|_{N+2} B_{N,n} + S\|j_{K+2}^{N+2}x\|_{K+2} i_N^K B_{K,k}. \tag{17}$$

Choose orthonormal systems $(e_i)_{i\in\mathbb{N}} \subseteq E_{N+1}$ and $(f_i)_{i\in\mathbb{N}} \subseteq E_{K+1}$ such that

$$j_{K+1}^{N+1}x = \sum_i a_i \langle x, e_i \rangle_{N+1} f_i \quad \forall\, x \in E_{N+1}.$$

Fix $\varphi \in W(U_{M,m}, U_M) \subseteq L(X_M', E_M')$. For arbitrary $u \in U_{M,m}$, $i \in \mathbb{N}$ we have

$$|e_i \circ (j_M^{N+1})' \circ \varphi(u)| = |\varphi(u)(j_M^{N+1}(e_i))| \leq \|j_M^{N+1}(e_i)\|_M.$$

We have proved that $i_N^M(e_i \circ (j_M^{N+1})' \circ \varphi) \in \|j_M^{N+2}(j_{N+2}^{N+1}e_i)\|_M i_N^M B_{M,m}$. By (17),

$$i_N^M(e_i \circ (j_m^{N+1})' \circ \varphi) = \chi_i + i_n^K \psi_i, \tag{18}$$

where

$$\chi_i \in \varepsilon\|j_{N+2}^{N+1}e_i\|_{N+2} B_{N,n}, \quad \psi_i \in S\|j_{K+2}^{N+2}j_{N+2}^{N+1}e_i\|_{K+2} B_{K,k} = S\|j_{K+2}^{N+1}e_i\|_{K+2} B_{K,k}.$$

We define two maps: first,

$$\chi(u) := \sum_i \chi_i(u)(j_{N+1}^N)'(e_i^*)$$

for $u \in X_N'$ where $e_i^*(x) := \langle x, e_i \rangle_{N+1}$, $x \in E_{N+1}$, second,

$$\psi(v) := \sum_i a_i^{-1} \psi_i(v)(j_{K+1}^K)'(f_i^*),$$

where the sum runs over all i such that $a_i \neq 0$, $v \in X_K'$ and $f_i^*(x) := \langle x, f_i \rangle_{K+1}$ for $x \in E_{K+1}'$.

We will show that χ is a well-defined element of a multiple of $W(U_{N,n}, U_N)$. Fix $x \in B_N$ and $u \in U_{N,n}$. Then, by Schwartz inequality,

$$|\chi(u)(x)| \leq \sum_i |\chi_i(u)||\langle j_{N+1}^N x, e_i \rangle_{N+1}| \leq \varepsilon \sum_i \|j_{N+2}^{N+1}e_i\|_{N+2}|\langle j_{N+1}^N x, e_i \rangle_{N+1}|$$
$$\leq \varepsilon\sigma(j_{N+2}^{N+1})\|j_{N+1}^N x\|_{N+1} \leq \varepsilon\sigma(j_{N+2}^{N+1}),$$

where σ denotes the Hilbert-Schmidt norm of operators. The above estimates imply that the series in the definition of χ is convergent and

$$\chi \in \varepsilon\sigma(j_{N+2}^{N+1})W(U_{N,n}, U_N). \tag{19}$$

Fix $v \in U_{K,k}$ and $z \in B_K$. Similarly as above we get

$$|\psi(v)(z)| \leq \sum_i (a_i)^{-1} |\psi_i(v)| |\langle j_{K+1}^K z, f_i \rangle_{K+1}| \leq$$

$$\leq S \sum_i (a_i)^{-1} \|j_{K+2}^{N+1} e_i\|_{K+2} |\langle j_{K+1}^K z, f_i \rangle_{K+1}| \leq$$

$$\leq S \sum_i \|j_{K+2}^{K+1} f_i\|_{K+2} |\langle j_{K+1}^K z, f_i \rangle_{K+1}| \leq S\sigma(j_{K+2}^{K+1}).$$

This implies

$$\psi \in S\sigma(j_{K+2}^{K+1}) W(U_{K,k}, U_K). \tag{20}$$

By (19) and (20), in order to prove (15) it suffices to show

$$I_N^M \varphi = \chi + I_N^K \psi.$$

This follows from an easy consequence of (18):

$$(I_N^M \varphi)(u)(x) = \chi(u)(x) + (I_N^K \psi)(u)(x) \quad \text{for every } u \in X_N' \text{ and } x \in E_N.$$

Case (d): $E = k_1(v)$ is a Köthe coechelon space.

Let us recall that $\|x\|_\nu := \sum_i v_\nu(i)|x_i|$ and that $v_\nu(i) > 0$ for each $\nu, i \in \mathbb{N}$. Evaluating (G_ε) for $x = e_i \in E_N$, where $N = \nu$, $K = \kappa$ and $M = \mu$ we obtain:

$$\forall\, N \,\exists\, M \geq N \,\forall\, K \geq M \,\exists\, n \,\forall\, m \geq n, \varepsilon > 0 \,\exists\, k, S \,\forall\, y \in X_N' \,\forall\, i \in \mathbb{N}:$$
$$\|y \circ i_N^M\|_{M,m}^* v_M(i) \leq \varepsilon \|y\|_{N,n}^* v_N(i) + S\|y \circ i_N^K\|_{K,k}^* v_K(i).$$

By Lemma 3.5 changing ε and S suitably we get:

$$v_M(i) i_N^M B_{M,m} \subseteq \varepsilon v_N(i) B_{N,n} + S v_K(i) i_N^K B_{K,k}. \tag{21}$$

Let $f \in W(U_{M,m}, U_M)$. Observe $E_M' = l_\infty(1/v_M)$, then $f(z) = (f_i(z))_{i \in \mathbb{N}} \in U_M$ for every $z \in U_{M,m}$ and $|f_i(z)| \leq v_M(i)$. Therefore $f_i \in v_M(i) B_{M,m}$ for every $i \in \mathbb{N}$. By (21), we get

$$i_N^M f_i = \varepsilon v_N(i) g_i + S v_K(i) i_N^K h_i \quad \text{for every } i \in \mathbb{N},$$

where $g_i \in B_{N,n}$ and $h_i \in B_{K,k}$. We define

$$g : X_N' \to E_N' = l_\infty(1/v_N), \quad g(y) := (v_N(i) g_i(y))_{i \in \mathbb{N}},$$
$$h : X_K' \to E_K' = l_\infty(1/v_K), \quad h(y) := (v_K(i) h_i(y))_{i \in \mathbb{N}}.$$

Finally, it is easy to check that $g \in W(U_{N,n}, U_N)$, $h \in W(U_{K,k}, U_K)$ and $I_N^M f = \varepsilon g + S I_N^K h$. This completes the proof by (15). $\square$

4 Splitting results for special spaces

In the present section we obtain a more natural splitting result and apply it to sequence spaces. We define the condition $(P\underline{A})$ for a PLS-space X as follows:

$$\forall \ N \ \exists \ M \ \forall \ K \ \exists \ n \ \forall \ m \ \exists \ \theta \in]0,1[\ \exists \ k, C \ \forall \ y \in X'_N;$$

$$\|y \circ i_N^M\|_{M,m}^* \le C \max \left(\|y \circ i_N^K\|_{K,k}^* {}^{(1-\theta)}, \|y\|_{N,n}^* {}^{(1-\theta)} \right) \|y\|_{N,n}^* {}^\theta \qquad (22)$$

or, equivalently,

$$\forall \ N \ \exists \ M \ \forall \ K \ \exists \ n \ \forall \ m \ \exists \ \eta > 0 \ \exists \ k, C, r_0 > 0 \ \forall \ r < r_0 \ \forall \ y \in X'_N;$$

$$\|y \circ i_N^M\|_{M,m}^* \le C \left(r^\eta \|y \circ i_N^K\|_{K,k}^* + \frac{1}{r}\|y\|_{N,n}^* \right). \qquad (23)$$

Changing the quantifier in (22) for θ to be $\forall \theta \in]0,1[$ one gets the condition (PA). As above it is equivalent to

$$\forall \ N \ \exists \ M \ \forall \ K \ \exists \ n \ \forall \ m \ \forall \ \eta > 0 \ \exists \ k, C, r_0 > 0 \ \forall \ r < r_0 \ \forall \ y \in X'_N;$$

$$\|y \circ i_N^M\|_{M,m}^* \le C \left(r^\eta \|y \circ i_N^K\|_{K,k}^* + \frac{1}{r}\|y\|_{N,n}^* \right). \qquad (24)$$

The equivalence of the two forms of each condition can be proved identically as in [4, Lemma 5.1] for $(P\Omega)$ and $(P\overline{\overline{\Omega}})$. These conditions are PLS-versions of conditions $(\underline{A})$ and (A) (see [37] and [6]) which are dual to $(\underline{DN})$ and (DN) respectively [27, Sec. 29]. It is worth noting that $(P\underline{A})$ and $(P\Omega)$ (introduced in [4]) differ only by inequality $r < r_0$ and $r > r_0$, respectively. The same analogy holds between (PA) and $(P\overline{\overline{\Omega}})$.

We present now an analogue of the famous $(DN) - (\Omega)$ splitting theorem [27, 30.1].

Theorem 4.1 *Let E be an LS-space and let X be a PLS-space satisfying (b), (c) or (d), then*
$\mathrm{Ext}^1_{PLS}(E, X) = 0$ *whenever E' has $(\overline{\overline{\Omega}})$ and X has $(P\underline{A})$ or E' has (Ω) and X has (PA) .*

PROOF. By Theorem 3.1, it suffices to show that the pair (E, X) satisfies (G). Recall that $(\overline{\overline{\Omega}})$ for E' means

$$\forall \ N \ \exists \ M \ge N \ \forall \ K \ge M, \theta \in]0,1[\ \exists \ D \ \forall \ x \in E \quad \|x\|_M \le D\|x\|_N^\theta \|x\|_K^{1-\theta}.$$

Fix N and find M which is good for $(P\underline{A})$ and $(\overline{\overline{\Omega}})$. Then fix K, find n from $(P\underline{A})$ and fix m. Finally, find k and η from $(P\underline{A})$. We choose $\theta := \frac{\eta}{\eta+1}$ in $(\overline{\overline{\Omega}})$

. Take $x \in E_N$ and $r := \frac{\|x\|_M}{\|x\|_N}$. By $(\overline{\overline{\Omega}})$,

$$r^\eta = \left(\frac{\|x\|_M}{\|x\|_N} \right)^\eta \leq D \frac{\|x\|_K}{\|x\|_M}.$$

We substitute r into (23) to get

$$\|y \circ i_N^M\|_{M,m}^* \leq C \left(D \frac{\|x\|_K}{\|x\|_M} \|y \circ i_N^K\|_{K,k}^* + \frac{\|x\|_N}{\|x\|_M} \|y\|_{N,n}^* \right).$$

This completes the proof. The case $E' \in (\Omega)$ and $X \in (PA)$ is analogous. $\square$

In order to apply the above result we need examples of spaces satisfying conditions $(P\underline{A})$ and (PA) . The following proposition summarize elementary facts concerning $(P\underline{A})$ and (PA) .

Proposition 4.2 *Every Fréchet Schwartz space has (PA) and (P$\underline{A}$). An LS-space has (PA) or (P$\underline{A}$) if and only if it has (A) or ($\underline{A}$) respectively. The classes of spaces with (PA) and (P$\underline{A}$) are closed with respect of complete quotients and countable products. The condition (PA) implies (P$\underline{A}$) and the latter implies* $\mathrm{Proj}^1\, X = 0$.

The proof is similar to the proof of [4, Cor. 5.2, Prop. 5.3 and Prop. 5.4] so we omit it. Observe that duals of power series spaces have always $(\underline{A})$ and they have (A) only for infinite type spaces [27, Sec. 29]. Thus products of such spaces have correspondingly $(P\underline{A})$ and (PA) .

Now, we show which sequence spaces have (PA) or $(P\underline{A})$.

Theorem 4.3 *(a) The Köthe type PLS-space $\Lambda^p(A)$ for $1 \leq p \leq \infty$ has (PA) if and only if*

$$\forall\, N\ \exists\, M\ \forall\, K\ \exists\, n\ \forall\, m, \theta \in]0,1[\ \exists\, k, C\ \forall\, i \in \mathbb{N}$$
$$a_{M,m}(i) \geq C \min \left(a_{K,k}(i)^{(1-\theta)}, a_{N,n}(i)^{(1-\theta)} \right) a_{N,n}(i)^\theta$$

The same condition holds for (P$\underline{A}$) with a suitable change of quantifiers.
(b) The PLS-type power series space $\Lambda_{r,s}(\alpha, \beta)$ satisfies condition (P$\underline{A}$) if and only if either $s = \infty$ or the space is isomorphic to a product of an LS-space and a Fréchet space (this is equivalent to $\mathrm{Proj}^1\, \Lambda_{r,s}(\alpha,\beta) = 0$).
(c) The PLS-type power series space $\Lambda_{r,s}(\alpha, \beta)$ satisfies condition (PA) if and only if either $s = \infty$ or the space is isomorphic to a Fréchet space.

It is worth noting that both non-quasianalytic Roumieu classes $\mathscr{E}_{\{\omega\}}$ and spaces of Beurling (ultra-)distributions $\mathscr{D}'_{(\omega)}$ are isomorphic to Köthe type PLS-spaces

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[40], [36], see [7] for the definitions, (the first has $(P\underline{A})$ the second (PA)). The role of these new invariants and applications of our splitting result for spaces of real analytic functions and Roumieu quasianalytic classes of ultradifferentiable functions is explained in [5]. The kernels of surjective convolution operators on $\mathscr{D}'_{(\omega)}(\mathbb{R})$, $\mathscr{E}_{\{\omega\}}(\mathbb{R})$ or $\mathscr{E}_{\{\omega\}}(]-1,1[)$ give examples of PLS-type power series spaces (see [14, Th. 2.10], [29, 2.11], [28, Satz 3.2, 3.18], [25], comp. [4, Th. 2.2]), in the first case they have (PA) in the other two $(P\underline{A})$.

Proof of 4.3: (a): Necessity follows by taking y as unit vectors. For the proof of sufficiency, translate the condition as in the definition of (PA) into the condition with the parameter r:

$$\forall\ N\ \exists\ M\ \forall\ K\ \exists\ n\ \forall\ m,\eta > 0\ \exists\ k, C, r_0 > 0\ \forall\ r < r_0\ \forall\ i \in \mathbb{N}:$$
$$\text{such that } a_{N,l}(i) \neq 0 \text{ for all } l$$
$$\frac{1}{a_{M,m}(i)} \leq C \max\left(r^\eta \frac{1}{a_{K,k}(i)}, \frac{1}{r}\frac{1}{a_{N,n}(i)}\right).$$

Then prove that this condition holds for all vectors in X'_N instead of the unit vectors only.

(b): By Proposition 4.2, $(P\underline{A})$ implies $\mathrm{Proj}^1 = 0$, apply [42, 4.3]. Sufficiency for $s < \infty$ follows from Prop. 4.2, since the LS-space factor must be a dual to a Fréchet power series space and it has $(\underline{A})$ (see [27, Sec. 29]). Sufficiency for $s = \infty$ follows from (c) below.

(c): Necessity for $s < \infty$ follows from (b) above and the observation that the LS-factor is $\Lambda'_0(\gamma)$. If this factor is non-trivial then it does not satisfy (A) (see [27, Sec.29]). Sufficiency for $s < \infty$ follows from Prop. 4.2.

Assume that $s = \infty$. For an arbitrary N, choose $M := N+1$ and take arbitrary K. Fix $n = 1$, take arbitrary m and $\theta \in]0,1[$. We choose k so big that

$$\theta \leq \frac{s_k - s_m}{s_k - s_n} \qquad \text{and} \qquad \frac{r_K - r_N}{r_M - r_N} < \frac{s_k - s_n}{s_m - s_n}.$$

Let us observe that if $\frac{r_K - r_M}{r_K - r_N} \leq \theta$ then

$$\exp(-r_M\alpha_i + s_m\beta_i) \leq \exp((-r_K\alpha_i + s_k\beta_i)(1-\theta))\cdot\exp((-r_N\alpha_i + s_n\beta_i)\theta) \quad (25)$$

and

$$\|e_i\|^*_{M,m} \leq (\|e_i\|^*_{N,n})^\theta(\|e_i\|^*_{K,k})^{1-\theta}.$$

Now, assume that

$$\|e_i\|^*_{M,m} \geq \|e_i\|^*_{N,n}$$

then

$$-r_M\alpha_i + s_m\beta_i \geq -r_N\alpha_i + s_n\beta_i \quad \text{and} \quad \alpha_i \leq \frac{s_m - s_n}{r_M - r_N}\beta_i < \left(\frac{s_k - s_n}{r_K - r_N}\right)\beta_i.$$

Observe that the function

$$f(\theta) := -r_K\alpha_i(1-\theta) + s_k\beta_i(1-\theta) - r_N\alpha_i\theta + s_n\beta_i\theta$$

has negative derivative

$$f'(\theta) = (r_K - r_N)\alpha_i + (s_n - s_k)\beta_i < \left(\frac{s_k - s_n}{r_K - r_N}\right)(r_K - r_N)\beta_i + (s_n - s_k)\beta_i = 0.$$

Therefore, if the inequality (25) holds for big $\theta < 1$ then it holds for all $\theta \in]0, 1[$ and either

$$\|e_i\|_{M,m}^* \le \|e_i\|_{N,n}^* \quad \text{or} \quad \|e_i\|_{M,m}^* \le (\|e_i\|_{N,n}^*)^\theta(\|e_i\|_{K,k}^*)^{1-\theta}.$$

We conclude by the same method as in (a). $\square$

Sometimes (PA) is also a necessary splitting condition.

Theorem 4.4 *If α is stable and X is an ultrabornological PLS-space, then $\mathrm{Ext}^1_{PLS}((\Lambda_r^\infty(\alpha))', X) = 0$ if and only if X has (PA) .*

Remark. Clearly the same holds for $\prod_{n\in\mathbb{N}}\Lambda_r(\alpha^{(n)})$, for instance, $C^\infty(U) \simeq \prod_{n\in\mathbb{N}}\Lambda_\infty(\log j)$ for any smooth non-compact manifold U.

PROOF. Sufficiency follows from Theorem 4.1 since $\Lambda_r(\alpha)$ has (Ω) .

Necessity. We may assume that $\alpha_0 = 0$ and that $\alpha_j \le d\alpha_{j-1}$ for some $d > 1$ and every $j \in \mathbb{N}$. We apply (G) for $x = e_j$. We fix N and find $M \ge N$ from (G), then we fix K. We choose η_0 such that $\frac{r_K-r_M}{r_M-r_N} \ge \eta_0 d$. There is n such that for every m there is $k(m)$ such that

$$\|y\circ i_N^M\|_{M,m}^* \le S\left(\exp\left((r_M - r_K)\alpha_j\right)\|y \circ i_N^K\|_{K,k(m)}^* + \exp\left((r_M - r_N)\alpha_j\right)\|y\|_{N,n}^*\right).$$

Take $r \le \exp\left((r_N - r_M)\alpha_0\right) = 1$. There is j such that

$$(r_N - r_M)\alpha_j \le \log r \le (r_N - r_M)\alpha_{j-1}.$$

Now, $\exp\left((r_M - r_N)\alpha_j\right) \le \exp\left(d(r_M - r_N)\alpha_{j-1}\right) \le \frac{1}{r^d}$. Clearly, for $\eta < \eta_0$ we have
$$\exp\left((r_M - r_K)\alpha_j\right) \le \exp\left((r_N - r_M)\frac{r_M - r_K}{r_N - r_M}\alpha_j\right) \le r^{\eta d}.$$
We have proved

$$\forall\ N\ \exists\ M \ge N\ \forall\ K \ge M\ \exists\ n\ \exists\ \eta_0\ \forall\ m\ \exists\ k(m), S\ \forall\ \eta < \eta_0\ \forall\ r \in]0, 1[:$$
$$\|y \circ i_N^M\|_{M,m}^* \le S_m\left(r^\eta\|y \circ i_N^K\|_{K,k(m)}^* + \frac{1}{r}\|y\|_{N,n}^*\right).$$

Then

$$\|y \circ i_N^K\|_{K,k(m)}^* \leq \|y \circ i_N^M\|_{M,k(m)}^* \leq S_{k(m)} \left(r^\eta \|y \circ i_K^N\|_{K,k(k(m))}^* + \frac{1}{r}\|y\|_{N,n}^* \right).$$

Combining the two inequalities above we get

$$\|y \circ i_N^M\|_{M,m}^* \leq S_m(S_{k(m)} + 1) \left(r^{2\eta}\|y \circ i_N^K\|_{K,k(k(m))}^* + \frac{1}{r}\|y\|_{N,n}^* \right),$$

since $r^{\eta-1} < 1/r$. Repeating this procedure inductively we get

$$\|y \circ i_N^M\|_{M,m}^* \leq S_{p,m} \left(r^{p\eta}\|y \circ i_N^K\|_{K,\tilde{k}(p)}^* + \frac{1}{r}\|y\|_{N,n}^* \right),$$

where $\tilde{k}(p) = k \circ k \circ \cdots \circ k(m)$, p-times composition, $p \in \mathbb{N}$. This completes the proof. $\square$

Kunkle [22, Th.5. 14] proved that $\mathrm{Ext}^1_{PLS}(\Lambda^1_{\infty,s}(\alpha,\beta), \Lambda^\infty_{p,\infty}(\gamma,\delta)) = 0$ for any p and s. From our theory we conclude:

Corollary 4.5 *If either $s = \infty$ or $\Lambda_{r,s}(\beta,\gamma)$ is a Fréchet space then*

$$\mathrm{Ext}^1_{PLS}((\Lambda_r^\infty(\alpha))', \Lambda_{r,s}(\beta,\gamma)) = 0.$$

PROOF. The case $s = \infty$ follows from Theorem 4.3 (d), Theorem 4.1 and the propert (Ω) of $\Lambda_r(\alpha)$. The other case follows from [31, Th. 9.1]. $\square$

5   Parameter dependence of solutions of differential equations

As explained in the introduction, the parameter dependence problem for linear partial differential operators with constant coefficients is equivalent to the question if the partial differential operator

$$P(D) : \mathscr{D}'(\Omega, F) \to \mathscr{D}'(\Omega, F) \tag{26}$$

on the space of vector valued distributions is surjective for suitably chosen Fréchet spaces F. We prove that this is the case for Ω convex and any nuclear Fréchet space F with property (Ω) , for instance when F is isomorphic to one of the spaces $H(U), C^\infty(U), \Lambda_r(\alpha), C^\infty[0,1]$, etc., see [27, 29.11]. Our approach should be compared with [4, Section 3].

The positive solution for the holomorphic dependence was probably known to some specialists; Palamodov showed the authors the full proof without using

splitting of short exact sequences. For the sake of completeness we give a full proof based on Palamodov's theory of systems of linear partial differential equations and $(DN) - (\Omega)$ splitting theorem of Vogt and Wagner (see [27, 30.1]).

Theorem 5.1 *Let $\Omega \subseteq \mathbb{R}^d$ be a convex open set and let U be a Stein manifold. For every linear partial differential operator with constant coefficients $P(D)$ the following map is surjective*

$$P(D) : \mathscr{D}'(\Omega, H(U)) \to \mathscr{D}'(\Omega, H(U)).$$

PROOF. First, we assume that U is a convex open subset of $\mathbb{C}^d$. For the sake of notational simplicity we take $d = 1$. We have the following differential complex obtained from the free resolution of the corresponding $\mathscr{P}$-module:

$$0 \longrightarrow \ker \begin{pmatrix} \bar{\partial} \\ P(D) \end{pmatrix} \longrightarrow \mathscr{D}'(\Omega \times U) \xrightarrow{\begin{pmatrix} -\bar{\partial} \\ P(D) \end{pmatrix}} [\mathscr{D}'(\Omega \times U)]^2 \longrightarrow$$

$$\xrightarrow{(P(D), \bar{\partial})} \mathscr{D}'(\Omega \times U) \longrightarrow 0 \qquad ,$$

where $P(D)$ acts on first d-variables and $\bar{\partial}$ acts on the last two real variables in $U \subseteq \mathbb{C} = \mathbb{R}^2$. This complex is a particular case of the example given in [30, VII, 7.2, Ex. 4]. Since $\Omega \times U$ is convex the complex is exact by [30, VII, 8.1, Th. 1].

If $f \in \mathscr{D}'(\Omega \times U)$, $\bar{\partial} f = 0$, then the pair $\begin{pmatrix} 0 \\ f \end{pmatrix} \in [\mathscr{D}'(\Omega \times U)]^2$ belongs to the kernel of $(P(D), \bar{\partial})$, thus by exactness of the complex there is $g \in \mathscr{D}'(\Omega \times U)$ such that $-\bar{\partial} g = 0$, $P(D)g = f$. We have proved that $P(D) : \ker \bar{\partial} \to \ker \bar{\partial}$ is surjective.

By the very definition $\mathscr{D}(\Omega, H(U)) = L(\mathscr{D}(\Omega), H(U))$. Let us prove that

$$\ker \bar{\partial} = \{f \in \mathscr{D}'(\Omega \times U) : \bar{\partial} f = 0\} = L(\mathscr{D}(\Omega), H(U)).$$

Define a map $S_f : \mathscr{D}(\Omega) \to \mathscr{D}'(U)$ as follows:

$$\langle S_f(\varphi), \psi \rangle = \langle f, \varphi \psi \rangle$$

for $\psi \in \mathscr{D}(U)$. Since $\langle \bar{\partial} S_f(\varphi), \psi \rangle = -\langle f, \varphi \bar{\partial} \psi \rangle = -\langle f, \bar{\partial} \psi \varphi \rangle = 0$, we have $S_f(\mathscr{D}(\Omega)) \subseteq H(U)$.

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On the other hand if $S : \mathscr{D}(\Omega) \to H(U)$ then we define $f_S \in \mathscr{D}'(\Omega \times U)$ as follows

$$\langle f_S, \varphi\psi \rangle := \langle S(\varphi), \psi \rangle$$

for $\varphi \in \mathscr{D}(\Omega)$, $\psi \in \mathscr{D}(U)$. Clearly $\overline{\partial} f_S = 0$ because

$$\langle \overline{\partial} f_S, \varphi\psi \rangle = -\langle f_S, \varphi\overline{\partial}\psi \rangle = -\langle S(\varphi), \overline{\partial}\psi \rangle = \langle \overline{\partial} S(\varphi), \psi \rangle = 0.$$

Let U be an arbitrary Stein manifold. By [19, 5.3.9], U embeds properly into $\mathbb{C}^d$ for suitable d as a submanifold. Clearly, we have the following short exact sequence of Fréchet spaces:

$$0 \longrightarrow I(U) \longrightarrow H(\mathbb{C}^d) \overset{q}{\longrightarrow} H(U) \longrightarrow 0,$$

where $I(U) = \{f \in H(\mathbb{C}^d) : f|_U \equiv 0\}$. By [41] remark on page 195, $I(U)$ has (Ω) . Each $f \in H(U, \mathscr{D}'(\Omega))$ can be extended to $g \in H(\mathbb{C}^d, \mathscr{D}'(\Omega))$. Indeed,

$$H(U, \mathscr{D}'(\Omega)) \simeq H(U)\varepsilon\mathscr{D}'(\Omega) \simeq L(\mathscr{D}(\Omega), H(U))$$

and extendability is equivalent to the fact that every operator $T : \mathscr{D}(\Omega) \to H(U)$ lifts with respect to q. Since $\mathscr{D}(\Omega) \simeq \bigoplus_{N\in\mathbb{N}} s$ and s has (DN) the lifting follows from the $(DN) - (\Omega)$ splitting theorem [27, 30.1].

We get the conclusion combining this fact with surjectivity of

$$P(D) : \mathscr{D}'(\Omega, H(\mathbb{C}^d)) \to \mathscr{D}'(\Omega, H(\mathbb{C}^d)) \simeq H(\mathbb{C}^d, \mathscr{D}'(\Omega)). \qquad \square$$

For the smooth dependence we cannot use the idea from the first part of the proof above, but we can use the splitting theory as the following observation shows:

Proposition 5.2 *Let F be a Fréchet-Schwartz space, let $Y = \prod_{t\in\mathbb{N}} Y_t$ be a product of LS-spaces and let $T : Y \to Y$ be a surjective operator.*

(a) If $\mathrm{Ext}^1_{PLS}(F', \ker T) = 0$, then the map $T\otimes\mathrm{id} : Y\varepsilon F \to Y\varepsilon F$ is surjective.

(b) If $\mathrm{Ext}^1_{PLS}(F', Y_t) = 0$ for every t and $T \otimes \mathrm{id} : Y\varepsilon F \to Y\varepsilon F$ is surjective, then $\mathrm{Ext}^1_{PLS}(F', \ker T) = 0$.

(c) If either $Y_t \simeq \Lambda'_\infty(\beta_t)$ and F has (Ω) or $Y_t \simeq \Lambda'_0(\beta_t)$ and F has $(\overline{\overline{\Omega}})$, then

$$T \otimes \mathrm{id} : Y\varepsilon F \to Y\varepsilon F$$

is surjective if and only if $\mathrm{Ext}^1_{PLS}(F', \ker T) = 0$.

PROOF. Use [4, Prop. 3.3, 3.4] and the fact that if F has (Ω) or F has $(\overline{\overline{\Omega}})$ then

$\mathrm{Ext}^1_{PLS}(F', \Lambda'_\infty(\beta_t)) = 0$ or $\mathrm{Ext}^1_{PLS}(F', \Lambda'_0(\beta_t)) = 0$ respectively (see [41]). $\square$

Since $\mathscr{D}'(\Omega) \simeq [\Lambda'_\infty(\beta)]^{\mathbb{N}}$ (see [36] and [40]) we have

Corollary 5.3 *Let F be Fréchet Schwartz with (Ω) and let $T : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ be surjective, then $T \otimes \mathrm{id} : \mathscr{D}'(\Omega, F) \to \mathscr{D}'(\Omega, F)$ is surjective if and only if*

$$\mathrm{Ext}^1_{PLS}(F', \ker T) = 0.$$

Remark. For non-quasianalytic weights ω (see [7] or [4]) $\mathscr{D}'_{(\omega)}(\Omega) \simeq [\Lambda'_\infty(\beta)]^{\mathbb{N}}$ for suitable β [40], therefore we have the same result also for $\mathscr{D}'_{(\omega)}$ instead of $\mathscr{D}'(\Omega)$.

The following result is crucial for the application of our splitting results from Sec. 3 and 4.

Proposition 5.4 *Let $\Omega \subseteq \mathbb{R}^d$ be a convex open set, $P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ a linear partial differential operator with constant coefficients. Then $\ker P(D)$ has the property (PA) .*

PROOF. By Theorem 5.1, $P(D) : \mathscr{D}'(\Omega, H(\mathbb{D})) \to \mathscr{D}'(\Omega, H(\mathbb{D}))$ is surjective. By Corollary 5.3,
$$\mathrm{Ext}^1_{PLS}(H'(\mathbb{D}), \ker P(D)) = 0.$$
This completes the proof by Theorem 4.4. Observe that $P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ is surjective and thus $\mathrm{Proj}^1 \ker P(D) = 0$ while $H(\mathbb{D}) \simeq \Lambda_0(\alpha)$ has (Ω) . $\square$

Theorem 5.5 *Let $\Omega \subseteq \mathbb{R}^d$ be a convex open set, $P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ a linear partial differential operator with constant coefficients, then for every Fréchet nuclear space F or Köthe sequence Fréchet-Schwartz space $F = \lambda_\infty(A)$ the map*

$$P(D) : \mathscr{D}'(\Omega, F) \to \mathscr{D}'(\Omega, F)$$

is surjective whenever F has property (Ω) . In particular, this is the case for $F = C^\infty(U)$, U an arbitrary smooth manifold.

PROOF. Apply Corollary 5.3, Proposition 5.4 and Theorem 4.1. $\square$

The property (Ω) is not a necessary condition in Theorem 5.5. This follows from the example in [39, p. 190] and the following result, which is a consequence of [2, Th. 36]. One should observe that for F satisfying LB_∞, by

[39],
$$\mathscr{D}'(\Omega, F) = L(\mathscr{D}(\Omega), F) = \bigcup L(\mathscr{D}(\Omega), F_B) = \bigcup \mathscr{D}'(\Omega, F_B),$$
where F_B are arbitrary Banach spaces continuously embedded into F. Recall that the condition LB_∞ is very restrictive, see [39].

Proposition 5.6 *Let $F = \prod_{N \in \mathbb{N}} F_N$, F_N Fréchet spaces with property LB_∞ and $T : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ is surjective then the following map is surjective as well*
$$T \otimes \mathrm{id} : \mathscr{D}'(\Omega, F) \to \mathscr{D}'(\Omega, F).$$
The same result holds for $\mathscr{D}'_{(\omega)}$ instead of $\mathscr{D}'(\Omega)$.

Theorem 5.7 *If the convolution operator $T_\mu : \mathscr{D}'_{(\omega)}(\mathbb{R}) \to \mathscr{D}'_{(\omega)}(\mathbb{R})$ is surjective, then*
$$T_\mu : \mathscr{D}'_{(\omega)}(\mathbb{R}, F) \to \mathscr{D}'_{(\omega)}(\mathbb{R}, F)$$
is surjective for any Fréchet nuclear space F with property (Ω) or any Köthe sequence Fréchet-Schwartz space $F = \lambda_\infty(A)$ with property (Ω) .

PROOF. By [14, Th. 2.10], $\ker T_\mu \simeq \Lambda_{\infty,\infty}(\alpha, \beta)$. By Theorem 4.3, $\ker T_\mu$ has (PA) . The result follows from Theorem 4.1 and Corollary 5.3.   □

Similar results hold for $T_\mu : \mathscr{E}_{\{\omega\}}(\mathbb{R}) \to \mathscr{E}_{\{\omega\}}(\mathbb{R})$ or $T_\mu : \mathscr{E}_{\{\omega\}}(]-1, 1[) \to \mathscr{E}_{\{\omega\}}(]-1, 1[)$ and F with property $(\overline{\overline{\Omega}})$ (use [29, 2.11], [28, Satz 3.2, 3.18], [25] instead of [14]). Here $\mathscr{E}_{\{\omega\}}(\Omega)$ denotes the space of ultradifferentiable functions in the sense of Roumieu [7].

It is worth noting that for hypoelliptic operators one can drop the assumption of condition (Ω) in Theorem 5.5. Indeed, hypoellipticity means that $\ker P(D)$ is a Fréchet space. By [31, Th. 9.1], $\mathrm{Ext}^1_{PLS}(F', \ker P(D)) = 0$.

References

[1] K. D. Bierstedt, An introduction to locally convex inductive limits, in: *Functional Analysis and its Applications*, H. Hogbe-Nlend (ed.), World Sci., Singapore 1988, pp. 35–133.

[2] J. Bonet, P. Domański, Real analytic curves in Fréchet spaces and their duals, *Mh. Math.* **126** (1998), 13–36.

[3] J. Bonet, P. Domański, Parameter dependence of solutions of partial differential equations in spaces of real analytic functions, *Proc. Amer. Math. Soc.* **129** (2000), 495–503.

[4] J. Bonet, P. Domański, Parameter dependence of solutions of differential equations on spaces of distributions and the splitting of short exact sequences, *J. Funct. Anal.*, **230** (2006), 329–381.

[5] J. Bonet, P. Domański, The structure of spaces of quasianalytic functions of Roumieu type, Preprint 2006.

[6] J. Bonet, P. Domański, D. Vogt, Interpolation of vector valued real analytic functions, *J. London Math. Soc.* **66** (2002), 407–420.

[7] R. W. Braun, R. Meise, B. A. Taylor, Ultradifferentiable functions and Fourier analysis, *Results Math.* **17** (1990), 207–237.

[8] S. Dierolf, Über Vererbbarkeitseigenschaften in topologischen Vektorräumen, Dissertation, Ludwig-Maximilians-Universität, München 1973.

[9] S. Dierolf, U. Schwanengel, Examples of locally compact non-compact minimal topological groups, *Pac. J. Math.* **82** (1979), 349–355.

[10] P. Domański, Classical PLS-spaces: spaces of distributions, real analytic functions and their relatives, in: *Orlicz Centenary Volume, Banach Center Publications*, **64**, *Proceedings of the Conferences: Władysław Orlicz Centenary Conference and Function Spaces VII held in Poznań, July 21–25, 2003*, Z. Ciesielski, A. Pełczyński and L. Skrzypczak (Eds.), Institute of Mathematics, Polish Academy of Sciences, Warszawa 2004, pp. 51–70.

[11] P. Domański, L. Frerick, D. Vogt, Fréchet quotients of spaces of real analytic functions, *Studia Math.* **159** (2003), 229–245.

[12] P. Domański, D. Vogt, A splitting theory for the space of distributions, *Studia Math.* **140** (2000), 57–77.

[13] P. Domański, D. Vogt, Distributional complexes split for positive dimensions, *J. reine angew. Math.* **522** (2000), 63–79.

[14] U. Franken, R. Meise, Generalized Fourier expansions for zero-solutions of surjective convolution operators on $\mathscr{D}'(\mathbb{R})$ and $\mathscr{D}'_{(\omega)}(\mathbb{R})$, *Note Mat. (Lecce)* **10** Suppl. 1 (1990), 251–272.

[15] L. Frerick, A splitting theorem for nuclear Fréchet spaces, in: *Functional Analysis, Proc. of the First International Workshop held at Trier University*, S. Dierolf, P. Domański, S. Dineen (eds.), Walter de Gruyter, Berlin 1996, pp. 165–167.

[16] L. Frerick, D. Kunkle, J. Wengenroth, The projective limit functor for spectra of webbed spaces, *Studia Math.* **158** (2003), 117–129.

[17] L.Frerick, J.Wengenroth, A sufficient condition for the vanishing of the derived projective limit functor, *Arch. Math.*, **67** (1996), 296–301.

[18] L. Hörmander, *The Analysis of Linear Partial Differential Operators*, Springer, Berlin 1983.

[19] L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, 3rd ed., North-Holland, Amsterdam 1990.

[20] H. Jarchow, *Locally Convex Spaces*, B. G. Teubner, Stuttgart 1981.

[21] J. Krone, D. Vogt, The splitting relation for Köthe spaces, *Math. Z.* **190** (1984), 349–381.

[22] D. Kunkle, Splitting of power series spaces of (PLS)-type, *Dissertation*, Wuppertal 2001.

[23] F. Mantlik, Partial differential operators depending analytically on a parameter, *Ann. Inst. Fourier (Grenoble)* **41** (1991), 577–599.

[24] F. Mantlik, Linear equations depending differentiably on a parameter, *Int. Equat. Operat. Theory* **13** (1990), 231-250.

[25] R. Meise, Sequence space representations for zero-solutions of convolution equations on ultradifferentiable functions of Roumieu type, *Studia Math.* **92** (1989), 211–230.

[26] R. Meise, D. Vogt, A characterization of quasi-normable Fréchet spaces, *Math. Nachr.* **122** (1985), 141–150.

[27] R. Meise, D. Vogt, *Introduction to Functional Analysis*, Clarendon Press, Oxford 1997.

[28] T. Meyer, Surjektivität von Faltungsoperatoren auf Räumen ultradifferenzierbaren Funktionen vom Roumieu Typ, Dissertation, Düsseldorf 1992.

[29] T. Meyer, Surjectivity of convolution operators on spaces of ultradifferentiable functions of Roumieu type, *Studia Math.* **125** (1997), 101–129.

[30] V. P. Palamodov, *Linear Differential Operators with Constant Coefficients*, Nauka, Moscow 1967 (Russian), English transl., Springer, Berlin 1971.

[31] V. P. Palamodov, Homological methods in the theory of locally convex spaces, *Uspekhi Mat. Nauk* **26** (1) (1971), 3–66 (in Russian); English transl., *Russian Math. Surveys* **26** (1) (1971), 1–64.

[32] H. J. Petzsche, Some results of Mittag-Leffler-type for vector valued functions and spaces of class A, in: *Functional Analysis: Surveys and Recent Results* , K. D. Bierstedt, B. Fuchssteiner (eds.), North-Holland, Amsterdam 1980, pp. 183-204.

[33] W. Roelcke, Einige Permanenzeigenschaften bei topologischen Gruppen und topologischen Vektorräumen, Vortrag auf der Funktionalanalysistagung in Oberwolfach 1972.

[34] W. Roelcke, S. Dierolf, On the three space problem for topological vector spaces, *Collect. Math.* **32** (1981), 13–35.

[35] F. Trèves, Un théorème sur les équations aux dérivées partielles à coefficients constants dépendant de paramètres, *Bull. Soc. Math. France* **90** (1962), 471–486.

[36] M. Valdivia, Representaciones de los espacios $\mathscr{D}(\Omega)$ y $\mathscr{D}'(\Omega)$, *Rev. R. Acad. Sci. Exactas Fis. y Nat. (Madrid)* **72** (1978), 385–414.

[37] D. Vogt, Vektorwertige Distributionen als Randverteilungen holomorpher Funktionen, *manuscripta math.* **17** (1975), 267-290.

[38] D. Vogt, On the solvability of $P(D)f = g$ for vector valued functions, *RIMS Kokyoroku* **508** (1983), 168-181.

[39] D. Vogt, Frécheträume, zwischen denen jede stetige lineare Abbildung beschränkt ist, *J. Reine Angew. Math.* **345** (1983), 182-200.

[40] D. Vogt, Sequence space representations of spaces of test functions and distributions, in: *Functional Analysis, Holomorphy and Approximation Theory*, (eds. G. L. Zapata), Lecture Notes Pure Appl. Math. **83**, Marcel Dekker, New York 1983, pp. 405–443.

[41] D. Vogt, On the functors Ext^1(E, F) for Fréchet spaces, *Studia Math.* **85** (1987), 163–197.

[42] D. Vogt, Topics on projective spectra of LB-spaces, in: *Advances in the Theory of Fréchet Spaces*, T. Terzioğlu (ed.), Kluwer, Dordrecht 1989, pp. 11–27.

[43] D. Vogt, Regularity properties of LF-spaces, in: *Progress in Functional Analysis, Proc. Int. Functional Analysis Meeting on the Occasion of the 60-th Birthday of M. Valdivia, Peniscola 1990*, K. D. Bierstedt, J. Bonet, J. Horváth, M. Maestre (eds.), North-Holland, Amsterdam 1992, pp. 57–84.

[44] D. Vogt, Fréchet valued real analytic functions, *Bull. Soc. Roy. Sc. Liège* **73** (2004), 155–170.

[45] J. Wengenroth, A splitting theorem for subspaces and quotients of $\mathscr{D}'$, *Bull. Pol. Acad. Sci. Math.* **49** (2001), 349–354.

[46] J. Wengenroth, *Derived Functors in Functional Analysis*, Lecture Notes Math. **1810**, Springer, Berlin 2003.

[47] G. Wiechert, Dualitäts- und Strukturtheorie der Kerne linearer Differentialoperatoren, Dissertation Wuppertal (1982).