# Differences of composition operators between weighted Banach spaces of holomorphic functions

José Bonet, Mikael Lindström and Elke Wolf

**Abstract**

We consider differences of composition operators between given weighted Banach spaces $H_v^\infty$ or $H_v^0$ of analytic functions with weighted sup-norms and give estimates for the distance of these differences to the space of compact operators. We also study boundedness and compactness of the operators. Some examples illustrate our results.

**MSC 2000:** 47B33, 47B38.

## Introduction

Let $v$ and $w$ be strictly positive bounded continuous functions (*weights*) on the open unit disk $D$ in the complex plane. In this note we are interested in operators defined on Banach spaces of analytic functions of the following form:

$$\begin{aligned} H_v^\infty &:= \{f \in H(D);\ \|f\|_v = \sup_{z\in D} v(z)|f(z)| < \infty\}, \\ H_v^0 &:= \{f \in H(D);\ \lim_{|z|\to 1^-} v(z)|f(z)| = 0\}, \end{aligned}$$

endowed with the norm $\|\cdot\|_v$. Here $H(D)$ denotes the space of all analytic functions. These spaces appear in the study of growth conditions of analytic functions and have been studied in various articles, see e.g. [22], [23], [16], [1], [17], [18], [2].
Let $\phi, \psi : D \to D$ be analytic mappings. Each such map induces through composition a linear composition operator $C_\phi(f) = f \circ \phi$ resp. $C_\psi(f) = f \circ \psi$ between spaces of holomorphic functions of the type defined above. We will consider differences of composition operators $(C_\phi - C_\psi)(f) = f \circ \phi - f \circ \psi$ acting on these spaces of holomorphic functions.

Composition operators have been studied on various spaces of analytic functions. We refer the reader to the excellent monographs [7] and [21], and the article [15]. The case of operators defined on weighted Banach spaces of the type defined above was treated e.g. in [5], [4] and [6]. Differences of composition operators have been investigated more recently; see [19], [13], [14], [10] and [20]. In this article we are mainly interested in finding an expression for the essential norm $||C_\phi - C_\psi||_e$, i.e. the distance of $C_\phi - C_\psi$ to the space of compact operators, when $C_\phi - C_\psi$ is a bounded operator from $H_v^\infty$ into $H_w^\infty$; compare with [10] and [12] for the case of $H^\infty$. It is known that if $||\varphi||_\infty < 1$, then $C_\varphi$ is a compact operator from $H_v^\infty$ into $H_w^\infty$. Therefore we are interested in the case $\max\{||\phi||_\infty, ||\psi||_\infty\} = 1$. In our investigation we also study boundedness and compactness of $C_\phi - C_\psi$. It turns out that we obtain similar conditions to those obtained in [5] and [4], at least when the weight $v$ is radial and satisfies certain natural conditions; see the details below.

## Notations and definitions

We refer the reader to [7], [9], [11] and [21] for notation on composition operators and spaces of analytic functions on the unit disc. The compact open topology on the space $H(D)$ will be denoted by *co*. The closed unit ball of $H_v^\infty$ resp. $H_v^0$ is denoted by $B_v^\infty$ resp. $B_v^0$. The formulation of many results on weighted spaces of analytic functions and on operators between them requires the so-called *associated weights* (see [3]). For a weight $v$ the associated weight $\tilde{v}$ is defined as follows

$$\tilde{v}(z) := \frac{1}{\sup\{|f(z)|;\ f \in H_v^\infty, \|f\|_v \leq 1\}} = \frac{1}{\|\delta_z\|_{H_v^\infty{}'}},\ z \in D,$$

where $\delta_z$ denotes the point evaluation of z. The associated weights are also continuous and $\tilde{v} \geq v > 0$ (see [3]). Furthermore, for each $z \in D$ there is $f_z \in H_v^\infty$, $||f_z||_v \leq 1$, such that $|f_z(z)| = \frac{1}{\tilde{v}(z)}$. A weight is called *essential* if there is a constant $C > 0$ with

$$v(z) \leq \tilde{v}(z) \leq Cv(z) \text{ for every } z \in D.$$

For examples of essential weights and conditions when weights are essential see [3], [5] and [4]. Especially interesting are *radial weights* $v$, i.e. weights which satisfy $v(z) = v(|z|)$ for every $z \in D$. Every radial weight which is non-increasing with respect to $|z|$ and such that $\lim_{|z|\to 1} v(z) = 0$ is called a *typical weight*. If the weight $v$ is typical, then the unit ball $B_v^\infty$ coincides with the closure of $B_v^0$ for the compact open topology. *In the sequel every radial weight is assumed to be non-increasing.*

In order to handle differences of composition operators we need the so called *pseudohyperbolic metric*. Recall that for any $z \in D$, $\varphi_z$ is the Möbius transformation of $D$ which interchanges the origin and $z$, namely,

$$\varphi_z(w) = \frac{z - w}{1 - \bar{z}w}, \quad w \in D.$$

The pseudohyperbolic distance $\rho(z, w)$ for $z, w \in D$ is defined by $\rho(z, w) = |\varphi_z(w)| = \left|\frac{z-w}{1-\bar{z}w}\right|$. We refer the reader to [9] for more details. According to [8] we define $\rho_v(z, p) := \sup\{|f(z)|\tilde{v}(z);\ f \in B_v^\infty,\ f(p) = 0\}$. Note that for any $z, p \in D$,

$$\rho(z, p) \leq \rho_v(z, p).$$

Indeed, let $f(p) = 0$, $f \in H^\infty$, $||f||_\infty \leq 1$. For each $z \in D$ there is $g_z \in H_v^\infty$, $||g_z||_v \leq 1$, such that $|g_z(z)|\tilde{v}(z) = 1$. Hence $|f(z)| = |f(z)g_z(z)|\tilde{v}(z) \leq \rho_v(z, p)$.

In case $v$ is a radial weight such that the following condition (which is due to Lusky [17]) holds

$$(L1) \quad \inf_k \frac{v(1 - 2^{-k-1})}{v(1 - 2^{-k})} > 0,$$

then it is proved in [8] that $\rho$ is equivalent to $\rho_v$. Several conditions equivalent to (L1) can be seen in [8]. In particular it is equivalent to a condition considered in [23].

An operator $T \in L(E, F)$ from the Banach space $E$ to the Banach space $F$ is called *compact* if it maps the closed unit ball of $E$ onto a relatively compact set in $F$. We recall that operators $T : E \to F$ which take weakly null sequences in $E$ to norm null sequences in $F$ are said to be *completely continuous*. The essential norm of a continuous linear operator $T$ is defined by $||T||_e := \inf\{||T - K|| : K \text{ is compact}\}$. Since $||T||_e = 0$ if and only if $T$ is compact, the estimates on $||T||_e$ lead to conditions for $T$ to be compact.

## Results

We start with an auxiliary result.

**Lemma 1** *Let $v$ be a radial weight satisfying condition (L1) and let $f \in H_v^\infty$. Then there exists a constant $C_v$ (depending only on the weight $v$) such that*

$$|f(z) - f(p)| \leq C_v||f||_v \max\left\{\frac{\rho(z, p)}{v(z)}, \frac{\rho(z, p)}{v(p)}\right\}$$

*for all $z, p \in D$.*

**Proof.** By Lemma 1 (a) in [8], there are $0 < s < 1$ and constant $0 < C < \infty$ such that $v(z)/v(p) \leq C$ for all $z, p \in D$ with $\rho(z,p) \leq s$. Hence it follows by Lemma 14 in [8] that

$$|f(z) - f(p)|v(z) \leq \frac{4C}{s}||f||_v \rho(z,p)$$

for all $z, p \in D$ with $\rho(z,p) \leq s/2$. If $\rho(z,p) > s/2$, then

$$|f(z) - f(p)| \min\{v(z), v(p)\} \leq 2||f||_v \leq \frac{4||f||_v}{s}\rho(z,p).$$

Therefore we conclude

$$|f(z) - f(p)| \min\{v(z), v(p)\} \leq C_v||f||_v \rho(z,p)$$

for all $z, p \in D$, from which the assertion follows. □

Now we characterize bounded operators $C_\phi - C_\psi$. Recall that not every composition operator $C_\varphi$ is bounded on $H_v^\infty$; see [5].

**Proposition 2** *Let $v$ and $w$ be weights. If $C_\phi - C_\psi : H_v^\infty \to H_w^\infty$ is bounded, then*

$$\max\left\{\sup_{z\in D} \frac{w(z)}{\tilde{v}(\phi(z))}\rho(\phi(z), \psi(z)), \sup_{z\in D} \frac{w(z)}{\tilde{v}(\psi(z))}\rho(\phi(z), \psi(z))\right\} < \infty.$$

*If $v$ also is radial and satisfies condition $(L1)$, then*

$$\max\left\{\sup_{z\in D} \frac{w(z)}{\tilde{v}(\phi(z))}\rho(\phi(z), \psi(z)), \sup_{z\in D} \frac{w(z)}{\tilde{v}(\psi(z))}\rho(\phi(z), \psi(z))\right\} < \infty$$

*implies the boundedness of $C_\phi - C_\psi : H_v^\infty \to H_w^\infty$.*

**Proof.** Assume that $C_\phi - C_\psi : H_v^\infty \to H_w^\infty$ is bounded. Hence we obtain

$$\sup_{z\in D} \frac{w(z)}{\tilde{v}(\phi(z))}\rho(\phi(z), \psi(z)) \leq \sup_{z\in D} \frac{w(z)}{\tilde{v}(\phi(z))}\rho_v(\phi(z), \psi(z)) \leq$$

$$\leq \sup_{z\in D} \frac{w(z)}{\tilde{v}(\phi(z))}\tilde{v}(\phi(z)) \sup\{|f(\phi(z)) - f(\psi(z))|; f \in B_v^\infty\} = ||C_\phi - C_\psi|| < \infty.$$

Similarly, $\sup_{z\in D} \frac{w(z)}{\tilde{v}(\psi(z))}\rho(\phi(z), \psi(z)) < \infty$.

For the converse implication we first notice that $v$ is essential by Proposition 2 (b) in [8]. Now we apply Lemma 1, so

$$\begin{aligned} ||C_\phi - C_\psi|| &= \sup_{z\in D} w(z) \sup\{|f(\phi(z)) - f(\psi(z))|;\ f \in B_v^\infty\} \\ &\leq \sup_{z\in D} w(z) C_v \max\left\{\frac{\rho(\phi(z), \psi(z))}{v(\phi(z))}, \frac{\rho(\phi(z), \psi(z))}{v(\psi(z))}\right\} < \infty, \end{aligned}$$

and $C_\phi - C_\psi : H_v^\infty \to H_w^\infty$ is bounded. □

Since $C_\phi - C_\psi : (H(D), co) \to (H(D), co)$ is continuous, we immediately get the following result:

**Proposition 3** *Let $v$ be a weight such that $\overline{B_v^0}^{co} = B_v^\infty$. If $C_\phi - C_\psi : H_v^0 \to H_w^0$ is bounded, then $C_\phi - C_\psi : H_v^\infty \to H_w^\infty$ is bounded.*

**Example 4** We give an example of non-bounded composition operators such that their difference is bounded.

Choose $w(z) = 1$ and $v(z) = 1-|z| = \tilde{v}(z)$ which are radial weights on $D$. Obviously, $v$ satisfies condition $(L1)$. Moreover, select, $\phi(z) = \frac{z+1}{2}$ and $\psi(z) = \frac{z+1}{2} + t(z-1)^3$, $z \in D$, such that $t$ is real and $|t|$ so small that $\psi$ maps $D$ into $D$. By [5] Proposition 2.1, $C_\phi : H_v^\infty \to H_w^\infty$ is not bounded, because for $z = r \in \mathbb{R}$ we have $\frac{w(r)}{\tilde{v}(\phi(r))} = \frac{2}{1-r} \to \infty$ if $r \to 1$. The fact that $C_\psi : H_v^\infty \to H_w^\infty$ is not bounded follows in an analogous way: For $z = r \in \mathbb{R}$ we obtain $\frac{w(r)}{\tilde{v}(\psi(r))} = \frac{1}{1-\frac{r+1}{2}-t(r-1)^3} \to \infty$ if $r \to 1$. By [19] Example 1 we know $\rho(\phi(z), \psi(z)) \le \frac{|t|}{\delta}|z-1|$, where $\delta$ is a constant. This yields

$$\begin{aligned}
\sup_{z\in D} \frac{w(z)}{\tilde{v}(\phi(z))}\rho(\phi(z),\psi(z)) &\le \sup_{z\in D} \frac{1}{1-|\frac{z+1}{2}|}\frac{|t|}{\delta}|z-1| < \infty \text{ and} \\
\sup_{z\in D} \frac{w(z)}{\tilde{v}(\psi(z))}\rho(\phi(z),\psi(z)) &\le \sup_{z\in D} \frac{1}{1-|\frac{z+1}{2}+t(z-1)^3|}\frac{|t|}{\delta}|z-1| < \infty
\end{aligned}$$

Hence, $C_\phi - C_\psi : H_v^\infty \to H_w^\infty$ is bounded.

**Example 5** We give a non-trivial example of a non-bounded difference of composition operators.

Choose $\phi(z) = \frac{z+1}{2}$, $\psi(z) = \frac{z+1}{2} + t(z-1)^3$, where $t$ is real and $|t|$ is so small that $\psi$ maps $D$ into $D$. Select now $w(z) = v(z) = e^{-\frac{1}{1-|z|}} = \tilde{v}(z)$, which are radial weights not satisfying $(L1)$. By [5] Proposition 2.1, $C_\phi : H_v^\infty \to H_w^\infty$ is not bounded since for $z = r \in \mathbb{R}$ we have $\frac{w(r)}{\tilde{v}(\phi(r))} = e^{-\frac{1}{1-r}+\frac{1}{1-\frac{r+1}{2}}} = e^{-\frac{1}{1-r}+\frac{2}{1-r}} = e^{\frac{1}{1-r}} \to \infty$ if $r \to 1$. Analogously $C_\psi : H_v^\infty \to H_w^\infty$ is not bounded since for $z = r \in \mathbb{R}$ we have $\frac{w(r)}{\tilde{v}(\psi(r))} = e^{-\frac{1}{1-r}+\frac{1}{1-\frac{r+1}{2}-t(r-1)^3}} = e^{-\frac{1}{1-r}+\frac{2}{1-r-2t(r-1)^3}} \to \infty$ if $r \to 1$. By [19] Example 1 we know $\rho(\phi(z), \psi(z)) \le \frac{|t|}{\delta}|z-1|$, where $\delta > 0$ is a constant. Since $|\phi(z)| \to 1$ or $|\psi(z)| \to 1$ is equivalent to $z \to 1$ and $\lim_{z\to 1} \frac{|t|}{\delta}|z-1| = 0$, we get

$$\lim_{|\phi(z)|\to 1} \rho(\phi(z),\psi(z)) = \lim_{|\psi(z)|\to 1} \rho(\phi(z),\psi(z)) = 0.$$

Now, for $z = r \in \mathbb{R}$, we have

$$\begin{aligned}
\frac{w(r)}{\tilde{v}(\phi(r))}\rho(\phi(r),\psi(r)) &= e^{\frac{1}{1-r}}\left|\frac{t(r-1)^3}{1-(\frac{r+1}{2})(\frac{r+1}{2}+t(r-1)^3)}\right| \\
&= e^{\frac{1}{1-r}}|t|\left|\frac{(r-1)^3}{1-(\frac{(r+1)^2}{4})-(\frac{r+1}{2}t(r-1)^3)}\right|
\end{aligned}$$

and $\frac{w(r)}{\tilde{v}(\phi(r))}\rho(\phi(r),\psi(r)) \to \infty$ for $r \to 1$. Hence $C_\phi - C_\psi : H_v^\infty \to H_w^\infty$ is not bounded.

The proof of our next result exploits a method presented in [4].

**Theorem 6** *Let $v$ and $w$ be radial weights such that $v$ is typical and satisfies condition $(L1)$. There is a constant $C_v > 0$ such that, if $\phi, \psi : D \to D$ are analytic maps such that $\max\{||\phi||_\infty, ||\psi||_\infty\} = 1$ and $C_\phi - C_\psi : H_v^\infty \to H_w^\infty$ is bounded, then*

$$\max\left\{\limsup_{|\phi(z)|\to 1}\frac{w(z)}{\tilde{v}(\phi(z))}\rho(\phi(z),\psi(z)), \limsup_{|\psi(z)|\to 1}\frac{w(z)}{\tilde{v}(\psi(z))}\rho(\phi(z),\psi(z))\right\} \le ||C_\phi - C_\psi||_e$$

$$\le C_v \max\left\{\limsup_{|\phi(z)|\to 1}\frac{w(z)}{\tilde{v}(\phi(z))}\rho(\phi(z),\psi(z)), \limsup_{|\psi(z)|\to 1}\frac{w(z)}{\tilde{v}(\psi(z))}\rho(\phi(z),\psi(z))\right\}.$$

**Proof.** We first prove the lower estimate of the essential norm by contradiction. Assume we can find $b > c > d > 0$, a compact operator $K : H_v^\infty \to H_w^\infty$ and a sequence $(z_n) \in D$ with $|\phi(z_n)| \to 1$ such that

$$\frac{w(z_n)}{\tilde{v}(\phi(z_n))}\rho(\phi(z_n), \psi(z_n)) \geq b > c > d > ||C_\phi - C_\psi - K|| \text{ for all } n.$$

Now we select an increasing sequence $(\alpha(n))_n$ of natural numbers going to infinity such that $|\phi(z_n)|^{\alpha(n)} \geq c/b$ for all $n$. Since $v$ is typical, it follows that for every $n$ we can find $f_n \in B_v^0$ such that $|f_n(\phi(z_n)| \geq \frac{1}{\tilde{v}(\phi(z_n))}\frac{d}{c}$.

Set $h_n(z) := z^{\alpha(n)}\varphi_{\psi(z_n)}(z)\ f_n(z)$. Thus, $h_n \in H_v^0$ with $||h_n||_v \leq 1$. Moreover $(h_n)$ converges to zero in the compact open topology, and consequently $h_n \to 0$ weakly in $H_v^0$; see e.g. [25]. Since the operator $K$ is compact, $\lim_{n\to\infty} ||Kh_n||_w = 0$. Thus, for each $n$,

$$c > ||C_\phi - C_\psi - K|| \geq ||(C_\phi - C_\psi)h_n||_w - ||Kh_n||_w,$$

and we conclude that

$$d > ||C_\phi - C_\psi - K|| \geq \limsup_n ||(C_\phi - C_\psi)h_n||_w = \limsup_n ||h_n \circ \phi - h_n \circ \psi||_w \geq$$

$$\geq \limsup_n w(z_n)|h_n(\phi(z_n)) - h_n(\psi(z_n))| =$$

$$= \limsup_n w(z_n)|\phi(z_n)|^{\alpha(n)}|\varphi_{\psi(z_n)}(\phi(z_n))\ f_n(\phi(z_n))| \geq$$

$$\geq \frac{d}{c}\limsup_n \frac{w(z_n)}{\tilde{v}(\phi(z_n))}\rho(\psi(z_n), \phi(z_n))|\phi(z_n)|^{\alpha(n)} \geq c\frac{d}{c},$$

which is a contradiction.

We now prove the upper estimate. Take the sequence of linear operators $C_k : H(D) \to H(D)$, $k \in \mathbb{N}$, defined by $C_k f(z) = f(\frac{k}{k+1}z)$, which are continuous for the compact open topology and $C_k f \to f$ uniformly on every compact subset of $D$. Moreover, the operators $C_k : H_v^\infty \to H_v^\infty$ are well-defined and compact with $||C_k|| \leq 1$.

For fixed $k \in \mathbb{N}$ we have,

$$||C_\phi - C_\psi||_e \leq ||C_\phi - C_\psi - (C_\phi - C_\psi)C_k|| = ||(C_\phi - C_\psi)(Id - C_k)||.$$

Let $f \in H_v^\infty$ with $||f||_v \leq 1$ and fix an arbitrary $r \in (0, 1)$. Set $g_k := (Id - C_k)f$. Then $g_k \in H_v^\infty$ and $||g_k||_v \leq 2$. Hence

$$||(C_\phi - C_\psi)g_k||_w \leq$$

$$\leq \sup_{\{z:|\phi(z)|\leq r \text{ and } |\psi(z)|\leq r\}} |g_k(\phi(z)) - g_k(\psi(z))|w(z) +$$

$$+ \sup_{\{z:|\phi(z)|>r \text{ or } |\psi(z)|>r\}} |g_k(\phi(z)) - g_k(\psi(z))|w(z) \leq$$

$$\leq \sup_{\{z:|\phi(z)|\leq r\}} |g_k(\phi(z))|w(z) + \sup_{\{z:|\psi(z)|\leq r\}} |g_k(\psi(z))|w(z)$$

$$+ \sup_{\{z:|\phi(z)|>r \text{ or } |\psi(z)|>r\}} |g_k(\phi(z)) - g_k(\psi(z))|w(z).$$

The sequence of operators $(Id - C_k)_k$ satisfies $\lim_k(Id - C_k)g = 0$ for each $g$ in $H(D)$, and the space $H(D)$ endowed with the compact open topology $co$ is a Fréchet space. By the Banach-Steinhaus theorem, $(Id - C_k)_k$ converges to zero uniformly on the compact subsets of $(H(D), co)$. Since the closed unit ball of $H_v^\infty$ is a compact subset of $(H(D), co)$ we obtain that

$$\lim_k \sup_{||f||_v\leq 1} \sup_{|\xi|\leq r} |((Id - C_k)f)(\xi)| = 0.$$

By Lemma 1,

$$|f(\phi(z)) - f(\psi(z))|w(z) \le C_v \max\left\{\frac{w(z)\rho(\phi(z),\psi(z))}{v(\phi(z))}, \frac{w(z)\rho(\phi(z),\psi(z))}{v(\psi(z))}\right\}$$

for all $z \in D$ and $f \in H_v^\infty$, $||f||_v \le 1$. Since $v$ is non-increasing we conclude from this

$$\lim_k ||(C_\phi - C_\psi)(Id - C_k)|| \le$$

$$\le 2C_v \max\left\{\sup_{\{z:|\phi(z)|>r\}} \frac{w(z)\rho(\phi(z),\psi(z))}{v(\phi(z))}, \sup_{\{z:|\psi(z)|>r\}} \frac{w(z)\rho(\phi(z),\psi(z))}{v(\psi(z))}\right\}.$$

Consequently,

$$||C_\phi - C_\psi||_e \le$$

$$2C_v \max\left\{\lim_{r\to 1} \sup_{\{z:|\phi(z)|>r\}} \frac{w(z)\rho(\phi(z),\psi(z))}{v(\phi(z))}, \lim_{r\to 1} \sup_{\{z:|\psi(z)|>r\}} \frac{w(z)\rho(\phi(z),\psi(z))}{v(\psi(z))}\right\}.$$

Since every radial weight with condition $(L1)$ is essential (see Prop. 2 in [8]), we are done. □

**Corollary 7** *Let $v$ and $w$ be radial weights such that $v$ is typical and satisfies condition $(L1)$. Then $C_\phi - C_\psi : H_v^\infty \to H_w^\infty$ is compact if and only if*

$$\limsup_{|\phi(z)|\to 1} \frac{w(z)}{\tilde{v}(\phi(z))}\rho(\phi(z),\psi(z)) = \limsup_{|\psi(z)|\to 1} \frac{w(z)}{\tilde{v}(\psi(z))}\rho(\phi(z),\psi(z)) = 0.$$

**Proof.** If $C_\phi - C_\psi$ is compact, then the conditions are satisfied by Theorem 6. Conversely, Theorem 6 implies the compactness of $C_\phi - C_\psi$ as soon as we know that $C_\phi - C_\psi$ is bounded. But by assumption we can choose $r < 1$ such that

$$\max\left\{\sup_{|\phi(z)|>r} \frac{w(z)}{\tilde{v}(\phi(z))}\rho(\phi(z),\psi(z)), \sup_{|\psi(z)|>r} \frac{w(z)}{\tilde{v}(\psi(z))}\rho(\phi(z),\psi(z))\right\} \le 1.$$

Hence the boundedness follows from

$$\max\left\{\sup_{z\in D} \frac{w(z)}{\tilde{v}(\phi(z))}\rho(\phi(z),\psi(z)), \sup_{z\in D} \frac{w(z)}{\tilde{v}(\psi(z))}\rho(\phi(z),\psi(z))\right\}$$

$$\le \max\left\{1, \sup_{z\in D} \frac{w(z)}{\tilde{v}(r)}\right\}.$$

□

Corollary 7 and the proof of the lower estimate in Theorem 6 permit us to obtain the following consequence.

**Corollary 8** *Let $v$ and $w$ be radial weights such that $v$ is typical and satisfies condition $(L1)$. Then $C_\phi - C_\psi : H_v^\infty \to H_w^\infty$ is completely continuous if and only if $C_\phi - C_\psi$ is compact.*

**Theorem 9** *Let $v$ and $w$ be typical weights such that $v$ satisfies condition $(L1)$. There is a constant $C_v > 0$ such that, if $\phi, \psi : D \to D$ are analytic maps such that $\max\{||\phi||_\infty, ||\psi||_\infty\} = 1$ and $C_\phi - C_\psi : H_v^0 \to H_w^0$ is bounded, then*

$$\max\left\{\limsup_{|z|\to 1} \frac{w(z)}{\tilde{v}(\phi(z))}\rho(\phi(z),\psi(z)), \limsup_{|z|\to 1} \frac{w(z)}{\tilde{v}(\psi(z))}\rho(\phi(z),\psi(z))\right\} \le ||C_\phi - C_\psi||_e$$

$$\le C_v \max\left\{\limsup_{|\phi(z)|\to 1} \frac{w(z)}{\tilde{v}(\phi(z))}\rho(\phi(z),\psi(z)), \limsup_{|\psi(z)|\to 1} \frac{w(z)}{\tilde{v}(\psi(z))}\rho(\phi(z),\psi(z))\right\}.$$

**Proof of Theorem 9.** The difference with the proof of the lower bound of Theorem 6 is that now we get $b > c > d > 0$, a compact operator $K : H_v^0 \to H_w^0$ and a sequence $(z_n) \in D$ with $|z_n| \to 1$ such that

$$\frac{w(z_n)}{\tilde{v}(\phi(z_n))}\rho(\phi(z_n), \psi(z_n)) \geq b > c > d > ||C_\phi - C_\psi - K|| \text{ for all } n.$$

We can asssume that $\phi(z_n) \to z_0$ for some $z_0$ with $|z_0| \leq 1$. If $|z_0| \neq 1$, then $0 = \lim_n w(z_n) \geq b\, v(z_0) > 0$, which is a contradiction. Therefore $|\phi(z_n)| \to 1$ and we can continue as in the proof of Theorem 6. Notice also that in the proof of the upper bound the operators $C_k : H_v^0 \to H_v^0$ are well-defined since $v$ is typical.

**Example 10** We select $\phi(z) = \frac{z+1}{2}$, $\psi(z) = \frac{z+1}{2} + t(z-1)^3$, where the real number $t$ is so small that $\psi$ is a self-map on $D$. Moreover we choose $w(z) = 1 - |z|$ and $v(z) = (1-|z|)^3 = \tilde{v}(z)$. Now $C_\phi, C_\psi : H_v^\infty \to H_w^\infty$ are not bounded since for $r \in \mathbb{R}$ we have $\frac{w(r)}{\tilde{v}(\phi(r))} = \frac{8}{(1-r)^2} \to \infty$ and $\frac{w(r)}{\tilde{v}(\psi(r))} = \frac{1-r}{(1-(\frac{r+1}{2}+t(r-1)^3)^3} \to \infty$ if $r \to 1$. It follows from Proposition 2 that the operator $C_\phi - C_\psi : H_v^\infty \to H_w^\infty$ is bounded (see Example 4). But it is not compact, since

$$\frac{w(r)}{\tilde{v}(\phi(r))}\rho(\phi(r), \psi(r)) = \frac{8}{(1-r)^2}\left|\frac{-t(r-1)^3}{1 - \frac{r+1}{2}(\frac{r+1}{2} + t(r-1)^3)}\right| \to 8|t| \text{ if } r \to 1.$$

For examples of compact and non-compact differences of composition operators $C_\phi - C_\psi : H^\infty \to H^\infty$, see [19] Example 1. The change of the behaviour of the operator $C_\phi - C_\psi$ depending on the weights $v$ and $w$ is emphasized in our last example.

**Example 11** We consider $\phi(z) = \frac{z+1}{2}$, $\psi(z) = \frac{z-1}{2}$, $z \in D$, which are both analytic self maps of the unit disk. By definition we obtain $\rho(\phi(z), \psi(z)) = \left|\frac{1}{1 - \overline{\frac{z+1}{2}}\frac{z-1}{2}}\right|$. Hence $\lim_{|\phi(z)| \to 1} \rho(\phi(z), \psi(z)) = \lim_{|\psi(z)| \to 1} \rho(\phi(z), \psi(z)) = 1$.

(a) Select $w(z) = 1 - |z| = v(z) = \tilde{v}(z)$. Obviously $v$ is typical and satisfies $(L1)$. By Theorem 6 we get

$$\limsup_{|\phi(z)| \to 1} \frac{w(z)}{\tilde{v}(\phi(z))}\rho(\phi(z), \psi(z)) = \limsup_{|\phi(z)| \to 1} \frac{1-|z|}{1 - |\frac{z+1}{2}|}\left|\frac{1}{1 - \overline{\frac{z+1}{2}}\frac{z-1}{2}}\right| = 1$$

and

$$\limsup_{|\psi(z)| \to 1} \frac{w(z)}{\tilde{v}(\psi(z))}\rho(\phi(z), \psi(z)) = \limsup_{|\psi(z)| \to 1} \frac{1-|z|}{1 - |\frac{z-1}{2}|}\left|\frac{1}{1 - \overline{\frac{z+1}{2}}\frac{z-1}{2}}\right| = 1$$

Hence

$$1 \leq \|C_\phi - C_\psi\|_e \leq C_v.$$

We conclude that $C_\phi - C_\psi : H_v^\infty \to H_w^\infty$ is bounded, but not compact.

(b) Choose $w(z) = 1$ and $v(z) = 1 - |z|$. We get

$$\sup_{z \in D} \frac{w(z)}{\tilde{v}(\phi(z))}\rho(\phi(z), \psi(z)) = \sup_{z \in D} \frac{1}{1 - |\frac{z+1}{2}|}\left|\frac{1}{1 - \overline{\frac{z+1}{2}}\frac{z-1}{2}}\right| = \infty.$$

Hence $C_\phi - C_\psi : H_v^\infty \to H_w^\infty$ is not bounded.

(c) Consider $w(z) = 1 - |z|$, $v(z) = 1$ to obtain

$$\limsup_{|\phi(z)|\to 1} \frac{w(z)}{\tilde{v}(\phi(z))}\rho(\phi(z),\psi(z)) = \limsup_{|\phi(z)|\to 1}(1-|z|)\left|\frac{1}{1-\overline{\frac{z+1}{2}}\frac{z-1}{2}}\right| = 0$$

and

$$\limsup_{|\psi(z)|\to 1} \frac{w(z)}{\tilde{v}(\psi(z))}\rho(\phi(z),\psi(z)) = \limsup_{|\psi(z)|\to 1}(1-|z|)\left|\frac{1}{1-\overline{\frac{z+1}{2}}\frac{z-1}{2}}\right| = 0.$$

Since the upper estimate in Theorem 6 is valid without the assumption that $v$ is typical, we conclude that $\|C_\phi - C_\psi\|_e = 0$, and the operator is compact.

**Acknowledgement.** We thank R. Mortini for the discussions we had with him on this topic. The research of J. Bonet was partially supported by MEC and FEDER project MTM2004-02262. The work of M. Lindström was partially supported by the Academy of Finland Project 205644.

# References

[1] K.D. Bierstedt, W.H. Summers, Biduals of weighted Banach spaces of analytic functions, *J. Austral. Math. Soc. (Series A)* **54** (1993), 70-79.

[2] K.D. Bierstedt, J. Bonet, A. Galbis, Weighted spaces of holomorphic functions on bounded domains, *Michigan Math. J.* **40** (1993), 271-297.

[3] K.D. Bierstedt, J. Bonet, J. Taskinen, Associated weights and spaces of holomorphic functions, *Studia Math.* **127** (1998), 137-168.

[4] J. Bonet, P. Domański, M. Lindström, Essential norm and weak compactness of composition operators on weighted Banach spaces of analytic functions, *Canad. Math. Bull.* **42**, no. 2, (1999), 139-148.

[5] J. Bonet, P. Domański, M. Lindström, J. Taskinen, Composition operators between weighted Banach spaces of analytic functions, *J. Austral. Math. Soc. (Serie A)* **64** (1998), 101-118.

[6] M.D. Contreras, A.G. Hernández-Díaz, Weighted composition operators in weighted Banach spaces of analytic functions, *J. Austral. Math. Soc. (Serie A)* **69** (2000) 41-60.

[7] C. Cowen, B. MacCluer, *Composition Operators on Spaces of Analytc Functions*, CRC Press, Baca Raton, 1995.

[8] P. Domański, M. Lindström, Sets of interpolation and sampling for weighted Banach spaces of holomorphic functions, *Ann. Pol. Math.* **79**, no.3, (2002), 233-264.

[9] J. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981.

[10] P. Gorkin, R. Mortini, D. Suárez, Homotopic composition operators on $H^\infty(B^n)$, Amer. Math. Soc. Contemp. Math. **328** (2003) 177-188.

[11] K. Hoffman, *Banach spaces of analytic functions*, Prentice Hall, Englewood Cliffs, 1967.

[12] T. Hosokowa, K. Izuchi, Essential norms of differences of composition operators on $H^\infty$, *J. Math. Soc. Japan* **57** (2005) 669-690.

[13] T. Hosokowa, K. Izuchi, D. Zheng, Isolated points and essential components of composition operators on $H^\infty$, *Proc. Amer. Math. Soc.* **130** (2002) 1765-1773.

[14] T. Hosokowa, K. Izuchi, D. Zheng, Topological structure of the space of weighted composition operators on $H^\infty$, *Integral Equations Oper. Theory* **53** (2005) 509-526.

[15] H. Jarchow, Some functional analytic properties of composition operators, *Quaestiones Math.* **18** (1995), 229-256.

[16] W. Kaballo, Lifting-Probleme für $H^\infty$-Funktionen, *Arch. Math* **34** (1980), 540-549.

[17] W. Lusky, On the structure of $Hv_0(D)$ and $hv_0(D)$, *Math. Nachr.* **159** (1992), 279-289.

[18] W. Lusky, On weighted spaces of harmonic and holomorphic functions, *J. London Math. Soc.* **51** (1995), 309-320.

[19] B. MacCluer, S. Ohno, R. Zhao, Topological structure of the space of compositon operators on $H^\infty$, *Integral Equations Operator Theory* **40** (2001), no. 4, 481-494.

[20] J. Moorhouse, Compact differences of composition operators, *J. Funct. Anal.* **219** (2005) 70-92.

[21] J.H. Shapiro, *Composition Operators and Classical Function Theory*, Springer, 1993.

[22] A.L. Shields, D.L. Williams, Bounded projections, duality, and multipliers in spaces of harmonic functions, *J. Reine Angew. Math.* **299/300** (1978), 256-279.

[23] A.L. Shields, D.L. Williams, Bounded projections and the growth of harmonic conjugates in the disc, *Michigan Math. J.* **29** (1982), 3-25.

[24] P. Wojtaszczyk, *Banach spaces for analysts*, Cambridge, 1991.

[25] E. Wolf, Weighted Fréchet spaces of holomorphic functions, *Studia Math.* (to appear).

**Authors' Addresses**.

J. Bonet: Departamento de Matemática Aplicada and IMPA-UPV, Universidad Politécnica de Valencia, E-46071 Valencia, SPAIN
e-mail: jbonet@mat.upv.es

M. Lindström: Department of Mathematics, Abo Akademi University, FIN-20500 Abo, FINLAND
e-mail: mlindstr@abo.fi

E. Wolf: Institute of Mathematics, University of Paderborn, D-33095 Paderborn, Germany.
e-mail: lichte@math.uni-paderborn.de