Convolution operators on quasianalytic classes of Roumieu type

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Abstract. Extending previous work of Braun, Meise, and Vogt, and of Meyer, we characterize those convolution operators that are surjective on the space \(E_{\omega}(\mathbb{R})\) of all quasianalytic \(\{\omega\}\)-ultradifferentiable functions of Roumieu type. We also investigate \(\{\omega\}\)-ultraddifferential operators on \(E_{\omega}[a,b]\) for compact intervals.

1. Introduction

For a weight function \(\omega\) let \(E_{\omega}(\mathbb{R})\) denote the space of all \(\{\omega\}\)-ultradifferentiable functions of Roumieu type on \(\mathbb{R}\). Then each \(\mu \in E'_{\omega}(\mathbb{R})\) induces a convolution operator \(T_\mu : E_{\omega}(\mathbb{R}) \to E_{\omega}(\mathbb{R})\). If \(\omega\) is non-quasianalytic, i.e., if \(E_{\omega}(\mathbb{R})\) contains non-trivial functions with compact support, then Braun, Meise, and Vogt [7] characterized those convolution operators \(T_\mu\) that are surjective on \(E_{\omega}(\mathbb{R})\). Though the arguments that were used in [7] rely heavily on the existence of fundamental solutions for surjective convolution operators, Meyer [21] proved a similar result for convolution operators \(T_\mu\) for which \(\mu \in E'_{\omega}(\mathbb{R})\) is supported by the origin, even for quasianalytic weight functions \(\omega\). In both articles, the proofs are based on properties of the projective limit functor due to Palamodov [24] and the sequence space representation for the kernels of slowly decreasing convolution operators \(T_\mu\) given by Meise [15].

In the present paper we show in Theorem 3.10 that the characterization, given in [7] also holds for quasianalytic weight functions \(\omega\). More precisely, we prove that for each weight function \(\omega\) and \(\mu \in E'_{\omega}(\mathbb{R})\) the convolution operator \(T_\mu\) is surjective on \(E_{\omega}(\mathbb{R})\) if and only if the Fourier-Laplace transform \(\hat{\mu}\) of \(\mu\) is \(\{\omega\}\)-slowly decreasing and the zero set \(V(\hat{\mu})\) of \(\hat{\mu}\) can be decomposed as \(V(\hat{\mu}) = V_0 \cup V_1\) such that

\[
\lim_{|a| \to \infty, a \in V_0} \frac{|\text{Im} \ a|}{\omega(a)} = 0 \quad \text{and} \quad \liminf_{|a| \to \infty, a \in V_1} \frac{|\text{Im} \ a|}{\omega(a)} > 0.
\]

The proof uses the better understanding of the slowly decreasing conditions that was achieved by Monn [22], Bonet, Galbis, and Meise [2], and Bonet, Galbis, and Monm [3] together with results about the derived functor of the projective limit functor and about \((LF)\)-spaces, due to Vogt [29] and to Wengenroth [31]. Applying the Fourier-Laplace transform and methods from Meise [14] and [15] again together with a recent result of Vogt [30] and Bonet and Domanski [1], we also show that a convolution operator \(T_\mu\) acting surjectively on \(E_{\omega}(\mathbb{R})\) admits a continuous linear right inverse only if

\[
\lim_{|a| \to \infty, a \in V(\hat{\mu})} |\text{Im} \ a|/\omega(a) = 0.
\]

We also investigate \(\{\omega\}\)-ultradifferentiable operators \(T_\mu\) on \(E_{\omega}(\mathbb{R})\) and on \(E_{\omega}[a,b]\) for compact intervals \([a,b]\) with \(a < b\) and we show that such an operator is slowly decreasing if and only if \(T_{\mu,[a,b]} : E_{\omega}[a,b] \to E_{\omega}(\mathbb{R})\) is surjective for all \(a,b \in \mathbb{R}\) with \(a < b\). Whenever this condition is satisfied then \(\ker T_{\mu,[a,b]}\) is isomorphic to the strong dual of a nuclear power series space of finite type. If in addition \(\lim_{|\zeta| \to \infty, \zeta \in V(\hat{\mu})} |\text{Im} \ \zeta|/\omega(\zeta) = 0\) then the restriction map \(\varrho : \ker T_\mu \to \ker T_{\mu,[a,b]}\) is an isomorphism for each \(a < b\).

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2. Preliminaries

In this section we introduce the notation that will be used throughout the entire paper.

2.1. Weight functions. A function \( \omega : \mathbb{R} \to [0, \infty] \) is called a weight function if it is continuous, even, increasing on \([0, \infty[\), and if it satisfies \( \omega(0) = 0 \) and also the following conditions:

*(a)* There exists \( K \geq 1 \) such that \( \omega(2t) \leq K \omega(t) + K \).

*(b)* \( \omega(t) = o(t) \) as \( t \) tends to infinity.

*(c)* \( \log(t) = o(\omega(t)) \) as \( t \) tends to infinity.

*(d)* \( \varphi : t \mapsto \omega(e^t) \) is convex on \([0, \infty[\).

If a weight function \( \omega \) satisfies

\[
\int_1^\infty \frac{\omega(t)}{t^2} \, dt = \infty
\]

then it is called a quasianalytic weight. Otherwise it is called non-quasianalytic.

A weight function \( \omega \) satisfies the condition \( (\alpha_1) \) if

\[
\sup_{\lambda \geq 1} \limsup_{t \to \infty} \frac{\omega(\lambda t)}{\lambda \omega(t)} < \infty.
\]

This condition was introduced by Petzsche and Vogt \([25]\) and is equivalent to the existence of \( C_1 > 0 \) such that for each \( W \geq 1 \) there exists \( C_2 > 0 \) such that

\[
\omega(Wt + W) \leq WC_1 \omega(t) + C_2, \quad t \geq 0.
\]

The radial extension \( \tilde{\omega} \) of a weight function \( \omega \) is defined as

\[
\tilde{\omega} : \mathbb{C}^n \to [0, \infty[, \quad \tilde{\omega}(z) := \omega(|z|).
\]

It will also be denoted by \( \omega \) in the sequel, by abuse of notation. The Young conjugate of the function \( \varphi = \varphi_\omega \), which appears in \( (\delta) \), is defined as

\[
\varphi^*(x) := \sup\{xy - \varphi(y) : y > 0\}, \quad x \geq 0.
\]

2.2. Example. The following functions are easily seen to be weight functions:

1. \( \omega(t) := |t|(|\log(e + |t|)|)^{-\alpha}, \quad \alpha > 0 \).

2. \( \omega(t) := |t|^\alpha, \quad 0 < \alpha < 1 \).

3. \( \omega(t) = \max(0, (\log t)^s), \quad s > 1 \).

2.3. Ultradifferentiable functions defined by weight functions. Let \( \omega \) be a given weight function. For a compact subset \( K \) of \( \mathbb{R}^N \) and \( m \in \mathbb{N} \) denote by \( C^\infty(K) \) the space of all \( C^\infty \)-Whitney jets on \( K \), define

\[
\mathcal{E}^m_{\omega}(K) := \{ f \in C^\infty(K) : \|f\|_{K,m} := \sup_{x \in K} \sup_{\alpha \in \mathbb{N}^n_0} |f^{(\alpha)}(x)| \exp\left(-\frac{1}{m} \varphi^*(m|\alpha|)\right) < \infty \},
\]

and let

\[
\mathcal{E}_{\omega}(K) := \text{ind}_{m \to \infty} \mathcal{E}^m_{\omega}(K)
\]

which is a (DFN)-space.

For an open set \( G \) in \( \mathbb{R}^N \), define the space \( \mathcal{E}_{\omega}(G) \) of all \( \omega \)-ultradifferentiable functions of Roumieu type on \( G \) as

\[
\mathcal{E}_{\omega}(G) := \{ f \in C^\infty(G) : \text{For each } K \subset G \text{ compact there is } m \in \mathbb{N} \text{ so that } \|f\|_{K,m} < \infty \}.
\]

It is endowed with the topology given by the representation

\[
\mathcal{E}_{\omega}(G) = \text{proj}_{K} \mathcal{E}_{\omega}(K),
\]

where \( K \) runs over all compact subsets of \( G \).

Note that \( \mathcal{E}_{\omega}(G) \) is a countable projective limit of (DFN)-spaces, which is ultrabornological, reflexive and complete. This follows from Rösner \([26]\), Satz 3.25 and Vogt \([30]\), Theorem 3.4.
The space \( \mathcal{E}_\omega(G) \) of all \( \omega \)-ultradifferentiable functions of Beurling type on \( G \) is defined as
\[
\mathcal{E}_\omega(G) := \{ f \in C^\infty(G) : \text{for each } K \subset G \text{ compact and } m \in \mathbb{N} \\
p_{K,m}(f) := \sup_{x \in K} \sup_{\alpha \in \mathbb{N}_0^n} |f^{(\alpha)}(x)| \exp \left(-m\omega^*(\frac{|\alpha|}{m})\right) < \infty\}.
\]

It is easy to check that \( \mathcal{E}_\omega(G) \) is a Fréchet space if we endow it with the locally convex topology given by the semi-norms \( p_{K,m} \).

If a statement holds in the Beurling and the Roumieu case then we will use the notation \( \mathcal{E}_\ast(G) \). It means that in all cases \( * \) can be replaced either by \( (\omega) \) or by \( \{\omega\} \).

2.4. Definition. Let \( \omega \) be a weight function and \( G \) an open convex set in \( \mathbb{R}^N \).
(a) We define the space \( A(\omega) \) by
\[
A(\omega) := \{ f \in H(\mathbb{C}) : \exists n \in \mathbb{N} : \|f\|_n := \sup_{z \in \mathbb{C}} |f(z)| \exp(-n\omega(z)) < \infty \}
\]
and endow it with its natural (LB)-topology. Then \( A(\omega) \) is an (DFN)-space. We also define the Fréchet space
\[
A(\omega) := \{ f \in H(\mathbb{C}) : \forall n \in \mathbb{N} : \|f\|_n := \sup_{z \in \mathbb{C}} |f(z)| \exp(-\frac{1}{n}\omega(z)) < \infty \}.
\]

(b) For each compact set \( K \) in \( G \), the support functional of \( K \) is defined as
\[
h_K : \mathbb{R}^N \rightarrow \mathbb{R}, \quad h_K(x) := \sup\{\langle x, y \rangle : y \in K\}.
\]

(c) For \( K \) as in (b) and \( \lambda > 0 \) let
\[
A(K, \lambda) := \{ f \in H(\mathbb{C}^N) : \|f\|_{K,\lambda} := \sup_{z \in \mathbb{C}^N} |f(z)| \exp(-h_K(\text{Im} z) - \lambda \omega(|z|)) < \infty \}
\]
and define
\[
A(\omega)(\mathbb{C}^N, G) := \text{ind}_{K,n} A(K, n)
\]
\[
A(\omega)(\mathbb{C}^N, G) := \text{ind}_{K,n} A(K, n).
\]

It is easy to check that \( A(K, \lambda) \) is a Banach space, that \( A(\omega)(\mathbb{C}^N, G) \) is an (LB)-space, that \( A(\omega) \) is a Fréchet space, and that \( A(\omega)(\mathbb{C}^N, G) \) is an (LF)-space.

2.5. The Fourier-Laplace Transform. Let \( \omega \) be a weight function and let \( G \) be an open convex set in \( \mathbb{R}^N \). For each \( u \in \mathcal{E}_\ast(G)' \) it is easy to check that
\[
\hat{u} : \mathbb{C}^N \rightarrow \mathbb{C}, \quad \hat{u}(z) := u_x(e^{-i(x,z)})
\]
is an entire function which belongs to \( A_\ast(\mathbb{C}^N, G) \) and that
\[
\mathcal{F} : \mathcal{E}_\ast(G) \rightarrow A_\ast(\mathbb{C}^N, G), \quad \mathcal{F}(u) := \hat{u},
\]
is linear and continuous.

The following result was proved for \( N = 1 \) by Meyer [20] and for general \( N \) in the Roumieu case by Rösser [26]. For a unified proof we refer to Heinrich and Meise [10], Theorems 3.6 and 3.7.

2.6. Theorem. For each weight function \( \omega \) satisfying \( \omega(t) = o(t) \) as \( t \) tends to infinity and each convex open set \( G \subset \mathbb{R}^N \) the Fourier-Laplace transform
\[
\mathcal{F} : \mathcal{E}_\ast(G) \rightarrow A_\ast(\mathbb{C}^N, G)
\]
is a linear topological isomorphism.

2.7. Convolution Operators. For \( \mu \in \mathcal{E}_\ast(\mathbb{R})' \), \( \mu \neq 0 \), and \( \varphi \in \mathcal{E}_\ast(\mathbb{R}) \) we define
\[
\hat{\mu}(\varphi) := \mu(\hat{\varphi}), \quad \hat{\varphi}(x) := \varphi(-x), \quad x \in \mathbb{R}.
\]
The convolution operator \( T_\mu : \mathcal{E}_\ast(\mathbb{R}) \rightarrow \mathcal{E}_\ast(\mathbb{R}) \) is defined by
\[
T_\mu(f) := \hat{\mu} \ast f, \quad (T_\mu(f))(x) := \hat{\mu}(f(x - .)), \quad x \in \mathbb{R}.
\]
It is a well-defined, linear, continuous operator; see Meyer [20] and [21]. For \( g \in A_c(\mathbb{C}, \mathbb{R}) \) we define the multiplication operator \( M_g : A_c(\mathbb{C}, \mathbb{R}) \to A_c(\mathbb{C}, \mathbb{R}) \) by \( M_g(f) = gf \). It is well-known that for \( \mu \in \mathcal{E}_c(\mathbb{R}) \) we have on \( \mathcal{E}_c(\mathbb{R})' : F \circ T^\mu_n = M_\mu \circ F \).

2.8. **Definition.** Let \( X = \text{ind}_{n \to} X_n \) be an \((\text{LF})\) space.

(a) \( X \) is called sequentially retractive if for each convergent sequence \( (x_j)_{j \in \mathbb{N}} \) in \( X \) there exists \( n \in \mathbb{N} \) such that \( (x_j)_{j \in \mathbb{N}} \) lies in \( X_n \) and converges there.

(b) \( X \) is called boundedly stable if on each set which is bounded in some \( X_n \) all but finitely many of the step topologies coincide.

From Wengenroth [31], Theorem 6.4 and Corollary 6.7, we recall the following equivalences which we will use in section 3.

2.9. **Theorem.** Let \( X = \text{ind}_{n \to} X_n \) be an \((\text{LF})\) space and let \( (\| \cdot \|_{n,k})_{k \in \mathbb{N}} \) be a fundamental sequence of semi-norms for \( X_n \). Then the following assertions are equivalent:

1. \( X \) is sequentially retractive.
2. There exist absolutely convex zero neighborhoods \( U_n \) in \( X_n \) for \( n \in \mathbb{N} \) such that \( U_n \subset U_{n+1} \) and such that for each \( n \in \mathbb{N} \) there exists \( m \geq n \) such that \( X \) and \( X_m \) induces the same topology on \( U_n \).
3. \( X \) is boundedly stable and satisfies the condition \( (P^*_3) \), i.e.,

\[
\forall n \in \mathbb{N} \exists m \geq n \exists k \geq m \exists N \in \mathbb{N} \forall M \in \mathbb{N} \exists K \in \mathbb{N}, \ S > 0 \forall x \in X_n : \|x\|_{m,M} \leq S(\|x\|_{k,K} + \|x\|_{n,N}).
\]

If \( X_n \) is a Fréchet-Montel space for each \( n \in \mathbb{N} \) then (1)-(3) are also equivalent to

4. \( X \) is regular, i.e., for each bounded set \( B \) in \( X \) there exists \( n \in \mathbb{N} \) such that \( B \subset X_n \) and is bounded there.
5. \( X \) is complete.

2.10. **Corollary.** For each weight function \( \omega \) and for each convex open set \( \Omega \subset \mathbb{R}^N \) the \((\text{LF})\)-space \( A_{\omega}(\mathbb{C}^N, \Omega) = \text{ind}_{n \to} A_{\omega}(K_n) \) satisfies the equivalent conditions of Theorem 2.9.

**Proof.** Since \( A_{\omega}(K_n) \) is a Fréchet-Montel space for each \( n \in \mathbb{N} \), it follows that \( \text{ind}_{n \to} A_{\omega}(K_n) \) is boundedly stable. In the proof of Rösner [26], Satz 3.25, it is shown that the system \( (\| \cdot \|_{n,k})_{n,k \in \mathbb{N}} \), defined by \( \|f\|_{n,k} : \sup_{z \in \mathbb{C}} |f(z)| \exp(-n |\text{Im} z| - \frac{1}{2} \omega(z)) \) satisfies the condition \( (P^*_3) \). Hence condition 2.9 (3) is satisfied and the corollary follows from Theorem 2.9. See also Bonet and Domanski [1]. \( \square \)

2.11. **Definition.** Let \( \alpha = (\alpha_j)_{j \in \mathbb{N}} \) be an increasing, unbounded sequence in \([0, \infty[\). For \( R \in \{0, \infty\} \) the power series spaces \( \Lambda_R(\alpha) \) are defined as

\[
\Lambda_R(\alpha) := \{ x = (x_j)_{j \in \mathbb{N}} \in \mathbb{C}^N : \|x\|_r := \sum_{j=1}^{\infty} |x_j| \exp(r \alpha_j) < \infty \forall r < R \}.
\]

\( \Lambda_{\infty}(\alpha) \) is called a power series space of infinite type, while \( \Lambda_0(\alpha) \) is said to be of finite type. Note that \( \Lambda_R(\alpha) \) is a Fréchet-Schwartz space for each \( \alpha \) and each \( R \).

3. **Surjectivity**

In this section we characterize the surjectivity of the convolution operators \( T_\mu : \mathcal{E}_c(\mathbb{R}) \to \mathcal{E}_c(\mathbb{R}) \). We show that some of the equivalences in Braun, Meise, and Vogt [7], Theorem 3.8, in combination with Corollary 2.8, that were proved in the non-quasianalytic case also hold in the quasianalytic case. We also extend the characterization which Meyer [21] gave for convolution operators \( T_\mu \) for which \( \mu \in \mathcal{E}_c'(\mathbb{R}) \) is supported by the origin, to arbitrary convolution operators. We begin by recalling several slowly decreasing conditions.

3.1. **Definition.** Let \( \omega \) be a weight function.

(a) \( F \in A_{\omega}(\mathbb{C}^N, \mathbb{R}^N) \) is called \( \{\omega\}\)-slowly decreasing, if for each \( m \in \mathbb{N} \) there exists \( R > 0 \) such that for each \( x \in \mathbb{R}^N \) with \( |x| \geq R \) there exists \( \xi \in \mathbb{C}^N \) satisfying \( |x - \xi| \leq \omega(x)/m \) such that \( |F(\xi)| \geq \exp(-\omega(\xi)/m) \).
(b) \( F \in A_\omega(C^N, \mathbb{R}^N) \) is called \( (\omega) \)-slowly decreasing, if there exists \( C > 0 \) such that for each \( x \in \mathbb{R}, |x| \geq C \), there exists \( \xi \in C^N \) such that
\[
|x - \xi| \leq C\omega(x) \text{ and } |F(\xi)| \geq \exp(-C) \Im \xi - C\omega(\xi).
\]

The significance of the \( \{\omega\} \)-slowly decreasing condition is explained by the following result.

3.2. PROPOSITION. Let \( \omega \) be a weight function and let \( F \in A_\omega(C^N, \mathbb{R}^N) \) be given. Then the following assertions are equivalent:

(a) \( F \) is \( \{\omega\} \)-slowly decreasing.

(b) There exists a weight function \( \sigma = o(\omega) \) such that \( F \in A_\sigma(C^N, \mathbb{R}^N) \) and such that \( F \) is \( (\sigma) \)-slowly decreasing.

(c) The multiplication operator \( M_F : A_\omega(C^N, \mathbb{R}^N) \to A_\omega(C^N, \mathbb{R}^N) \), \( M_F(g) := Fg \), has closed range.

(d) \( M_F^{-1} : FA_\omega(C^N, \mathbb{R}^N) \to A_\omega(C^N, \mathbb{R}^N) \) is sequentially continuous.

PROOF. (a) \( \Rightarrow \) (b): This holds by Bonet, Galbis, and Meise [2], Lemma 3.2, since in their proof the non-quasianalyticity of the weight function \( \omega \) is not needed (see, e.g., Heinrich and Meise [10], Corollary 3.8).

(b) \( \Rightarrow \) (c): Since every principal ideal in \( H(C^N) \) is closed, it suffices to show that the following assertion holds:

\[
|F(z)| \leq A \exp(B |\Im z| + B\sigma(z)), \quad z \in C^N
\]

and there exists \( \kappa \in \mathbb{N} \) such that for each \( p \in \mathbb{N} \) there exists \( C_p > 0 \) such that

\[
|g(z)| \leq C_p \exp(\kappa |\Im z| + 1/p \omega(z)), \quad z \in C^N.
\]

Next note that with \( n = 1 \) we get from Bonet, Galbis, and Momma [3], Proposition 2 (c), that there exist \( k, m \in \mathbb{N} \) and \( R > 0 \) such that for each \( z \in C^N, |z| \geq R \), there exists \( \zeta \in C^N \) with \( |\zeta - z| \leq |\Im z| + k\sigma(z) \) such that \( |F(\zeta)| \geq \exp(-m |\Im \zeta| - m\sigma(\zeta)) \).

Now we apply Hörmander [11], Lemma 3.2, with \( r := |\Im z| + k\sigma(z) \) to get for \( |z| \geq R \):

\[
\left| \frac{g(z)}{F(z)} \right| \leq \frac{\sup_{|w - z| \leq 4r} |g(w)| \sup_{|w - z| \leq 4r} |F(w)|}{(\sup_{|w - z| \leq r} |F(w)|)^2}.
\]

Using the upper estimate (3.2) for \( F \) and the lower estimate for \( |F(\zeta)| \) it follows that

\[
\left| \frac{g(z)}{F(z)} \right| \leq (\sup_{|w - z| \leq 4r} |g(w)|) \exp((5B|\Im z| + 2m|\Im \zeta| + 4k\sigma(z) + B\sigma(5|z| + 4k\sigma(z)) + 2m\sigma(\zeta)) .
\]

Obviously, \( |\zeta - z| \leq |\Im z| + k\sigma(\zeta) \) implies \( |\Im z| + k\sigma(z) \) and \( \sigma(\zeta) \leq \sigma(2|z| + k\sigma(z)) \).

Since \( \sigma \) is a weight function, it is easy to check that this implies the existence of \( A_1 \geq A \) and \( B_1 \geq B_0 \) such that by (3.3) we get for each \( p \in \mathbb{N} \)

\[
\left| \frac{g(z)}{F(z)} \right| \leq (\sup_{|w - z| \leq 4r} |g(w)|) A_1 \exp(B_1|\Im z| + B_1\sigma(z)) \leq A_1 C_p \exp(B_1|\Im z| + (\kappa + 4)|\Im z| + B_1\sigma(z) + 1/p |\zeta - z| + 4k\sigma(z)).
\]

Since \( \omega \) is a weight function and since \( \sigma = o(\omega) \), it follows from this, that \( g/F \) is in \( A_\sigma(C^N, \mathbb{R}^N) \). Hence we proved that (3.1) and consequently that (c) holds.

(c) \( \Rightarrow \) (d): By Corollary 2.10, the (LF)-space \( A_\omega(C^N, \mathbb{R}^N) = \text{ind}_{n=1} A_n \) is sequentially retractive.

The continuous linear map \( M_F : A_\omega(C^N, \mathbb{R}^N) \to A_\omega(C^N, \mathbb{R}^N) \) has closed range by the present hypothesis. Hence \( \text{im}(M_F) \cap A_n = M_F^{-1}(A_n) \) is closed in \( A_n \) for each \( n \in \mathbb{N} \). This means that \( \text{im}(M_F) \)
is stepwise closed in the sense of Floret [9], Theorem 6.4. By this theorem $M_F^{-1} : FA(\omega)(C^N, \mathbb{R}^N) \to A[\omega](C^N, \mathbb{R}^N)$ is sequentially continuous. Hence (d) holds.

(d) $\Rightarrow$ (a): Note first that for each $\lambda > 0$ the spaces $A[\omega](C^N, \mathbb{R}^N)$ and $A(\lambda \omega)(C^N, \mathbb{R}^N)$ are equal. Therefore, we may assume that there exists $t_0 > 0$ such that $\omega(t) \leq t/2$ for $t \geq t_0$. Next choose $k \in \mathbb{N}$ so that $F \in A_k$, where $A_k := A(B(0, k))$ in the notation of 2.4. To argue by contraposition, we assume that $F$ is not $\{\omega\}$-slowly decreasing. Then there exist $k \in \mathbb{N}$ and an unbounded sequence $(x_j)_j \subseteq \mathbb{R}^N$ for which $(|x_j|)_j \subseteq \mathbb{N}$ is increasing and for which the following holds for each $j \in \mathbb{N}$

$$|F(\zeta)| \leq \exp(-\frac{1}{k} \omega(\zeta))$$

for all $\zeta \in C^N$ with $|\zeta - x_j| < \frac{1}{k} \omega(x_j)$. We claim that this implies the following assertion:

(3.6) There exists a sequence $(g_j)_j \subseteq A_1$ which is unbounded in $A_n$ for each $n \in \mathbb{N}$, while $(M_F g_j)_j \subseteq \mathbb{N}$ is a null-sequence in $A_{k+1}$.

Obviously, (3.6) implies that $M_F^{-1} : FA(\omega)(C^N, \mathbb{R}^N) \to A[\omega](C^N, \mathbb{R}^N)$ is not sequentially continuous. Hence (d) implies (a).

To prove (3.6) we argue similarly as in Momm [22] (see also [2], Proposition 3.4) and define for $j \in \mathbb{N}$ and $R > 0$ the function $h_{j, R} : C^N \to \mathbb{R}$ by $h_{j, R}(z) := |\Im z|$ for $z \in C^N \setminus B(x_j, R)$ and for $z \in B(x_j, R)$ by

$$h_{j, R}(z) := \sup \{v(z) : v \text{ is plurisubharmonic on } B(x_j, R) \text{ and for each } \xi \in \partial B(x_j, R) \colon \limsup_{\zeta \to \xi} v(\zeta) \leq |\Im \xi| \}.$$ Then $h_{j, R}$ is continuous and plurisubharmonic on $C^N$. Next let $K \geq 1$ be the constant from 2.1 (a), choose $p \in \mathbb{N}$, $p \geq 2$, so large that $2K/p \leq 1/\kappa$, let $R_j := \frac{x_j}{p}$, and define $\varphi_j := h_{j, R_j}$. Since $|x_j| \to \infty$, we may assume that for all $j \in \mathbb{N}$ the following holds:

$$2 \leq \frac{\omega(x_j)}{2p}, \quad \frac{1}{\omega(x_j)} \leq \frac{\omega(x_j)}{8p^2}, \quad |x_j| \geq t_0 \quad \text{and hence} \quad \frac{\omega(x_j)}{p} + 1 \leq \frac{|x_j|}{2}.$$ Using Hörmander’s solution of the $\overline{\partial}$-problem (see Hörmander [12], Theorem 4.4.4) it follows as in Momm [23], 1.8, that there exists a constant $C_N > 0$ such that for each $j \in \mathbb{N}$ there exists $f_j \in H(C^N)$ satisfying the following estimates

$$|f_j(x_j)| \geq \exp\left(\inf_{|w-x_j| \leq 1} \varphi_j(w) - C_N \log(1 + |x_j|^2)\right)$$

and

$$|f_j(z)| \leq C_N \exp\left(\sup_{|w-z| \leq 1} \varphi_j(w) + C_N \log(1 + |z|^2)\right), \quad z \in C^N.$$ Next note that for $z \in C^N \setminus B(x_j, R_j + 1)$ we have

$$\sup_{|w-z| \leq 1} \varphi_j(w) = \sup_{|w-z| \leq 1} |\Im w| \leq |\Im z| + 1.$$ From this estimate and (3.9) we get for each $j \in \mathbb{N}$ and each $m \in \mathbb{N}$

$$\sup_{z \in C} |f_j(z)| \exp(-|\Im z| - \frac{1}{m} \omega(z)) < \infty.$$ Hence $f_j \in A_1$ for each $j \in \mathbb{N}$. Therefore, also the sequence $(g_j)_j \subseteq \mathbb{N}$ defined by

$$g_j := \exp\left(-\frac{\omega(x_j)}{8p}\right)f_j, \quad j \in \mathbb{N},$$

is in $A_1$. To show that it is not bounded in $A_n$ for any $n \in \mathbb{N}$, note that the function

$$v_j(z) := \frac{1}{2R_j^2} \left( |\Im z|^2 - |\Re z|^2 + R_j^2 \right)$$
is harmonic and satisfies \( v_j(z) \leq |\text{Im} \, z| \) for \( z \in \partial B(x_j, R_j) \), since \( x_j \in \mathbb{R}^N \). By the definition of \( \varphi_j \), this implies \( \varphi_j \geq v_j \) on \( B(x_j, R_j) \) and consequently by (3.7)

\[
\inf_{|w-z| \leq 1} \varphi_j(w) \geq \inf_{|w-z| \leq 1} v_j(w) \geq \frac{1}{2R_j} \frac{(1 + R_j^2)}{2} = \frac{R_j}{2} - \frac{1}{2R_j} = \frac{\omega(x_j)}{2p} - \frac{2p}{\omega(x_j)} \geq \frac{\omega(x_j)}{4p}.
\]

Since \( \log(1 + t^2) = o(\omega(t)) \) for \( t \) tending to infinity, there exists \( \delta > 0 \) such that \( \exp(-C_N \log(1 + |x_j|^2)) \geq \delta \exp(-\omega(x_j)/32) \) for each \( j \in \mathbb{N} \). Therefore, it follows from (3.8) that for each \( n \in \mathbb{N} \) and each \( m \in \mathbb{N} \) with \( m \geq 16p \) we have for each \( j \in \mathbb{N} \):

\[
\sup_{z \in \mathbb{C}^N} |g_j(z)| \exp(-n|\text{Im} \, z| - \frac{1}{m} \omega(z)) \geq \exp((-\frac{1}{8p} - \frac{1}{m} + \frac{1}{4p} \omega(x_j) - \log(1 + (x_j)^2)) \geq \frac{1}{32p} \omega(x_j)).
\]

This shows that \((g_j)_{j \in \mathbb{N}}\) is unbounded in \( A_n \) for any \( n \in \mathbb{N} \).

To prove that \((M_F(g_j))_{j \in \mathbb{N}}\) is a null-sequence in \( A_{k+1} \), note first that for \( z \in \mathbb{C}^N \setminus B(x_j, R_j + 1) \) we get from (3.10) and (3.9) that for each \( m \in \mathbb{N} \) we have

\[
|F(z)f_j(z)| \leq ||F||_{|\text{Im} \, K|, 1/m} \exp(k|\text{Im} \, z| + \frac{1}{m} \omega(z)) \exp(|\text{Im} \, z| + 1)
\]

\[
\leq ||F||_{|\text{Im} \, K|, 1/m} \exp((k + 1)|\text{Im} \, z| + \frac{1}{m} \omega(z)).
\]

To estimate \( Ff_j \) in \( B(x_j, R_j + 1) \), fix \( z \in B(x_j, R_j + 1) \). Then we have by the maximum principle and (3.7)

\[
\sup_{|w-z| \leq 1} \varphi_j(w) \leq \sup_{|w-z| \leq R_j + 2} \varphi_j(w) \leq \sup_{|w-z| \leq R_j + 2} |\text{Im} \, w| \leq R_j + 2 = \frac{\omega(x_j)}{p} + 2 \leq \frac{3\omega(x_j)}{2p}
\]

and also

\[
|\text{Re} \, z| \geq |x_j| - R_j - 1 = |x_j| - \frac{\omega(x_j)}{p} - 1 \geq \frac{|x_j|}{2}.
\]

Since \( \omega \) satisfies 2.1 (a), the last estimate implies \( \omega(x_j) \leq \omega(2\text{Re} \, z) \leq K(\omega(z) + K) \) and consequently

\[
\sup_{|w-z| \leq 1} \varphi_j(w) \leq \frac{3K\omega(z)}{2p} + \frac{3K}{2p}.
\]

From this, (3.5), and (3.9) we get the existence of \( C' \) such that for each \( j \in \mathbb{N} \):

\[
|F(z)f_j(z)| \leq C_N \exp(-\frac{1}{k} \omega(z) + \frac{3K\omega(z)}{2p} + \frac{3K}{2p} + C_N \log(1 + |z|^2))
\]

\[
\leq C' \exp((\frac{2K}{p} - \frac{1}{k}) \omega(z)) \leq C'.
\]

From (3.11) and (3.12) it follows that \((Ff_j)_{j \in \mathbb{N}}\) is bounded in \( A_{k+1} \). Since \((\exp(-\omega(x_j)/8p))_{j \in \mathbb{N}}\) is a null-sequence, we proved that \((M_F(g_j))_{j \in \mathbb{N}}\) is a null-sequence in \( A_{k+1} \). Hence the proof of (3.6) and also the one of the proposition is complete.

3.3. Corollary. Let \( \omega \) be a weight function and let \( F \in A_{\omega}(\mathbb{C}, \mathbb{R}) \) be given. Then the conditions (a) - (d) in Proposition 3.2 are equivalent to the following one:

(e) There exists a weight function \( \sigma \) satisfying \( \sigma = o(\omega) \) such that \( F \in A_{\omega}(\mathbb{C}, \mathbb{R}) \), and there exist \( \varepsilon, C, D > 0 \) such that for each component \( S \) of the set

\[
S(F, \varepsilon, C) := \{ z \in \mathbb{C} : |F(z)| < \varepsilon \exp(-C|\text{Im} \, z| - C\sigma(z)) \}
\]

the following estimates hold:

\[
\sup_{z \in S} |\text{Im} \, z| + C\sigma(z) \leq D(1 + \inf_{z \in S} |\text{Im} \, z| + \sigma(z)), \quad \sup_{z \in S} \omega(z) \leq D(1 + \inf_{z \in S} \omega(z)).
\]
Proof. To show that condition 3.2(b) implies the present condition (e), note that by Monm [22], Proposition 1, (e) follows from (b), except for the last estimate. This, however, follows from the diameter estimates given in the proof of Meise, Taylor, and Vogt [17], Lemma 2.3.

To show that (e) implies condition 3.2(c) let $V(F) := \{a \in \mathbb{C} : F(a) = 0\}$ and denote for each $a \in V(F)$ by $\text{ord}(F,a)$ the order of vanishing of $F$ at $a$. Then consider the map
\[
g : A_{\omega}(\mathbb{C}, \mathbb{R}) \to \prod_{a \in V(F)} \mathbb{C}^{\text{ord}(F,a)}, \quad g(a) := (g(a), g'(a), \ldots, g^{(\text{ord}(F,a)-1)}(a))_{a \in V(F)}.
\]
It is easy to check that $g$ is linear and continuous. Hence $I_{\text{loc}}(F) := \ker g$ is closed in $A_{\omega}(\mathbb{C}, \mathbb{R})$. Thus, (d) follows if we show that $FA_{\omega}(\mathbb{C}, \mathbb{R}) = \text{im}(MF) = I_{\text{loc}}(F)$. To do so, note first that obviously we have $\text{im}(MF) \subset I_{\text{loc}}(F)$. For the converse inclusion fix $g \in I_{\text{loc}}(F)$. Then there exists $k \in \mathbb{N}$ such that for each $m \in \mathbb{N}$ there is $C_m > 0$ such that
\[
|g(z)| \leq C_m \exp(k|\text{Im} z| + \frac{1}{m} \omega(z)), \quad z \in \mathbb{C}.
\]
By (e), we can choose $\sigma, \varepsilon, C, D$ according to (e). Then note that $g \in I_{\text{loc}}(F)$ implies $g/F \in H(\mathbb{C})$. Since $\sigma = o(\omega)$, we get for each $m \in \mathbb{N}$ the existence of $C_m'$ such that for each $z \in \mathbb{C} \setminus S_\sigma(F, \varepsilon, C)$ the following estimate holds
\[
\left|\frac{g(z)}{F(z)}\right| \leq C_m' \exp(k|\text{Im} z| + \frac{1}{m} \omega(z)) \exp(C|\text{Im} z| + C\sigma(z))
\]
\[
\leq C_m' \exp((k + C)|\text{Im} z| + \frac{2}{m} \omega(z)).
\]
Now note that from (3.13) and the estimates in (e) it follows by the maximum principle that for each $m \in \mathbb{N}$ there exists $C_m''$ such that for each component $s$ of $S_\sigma(F, \varepsilon, C)$ and each $z \in s$ we get the estimate
\[
\left|\frac{g(z)}{F(z)}\right| \leq C_m'' \exp((k + C)\sup_{\zeta \in s} |\text{Im} \zeta| + \frac{2}{m} \sup_{\zeta \in s} \omega(\zeta))
\]
\[
\leq C_m'' \exp((k + C)D(1 + |\text{Im} z| + \sigma(z)) + \frac{2D}{m}(1 + \omega(z)))
\]
\[
\leq C_m'' \exp((k + C)D|\text{Im} z| + \frac{3D}{m} \omega(z)).
\]
Obviously, (3.13) and (3.14) imply that $g/F \in A_{\omega}(\mathbb{C}, \mathbb{R})$. Hence $g = F(g/F) \in FA_{\omega}(\mathbb{C}, \mathbb{R})$. \hfill \Box

In order to apply Proposition 3.2 we recall the following sequence spaces from Meise [15], 1.4.

3.4. Definition. Let $\alpha = (\alpha_j)_{j \in \mathbb{N}}$ and $\beta = (\beta_j)_{j \in \mathbb{N}}$ be sequences of nonnegative real numbers and let $E = (E_j)_{j \in \mathbb{N}}$ be a sequence of Banach spaces. For $R > 0$ and $m \in \mathbb{N}$ let
\[
K(E, R, m) := \{x \in (x_j)_{j \in \mathbb{N}} \in \prod_{j=1}^{\infty} E_j : \|x\|_{R,m} := \sup_{j \in \mathbb{N}} \|x_j\|_{E_j} \exp(-Ra_j - \beta_j/m) < \infty\}
\]
and define the Fréchet space $K(E, R, \alpha, \beta)$ and the (LF)-space $K(E, \alpha, \beta)$ by
\[
K(E, R, \alpha, \beta) := \text{proj}_{-m} K(E, R, m) \quad \text{and} \quad K(E, \alpha, \beta) := \text{ind}_{k \to \text{proj}_{-m} K(E, k, m)}.
\]
If $E_j = \mathbb{C}$ for each $j \in \mathbb{N}$, then we write $K(\alpha, \beta)$ instead of $K(E, \alpha, \beta)$.

3.5. Remark. If $\lim_{j \to \infty} \beta_j = \infty$ then for each $k \in \mathbb{N}$ the space $\text{proj}_{-m} K(k, m)$ is a Fréchet-Schwartz space. Note that by Meise [15], Example 1.9 (2), the (LF)-space $K(\alpha, \beta)$ is in fact an (LB)-space, whenever $\lim inf_{j \to \infty} \alpha_j/\beta_j > 0$.

Because of Corollary 3.3, we get from Meise [15], Theorem 2.6, the following holds (for more details we refer to the proof of Proposition 4.7 below):

3.6. Theorem. Let $\omega$ be a weight function and let $F \in A_{\omega}(\mathbb{C}, \mathbb{R})$ be $\{\omega\}$-slowly decreasing. Then $A_{\omega}(\mathbb{C}, \mathbb{R})/FA_{\omega}(\mathbb{C}, \mathbb{R})$ is either finite dimensional or isomorphic to $K(\alpha, \beta)$, for the sequences $\alpha$ and $\beta$ defined as
\[
\alpha := (|\text{Im} a_j|)_{j \in \mathbb{N}}, \quad \beta := (\omega(\alpha_j))_{j \in \mathbb{N}},
\]
where \((a_j)_{j \in \mathbb{R}}\) is an enumeration of the points in \(V(F)\) with each point repeated as many times as the multiplicity of the zero of \(F\) at this point.

From Braun, Meise, and Vogt [7], Proposition 3.7, and Vogt [28], Theorem 4.3, we recall the following result.

3.7. Proposition. Let \(\alpha\) and \(\beta\) be sequences of nonnegative real numbers such that \(\lim_{j \to -\infty} \beta_j = \infty\). Then \(K(\alpha, \beta)\) is complete if and only if there exists \(\delta > 0\) such that each limit point of the set \(\{\alpha_j/\beta_j : j \in \mathbb{N}, \beta_j \neq 0\}\) is contained in \([0] \cup [\delta, \infty[\).

3.8. Lemma. Let \(E = \text{ind}_{n \to} E_n\) be an (LF)-space which is sequentially retractive and for which each \(E_n\) is a Fréchet-Schwartz space. Let \(S : E \to E\) be a continuous linear operator for which \(S(E) \cap E_n\) is closed in \(E_n\) for each \(n \in \mathbb{N}\). Then the following assertions are equivalent:

1. \(S\) is an injective topological homomorphism.
2. \(S' : E' \to E'\) is surjective.
3. \((\text{LF})\)-space \(E/S(E) := \text{ind}_{n \to} E_n/(S(E) \cap E_n)\) is sequentially retractive.
4. \(E/S(E)\) is complete.
5. \(E/S(E)\) is regular.

Proof. (1) \(\Leftrightarrow\) (2): This holds by Floret [9], Theorem 6.2.

(1) \(\Rightarrow\) (3): By the present hypothesis, we have the following short algebraically exact sequence of (LF)-spaces with continuous linear maps

\[
0 \to E \xrightarrow{S} E \xrightarrow{q} E/S(E) \to 0,
\]

where \(S(E)\) carries the topology defined in (3) and where \(q\) is the quotient map. Next note that by Wengenroth [31], Theorem 6.4, \(E\) is sequentially retractive if and only if \(E\) is acyclic, a concept explained in [31] and Vogt [29], Section 1. Hence it follows from (3.15) and [29], Theorem 1.5, that \(E/S(E)\) is acyclic and consequently sequentially retractive. Thus (3) holds.

(3) \(\Rightarrow\) (1): This implication follows from (3.15) by Vogt [29], Theorem 1.4, if we show the following:

\[
(3.15) \quad 0 \to E \xrightarrow{S} E \xrightarrow{q} E/S(E) \to 0,
\]

where \(S(E)\) carries the topology defined in (3) and where \(q\) is the quotient map. Next note that by Wengenroth [31], Theorem 6.4, \(E\) is sequentially retractive if and only if \(E\) is acyclic, a concept explained in [31] and Vogt [29], Section 1. Hence it follows from (3.15) and [29], Theorem 1.5, that \(E/S(E)\) is acyclic and consequently sequentially retractive. Thus (3) holds.

For each \(n \in \mathbb{N}\) there is \(m \in \mathbb{N}\) such that \(S^{-1}(E_n) \subset E_m\).

To show this, we define on \(S(E)\) the (LF)-topology \(\tau\) by \((S(E), \tau) := \text{ind}_{n \to}(S(E) \cap E_n)\). Then the map \(S : E \to (S(E), \tau)\) is injective and has closed graph. Consequently, it is an injective topological homomorphism. By the continuity of \(S^{-1} : (S(E), \tau) \to E\) and Grothendieck’s factorization theorem we get for each \(n \in \mathbb{N}\) the existence of \(m \in \mathbb{N}\) such that

\[
S^{-1}(E_n) = S^{-1}(S(E) \cap E_n) \subset E_m.
\]

Thus, (3.16) holds and consequently (3) holds.

(3) \(\Leftrightarrow\) (4) \(\Leftrightarrow\) (5): This follows from Theorem 2.9. \(\square\)

3.9. Theorem. Let \(\omega\) be a weight function and let \(F \in A_{(\omega)}(\mathbb{C}, \mathbb{R})\) be \(\{\omega\}\)-slowly decreasing. Then the following conditions are equivalent:

1. \(M_F : A_{(\omega)}(\mathbb{C}, \mathbb{R}) \to A_{(\omega)}(\mathbb{C}, \mathbb{R})\) is an injective topological homomorphism.
2. There exists \(\delta > 0\) such that each limit point of the set \(\{1/\text{Im}\omega(a) : a \in V(F), \omega(a) \neq 0\}\) is contained in \([0] \cup [\delta, \infty[\).

Proof. Note that \(A_{(\omega)}(\mathbb{C}, \mathbb{R}) = \text{ind}_{n \to} A_n\), where each \(A_n\) is a Fréchet-Schwartz space. By Corollary 2.10, \(A_{(\omega)}(\mathbb{C}, \mathbb{R})\) is sequentially retractive. Since \(F\) is \(\{\omega\}\)-slowly decreasing, it follows from Proposition 3.2 that \(M_F\) has closed range. Thus, the hypotheses of Lemma 3.8 are fulfilled for \(S = M_F\) and \(E = A_{(\omega)}(\mathbb{C}, \mathbb{R})\). Moreover, the open mapping theorem for (LF)-spaces implies that \(A_{(\omega)}(\mathbb{C}, \mathbb{R})/FA_{(\omega)}(\mathbb{C}, \mathbb{R})\) and \(\text{ind}_{n \to} A_n/(A_n \cap FA_{(\omega)}(\mathbb{C}, \mathbb{R}))\) are topologically equal. Hence Lemma 3.8 implies that condition (1) is equivalent to the completeness of \(A_{(\omega)}(\mathbb{C}, \mathbb{R})/FA_{(\omega)}(\mathbb{C}, \mathbb{R})\). By Theorem 3.6 the latter space is isomorphic to \(K(\gamma, \delta)\). From the definition of the sequences \(\gamma\) and \(\delta\) in Theorem 3.6 and Proposition 3.7 it now follows that \(A_{(\omega)}(\mathbb{C}, \mathbb{R})/FA_{(\omega)}(\mathbb{C}, \mathbb{R})\) is complete if and only if condition (2) holds. Hence we proved the equivalence of (1) and (2). \(\square\)

3.10. Theorem. Let \(\omega\) be a weight function and let \(\mu \in \mathcal{E}_{(\omega)}(\mathbb{R})^\prime, \mu \neq 0\), be given. Then the following assertions are equivalent:
(1) $T_\mu : \mathcal{E}_1(\omega)(\mathbb{R}) \to \mathcal{E}_1(\omega)(\mathbb{R})$ is surjective.
(2) The following two conditions are satisfied:
   (a) $\mu$ is $\{\omega\}$-slowly decreasing,
   (b) There exists $\delta > 0$ such that each limit point of the set $\{|\text{Im } a/\omega(a) : a \in V(\mu), \omega(a) \neq 0\}$ is contained in $\{0\} \cup [\delta, \infty].$

Proof. (1) $\Rightarrow$ (2): Since the space $\mathcal{E}_1(\omega)(\mathbb{R})$ is ultrabornological and webbed, the surjectivity of $T_\mu$ implies that $T_\mu$ is open or equivalently a surjective topological homomorphism. By a result of Grothendieck (see Köthe [13], 32, 4.3), $T_\mu^*(\mathcal{E}_1(\omega)(\mathbb{R})')$ is weakly closed in $\mathcal{E}_1(\omega)(\mathbb{R})'$ and hence closed. Because of $\mathcal{F} \circ T_\mu = M_\mu \circ \mathcal{F},$ this implies that $M_\mu$ has closed range. Therefore, $\mu$ is $\{\omega\}$-slowly decreasing by Proposition 3.2. Hence condition (a) holds.

Moreover, also the hypotheses of Lemma 3.8 are fulfilled for $E = A_{1,\omega}(\mathbb{C}, \mathbb{R})$ and $S = M_\mu,$ since $A_{1,\omega}(\mathbb{C}, \mathbb{R})$ is sequentially retractive by Corollary 2.10. From 2.7 we know that

\begin{equation}
F^t \circ M_\mu^t = (T_\mu^t)^t \circ F^t = T_\mu \circ F^t.
\end{equation}

This shows that $M_\mu^t$ is surjective. Hence $M_\mu$ is an injective topological homomorphism, by Lemma 3.8. Consequently, Theorem 3.9 implies that (b) holds.

(2) $\Rightarrow$ (1): By Theorem 3.9 the conditions (a) and (b) imply that $M_\mu : A_{1,\omega}(\mathbb{C}, \mathbb{R}) \to A_{1,\omega}(\mathbb{C}, \mathbb{R})$ is an injective topological homomorphism. Hence the Theorem of Hahn-Banach implies that $M_\mu^t$ is surjective.

Since the space $\mathcal{E}_1(\omega)(\mathbb{R})$ is reflexive, we get from (3.17) that $T_\mu$ is surjective. $\square$

Of course, one wants to know which surjective convolution operators $\mathcal{E}_1(\omega)(\mathbb{R})$ admit a continuous linear right inverse. We were only able to prove the following necessary condition, which is a characterization in the non-quasianalytic case by Braun, Meise, and Vogt [7], Theorem 4.2.

3.11. Proposition. Let $\omega$ be a quasianalytic weight function which satisfies the condition (a1), let $\mu \in \mathcal{E}_{1,\omega}(\mathbb{R}), \mu \neq 0$ be given, and assume that $T_\mu : \mathcal{E}_1(\omega)(\mathbb{R}) \to \mathcal{E}_1(\omega)(\mathbb{R})$ is surjective. If $T_\mu$ admits a continuous linear right inverse, then

$$\lim_{a \in V(\omega) \atop |a| \to \infty} |\text{Im } a|/\omega(a) = 0.$$ 

Proof. If we assume that the present condition does not hold then we can find a sequence $\{(a_j) \in \mathbb{N}\}$ in $V(\mu)$ and $\delta > 0$ with $|\text{Im } a_j| \geq \delta \omega(a_j)$ for each $j \in \mathbb{N}.$ Proceeding by recurrence, we extract a subsequence of $(a_j)_{j \in \mathbb{N}},$ which we denote in the same way, such that

(i) $|a_{j+1}| \geq 4|a_j|,$ and for $n(t) := \text{card}\{j : |a_j| \leq t\},$
(ii) $n(t) \log t = o(\omega(t))$ as $t \to \infty.$

Applying [6], 1.7 and 1.8 (a), we find a weight function $\sigma_0(t)$ such that $n(t) \log t = o(\sigma_0(t))$ and $\sigma_0(t) = o(\omega(t))$ as $t \to \infty.$ As in [7], 3.11, we define

$$F(z) := \prod_{j=1}^{\infty} \left(1 - \frac{z}{a_j}\right), \quad z \in \mathbb{C}.$$ 

By Rudin [27], Theorem 15.6, $F$ is an entire function such that its set of zeros consists of the sequence $(a_j)_{j},$ and satisfies the following conditions:

(1) There exists $C > 0 : |F(z)| \leq C \exp(\sigma_0(z)),$ $z \in \mathbb{C}.$
(2) There exists $\varepsilon_0 > 0$ such that $|F(\zeta)| \geq \varepsilon_0 \exp(-\sigma_0(\zeta))$ for all $\zeta \in \mathbb{C} \setminus \bigcup_{j=1}^{\infty} B(a_j, 1).$
(3) There exist $\varepsilon_0 > 0, K_0 > 0$ such that for all $\zeta \in \mathbb{C}$ with $1 \leq |\zeta - a_j| \leq 2$ for some $j$:

$$|F(z)| \geq \varepsilon_0 \exp(-K_0 \sigma_0(z)), \quad z \in \mathbb{C}.$$ 

This can be achieved by the arguments given in [4], proof of Lemma 3.5, arguments based on Braun, Meise, and Vogt [7], 3.11. In particular, $F$ is $(\sigma_0)$-slowly decreasing by (ii).

Since each $a_j$ is a zero of $\mu(z),$ it follows that $g(z) := \mu(z)/F(z)$ is an entire function. Since $F$ is $(\sigma_0)$-slowly decreasing, we conclude $g \in A_{1,\sigma_0}(\mathbb{C}, \mathbb{R}).$ On the other hand, $\sigma_0(t) = o(\omega(t)),$ hence $A_{1,\sigma_0}(\mathbb{C}, \mathbb{R}) \subset A_{1,\omega}(\mathbb{C}, \mathbb{R}),$ and the latter space is an algebra. This yields that $M_\sigma : A_{1,\omega}(\mathbb{C}, \mathbb{R}) \to A_{1,\omega}(\mathbb{C}, \mathbb{R}), \mu_\sigma(h) := gh,$ is a continuous linear operator.
By hypothesis, $M_\omega : A_{(\omega)}(\mathbb{C}, \mathbb{R}) \rightarrow A_{(\omega)}(\mathbb{C}, \mathbb{R})$ admits a continuous left inverse $L_\omega$. The operator $L_F := L_\omega \circ M_\omega : A_{(\omega)}(\mathbb{C}, \mathbb{R}) \rightarrow A_{(\omega)}(\mathbb{C}, \mathbb{R})$ is continuous and it is a left inverse of $M_F$, since $L_F M_F(h) = h$ for each $h \in A_{(\omega)}(\mathbb{C}, \mathbb{R})$.

We define, for an entire function $f \in H(\mathbb{C})$, $g(f) := (f(a_j))_j \in \mathbb{C}^N$. Proceeding as we did in the proof of [4], Lemma 3.8 (a proof based on the method of the proof of Meise [14], Theorem 3.7), we conclude from properties (1), (2), and (3) of $F$ that

$$M_F A_{(\omega)}(\mathbb{C}, \mathbb{R}) = \ker g \cap A_{(\omega)}(\mathbb{C}, \mathbb{R}),$$

hence this principal ideal is closed, and the quotient $A_{(\omega)}(\mathbb{C}, \mathbb{R})/M_F A_{(\omega)}(\mathbb{C}, \mathbb{R})$ coincides with the sequence (LF)-space $G := K(\alpha, \beta)$ for $\alpha := (| \text{Im} a_j|)_j \in \mathbb{N}$ and $\beta := (\omega(a_j))_j \in \mathbb{N}$. Since $M_F : A_{(\omega)}(\mathbb{C}, \mathbb{R}) \rightarrow A_{(\omega)}(\mathbb{C}, \mathbb{R})$ has a continuous linear left inverse, we conclude that $G$ is isomorphic to a complemented subspace of $A_{(\omega)}(\mathbb{C}, \mathbb{R})$.

We now show that the (LF)-space $G$ coincides algebraically and topologically with the (LB)-sequence space

$$E := \{y \in \mathbb{C}^N : \exists m : \|y\|_m := \sup_{j \in \mathbb{N}} |y_j| \exp(-m|\text{Im} a_j|) < \infty\}.$$

Indeed, it is clear that $E \subset G$. On the other hand, if $x \in G$, there is $n \in \mathbb{N}$ such that for $k = 1$ we can find $C_1 > 0$ with

$$|x_j| \leq C_1 \exp(n|\text{Im} a_j| + \omega(a_j))$$

for each $j \in \mathbb{N}$. Since $|\text{Im} a_j| \geq \delta \omega(a_j)$ for each $j$, we select $m \in \mathbb{N}$, $m > n + \delta^{-1}$, we get

$$|x_j| \leq C_1 \exp(m|\text{Im} a_j|).$$

By the closed graph theorem $E = G$ also topologically.

This implies that $G$ is isomorphic to the dual of the power series space $\Lambda_\infty((| \text{Im} a_j|)_j \in \mathbb{N})$ of infinite type and is complemented in $A_{(\omega)}(\mathbb{C}, \mathbb{R}) \cong E_{(\omega)}(\mathbb{R})'$. This yields that $\Lambda_\infty((| \text{Im} a_j|)_j \in \mathbb{N})$ is isomorphic to a complemented subspace of $E_{(\omega)}(\mathbb{R})$. Since $G$ satisfies $(a_1)$, this implies by Vogt [30] or Bonet and Domanski [1], Corollary 2.5, that $\Lambda_\infty((| \text{Im} a_j|)_j \in \mathbb{N})$ has property $(\mathcal{P})$. This, however, is a contradiction. □

4. Ultradifferential operators on compact intervals

In this section we show that the surjectivity of $\{\omega\}$-ultradifferential operators on $E_{(\omega)}[a,b]$ is characterized by $\tilde{\mu}$ being $\{\omega\}$-slowly decreasing.

4.1. Definition. Let $\omega$ be a weight function and assume that for $\mu \in E'_{(\omega)}(\mathbb{R})$ its Fourier-Laplace transform $\hat{\mu}$ is in $A_{(\omega)}$. Then the operator $T_\mu$ will be called an $\{\omega\}$-ultradifferential operator since for each $f \in E_{(\omega)}(\mathbb{R})$ we have

$$T_\mu(f) = \sum_{j=0}^{\infty} \hat{\mu}(0)^j f^{(j)}.$$

4.2. Definition. For a weight function $\omega$ and for $R > 0$ define the Fréchet space $A_{(\omega,R)}$ of entire functions by

$$A_{(\omega,R)} := \text{proj}_{m \rightarrow} A([-R, R], \frac{1}{m}).$$

We also define the space

$$A_{(\omega,R)} := \text{ind}_{n \rightarrow} A([-R, R], n),$$

which is a (DFN)-space.

4.3. Remark. By Rösner [26], 2.19, for each weight function $\omega$ and each $R > 0$, the Fourier-Laplace transform $F : E_{(\omega)}[-R, R] \rightarrow A_{(\omega,R)}$ is a linear topological isomorphism.

4.4. Proposition. Let $\omega$ be a weight function. For $F \in A_{(\omega)}, F \neq 0$, the following conditions are equivalent:

1. $F$ is $\{\omega\}$-slowly decreasing.
2. For each $R > 0$ and each $g \in A_{(\omega,R)}$ which satisfies $g/F \in H(\mathbb{C})$, the function $g/F$ is in $A_{(\omega,R)}$.  

(3) For each $R > 0$ the multiplication operator
\[ M_F : A_{(\omega,R)} \to A_{(\omega,R)}, \quad M_F(g) := Fg, \]
has closed range.

(4) For each $R > 0$ the map $M_F$ defined in (3) is an injective topological homomorphism.

**Proof.** (1) ⇒ (2): Note first that a standard application of Braun, Meise, and Taylor [6], Lemma 1.7, implies the existence of a weight function $\sigma_1$ satisfying $\sigma_1 = o(\omega)$ such that $F \in A_{(\sigma_1)}$ for each weight function $\sigma$ which satisfies $\sigma_1 = o(\sigma)$. Since $g \in A_{(\omega,R)}$, we can find a weight function $\sigma_2$ and $C_2 > 0$ such that $\sigma_2 = o(\omega)$ and such that
\[ |g(z)| \leq C_2 \exp(R) \Im z + \sigma_2(z), \quad z \in \mathbb{C}. \]
Next note that because of the hypothesis (1) it follows from Proposition 3.2 that there exists a weight function $\sigma_3$ with $\sigma_3 = o(\omega)$ such that $F \in A_{(\sigma_1)}$ and $F$ is $(\sigma_3)$-slowly decreasing. Now choose a weight function $\sigma$ which satisfies $\sigma = o(\omega)$ and $\max(\sigma_1, \sigma_2, \sigma_3) \leq \sigma$. Then we have $g \in A_{(\sigma,R)}$, $F \in A_{(\sigma)}$ and that $F$ is $(\sigma)$-slowly decreasing. Since $g/F \in H(\mathbb{C})$ by hypothesis, it follows from [5], Lemma 4.6, that $g/F \in A_{(\sigma,R)} \subset A_{(\omega,R)}$. Hence we showed that (2) holds.

(2) ⇒ (3): Obviously, the inclusion map $J : A_{(\omega,R)} \to H(\mathbb{C})$ is linear and continuous and the principal ideal $FH(\mathbb{C})$ is closed in $H(\mathbb{C})$. Hence $J^{-1}(FH(\mathbb{C}))$ is closed in $A_{(\omega,R)}$. Because of $J^{-1}(FH(\mathbb{C})) = FA_{(\omega,R)} = MF(A_{(\omega,R)})$, this implies that (3) holds.

(3) ⇒ (4): Since $M_F$ is injective and since $A_{(\omega,R)}$ is a Fréchet space, this follows from the closed range theorem (see Meise and Vogt [19], 26.3).

(4) ⇒ (1): If we show that $M_F^{-1} : FA_{(\omega)}(\mathbb{C}, \mathbb{R}) \to A_{(\omega)}(\mathbb{C}, \mathbb{R})$ is sequentially continuous then it follows from Proposition 3.2 (d) that (1) holds. To do so, let $(Fh_j)_{j\in\mathbb{N}}$ be a sequence in $FA_{(\omega)}(\mathbb{C}, \mathbb{R})$ that satisfies $\lim_{j \to \infty} Fh_j = 0$. By Corollary 2.10, the inductive limit $A_{(\omega)}(\mathbb{C}, \mathbb{R}) = \underset{n \to \infty}{\text{ind}} A_{(\omega,n)}$ is sequentially retractive. Hence there exists $n \in \mathbb{N}$ such that $(Fh_j)_{j\in\mathbb{N}}$ is in fact a sequence in $A_{(\omega,n)}$ and converges to 0 in this space. Now (2) implies that $(h_j)_{j\in\mathbb{N}}$ converges to zero in $A_{(\omega,n)}$ and consequently in $A_{(\omega)}(\mathbb{C}, \mathbb{R})$.

4.5. **Corollary.** Let $\omega$ be a weight function and let $T_\mu \neq 0$ be an $\{\omega\}$-ultradifferentiable operator. Then the Fourier-Laplace transform $\hat{\mu}$ of $\mu$ is slowly decreasing if and only if for each $a, b \in \mathbb{R}$ with $a < b$ the convolution operator
\[ T_{\mu,[a,b]} : \mathcal{E}_{(\omega)}[a,b] \to \mathcal{E}_{(\omega)}[a,b] \]
is surjective.

**Proof.** Since $T_\mu$ commutes with translations, it is enough to prove the corollary for $[a,b] = [-R,R]$ and each $R > 0$. Since $\mathcal{E}_{(\omega)}[-R,R]$ is a (DFN)-space the strong dual of which is isomorphic to $A_{(\omega,R)}$ via Fourier-Laplace transform (by Remark 4.3) and since $\mathcal{F} \circ T_{\mu,[a,b]} = \mathcal{F} \circ T_{\mu} = M_\mu \circ \mathcal{F}$, the corollary follows from the closed range theorem (see, e.g., Meise and Vogt [18], 26.3).

4.6. **Lemma.** Let $\omega$ be a weight function and assume that $F \in A_{(\omega)}$ is $\{\omega\}$-slowly decreasing. Then there exists a weight function $\sigma$ satisfying $\sigma = o(\omega)$ such that $F \in A_{(\sigma)}$. Moreover, there exist $\varepsilon_0, C_0$, and $D > 0$ such that each connected component $S$ of the set
\[ S_\sigma(F, \varepsilon_0, C_0) := \{ z \in \mathbb{C} : |F(z)| < \varepsilon_0 \exp(-C_0 \sigma(z)) \} \]
satisfies
\[ \text{diam } S \leq D \inf_{z \in S} \sigma(z) + D \text{ and } \text{sup}_{z \in S} \omega(z) \leq D \inf_{z \in S} \omega(z) + D. \]

**Proof.** By Proposition 3.2 there exists a weight function $\sigma_1$ satisfying $\sigma_1 = o(\omega)$ such that $F \in A_{(\sigma_1)}(\mathbb{C}, \mathbb{R})$ and $F$ is $(\sigma_1)$-slowly decreasing. From Braun, Meise, and Taylor [6], Lemma 1.7, we get the existence of a weight function $\sigma_2$ satisfying $\sigma_2 = o(\omega)$ and $F \in A_{(\sigma_2)}$. Hence we can choose a weight function $\sigma$ which satisfies $\max(\sigma_1, \sigma_2) \leq \sigma$ and $\sigma = o(\omega)$. Then $F \in A_{(\sigma)}$ and $F$ is $(\sigma)$-slowly decreasing. Thus $F$ satisfies the hypotheses of [5], Lemma 4.2. Therefore, [5], Lemma 4.3, implies the existence of positive numbers $\varepsilon_0, C_0$, and $D$ such that for each component $S$ of $S_\sigma(F, \varepsilon_0, C_0)$ we have $\text{diam } S \leq D \inf_{z \in S} \sigma(z) + D$. To show that we also have
\[ \text{sup}_{z \in S} \omega(z) \leq D \inf_{z \in S} \omega(z) + D \]
(4.1)
for each component $S$ of $S_\sigma(F, \varepsilon_0, C_0)$, provided that $D > 0$ is large enough, we remark that the following was shown in the proof of [5], Lemma 4.3: There exist $m \in \mathbb{N}$ and $R_0 \geq 1$ such that for each $z_0 \in S_\sigma(F, \varepsilon_0, C_0)$ satisfying $|z_0| \geq R_0$ the connected component $S$ of $S_\sigma(F, \varepsilon_0, C_0)$ which contains $z_0$ satisfies

$$\text{diam } S \leq 4m\sigma(z_0).$$

It is no restriction to assume that $R_0$ is so large that from 2.1 $(\alpha)$ and $(\beta)$ and $\sigma = o(\omega)$ we get the existence of $L$ and $K_0 \geq 1$ such that

$$\sigma(t) \leq \omega(t) \leq Lt \text{ and } \omega(2t) \leq K_0\omega(t), \ t \geq R_0.$$ 

Next we fix a component $S$ of $S_\sigma(F, \varepsilon_0, C_0)$ such that $S \cap (\mathbb{C} \setminus B(0, R_0)) \neq \emptyset$ and we choose $z_0 \in S$ with $|z_0| \geq R_0$ as well as $z_1, z_2 \in \overline{S}$ such that

$$\inf_{z \in S} \omega(z) = \omega(z_1) \text{ and } \sup_{z \in S} \omega(z) = \omega(z_2).$$

In the proof of [5], Lemma 4.3, it was shown that then $|z_0| \leq 2|z_1|$. By our choices, this implies

$$|z_2| \leq |z_2 - z_1| + |z_1| \leq \text{diam } S + |z_1| \leq 4m\sigma(z_0) + |z_1| \leq 4m\omega(2|z_1|) + |z_1| \leq 4mK_0\omega(z_1) + |z_1| \leq (4mLK_0 + 1)|z_1|.$$ 

Since $\omega$ satisfies 2.1 $(\alpha)$, this estimate implies the existence of $K_1 \geq 1$ such that

$$\sup_{z \in S} \omega(z) = \omega(z_2) \leq \omega((4mLK_0 + 1)|z_1|) \leq K_1\omega(z_1) = K_1 \inf_{z \in S} \omega(z).$$

Since there are only finitely many components $S$ of $S_\sigma(F, \varepsilon_0, C_0)$ which are contained in $B(0, R_0)$, we proved (4.1), provided that we choose $D > 0$ large enough. 

4.7. PROPOSITION. Let $\omega$ be a weight function and let $F \in A_{\{\omega\}}$ be $\{\omega\}$-slowly decreasing. For $R > 0$ denote by $q_R : A_{\{\omega\}} \rightarrow A_{\{\omega_R\}} / FA_{\{\omega_R\}}$ and by $q : A_{\{\omega\}}(\mathbb{C}, \mathbb{R}) \rightarrow A_{\{\omega\}}(\mathbb{C}, \mathbb{R}) / FA_{\{\omega\}}(\mathbb{C}, \mathbb{R})$ the corresponding quotient maps. Let $J_R : A_{\{\omega_R\}} \rightarrow A_{\{\omega\}}(\mathbb{C}, \mathbb{R})$ be the inclusion map. Then for each $R > 0$ the map $J_R$ induces a continuous linear injection $j_R : A_{\{\omega_R\}} / FA_{\{\omega_R\}} \rightarrow A_{\{\omega\}}(\mathbb{C}, \mathbb{R}) / FA_{\{\omega\}}(\mathbb{C}, \mathbb{R})$ which satisfies $j_R \circ q_R = J_R \circ q$.

PROOF. Fix $R > 0$ and note that $FA_{\{\omega_r\}}$ is a closed linear subspace of $A_{\{\omega_r\}}$ by Proposition 4.4, while $FA_{\{\omega\}}(\mathbb{C}, \mathbb{R})$ is closed in $A_{\{\omega\}}(\mathbb{C}, \mathbb{R})$ by Proposition 2.4. Next note that the result holds trivially if $F$ has only finitely many zeros. Therefore, we assume from now on that $V(F) := \{a \in \mathbb{C} : F(a) = 0\}$ is an infinite set. Then we choose a weight function $\sigma$ and positive numbers $\varepsilon_0, C_0$, and $D$ according to Lemma 4.6 and we label the connected components $S$ of $S_\sigma(F, \varepsilon_0, C_0)$ which satisfy $S \cap V(F) \neq \emptyset$ in such a way that the sequence $\beta$, defined by

$$\beta_j := \sup_{z \in S_j} \omega(z), \ j \in \mathbb{N},$$

is increasing. Also, we define the sequence $\alpha$ by

$$\alpha_j := \sup_{z \in S_j} |\text{Im } z|, \ j \in \mathbb{N},$$

Then we define the sequence $E = (E_j)_{j \in \mathbb{N}}$ by

$$E_j := \prod_{b \in S_j \cap V(F)} C^{\text{ord}(F,b)}, \ j \in \mathbb{N},$$

and we let

$$\varrho_j : H^\infty(S_j) \rightarrow E_j, \ \varrho_j(f) := \left(\frac{1}{R!} f^{(k)}(b)\right)_{0 \leq k < \text{ord}(F,b)} |_{b \in S_j \cap V(F)}.$$ 

We endow $E_j$ with the quotient norm

$$\|\varrho_j(g)\| := \inf\{\|h\|_{H^\infty(S_j)} : \varrho_j(h) = \varrho_j(g), \ g \in H^\infty(S_j)\}.$$ 

Then $\varrho_j$ is linear, continuous, and surjective. If $f \in A_{\{\omega_R\}}$, then for each $m \in \mathbb{N}$ there exists $C_m > 0$ such that

$$|f(z)| \leq C_m \exp(R |\text{Im } z| + \frac{1}{m} \omega(z)), \ z \in \mathbb{C}.$$
Obviously, this implies that for each $m \in \mathbb{N}$ and each $j \in \mathbb{N}$ we have
\[ \|f|_{S_j}\|_{H^\infty(S_j)} \leq C_m \exp(Ra_j + \frac{1}{m} \beta_j). \]
Hence the map
\[ \varrho^R : A_{(\omega,R)} \to K(\mathbb{E}, R, \alpha, \beta), \quad \varrho^R(f) := (\varrho_j(f|_{S_j}))_{j \in \mathbb{N}} \]
is well-defined, linear, and continuous. By the definition of the spaces $A_{(\omega,R)}(\mathbb{C}, \mathbb{E}) = \text{ind}_{n \to \infty} A_{(\omega,n)}$ and $K(\mathbb{E}, R, \alpha, \beta) = \text{ind}_{n \to \infty} K(\mathbb{E}, n, \alpha, \beta)$ also the map
\[ \varrho : A_{(\omega)}(\mathbb{C}, \mathbb{E}) \to K(\mathbb{E}, R, \alpha, \beta), \quad \varrho(f) := (\varrho_j(f|_{S_j}))_{j \in \mathbb{N}} \]
is well-defined, linear, and continuous.

Next we claim that $\ker \varrho^R = FA_{(\omega,R)}$ and $\ker \varrho = FA_{(\omega)}(\mathbb{C}, \mathbb{E})$. Obviously, $FA_{(\omega,R)}$ is contained in $\ker \varrho^R$. To prove the converse inclusion, fix $g \in \ker \varrho^R$. Then $g/F$ is in $H(\mathbb{C})$. By Proposition 4.4 this implies that $g \in FA_{(\omega,R)}$. Since $A_{(\omega)}(\mathbb{C}, \mathbb{E}) = \text{ind}_{n \to \infty} A_{(\omega,n)}$, this implies $\ker \varrho = FA_{(\omega)}(\mathbb{C}, \mathbb{E})$.

To show that $\varrho^R$ is surjective, fix $y = (y_j)_{j \in \mathbb{N}}$ in $K(\mathbb{E}, R, \alpha, \beta)$. By the definition of the norm in $E_j$, we can choose $\lambda_j \in H^\infty(S_j)$ satisfying
\[ \varrho_j(\lambda_j) = y_j \quad \text{and} \quad \|y_j\|_{H^\infty(S_j)} \leq 2 \|y_j\|, \quad j \in \mathbb{N}. \]
Then we define
\[ \lambda : S_\sigma(F, \varepsilon_0, C_0) \to \mathbb{C}, \quad \lambda(z) = \lambda_j(z) \quad \text{if} \quad z \in S_j \quad \text{and} \quad \lambda(z) = 0 \quad \text{if} \quad z \in \mathbb{C} \setminus \bigcup_{j=1}^\infty S_j \]
and we claim that for each $m \in \mathbb{N}$ there exist $p \in \mathbb{N}$ and $C_m > 0$ such that
\[ \|\lambda(z)\| \exp(-R|\text{Im } z|) - \frac{1}{m} \omega(z)) \leq C_m \|y\|_{R,p}. \]
To prove this, fix $m \in \mathbb{N}$ and choose $p \geq 2Dm$. Since $\sigma = o(\omega)$, there exists $C_m > 0$ such that
\[ 2 \exp(RD\sigma(t) + (R + 1)D) \leq C_m \exp(\frac{D}{p} \omega(t)) \quad \text{for} \quad t \geq 0. \]
Then we get for each $j \in \mathbb{N}$ and each $z \in S_j$ the following estimate
\[ |\lambda_j(z)| \leq 2\|y_j\| \leq 2\|y\|_{R,p} \exp(R\beta_j + \frac{1}{p} \alpha_j) \]
\[ \leq 2\|y\|_{R,p} \exp(R|\text{Im } z| + R \text{diam } S_j + \frac{1}{p} (D\omega(z) + D)) \]
\[ \leq 2\|y\|_{R,p} \exp(R|\text{Im } z| + RD\sigma(z) + (R + 1)D + \frac{D}{p} \omega(z)) \]
\[ \leq C_m \|y\|_{R,p} \exp(R|\text{Im } z| + \frac{1}{m} \omega(z)), \]
which implies (4.2).

Next note that by Lemma 4.6 there exists $B > 0$ such that
\[ |F(z)| \leq B \exp(B\sigma(z)), \quad z \in \mathbb{C}. \]
Hence it follows from the proof of [5], Lemma 4.7, that there exist $\varepsilon_1, C_1 > 0$, $\chi \in C^\infty(\mathbb{C})$ and $A_0, B_0 > 0$ such that
\[ (4.3) \]
\[ 0 \leq \chi \leq 1, \quad \chi \equiv 1 \text{ on } S_\sigma(F, \varepsilon_1, C_1), \quad \text{Supp } \chi \subset S_\sigma(F, \varepsilon_0, C_0), \quad \text{and} \quad \left| \frac{\partial \chi}{\partial \xi} \right| \leq A_0 \exp(B_0\sigma(z)), \quad z \in \mathbb{C}. \]
Now define
\[ v := - \frac{1}{F} \frac{\partial}{\partial \xi}(\chi \lambda) = - \frac{1}{F} \frac{\partial \chi}{\partial \xi} \lambda \]
and note that $v$ is in $C^\infty(\mathbb{C})$ and vanishes on $S_\sigma(F, \varepsilon_1, C_1)$. Moreover, we get from (4.2) and (4.3) that for each $m \in \mathbb{N}$ there exist $p \in \mathbb{N}$ and $C_m > 0$ such that for each $z \in \mathbb{C}$ we have
\[ |v(z)| \leq \frac{1}{\varepsilon_1} A_0 C_m \|y\|_{R,p} \exp(R|\text{Im } z| + \frac{1}{m} \omega(z) + (B_0 + C_1)\sigma(z)). \]
Using Lemma 1.7 of Braun, Meise, and Taylor [6], we get the existence of a weight function \( \tau \geq \sigma \) and of \( A_1 > 0 \) such that
\[
|v(z)| \leq A_1 \exp(R \text{ Im } z + \tau(z)), \quad z \in C.
\]
Since \( \tau \) satisfies condition 2.1 (\( \gamma \)), this estimate implies
\[
\int_C (|v(z)| \exp(-R \text{ Im } z - 2\tau(z)))^2 dz < \infty.
\]
By Hörmander [12], Theorem 4.4.2, there exists \( g \in L^2_{\text{loc}}(C) \) which satisfies \( \frac{\partial g}{\partial z} = v \) and
\[
\int (|g(z)| \exp(-R \text{ Im } z - 2\tau(z) - \log(1 + |z|^2)))^2 dz < \infty. \tag{4.4}
\]
Since \( v \) is a \( C^\infty \)-function on \( C \) and since \( \frac{\partial}{\partial z} \) is elliptic, \( g \) belongs to \( C^\infty(C) \). By the choice of \( v \), we now get that \( f := \chi + gF \in C^\infty(C) \) and \( \frac{\partial f}{\partial z} = 0 \), i.e., \( f \in H(C) \). Now the estimates for \( \lambda \) in (4.2), for \( g \) in (4.4), and for \( F \) imply a weighted \( L^2 \)-estimate for \( f \) which can be converted by standard arguments to a sup-estimate which shows that \( f \) is in fact in \( A_{(\omega,R)} \). By the definition of \( f \) and \( \lambda \), we get
\[
g(f) = \left( g_j(f|S_j) \right)_{j \in \mathbb{N}} = \left( \gamma_j(\lambda_j) \right)_{j \in \mathbb{N}} = y.
\]
Hence we proved that \( g^R : A_{(\omega,R)} \to K(\mathbb{E}, R, \alpha, \beta) \) is surjective. Since \( K(\mathbb{E}, \alpha, \beta) = \text{id}_{C, \mathbb{R}} \) and \( A_{(\omega,R)}(\mathbb{C}, \mathbb{R}) = \text{id}_{C, \mathbb{R}} \), we also get that \( g : A_{(\omega)}(\mathbb{C}, \mathbb{R}) \to \mathbb{E} \) is surjective. Since \( \ker g^R = FA_{(\omega,R)} \) and \( \ker g = FA_{(\omega)}(\mathbb{C}, \mathbb{R}) \), classical open mapping theorems show that we can identify \( A_{(\omega,R)}/FA_{(\omega,R)} \) with \( K(\mathbb{E}, R, \alpha, \beta) \) and \( A_{(\omega)}(\mathbb{C}, \mathbb{R})/FA_{(\omega)}(\mathbb{C}, \mathbb{R}) \) with \( K(\mathbb{E}, \alpha, \beta) \). If we do this \( g \) and \( g^R \) are the corresponding quotient maps. Now note that by the definition of the maps \( g^R \) and \( g \), the following diagram, where \( j_R : K(\mathbb{E}, R, \alpha, \beta) \to K(\mathbb{E}, \alpha, \beta) \) denote the inclusion, is commutative
\[
A_{(\omega,R)}(\mathbb{C}, \mathbb{R})^R \xrightarrow{\text{id}} K(\mathbb{E}, R, \alpha, \beta) \\
\downarrow j_R \quad \quad \quad \quad \quad \quad \quad \quad \downarrow j_R \\
A_{(\omega)}(\mathbb{C}, \mathbb{R}) \xrightarrow{\theta} K(\mathbb{E}, \alpha, \beta).
\]
Thus the proof is complete. \( \square \)

4.8. Remark. Under the hypotheses of Proposition 4.7 we proved that for each \( R > 0 \) the space \( A_{(\omega,R)}/FA_{(\omega,R)} \) is topologically isomorphic to the Fréchet space \( K(\mathbb{E}, R, \alpha, \beta) \), as the proof of 4.7 shows.

4.9. Corollary. Let \( \omega \) be a weight function, let \( F \) be \( \{\omega\} \)-slowly decreasing, and assume that
\[
\lim_{|a| \to \infty, a \in V(F)} |\text{ Im } a|/|\omega(a)| = 0. \quad \text{Then for each } R > 0 \text{ the map } j_R, \text{ defined in Proposition 4.7,}
\]
\[
j_R : A_{(\omega,R)}/FA_{(\omega,R)} \to A_{(\omega)}(\mathbb{C}, \mathbb{R})/FA_{(\omega)}(\mathbb{C}, \mathbb{R}) \text{ is surjective and hence a linear topological isomorphism.}
\]

Proof. From the proof of Proposition 4.7 and the open mapping theorem it follows that we only have to show that \( K(\mathbb{E}, \alpha, \beta) \subset K(\mathbb{E}, R, \alpha, \beta) \). In fact we will show that \( K(\mathbb{E}, \alpha, \beta) \subset K(\mathbb{E}, 0, \alpha, \beta) \). To do so we fix \( y \in K(\mathbb{E}, \alpha, \beta) \). Then there exists \( n \in \mathbb{N} \) such that for each \( m \in \mathbb{N} \) there exists \( C_m > 0 \) such that for each \( j \in \mathbb{N} \)
\[
\|y_j\|_j \leq C_m \exp(n\alpha_j + \frac{1}{2m} \beta_j).
\]
Next choose a weight function \( \sigma = o(\omega) \) so that the assertions of Lemma 4.6 hold and for each \( j \in \mathbb{N} \) choose \( a_j \in S_j \). (If \( V(F) \) is finite, there is nothing to prove). Then we get from Lemma 4.6
\[
\alpha_j = \sup_{z \in S_j} |\text{ Im } z| \leq |\text{ Im } a_j| + \text{ diam } S_j \leq |\text{ Im } a_j| + D \text{ Im } a_j + D.
\]
Since \( \lim_{|a| \to \infty, a \in V(F)} |\text{ Im } a|/\omega(a) = 0 \), for each \( m \in \mathbb{N} \) there exists \( D_m > 0 \) such that
\[
|\text{ Im } a| \leq \frac{1}{4mn} \omega(a) + D_m, \quad a \in V(F)
\]
and we can choose \( K_m > 0 \) such that
\[
D \text{ Im } a + D \leq \frac{1}{4mn} \omega(t) + K_m, \quad t \geq 0.
\]
Then we get
\[ na_j + \frac{1}{2m} \beta_j \leq \frac{1}{4m} \omega(a_j) + \frac{1}{4m} \omega(a_j) + nD_m + K_m \leq \frac{1}{2m} \beta_j + nD_m + K_m \]
and hence
\[ \|y_j\| \leq C_m \exp(nD_m + K_m) \exp\left(\frac{1}{m} \beta_j\right), \quad j \in \mathbb{N}. \]
This shows that \( y \) is in fact in \( K(\mathbb{E}, 0, \alpha, \beta) \).

4.10. Proposition. Let \( \omega \) be a weight function and let \( T_\mu \neq 0 \) be an \( \{\omega\} \)-ultradifferentiable operator. If the Fourier-Laplace transform \( \hat{\mu} \) of \( \mu \) is slowly decreasing, then for each \( a, b \in \mathbb{R} \) with \( a < b \) the following assertions hold:

1. \( \ker T_{\mu,[a,b]} \) is isomorphic to \( \Lambda_0(\gamma)' \), where \( \gamma = (\omega(a_j))_{j \in \mathbb{N}} \) and where \( (a_j)_{j \in \mathbb{N}} \) counts the zeros of \( \hat{\mu} \) with multiplicities in such a way that \( (\omega(a_j))_{j \in \mathbb{N}} \) is increasing.

2. If \( \lim_{z \to -\infty, z \in \mathcal{V}(\hat{\mu})} |\Im(z)/\omega(z)| = 0 \) then the map \( \vartheta_{[a,b]} : \ker T_\mu \to \ker T_{\mu,[a,b]} \), \( \vartheta_{[a,b]}(f) := f|_{[a,b]} \), is an isomorphism.

Proof. Since \( T_\mu \) commutes with translations, it suffices to consider intervals of the form \( [-R,R] \) for \( R > 0 \). By Proposition 4.7 the short sequence
\[ 0 \to A(\omega, R) \xrightarrow{M_\mu} \tilde{A}(\omega, R) \xrightarrow{\vartheta_\mu} A(\omega, R)/\hat{\mu}A(\omega, R) \to 0 \]
of Fréchet-Schwartz spaces and continuous linear maps is exact. Hence its dual sequence is exact, too, by Meise and Vogt [18], Proposition 26.24. Since the spaces \( \mathcal{E}_\omega([-R,R]) \) are reflexive, it follows from Remark 4.3 and \( \hat{\mu}A(\omega, R) = \text{im} M_\mu = (\text{ker} M_\mu)^\perp \) that up to Fourier-Laplace transform the dual sequence can be identified with
\[ 0 \to \ker T_{\mu,[0,R]} \to \mathcal{E}_\omega([-R,R]) \xrightarrow{T_{\mu,[0,R]}} \mathcal{E}_\omega([-R,R]) \to 0. \]
Hence we get from Remark 4.8 that \( \ker T_{\mu,[0,R]} \) is isomorphic to \( (A(\omega, R)/\hat{\mu}A(\omega, R))' \cong (K(\mathbb{E}, R, \alpha, \beta))' \). Now note that \( K(\mathbb{E}, R, \alpha, \beta) \) is a nuclear Fréchet space which is isomorphic to \( K(\mathbb{E}, 0, \alpha, \beta) = \Lambda_0(\mathbb{E}, \beta) \) by an obvious diagonal transform. Now (1) follows from Meise [14], Proposition 1.4, by the definition of the sequence \( \mathbb{E} \) and the diameter estimates for the sets \( S_j \) in the proof of Proposition 4.7.

To prove (2), note that by the arguments in Meise [15], 3.4, we have \( (\ker T_\mu)' \cong \mathcal{E}_\omega'(\mathbb{R})/(\ker T_\mu)^\perp \cong A(\omega, \mathbb{C}, \mathbb{R})/\hat{\mu}A(\omega, \mathbb{C}, \mathbb{R}) \) via Fourier-Laplace transform. Hence for each \( R > 0 \) we have the following commutative diagram with exact rows:
\[
\begin{array}{ccccccc}
0 & \to & \ker T_\mu & \to & \mathcal{E}_\omega(\mathbb{R}) & \xrightarrow{T_\mu} & \mathcal{E}_\omega(\mathbb{R}) & \to & 0 \\
\downarrow \vartheta_{[-R,R]} & & \downarrow \vartheta_{[-R,R]} & & \downarrow \vartheta_{[-R,R]} & & \downarrow \vartheta_{[-R,R]} & & \downarrow \vartheta_{[-R,R]} \\
0 & \to & \ker T_{\mu,[0,R]} & \to & \mathcal{E}_\omega([-R,R]) & \xrightarrow{T_{\mu,[0,R]}} & \mathcal{E}_\omega([-R,R]) & \to & 0.
\end{array}
\]
If we dualize it and apply the Fourier-Laplace transform, the dual map of \( \vartheta_{[-R,R]} : \ker T_\mu \to \ker T_{\mu,[0,R]} \) corresponds to the map \( j_R : A(\omega, R)/\hat{\mu}A(\omega, R) \to A(\omega, \mathbb{C}, \mathbb{R})/\hat{\mu}A(\omega, \mathbb{C}, \mathbb{R}) \), defined in Proposition 4.7. As we showed in the proof of 4.7, \( j_R \) becomes the inclusion of \( K(\mathbb{E}, R, \alpha, \beta) \) in \( K(\mathbb{E}, \alpha, \beta) \), if we identify the corresponding quotient spaces with these vector-valued sequence spaces. Since \( \lim_{z \to -\infty, z \in \mathcal{V}(\hat{\mu})} |\Im(z)/\omega(z)| = 0 \) holds by hypothesis, it follows easily that
\[ K(\mathbb{E}, R, \alpha, \beta) = K(\mathbb{E}, 0, \alpha, \beta) = K(\mathbb{E}, \alpha, \beta) \]
as sets but also as locally convex spaces. Therefore, \( j_R \) is a linear topological isomorphism. Next note that \( \ker T_{\mu,[0,R]} \) is reflexive as closed subspace of a (DFS)-space. To see that also \( \ker T_\mu \) is reflexive, we argue as follows. By Theorem 3.10, the present hypotheses imply that \( T_\mu : \mathcal{E}_\omega(\mathbb{R}) \to \mathcal{E}_\omega(\mathbb{R}) \) is surjective. Since \( \text{Proj}^1 \mathcal{E}_\omega(\mathbb{R}) = 0 \) by Meyer [21], Theorem 3.7, (or Rössner [26], Satz 3.25) it follows from the long exact sequence theorem (see Wengenroth [31], Corollary 3.1.5) that \( \text{Proj}^1 \ker T_\mu = 0 \). Hence \( \ker T_\mu \) is ultrabornological by Wengenroth [31], Theorem 3.3.4. Therefore, the semi-reflexive space \( \ker T_\mu \) is reflexive. Hence \( \vartheta_{[-R,R]} : \ker T_\mu \to \ker T_{\mu,[0,R]} \) is an isomorphism, too.

4.11. Remark. If \( \omega \) is non-quasianalytic and \( T_\mu \) is a convolution operator on \( \mathcal{E}_\omega(\mathbb{R}) \) which is surjective, then Theorem 4.2 in Braun, Meise, and Vogt [7] shows that \( T_\mu \) admits a continuous linear right inverse if and only if \( \lim_{|\alpha| \to \infty, \alpha \in \mathcal{V}(\hat{\mu})} |\Im(\alpha)/\omega(\alpha)| = 0 \). In the quasianalytic case, so far we only have
the necessity of this condition by Proposition 3.11. For \( \{\}\)-ultradifferential operators, the sufficiency of this condition will follow from Proposition 4.10 as soon as one knows that for some \( R > 0 \) the operator \( T_{\omega,\{-R,R\}} \) admits a continuous linear right inverse. Because then one can apply the formal arguments that were used in [5], Corollary 4.11, in the Beurling case and which were first applied by Domanski and Vogt [8], Theorem 4.7, in the real-analytic case. However, it is still open, whether \( T_{\omega,\{-R,R\}} \) admits a continuous linear right inverse. The main difficulty is that the linear topological structure of \( \mathcal{E}_{\omega}[-R,R] \) or equivalently of \( A_{\omega,R} \) is not known.

**Problem:** Is \( A_{\omega,R} \) isomorphic to a power series space of finite type?

**Remark.** It follows easily from Meise and Taylor [16], Lemma 1.10, that \( A_{\omega,R} \) has the property (DN). If \( \omega \) is non-quasianalytic then [16], Corollary 6.4, in connection with [18], Proposition 29.18, shows that \( A_{\omega,R} \) is isomorphic to a power series space of finite type.

### References


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