Topological structure of the set of weighted composition operators on weighted Bergman spaces of infinite order

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Abstract

We consider the topological space of all weighted composition operators on weighted Bergman spaces of infinite order endowed with the operator norm. We show that the set of compact composition operators is path connected. Furthermore, we find conditions to ensure that two composition operators are in the same path connected component if the difference of them is compact. Moreover, we compare the topologies induced by $L(H^\infty)$ and $L(H_v^\infty)$ on the space of bounded composition operators and give a sufficient condition for a composition operator to be isolated.

Keywords: weighted composition operator, weighted Bergman space of infinite order

MSC2000: 47B33, 47B38

1. Introduction

Let ϕ and ψ be analytic maps on the open unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$ such that $\phi(\mathbb{D}) \subset \mathbb{D}$. These maps define on the space $H(\mathbb{D})$ of analytic functions on $\mathbb{D}$ the so-called *weighted composition operator* ψC_ϕ by $(\psi C_\phi)(f) = \psi(f \circ \phi)$. Operators of this type have been studied on various spaces of analytic functions. For an introduction we refer the reader to the excellent monographs [24] and [8].

We are interested in weighted composition operators acting in the following setting. We say that a function v which is continuous, strictly positive, bounded and radial (i.e. $v(z) = v(|z|)$ for every $z \in \mathbb{D}$) is a *weight*. We consider operators defined on the weighted Banach spaces of holomorphic functions

$$H_v^\infty := \{f \in H(\mathbb{D});\ \|f\|_v = \sup_{z \in \mathbb{D}} v(z)|f(z)| < \infty\}$$

and on the smaller spaces

$$H_v^0 := \{f \in H_v^\infty;\ \lim_{|z| \to 1} v(z)|f(z)| = 0\}$$

both endowed with the norm $\|.\|_v$. Such spaces appear in the study of growth conditions of analytic functions and have been investigated in various articles, see e.g. [14], [1], [18], [17], [3], [5], [6].

The problem of the topological structure of the space of (weighted) composition operators has been considered on several spaces of analytic functions, starting with the paper by MacCluer, Ohno and Zhao [19]; see also [13], [11] and [12]. In [13] Hosokawa and Ohno showed that the set of compact composition operators on the little Bloch space is path connected. Moreover, they also analyzed the case of compact differences of composition operators. See also [20]. In this paper we will prove that the set of compact composition operators on the space H_v^∞ is path connected and we investigate the behaviour of pairs of weighted composition operators with a compact difference. We also study the topologies induced by $L(H^\infty)$ and $L(H_v^\infty)$ on the space of bounded composition operators and obtain a sufficient condition for a composition operator to be isolated.

2. Notation and auxiliary results

For notation and more information on composition operators we refer the reader to the excellent monographs [8] and [24]. A radial weight v is called *typical* if it is non-increasing with respect to $|z|$ and satisfies $\lim_{|z|\to 1} v(z) = 0$. The so called *associated weights* (see [3]) are an important tool to handle problems in the setting of weighted spaces of analytic functions. For a weight v the associated weight $\tilde{v}$ is defined by

$$\tilde{v}(z) := \frac{1}{\sup\{|f(z)|;\ f \in H_v^\infty, \|f\|_v \leq 1\}} = \frac{1}{\|\delta_z\|_{H_v^\infty{}'}},\ z \in \mathbb{D},$$

where δ_z denotes the point evaluation of z. By [3] we know that associated weights are continuous, $\tilde{v} \geq v > 0$ and that for each $z \in \mathbb{D}$ we can find $f_z \in H_v^\infty$, $||f_z||_v \leq 1$, such that $|f_z(z)| = \frac{1}{\tilde{v}(z)}$. It is well known that $H_{\tilde{v}}^\infty$ is isometrically isomorphic to H_v^∞. We are especially interested in radial weights which satisfy the following condition which is due to W. Lusky (see [17])

$$(L1) \qquad \inf_k \frac{v(1-2^{-k-1})}{v(1-2^{-k})} > 0.$$

The standard weights $v_p(z) = (1-|z|^2)^p$, $p > 0$, are weights which have $(L1)$. It is known (see [9] and [16]) that the Lusky condition $(L1)$ is equivalent to the following conditions:
(U) there is $\alpha > 0$ such that $\frac{v(z)}{(1-|z|)^\alpha}$ is increasing near the boundary of $\mathbb{D}$.
(A) there are $0 < r < 1$ and $1 < C < \infty$ with $\frac{v(z)}{v(p)} \leq C$ for all $p, z \in \mathbb{D}$ with $\rho(p,z) \leq r$.
For a detailed proof of this equivalence we refer the reader to [15]. If a weight v satisfies condition (L1), then there is $C > 0$ such that $v \leq \tilde{v} \leq Cv$ on $\mathbb{D}$. See [9].
We also need some geometric data of the unit disk. The *pseudohyperbolic metric* is given by

$$\rho(z,a) := |\sigma_a(z)|,\ \text{where } \sigma_a(z) := \frac{a-z}{1-\bar{a}z}$$

is the automorphism of $\mathbb{D}$ which changes 0 and a.

Let $\phi^{(1)}, \phi^{(2)}, \phi$ be analytic self-maps of the open unit disk $\mathbb{D}$, $\psi^{(1)}, \psi^{(2)}, \psi \in H(\mathbb{D})$ and $\mathcal{C}_w(X)$ be the space of all weighted composition operators on the Banach space X. *We write* $\psi^{(1)}C_{\phi^{(1)}} \sim_{H_v^\infty} \psi^{(2)}C_{\phi^{(2)}}$ *if and only if the map* $s \mapsto ((1-s)\psi^{(1)} + s\psi^{(2)})C_{(1-s)\phi^{(1)}+s\phi^{(2)}}$ *is continuous from* $[0,1]$ *into* $\mathcal{C}_w(H_v^\infty)$.
Furthermore, we put $\phi_s(z) := \phi(sz)$ and $\psi_s(z) := \psi(sz)$ for every $0 \leq s \leq 1$ and for every $z \in \mathbb{D}$ as well as

$$(\phi,\psi)^\#(z) := \frac{\psi(z)v(z)}{\tilde{v}(\phi(z))}.$$

Let v be a strictly decreasing, radial and typical weight. Since v is decreasing and $|sz| \leq |z|$ for every $0 \leq s \leq 1$ and every $z \in \mathbb{D}$, we have

$$(\phi,\psi)_s{}^\#(z) := (\phi_s,\psi_s)^\#(z) = \frac{\psi(sz)v(z)}{\tilde{v}(\phi(sz))} \leq \frac{\psi(sz)v(sz)}{\tilde{v}(\phi(sz))} = (\phi,\psi)^\#(sz)$$

for every $0 \leq s \leq 1$ and every $z \in \mathbb{D}$. Recall that $||\psi C_\phi|| = \sup_{z\in\mathbb{D}} \frac{|\psi(z)|v(z)}{\tilde{v}(\phi(z))}$ and the essential norm $||\psi C_\phi||_e = \limsup_{|\phi(z)|\to 1} \frac{|\psi(z)|v(z)}{\tilde{v}(\phi(z))}$, when $\psi C_\phi : H_v^\infty \to H_v^\infty$ is bounded. If $\psi \in H_v^0$, then $||\psi C_\phi||_e = \limsup_{|z|\to 1} \frac{|\psi(z)|v(z)}{\tilde{v}(\phi(z))}$. It is clear that $\psi_s C_{\phi_s} : H_v^\infty \to H_v^\infty$ is always compact when $s < 1$. For these results, see [21], [4].

Finally, let us collect some auxiliary results which will be needed during the paper.

The following lemma is taken from [7] Lemma 1 and [15] Lemma 1, see also [9] Lemma 14.

Lemma 1 *Let v be a radial weight on $\mathbb{D}$ satisfying $(L1)$ such that v is continuously differentiable respect to $|z|$. Then there exists a constant $C < \infty$ such that if $f \in H_v^\infty$, then*

$$\begin{aligned} |f(z) - f(p)| &\leq C\|f\|_v \max\left\{\frac{1}{\tilde{v}(z)}, \frac{1}{\tilde{v}(p)}\right\} \rho(z,p) \ \textit{and} \\ |f(z)v(z) - f(p)v(p)| &\leq C\|f\|_v \rho(z,p) \ \textit{ for all }\ z, p \in \mathbb{D}. \end{aligned}$$

The next result follows from [9] Lemma 14, [15] Lemma 1 and the proof of Lemma 2 in [15].

Lemma 2 *Let v be a radial weight on $\mathbb{D}$ satisfying the Lusky condition $(L1)$ and such that v is continuously differentiable with respect to $|z|$. Then there are $M > 0$ and $r_0 > 0$ such that, for each $p, z \in \mathbb{D}$ with $\rho(p, z) \le r_0$ we get*

(a) $\frac{\tilde{v}(z)}{\tilde{v}(p)} \le M$,

(b) $\left|1 - \frac{\tilde{v}(z)}{\tilde{v}(p)}\right| \le M\rho(z, p)$.

Let us now estimate the norm of the difference of two weighted composition operators. We have that

$$||\psi^{(1)}C_{\phi^{(1)}} - \psi^{(2)}C_{\phi^{(2)}}||_{L(H_v^\infty)} = \sup_{||f||_v \le 1} \sup_{z\in\mathbb{D}} |\psi^{(1)}(z)f(\phi^{(1)}(z)) - \psi^{(2)}(z)f(\phi^{(2)}(z))|v(z).$$

This can be splitted into two parts by adding and subtracting $\frac{\psi^{(1)}(z)v(z)}{\tilde{v}(\phi^{(1)}(z))} f(\phi^{(2)}(z))\tilde{v}(\phi^{(2)}(z))$ and then, by Lemma 1, we can deduce

Lemma 3 *Let v be a radial weight on $\mathbb{D}$ satisfying the Lusky condition $(L1)$ and such that v is continuously differentiable with respect to $|z|$. Moreover let $\psi^{(1)}, \psi^{(2)} \in H(\mathbb{D})$ and $\phi^{(1)}, \phi^{(2)}$ be analytic self-maps of $\mathbb{D}$. Then there is a constant $C < \infty$ such that*

$$\begin{aligned} ||\psi^{(1)}C_{\phi^{(1)}} - \psi^{(2)}C_{\phi^{(2)}}||_{L(H_v^\infty)} \;&\le\; \sup_{z\in\mathbb{D}} \big(C|(\phi^{(1)}, \psi^{(1)})^{\#}(z)|\; \rho(\phi^{(1)}(z), \phi^{(2)}(z)) \\ &+\; |(\phi^{(1)}, \psi^{(1)})^{\#}(z) - (\phi^{(2)}, \psi^{(2)})^{\#}(z)|\big). \end{aligned}$$

Since v is a radial weight, the following theorem can be derived from [15] Theorem 3.

Theorem 4 *Let v be a radial weight such that v is continuously differentiable with respect to $|z|$, $v = \tilde{v}$ and satisfies the Lusky condition $(L1)$. Let $\psi^{(1)}, \psi^{(2)} \in H_v^0$. If $\phi^{(1)}, \phi^{(2)} : \mathbb{D} \to \mathbb{D}$ are analytic maps such that and $\psi^{(1)}C_{\phi^{(1)}}, \psi^{(2)}C_{\phi^{(2)}} : H_v^\infty \to H_w^\infty$ are bounded, then the operator $\psi^{(1)}C_{\phi^{(1)}} - \psi^{(2)}C_{\phi^{(2)}} : H_v^\infty \to H_v^\infty$ is compact if and only if*

(a) $\lim_{|z|\to 1} v(z)\frac{|\psi^{(1)}(z)|}{v(\phi^{(1)}(z))}\rho(\phi^{(1)}(z), \phi^{(2)}(z)) = 0$,

(b) $\lim_{|z|\to 1} v(z)\frac{|\psi^{(2)}(z)|}{v(\phi^{(2)}(z))}\rho(\phi^{(1)}(z), \phi^{(2)}(z)) = 0$,

(c) $\lim_{|z|\to 1} v(z)\left|\frac{\psi^{(1)}(z)}{v(\phi^{(1)}(z))} - \frac{\psi^{(2)}(z)}{v(\phi^{(2)}(z))}\right| = 0$.

3. Connectedness of compact composition operators

In this section we show that the set of compact composition operators with weights in H_v^0 is path connected.

Proposition 5 *Let v be a typical, decreasing weight on $\mathbb{D}$ satisfying $(L1)$ such that v is continuously differentiable with respect to $|z|$, $\psi \in H_v^0$ and ϕ an analytic self-map of $\mathbb{D}$. Moreover, suppose that $\psi C_\phi : H_v^\infty \to H_v^\infty$ is compact. Then the map $[0, 1] \to (C(\mathbb{D}), \|.\|_\infty), s \mapsto (\phi, \psi)_s{}^{\#}$ is continuous.*

Proof. We have to show that if $t \to s$, then $I := \sup_{z\in\mathbb{D}} \left|\frac{\psi(sz)v(z)}{\tilde{v}(\phi(sz))} - \frac{\psi(tz)v(z)}{\tilde{v}(\phi(tz))}\right| \to 0$.
First, we consider the case $s < 1$. If $t \to s$ and $s < 1$, we can find $0 < s_0 < 1$ such that $s, t \le s_0$. Moreover, there is $0 < R < 1$ such that $|\phi(tz)| \le R$ for every $z \in \mathbb{D}$ and every $t \le s_0$. We have

$$I \le \sup_{z\in\mathbb{D}} v(z)|\psi(sz)| \left|\frac{1}{\tilde{v}(\phi(sz))} - \frac{1}{\tilde{v}(\phi(tz))}\right| + \sup_{z\in\mathbb{D}} \frac{v(z)}{\tilde{v}(\phi(tz))} |\psi(sz) - \psi(tz)| := I_1 + I_2.$$

Using Lemma 1 and the fact that $\rho(\phi(tz), \phi(sz)) \leq \rho(tz, sz)$ we obtain,

$$I_1 \leq ||\psi||_v \sup_{z\in\mathbb{D}} \left| \frac{\tilde{v}(\phi(tz)) - \tilde{v}(\phi(sz))}{\tilde{v}(\phi(sz))\tilde{v}(\phi(tz))} \right| \leq \frac{||\psi||_v}{\tilde{v}^2(R)} \sup_{z\in\mathbb{D}} |\tilde{v}(\phi(tz)) - \tilde{v}(\phi(sz))| \leq \frac{||\psi||_v C\tilde{v}(0)}{\tilde{v}^2(R)} \rho(tz, sz).$$

Moreover, with Lemma 1, we get

$$I_2 \leq \frac{C||\psi||_v}{\tilde{v}(R)} \rho(tz, sz).$$

Since $s < 1$, $\rho(tz, sz) \leq \frac{|t-s|}{1-s} \to 0$, when $t \to s$, and the claim follows.
Next, we treat the case $s = 1$. Fix $\varepsilon > 0$. Since ψC_ϕ is compact and $\psi \in H_v^0$, there is $0 < r < 1$ such that

$$|(\phi, \psi)^{\#}(z)| = \left| \frac{\psi(z)v(z)}{\tilde{v}(\phi(z))} \right| < \frac{\varepsilon}{4} \text{ for every } z \in \mathbb{D} \text{ with } |z| > r.$$

Fix $r < r' < 1$. If $t > \frac{r}{r'}$ and $|z| > r'$, then $|tz| > r$ and we have

$$|(\phi, \psi)_t^{\#}(z)| \leq |(\phi, \psi)^{\#}(tz)| < \frac{\varepsilon}{4} \text{ for every } t > \frac{r}{r'} \text{ and every } |z| > r'.$$

Put now $\delta_1 := 1 - \frac{r}{r'}$. For $t > 1 - \delta_1$ we obtain

$$\sup_{|z|>r'} |(\phi, \psi)^{\#}(z) - (\phi, \psi)_t^{\#}(z)| \leq \sup_{|z|>r'} |(\phi, \psi)^{\#}(z)| + \sup_{|z|>r'} |(\phi, \psi)_t^{\#}(z)| < \frac{\varepsilon}{2}.$$

For $|z| \leq r'$, we can find $\delta_2 > 0$ such that

$$\sup_{|z|\leq r'} |(\phi, \psi)^{\#}(z) - (\phi, \psi)_t^{\#}(z)| < \frac{\varepsilon}{2}$$

for $t > 1 - \delta_2$ by the first part of the proof. Finally, $t > 1 - \delta = \min\{\delta_1, \delta_2\}$ implies

$$\sup_{z\in\mathbb{D}} |(\phi, \psi)^{\#}(z) - (\phi, \psi)_t^{\#}(z)| < \varepsilon.$$

□

The set of compact weighted composition operators is starlike. We prove that two compact weighted composition operators can be linked by an arc of a special form, thus concluding that the set of compact composition operators is connected.

Theorem 6 *Let v be a radial, typical weight satisfying $(L1)$ and $\psi^{(1)}, \psi^{(2)} \in H_v^0$. If the operators $\psi^{(1)}C_{\phi^{(1)}}$ and $\psi^{(2)}C_{\phi^{(2)}}$ are compact, then*

$$\psi^{(1)}C_{\phi^{(1)}} \sim_{H_v^\infty} \psi^{(1)}(0)C_{\phi^{(1)}(0)} \sim_{H_v^\infty} \psi^{(2)}(0)C_{\phi^{(2)}(0)} \sim_{H_v^\infty} \psi^{(2)}C_{\phi^{(2)}}.$$

In particular the set of compact composition operators is path connected.

Proof. Fix two analytic self-maps $\phi^{(1)}, \phi^{(2)}$ of $\mathbb{D}$ and $\psi^{(1)}, \psi^{(2)} \in H_v^0$ such that the corresponding weighted composition operators $\psi^{(1)}C_{\phi^{(1)}}$ and $\psi^{(2)}C_{\phi^{(2)}}$ are compact on H_v^∞. We want to show that

$$\psi^{(1)}C_{\phi^{(1)}} \sim_{H_v^\infty} \psi^{(1)}(0)C_{\phi^{(1)}(0)} \sim_{H_v^\infty} \psi^{(2)}(0)C_{\phi^{(2)}(0)} \sim_{H_v^\infty} \psi^{(2)}C_{\phi^{(2)}},$$

where $(\psi^{(i)}(0)C_{\phi^{(i)}(0)})f(z) := \psi^{(i)}(0)f(\phi^{(i)}(0))$ for every $f \in H_v^\infty$ and every $z \in \mathbb{D}$. First, we note $\psi^{(1)}(0)C_{\phi^{(1)}(0)} \sim_{H_v^\infty} \psi^{(2)}(0)C_{\phi^{(2)}(0)}$. Put

$$p_s := (1-s)\phi^{(1)}(0) + s\phi^{(2)}(0) \text{ and } q_s := (1-s)\psi^{(1)}(0) + s\psi^{(2)}(0) \text{ for every } s \in [0, 1].$$

Then it is easily seen that $\{q_s C_{p_s} : s \in [0, 1]\}$ is a continuous curve from $\psi^{(1)}(0)C_{\phi^{(1)}(0)}$ to $\psi^{(2)}(0)C_{\phi^{(2)}(0)}$. Next, we will prove $\psi^{(i)}C_{\phi^{(i)}} \sim_{H_v^\infty} \psi^{(i)}(0)C_{\phi^{(i)})(0)}$ for $i = 1, 2$. By assumption,

$\phi^{(i)}$ is an analytic self-map of $\mathbb{D}$ and $\psi^{(i)} \in H_v^0$ such that $\psi^{(i)} C_{\phi^{(i)}}$ is compact on H_v^∞. Then $\psi_s^{(i)} C_{\phi_s^{(i)}}$ is also compact for all $s \in [0,1]$. We will show that

$$||\psi_s^{(i)} C_{\phi_s^{(i)}} - \psi_t^{(i)} C_{\phi_t^{(i)}}||_{L(H_v^\infty)} \to 0 \text{ when } t \to s.$$

By Lemma 3,

$$||\psi_s^{(i)} C_{\phi_s^{(i)}} - \psi_t^{(i)} C_{\phi_t^{(i)}}||_{L(H_v^\infty)}$$
$$\leq C \sup_{z \in \mathbb{D}} |(\phi_s^{(i)}, \psi_s^{(i)})^{\#}(z)| \; \rho(\phi_s^{(i)}(z), \phi_t^{(i)}(z)) + ||(\phi_s^{(i)}, \psi_s^{(i)})^{\#} - (\phi_t^{(i)}, \psi_t^{(i)})^{\#}||_\infty.$$

Because of Proposition 5, it is enough to show that $\sup_{z \in \mathbb{D}} |(\phi_s^{(i)}, \psi_s^{(i)})^{\#}(z)| \rho(\phi_s^{(i)}(z), \phi_t^{(i)}(z)) \to 0$ when $t \to s$.

If $s < 1$, then $\sup_{z \in \mathbb{D}} |(\phi_s^{(i)}, \psi_s^{(i)})^{\#}(z)| \leq ||\psi^{(i)} C_{\phi^{(i)}}||_{L(H_v^\infty)} < \infty$ and $\sup_{z \in \mathbb{D}} \rho(\phi_s^{(i)}(z), \phi_t^{(i)}(z)) \leq \frac{|t-s|}{1-s} \to 0$, as $t \to s$, and we are done.

If $s = 1$, we fix $\varepsilon > 0$. Since $\psi^{(i)} C_{\phi^{(i)}}$ is compact on H_v^∞ and $\psi^{(i)} \in H_v^0$, we can find $0 < r < 1$ such that $\frac{|\psi^{(i)}(z)| v(z)}{\tilde{v}(\phi^{(i)}(z))} < \frac{\varepsilon}{C}$ for every $|z| > r$. Since $\rho(\phi^{(i)}(tz), \phi^{(i)}(z)) \leq \rho(tz, z) = \frac{|z|(1-t)}{1-t|z|^2}$, it follows that $\sup_{|z| \leq r} \rho(\phi^{(i)}(z), \phi_t^{(i)}(z)) \leq \frac{r(1-t)}{1-tr^2} \to 0$, when $t \to 1$. Hence

$$\lim_{t \to 1} ||\psi^{(i)} C_{\phi^{(i)}} - \psi_t^{(i)} C_{\phi_t^{(i)}}||_{L(H_v^\infty)} \leq C \, \frac{\varepsilon}{C} = \varepsilon.$$

□

4. Connectedness and compact differences

For the proof of our main result, Theorem 8, we need the following crucial estimate.

Lemma 7 *Let v be a radial weight satisfying Lusky's condition $(L1)$ and such that $\tilde{v}$ is continuously differentiable with respect to $|z|$. Let $\phi^{(1)}, \phi^{(2)}$ be analytic self-maps of $\mathbb{D}$, $\psi^{(1)}, \psi^{(2)} \in H(\mathbb{D})$ and $s \in [0,1]$. There are $M > 0$ and $r_0 \in]0,1[$ such that, for each $z \in \mathbb{D}$ with $\rho(\phi^{(1)}(z), \phi^{(2)}(z)) \leq r_0$ we get*

$$|((1-s)\phi^{(1)} + s\phi^{(2)}, (1-s)\psi^{(1)} + s\psi^{(2)})^{\#}(z) - (\phi^{(1)}, \psi^{(1)})^{\#}(z)|$$
$$\leq s \; M \; |(\phi^{(1)}, \psi^{(1)})^{\#}(z) - (\phi^{(2)}, \psi^{(2)})^{\#}(z)|$$
$$+ \; |(\phi^{(1)}, \psi^{(1)})^{\#}(z)| \left(\frac{M(1-s)\rho(\phi^{(1)}(z), \phi^{(2)}(z))}{1 - s\rho(\phi^{(1)}(z), \phi^{(2)}(z))} + s \; M^2 \rho(\phi^{(1)}(z), \phi^{(2)}(z)) \right).$$

Proof. Let $r_0 > 0$ and $M > 0$ be as in Lemma 2. First, we notice that

$$((1-s)\phi^{(1)} + s\phi^{(2)}, (1-s)\psi^{(1)} + s\psi^{(2)})^{\#}(z) = (1-s) \frac{\tilde{v}(\phi^{(1)}(z))}{\tilde{v}((1-s)\phi^{(1)}(z) + s\phi^{(2)}(z))} (\phi^{(1)}, \psi^{(1)})^{\#}(z)$$
$$+ s \frac{\tilde{v}(\phi^{(2)}(z))}{\tilde{v}((1-s)\phi^{(1)}(z) + s\phi^{(2)}(z))} (\phi^{(2)}, \psi^{(2)})^{\#}(z).$$

Hence

$$|((1-s)\phi^{(1)} + s\phi^{(2)}, (1-s)\psi^{(1)} + s\psi^{(2)})^{\#}(z) - (\phi^{(1)}, \psi^{(1)})^{\#}(z)|$$
$$\leq |(\phi^{(1)}, \psi^{(1)})^{\#}(z) - (\phi^{(2)}, \psi^{(2)})^{\#}(z)| \frac{s\tilde{v}(\phi^{(2)}(z))}{\tilde{v}((1-s)\phi^{(1)}(z) + s\phi^{(2)}(z))}$$
$$+ \left| (\phi^{(1)}, \psi^{(1)})^{\#}(z)| \cdot |1 - (1-s) \frac{\tilde{v}(\phi^{(1)}(z))}{\tilde{v}((1-s)\phi^{(1)}(z) + s\phi^{(2)}(z))} - s \frac{\tilde{v}(\phi^{(2)}(z))}{\tilde{v}((1-s)\phi^{(1)}(z) + s\phi^{(2)}(z))} \right|$$
$$=: I_1 + I_2.$$

Assume now that $z \in \mathbb{D}$ satisfies $\rho(\phi^{(1)}(z), \phi^{(2)}(z)) \leq r_0$. To estimate I_1, observe that the convex combination $(1-s)\phi^{(1)}(z)+s\phi^{(2)}(z)$ also satisfies $\rho(\phi^{(1)}(z), (1-s)\phi^{(1)}(z)+s\phi^{(2)}(z)) \leq r_0$. Therefore, by Lemma 2,

$$\frac{\tilde{v}(\phi^{(2)}(z))}{\tilde{v}((1-s)\phi^{(1)}(z)+s\phi^{(2)}(z))} \leq M$$

and

$$I_1 \leq sM|(\phi^{(1)}, \psi^{(1)})^{\#}(z) - (\phi^{(2)}, \psi^{(2)})^{\#}(z)|.$$

Now,

$$\begin{aligned} I_2 \quad &\leq \quad sM|(\phi^{(1)}, \psi^{(1)})^{\#}(z)| \left|1 - \frac{\tilde{v}(\phi^{(1)}(z))}{\tilde{v}((1-s)\phi^{(1)}(z)+s\phi^{(2)}(z))}\right| \\ &+ \quad s|(\phi^{(1)}, \psi^{(1)})^{\#}(z)| \left|\frac{\tilde{v}(\phi^{(2)}(z)) - \tilde{v}(\phi^{(1)}(z))}{\tilde{v}((1-s)\phi^{(1)}(z)+s\phi^{(2)}(z))}\right|. \end{aligned}$$

By Lemma 2, we get

$$\begin{aligned} \left|1 - \frac{\tilde{v}(\phi^{(1)}(z))}{\tilde{v}((1-s)\phi^{(1)}(z)+s\phi^{(2)}(z))}\right| \quad &\leq \quad M\rho(\phi^{(1)}(z), (1-s)\phi^{(1)}(z)+s\phi^{(2)}(z)) \\ &\leq \quad M\frac{(1-s)\rho(\phi^{(1)}(z), \phi^{(2)}(z))}{1-s\rho(\phi^{(1)}(z), \phi^{(2)}(z))}. \end{aligned}$$

On the other hand, again using Lemma 2,

$$\begin{aligned} \left|\frac{\tilde{v}(\phi^{(2)}(z)) - \tilde{v}(\phi^{(1)}(z))}{\tilde{v}((1-s)\phi^{(1)}(z)+s\phi^{(2)}(z))}\right| \quad &\leq \quad \frac{\tilde{v}(\phi^{(2)}(z))}{\tilde{v}((1-s)\phi^{(1)}(z)+s\phi^{(2)}(z))}\left|1 - \frac{\tilde{v}(\phi^{(1)}(z))}{\tilde{v}(\phi^{(2)}(z))}\right| \\ &\leq \quad M^2\rho(\phi^{(1)}(z), \phi^{(2)}(z)). \end{aligned}$$

The conclusion follows from the estimates. □

The set of weighted composition operators is starlike. We prove below that two weighted composition operators satisfying certain conditions can be linked by an arc of a special form. As a consequence we obtain in Corollary 9 a result on composition operators on the Bloch space.

Theorem 8 *Let v be a radial, typical weight satisfying (L1) such that $v = \tilde{v}$ and v is continuously differentiable with respect to $|z|$ and $\psi^{(1)}, \psi^{(2)} \in H_v^0$. Assume that $\psi^{(1)}C_{\phi^{(1)}}$, $\psi^{(2)}C_{\phi^{(2)}}$ are bounded, $\lim_{|z|\to 1}\rho(\phi^{(1)}(z), \phi^{(2)}(z)) = 0$ and that $\lim_{|z|\to 1} v(z)\left|\frac{\psi^{(1)}(z)}{v(\phi^{(1)}(z))} - \frac{\psi^{(2)}(z)}{v(\phi^{(2)}(z))}\right| = 0$. Then the map $s \mapsto ((1-s)\psi^{(1)}+s\psi^{(2)})C_{(1-s)\phi^{(1)}+s\phi^{(2)}}$ is continuous from $[0,1]$ into $C_w(H_v^\infty)$.*

Proof. Let $r_s := (1-s)\phi^{(1)}+s\phi^{(2)}$ and $q_s := (1-s)\psi^{(1)}+s\psi^{(2)}$ for every $s \in [0,1]$. We have to show that for every $\varepsilon > 0$ there is $\delta > 0$ such that $|s-t| < \delta$ implies $\|q_sC_{r_s} - q_tC_{r_t}\|_{L(H_v^\infty)} < \varepsilon$.

First, by Lemma 3 we obtain

$$\begin{aligned} \|q_sC_{r_s} - q_tC_{r_t}\|_{L(H_v^\infty)} \quad &\leq \quad C\sup_{z\in\mathbb{D}}|(r_s, q_s)^{\#}(z)|\rho(r_s(z), r_t(z)) \\ &+ \quad \sup_{z\in\mathbb{D}}|(r_s, q_s)^{\#}(z) - (r_t, q_t)^{\#}(z)| =: T_1 + T_2. \end{aligned}$$

Now, fix $\varepsilon > 0$. Next, we show that there is $C_1 > 0$ such that $\sup_{z\in\mathbb{D}} C|(r_s, q_s)^{\#}(z)| \leq C_1$. Fix $z \in \mathbb{D}$. If $\max\{|\phi^{(1)}(z)|, |\phi^{(2)}(z)|\} = |\phi^{(1)}(z)|$, we obtain the following estimate:

$$\begin{aligned} |(r_s, q_s)^{\#}(z)| \quad &\leq \quad \frac{((1-s)|\psi^{(1)}(z)| + s|\psi^{(2)}(z)|)v(z)}{v((1-s)\phi^{(1)}(z)+s\phi^{(2)}(z))} \\ &\leq \quad \frac{((1-s)|\psi^{(1)}(z)| + s|\psi^{(2)}(z)|)v(z)}{v(\phi^{(1)}(z))}. \end{aligned}$$

Otherwise in case $\max\{|\phi^{(1)}(z)|, |\phi^{(2)}(z)|\} = |\phi^{(2)}(z)|$, we get similarly

$$|(r_s, q_s)^{\#}(z)| \quad \le \quad \frac{((1-s)|\psi^{(1)}(z)| + s|\psi^{(2)}(z)|)v(z)}{v(\phi^{(2)}(z))}.$$

Hence

$$\begin{aligned}\sup_{z\in\mathbb{D}} |(r_s, q_s)^{\#}(z)| \le (1-s) \sup_{z\in\mathbb{D}} v(z)|\psi^{(1)}(z)| \max\left\{\frac{1}{v(\phi^{(1)}(z))}, \frac{1}{v(\phi^{(2)}(z))}\right\} \\ + s\sup_{z\in\mathbb{D}} v(z)|\psi^{(2)}(z)| \max\left\{\frac{1}{v(\phi^{(1)}(z))}, \frac{1}{v(\phi^{(2)}(z))}\right\} < \infty\end{aligned}$$

by the boundedness of the operators $\psi^{(1)}C_{\phi^{(1)}}$, $\psi^{(2)}C_{\phi^{(2)}}$, $\psi^{(1)}C_{\phi^{(2)}}$ and $\psi^{(2)}C_{\phi^{(1)}}$. Note that the assumption $\lim_{|z|\to 1} \rho(\phi^{(1)}(z), \phi^{(2)}(z)) = 0$ and the boundedness of $\psi^{(1)}C_{\phi^{(1)}}$ imply that $\psi^{(1)}C_{\phi^{(2)}}$ is bounded on H_v^∞. Thus we can find $C_1 > 0$ such that

$$C \sup_{z\in\mathbb{D}} |(r_s, q_s)^{\#}(z)| \le C_1.$$

Since we assume

$$\lim_{|z|\to 1} \rho(\phi^{(1)}(z), \phi^{(2)}(z)) = 0,$$

we can find a $1 > R_1 > 0$ such that $\rho(\phi^{(1)}(z), \phi^{(2)}(z)) \le r_0$ whenever $|z| > R_1$, and r_0 is the constant in Lemma 2. Therefore, we also have that $\sup_{z\in D} \rho(\phi^{(1)}(z), \phi^{(2)}(z)) \le \lambda < 1$ and thus the proof of [13] Lemma 4.1 yields that $\sup_{z\in\mathbb{D}} \rho(r_s(z), r_t(z)) \to 0$ if $s \to t$. Finally, this means that we can find $\delta_1 > 0$ such that if $|s-t| < \delta_1$, then

$$T_1 = C \sup_{z\in\mathbb{D}} |(r_s, q_s)^{\#}(z)| \rho(r_s(z), r_t(z)) < \frac{\varepsilon}{2}.$$

Let us now consider the term T_2. For every $z \in \mathbb{D}$ with $|z| > R_1$, so $\rho(\phi^{(1)}(z), \phi^{(2)}(z)) \le r_0$, we have by Lemma 7 that

$$\begin{aligned}|(r_s, q_s)^{\#}(z) - (r_t, q_t)^{\#}(z)| &\le |(r_s, q_s)^{\#}(z) - (\phi^{(1)}, \psi^{(1)})^{\#}(z)| + |(r_t, q_t)^{\#}(z) - (\phi^{(1)}, \psi^{(1)})^{\#}(z)| \\ &\le sM|(\phi^{(1)}, \psi^{(1)})^{\#}(z) - (\phi^{(2)}, \psi^{(2)})^{\#}(z)| \\ &+|(\phi^{(1)}, \psi^{(1)})^{\#}(z)| \left(\frac{M(1-s)\rho(\phi^{(1)}(z), \phi^{(2)}(z))}{1 - s\rho(\phi^{(1)}(z), \phi^{(2)}(z))} + sM^2\rho(\phi^{(1)}(z), \phi^{(2)}(z)\right) \\ &+tM|(\phi^{(1)}, \psi^{(1)})^{\#}(z) - (\phi^{(2)}, \psi^{(2)})^{\#}(z)| \\ &+|(\phi^{(1)}, \psi^{(1)})^{\#}(z)| \left(\frac{M(1-t)\rho(\phi^{(1)}(z), \phi^{(2)}(z))}{1 - t\rho(\phi^{(1)}(z), \phi^{(2)}(z))} + tM^2\rho(\phi^{(1)}(z), \phi^{(2)}(z)\right)\end{aligned}$$

By our assumptions, we can find $R_2 \ge R_1 > 0$ such that if $|z| > R_2$, then

$$\begin{aligned}\sup_{|z|>R_2} \max\{t, s\} C|(\phi^{(1)}, \psi^{(1)})^{\#}(z) - (\phi^{(2)}, \psi^{(2)})^{\#}(z)| &< \frac{\varepsilon}{10}, \\ \sup_{|z|>R_2} |(\phi^{(1)}, \psi^{(1)})^{\#}(z)| \left(\frac{M(1-t)\rho(\phi^{(1)}(z), \phi^{(2)}(z))}{1 - t\rho(\phi^{(1)}(z), \phi^{(2)}(z))} + tM^2\rho(\phi^{(1)}(z), \phi^{(2)}(z)\right) &< \frac{\varepsilon}{10}, \\ \sup_{|z|>R_2} |(\phi^{(1)}, \psi^{(1)})^{\#}(z)| \left(\frac{M(1-s)\rho(\phi^{(1)}(z), \phi^{(2)}(z))}{1 - s\rho(\phi^{(1)}(z), \phi^{(2)}(z))} + sM^2\rho(\phi^{(1)}(z), \phi^{(2)}(z)\right) &< \frac{\varepsilon}{10}.\end{aligned}$$

Moreover, if $|z| \le R_2$, we can find $\delta_2 > 0$ such that if $|s-t| < \delta_2$, then

$$\sup_{|z|\le R_2} |(r_s, q_s)^{\#}(z) - (r_t, q_t)^{\#}(z)| < \frac{\varepsilon}{10}.$$

Thus

$$\begin{aligned} T_2 &\le \sup_{|z|\le R_2} |(r_s,q_s)^{\#}(z)-(r_t,q_t)^{\#}(z)| + \sup_{|z|>R_2} sM|(\phi^{(1)},\psi^{(1)})^{\#}(z)-(\phi^{(2)},\psi^{(2)})^{\#}(z)| \\ &+ \sup_{|z|>R_2} |(\phi^{(1)},\psi^{(1)})^{\#}(z)| \left(\frac{M(1-s)\rho(\phi^{(1)}(z),\phi^{(2)}(z))}{1-s\rho(\phi^{(1)}(z),\phi^{(2)}(z))} + s\ M^2\rho(\phi^{(1)}(z),\phi^{(2)}(z))\right) \\ &\le \sup_{|z|>R_2} tM|(\phi^{(1)},\psi^{(1)})^{\#}(z)-(\phi^{(2)},\psi^{(2)})^{\#}(z)| \\ &+ \sup_{|z|>R_2} |(\phi^{(1)},\psi^{(1)})^{\#}(z)| \left(\frac{M(1-t)\rho(\phi^{(1)}(z),\phi^{(2)}(z))}{1-t\rho(\phi^{(1)}(z),\phi^{(2)}(z))} + t\ M^2\rho(\phi^{(1)}(z),\phi^{(2)}(z))\right) \\ &< \frac{\varepsilon}{2}. \end{aligned}$$

The claim follows. □

Remark. If the condition $\lim_{|z|\to 1}\rho(\phi^{(1)}(z),\phi^{(2)}(z)) = 0$ holds, then condition (c) in Theorem 4 is valid if and only if the difference $\psi^{(1)}C_{\phi^{(1)}} - \psi^{(2)}C_{\phi^{(2)}} : H_v^\infty \to H_v^\infty$ is compact.

The Bloch spaces are defined as follows

$$\mathcal{B} := \{f \in H(\mathbb{D});\ \|f\| = \sup_{z\in\mathbb{D}}(1-|z|)|f'(z)| < \infty\}$$

and

$$\mathcal{B}_0 := \{f \in \mathcal{B};\ \lim_{|z|\to 1}(1-|z|)|f'(z)| = 0\}.$$

Provided we identify functions that differ by a constant, $||\cdot||$ becomes a norm and $\mathcal{B}$ a Banach space. The map $S : \mathcal{B} \to H^\infty_{1-|z|}$ given by $S(f) = f'$ is an onto isometry.

Since $C_{\phi^{(1)}}$ and $C_{\phi^{(2)}}$ are always bounded operators on $\mathcal{B}$ and $C_{\phi^{(1)}} = S^{-1}\circ(\phi^{(1)})'C_{\phi^{(1)}}\circ S$, we get that $(\phi^{(1)})'C_{\phi^{(1)}}$, $(\phi^{(2)})'C_{\phi^{(2)}}$ are bounded on $H^\infty_{1-|z|}$ and

$$||C_{\phi^{(1)}} - C_{\phi^{(2)}}||_{L(\mathcal{B})} = ||(\phi^{(1)})'C_{\phi^{(1)}} - (\phi^{(2)})'C_{\phi^{(2)}}||_{L(H^\infty_{1-|z|})}.$$

Nieminen (see the proof of Theorem 3.2 in [22]) has proved that $\lim_{|z|\to 1}\rho(\phi^{(1)}(z),\phi^{(2)}(z)) = 0$ implies that $\lim_{|z|\to 1}(1-|z|)\left|\frac{(\phi^{(1)})'(z)}{1-|\phi^{(1)}(z)|} - \frac{(\phi^{(2)})'(z)}{1-|\phi^{(2)}(z)|)}\right| = 0$. Using Theorem 8 we obtain the following result that should be compared with Corollary 4.6 in [13].

Corollary 9 *Let $\phi^{(1)},\phi^{(2)} \in \mathcal{B}_0$ and $\lim_{|z|\to 1}\rho(\phi^{(1)}(z),\phi^{(2)}(z)) = 0$. Then the operators $C_{\phi^{(1)}}$ and $C_{\phi^{(2)}}$ belong to the same path component of the set of composition operators on $\mathcal{B}$.*

5. Comparing topologies

Let v be a typical weight satisfying condition (L1). Then by [6, Theorem 2.3], every composition operator C_φ is continuous. We let $\mathcal{C}$ denote the space of all continuous composition operators (on H_v^∞ or on H^∞). Our purpose is to compare the topologies τ_∞, τ_v and τ_w induced by the Banach spaces $L(H^\infty)$, $L(H_v^\infty)$ and $L(H_w^\infty)$ respectively on $\mathcal{C}$, where also w is a typical weight satisfying condition (L1). Our results complement those of Saksman and Sundberg in [23].

Proposition 10 *Let v be a radial weight on $\mathbb{D}$ satisfying the Lusky condition $(L1)$ and let ϕ, ψ be analytic self-maps of $\mathbb{D}$. Then we have*

(a) $\frac{1}{2}||C_\phi - C_\psi||_{L(H^\infty)} \le \sup_{z\in\mathbb{D}}\rho(\phi(z),\psi(z)) = \frac{4||C_\phi - C_\psi||_{L(H^\infty)}}{4+||C_\phi - C_\psi||^2_{L(H^\infty)}} \le ||C_\phi - C_\psi||_{L(H^\infty)}$.

(b) $||C_\phi - C_\psi||_{L(H_v^\infty)} \approx \max\left\{\sup_{z\in D}\frac{v(z)}{\tilde{v}(\phi(z))}\rho(\phi(z),\psi(z)), \sup_{z\in\mathbb{D}}\frac{v(z)}{\tilde{v}(\psi(z))}\rho(\phi(z),\psi(z))\right\}$.

Proof. (*a*) This follows from Proposition 4 in [19] (see also [10], Lemma 8).

(*b*) This follows from (the proof of) Proposition 2 in [7]. □

Since every composition operator C_ϕ is continuous on H_v^∞, we have that $||C_\phi||_{L(H_v^\infty)} = \sup_{z\in\mathbb{D}} \frac{v(z)}{\tilde{v}(\phi(z))} < \infty$. Let $(\psi_k)_k$ be a sequence of analytic self-maps of $\mathbb{D}$ such that C_{ψ_k} tends to C_ϕ in $L(H^\infty)$. By Proposition 9 (a) we have that $\sup_{z\in\mathbb{D}} \rho(\phi(z), \psi_k(z)) \to 0$ as $k \to \infty$. Further, using the first part of Lemma 2, we obtain $\frac{v(\phi(z))}{\tilde{v}(\psi_k(z))} \le M$ for all $z \in \mathbb{D}$ and k large enough. Therefore $\sup_{z\in\mathbb{D}} \frac{v(z)}{\tilde{v}(\psi_k(z))} \le M\ ||C_\phi||_{L(H_v^\infty)}$ for k large enough. Hence using Proposition 10 (b), we obtain the following estimate

$$||C_{\psi_k} - C_\phi\|_{L(H_v^\infty)} \le C \sup_{z\in\mathbb{D}} \rho(\phi(z), \psi_k(z)) \to 0, \quad \text{as} \quad k \to \infty.$$

Corollary 11 *Let v be a typical weight satisfying condition $(L1)$. Then τ_∞ is finer than τ_v.*

Corollary 12 *Let v be a typical weight on $\mathbb{D}$ satisfying condition $(L1)$. If there is an analytic self-map φ of $\mathbb{D}$ with $\|\varphi\|_\infty = 1$ and such that C_φ is compact on H_v^∞, then τ_v is strictly coarser than τ_∞.*

Proof. By assumption there is an analytic self-map φ of $\mathbb{D}$ with $\|\varphi\|_\infty = 1$ such that C_φ is compact on H_v^∞. Put $\varphi_k(z) := (1 - \frac{1}{k})\varphi(z)$ for every $z \in \mathbb{D}$. Since $||\varphi_k||_\infty \le 1 - \frac{1}{k}$, C_{φ_k} is compact on H^∞ for every $k \in \mathbb{N}$, but $C_\varphi \in L(H^\infty)$ is not compact on H^∞, as $\|\varphi\|_\infty = 1$.

On the other hand, $|\varphi_k(z)| \le |\varphi(z)|$, so $\tilde{v}(\varphi_k(z)) \ge \tilde{v}(\varphi(z))$ for all $z \in D$. Fix $\varepsilon > 0$ and let $f \in H_v^\infty$ with $||f||_v \le 1$. Then there are $0 < r_0 < 1$ and $k_0 \in \mathbb{N}$ such that

$$\sup_{z\in\mathbb{D}} v(z)|f(\varphi(z)) - f(\varphi_k(z))| \le C \max\left\{\sup_{z\in D} \frac{v(z)}{\tilde{v}(\varphi(z))}\rho(\varphi(z), \varphi_k(z)), \sup_{z\in\mathbb{D}} \frac{v(z)}{\tilde{v}(\varphi_k(z))}\rho(\varphi(z), \varphi_k(z))\right\}$$

$$\le C \sup_{|z|\le r_0} \frac{v(z)}{\tilde{v}(\varphi(z))}\rho(\varphi(z), \varphi_k(z)) + C \sup_{|z|>r_0} \frac{v(z)}{\tilde{v}(\varphi(z))}\rho(\varphi(z), \varphi_k(z))$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for every $k \ge k_0$ since by assumption C_φ is compact on H_v^∞ and $\sup_{|z|\le r_0} \rho(\varphi(z), \varphi_k(z)) \to 0$ if $k \to \infty$. Thus, $C_{\varphi_k} \to C_\varphi$ in $L(H_v^\infty)$ but $C_{\varphi_k} \not\to C_\varphi$ in $L(H^\infty)$. □

Corollary 13 *If v is a weight of the form $v(z) = (1 - |z|)^\alpha$, $z \in \mathbb{D}, \alpha > 0$, then τ_∞ is strictly finer than τ_v.*

Proof. By Corollary 11, the topology τ_∞ is finer than τ_v. Shapiro [24] pp. 49-50 constructs an analytic self-map φ of $\mathbb{D}$ with $\|\varphi\|_\infty = 1$ satisfying $\frac{1-|z|}{1-|\varphi(z)|} \to 0$ as $|z| \to 1$. We can apply [6, Corollary 3.4] to conclude that C_φ is compact on H_v^∞. The conclusion follows from Corollary 12. □

Observe that [6, Corollary 3.4] implies that there are the same compact composition operators on all the spaces H_v^∞ with $v(z) = (1 - |z|)^\alpha$, $z \in \mathbb{D}$, for all $\alpha > 0$.

Remark 14 By Corollary 12 it is obvious that if C_ϕ and C_{ϕ_1} are path connected in $\mathcal{C} \subset L(H_v^\infty)$, then they are also path connected in $\mathcal{C} \subset L(H^\infty)$, but in general the converse is not true, as the following example shows.
We choose ϕ and ϕ_1 such that ϕ touches $\partial\mathbb{D}$ in 1 and ϕ_1 touches $\partial\mathbb{D}$ in -1, $\|\phi\|_\infty = \|\phi_1\|_\infty = 1$ and $\lim_{|z|\to 1} \frac{1-|z|}{1-|\phi(z)|} = \lim_{|z|\to 1} \frac{1-|z|}{1-|\phi_1(z)|} = 0$ (see [24] pp. 49-50). Hence C_ϕ and C_{ϕ_1} are compact and thus in the same path component of $\mathcal{C}(H_v^\infty)$. But obviously, $\sup_{z\in\mathbb{D}} \rho(\phi(z), \phi_1(z)) = 1$, hence by [19] Theorem 2, C_ϕ and C_{ϕ_1} are not in the same path component of $\mathcal{C}(H^\infty)$.

A non negative function $g(x)$ defined on an interval I of the real line is called *almost decreasing* if there is $C > 0$ such that, if $x, y \in I$ satisfy $x < y$, then $g(y) \le Cg(x)$.

Theorem 15 *Let v and w be typical weights satisfying condition $(L1)$ such that w/v is almost decreasing with respect to $|z|$. Then τ_v is finer than τ_w on the set $\mathcal{C}_0$ of composition operators C_φ such that $\varphi(0) = 0$.*

Proof. Select a sequence $(\psi_k)_k$ of analytic self-maps of $\mathbb{D}$ such that C_{ψ_k} tends to C_ϕ in $L(H_v^\infty)$ and $\psi_k(0) = \phi(0) = 0$. Hence $|\phi(z)| \le |z|$ and $|\psi_k(z)| \le |z|$ for all $z \in \mathbb{D}$ and all k. Since w/v is almost decreasing, we can find an $M > 0$ such that for all $z \in \mathbb{D}$ and all k,

$$\frac{w(z)}{w(\phi(z))} \le M\frac{v(z)}{v(\phi(z))} \quad \text{and} \quad \frac{w(z)}{w(\psi_k(z))} \le M\frac{v(z)}{v(\psi_k(z))}.$$

Since the weights v and w are typical and satisfy condition (L1), we can replace the weights in the denominators by the corresponding associated weights in the inequalities above, increasing the constant M. By Proposition 10, we conclude

$$||C_{\psi_k} - C_\phi||_{L(H_w^\infty)} \le C_1 \max\left\{\sup_{z\in D} \frac{w(z)}{\tilde{w}(\phi(z))}\rho(\phi(z), \psi_k(z)), \sup_{z\in\mathbb{D}} \frac{v(z)}{\tilde{w}(\psi_k(z))}\rho(\phi(z), \psi_k(z))\right\} \le$$

$$C_1 M \max\left\{\sup_{z\in D} \frac{v(z)}{\tilde{v}(\phi(z))}\rho(\phi(z), \psi_k(z)), \sup_{z\in\mathbb{D}} \frac{v(z)}{\tilde{v}(\psi_k(z))}\rho(\phi(z), \psi_k(z))\right\} \le C_2 C_1 M ||C_{\psi_k} - C_\phi||_{L(H_v^\infty)},$$

from where the conclusion follows.

□

We conclude this section with a remark about composition operators $C_\phi \in \mathcal{C}$ which are isolated in $L(H_v^\infty)$. To do this fix a typical weight v satisfying condition $(L1)$ and, for $\varepsilon > 0$ set

$$E_{\phi,\varepsilon,v} := \{(z_n)_n \subset \mathbb{D};\ |\phi(z_n)| \to 1,\ \frac{v(z_n)}{\tilde{v}(\phi(z_n))} \ge \varepsilon\}.$$

Let $\overline{E}_{\phi,\varepsilon,v}$ denote the set of limit points of sequences in $E_{\phi,\varepsilon,v}$. Clearly $\overline{E}_{\phi,\varepsilon,v}$ is a subset of $\partial\mathbb{D}$.

Theorem 16 *Let v be a typical weight satisfying condition $(L1)$ such that v is continuously differentiable with respect to $|z|$. Moreover let ϕ be an analytic self-map of $\mathbb{D}$. If there is $\varepsilon > 0$ such that the set $\overline{E}_{\phi,\varepsilon,v}$ has Lebesgue measure strictly positive, then C_ϕ is isolated in the set of composition operators $\mathcal{C}$ in $L(H_v^\infty)$.*

Proof. It is well-known that if $\phi \ne \phi_1$, then the set $\{z \in \partial\mathbb{D};\ \phi(z) = \phi_1(z)\}$ is a null set for the Lebesgue measure. By assumption there is $\varepsilon > 0$ such that the set $E_{\phi,\varepsilon,v}$ has positive measure. Accordingly, there is a sequence $(z_n)_n \subset \mathbb{D}$, and a point $a \in \overline{E}_{\phi,\varepsilon,v}$ such that $z_n \to a$ and $\phi(z_n) \to b \in \partial\mathbb{D}$, but $\phi_1(z_n) \not\to b$. Therefore we can find a subsequence of $(z_n)_n$, which we still denote in the same way, such that $\phi_1(z_n) \to c \ne b$. Hence $\lim_{n\to\infty} \rho(\phi(z_n), \phi_1(z_n)) = 1$. Next, consider $h_n(z) := \varphi_{\phi_1(z_n)}(z) f_n(z)$ for every $z \in \mathbb{D}$ and every $n \in \mathbb{N}$, where f_n lies in the unit ball of H_v^∞ and satisfies $|f_n(\phi(z_n))| = \frac{1}{\tilde{v}(\phi(z_n))}$ for every $n \in \mathbb{N}$. Thus each h_n belongs to the unit ball of H_v^∞ and we obtain

$$\begin{aligned} \|C_\phi - C_{\phi_1}\| &\ge \limsup_{n\to\infty} v(z_n)|h_n(\phi(z_n)) - h_n(\phi_1(z_n))| \\ &= \limsup_{n\to\infty} \frac{v(z_n)}{\tilde{v}(\phi(z_n))}\rho(\phi(z_n), \phi_1(z_n)) \ge \varepsilon. \end{aligned}$$

Thus, the claim follows. □

Acknowledgment. The research of José Bonet was partially supported by FEDER and MEC Project MTM 2007-62643 and Generalitat Valenciana project Prometeo/2008/101. Part of the present work was done during a stay of Bonet at the University of Paderborn in August 2008. The support of the Alexander von Humboldt Foundation is greatly appreciated.

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