TOEPLITZ OPERATORS ON THE SPACE OF ANALYTIC FUNCTIONS WITH LOGARITHMIC GROWTH

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ABSTRACT. Continuous and compact Toeplitz operators for positive symbols are characterized on the space H_V^∞ of analytic functions with logarithmic growth on the open unit disc of the complex plane. The characterizations are in terms of the behaviour of the Berezin transform of the symbol. The space H_V^∞ was introduced and studied by Taskinen. The Bergman projection is continuous on this space in a natural way, which permits to define Toeplitz operators. Sufficient conditions for general symbols are also presented.

1. INTRODUCTION.

The purpose of this paper is to study Toeplitz operators T_φ in the space H_V^∞ of analytic functions on the unit disc which grow logarithmically at the boundary. Though the space H_V^∞ is quite far from the Bergman Hilbert space A^2, our results will be astonishingly similar to that basic case. For example, in the case of positive symbols φ we obtain a necessary and sufficient condition for the continuity in terms of a growth condition of the Berezin transform $\tilde{\varphi}$ of φ, see Theorem 3.1. Moreover, the difference between the A^2 and H_V^∞–cases is most typical: in the former, $\tilde{\varphi}$ must be bounded on $\mathbb{D}$ whereas in the latter a growth like a power of the logarithm of the boundary distance is allowed (see [22], Chapter 6, for the basic results for Toeplitz operators on A^2). In Theorem 3.2 we also characterize compact Toeplitz operators (with positive φ) on H_V^∞ again in terms of a Berezin–type condition. It is remarkable how beautifully the abstract definition of continuous and compact operators in countable inductive limit of Banach spaces is tranformed into concrete conditions on logarithmic weight families, see (3.1) and (3.2). For general not necessarily positive φ we are able to provide sufficient conditions for continuity and compactness, see Remark 3.4, Proposition 4.1 and Theorem 4.2.

The space H_V^∞ and the related space L_V^∞ were introduced and studied by the second author in [15] as substitutes of the usual sup–normed spaces H^∞ and L^∞, to avoid the well–known discontinuity problem of the Bergman projection P with respect to the sup–norm. These spaces are topological algebras and they have the properties that H_V^∞ is a closed subspace of L_V^∞ and that P is a continuous projection from L_V^∞ onto H_V^∞. In the article [15] the space H_V^∞ was introduced using weighted sup-seminorms and it was proved, as a consequence of abstract results due to Bierstedt, Meise and Summers [3], that it is in fact a countable inductive limit (in fact a union) of weighted Banach spaces of analytic functions on the disc with compact linking maps. This description is better adapted to the

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study of operators defined on the space, and is the one we outline in Section 2. The research of Taskinen [15] in this direction was continued by several authors [6, 8, 9, 16]. From the harmonic analysis point of view, the advantage of using the space H^∞_V is that very simple pointwise estimates can all the time be applied. The emerging logarithmic singularities are absorbed by the inductive limit. A reader who is not familiar with inductive limits may be satisfied by the observation that their presence only amounts to statements on weight functions like "for all powers of log" or "for some power of log" instead of the usual unique weight functions in the Banach space case. Properties of operators in inductive limits are explicitely formulated in terms of containment of sets of analytic functions satisfying certain weight estimates.

Toeplitz operators on spaces of analytic functions have been investigated since long. We refer the reader e.g. to the books by Böttcher and Silbermann [5] and by Zhu [22]. For Toeplitz operators on Bergman spaces, in particular on A^2 of various domains, see [1, 10, 12, 14, 17, 20, 21, 18]. As for other spaces, let us also mention the recent papers [7, 19].

2. Notation.

All function spaces are defined on the open unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$, unless otherwise stated. By dA we denote the normalized two–dimensional Lebesgue measure on $\mathbb{D}$, and by $L^p := L^p(dA)$ the space of p–integrable functions on the disc $\mathbb{D}$ with respect to the measure dA. Here $1 \le p \le \infty$. Moreover, A^p stands for the Bergman space, which is the closed subspace of L^p consisting of analytic functions.

The Bergman projection P is the integral operator

$$Pf(z) := \int_{\mathbb{D}} \frac{f(\zeta)}{(1 - z\bar{\zeta})^2} dA(\zeta), \qquad , \quad z \in \mathbb{D}, \tag{2.1}$$

defined at least for all $f \in L^1(dA)$. Given a function (symbol) $\varphi \in L^1_{\mathrm{loc}}(dA)$ we denote by M_φ the pointwise multiplication with φ, and by T_φ we denote the corresponding Toeplitz–operator

$$T_\varphi f(z) := PM_\varphi f(z) := \int_{\mathbb{D}} \frac{\varphi(\zeta) f(\zeta)}{(1 - z\bar{\zeta})^2} dA(\zeta). \tag{2.2}$$

The Berezin transform of a function $\varphi \in L^1$ is defined by

$$\tilde{\varphi}(z) = \int_{\mathbb{D}} \frac{\varphi(\zeta)(1 - |z|^2)^2}{|1 - z\bar{\zeta}|^4} dA(\zeta). \tag{2.3}$$

See more details in [22].

We recall the definitions of the spaces H^∞_V and L^∞_V from [15]; notice that we use here slightly different notations. First, denote

$$w(z) := 1 + |\log(1 - z)|. \tag{2.4}$$

This function is mainly used to define the radial weight $w(|z|)$. Sometimes it is however necessary to define the logarithm as an analytic function on $\mathbb{D}$: then the argument of $1 - z$ is understood to belong to the interval $]-\pi, \pi[$ for all $z \in \mathbb{D}$, and the definition becomes unambiguous.

The space H_V^∞ (respectively, L_V^∞) consists of analytic (resp. measurable) functions $f : \mathbb{D} \to \mathbb{C}$ such that for some $n \in \mathbb{N}$ and constant $C_n > 0$

$$|f(z)| \leq C_n w(|z|)^n \qquad \text{for (almost) all } z \in \mathbb{D}. \tag{2.5}$$

The space H_V^∞ is an (LB)–space, i.e., countable inductive limits of Banach spaces. In fact, it is a complete (LB)-space. We refer the reader to [11, 13] for locally convex inductive limits. A convenient way to describe the topology of H_V^∞ is obtained with a projective description of the inductive limit topology. This can be done as a consequence of the work of Bierstedt, Meise and Summers [3]. More precisely, it allows us to define the topology by means of the following family of weighted sup-seminorms:

$$\|f\|_v := \sup_{z\in\mathbb{D}} |f(z)|v(z) \quad , \quad v \in V, \tag{2.6}$$

where V consists of all continuous, positive, radial functions $v : \mathbb{D} \to \mathbb{R}$ such that for all $n \in \mathbb{N}$,

$$v(z) \leq C_n w(z)^{-n} \tag{2.7}$$

More details, results, examples and motivation about weighted inductive limits of Banach spaces of analytic functions can be seen in [2] and the references therein.

We still denote

$$B_n := \{f \in H_V^\infty \mid \sup_{z\in\mathbb{D}} |f(z)| \leq w(|z|)^n\}; \tag{2.8}$$

in the same way (using "ess sup") we define the subsets B_n^L of L_V^∞. The sets B_n are bounded and even precompact in H_V^∞. Every bounded subset of H_V^∞ is contained in a multiple of some B_n. The same holds for the sets B_n^L in L_V^∞, except that bounded sets need not be precompact in this case.

We collect in the following proposition the results of [15] that are needed later. See also [4].

Proposition 2.1. (1). *If $v \in V$, then the pointwise product $w^k v$ also belongs to V, for every k.*
(2). *The mapping P is a continuous projection from L_V^∞ onto H_V^∞.*
(3). *In addition, $P(B_n^L) \subset C_n B_{n+1}$ for all n.*

A linear operator from a complete locally convex space into itself is called compact (respectively, bounded), if it maps a neighbourhood of zero into a precompact (resp. bounded) set. Recall that in a complete space a subset is precompact if and only if it is relatively compact. See [11].

Remark 2.2. (1). A linear operator T between two (LB)–spaces is continuous, if and only if it maps bounded sets into bounded sets. In our case this means that $T_\varphi : H_V^\infty \to H_V^\infty$ is continuous if and only if for every $n \in \mathbb{N}$ one can find $C_n > 0$ and $m \in \mathbb{N}$ such that $T_\varphi(B_n) \subset C_n B_m$.
(2). One can also show that $T_\varphi : H_V^\infty \to H_V^\infty$ is compact, if and only if there exists $m \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ one can find a constant C_n with $T_\varphi(B_n) \subset C_n B_m$.

The symbols C, c (respectively, C_n) denote strictly positive absolute (resp. depending on n) constants, which may vary from place to place but not in the same sequence of inequalities.

Definition 2.3. Denote for all $L \in \mathbb{N}$ and $\lambda \in]-2^L, 2^L] \cap \mathbb{Z} =: I_L$

$$
\begin{aligned}
D_{L,\lambda} := \{ z := re^{i\theta} \ \ | \ \ & 1 - 2^{L-1} < r \leq 1 - 2^{-L} \\
& \text{and } \pi(\lambda - 1)2^{-L} < \theta \leq \pi \lambda 2^{-L} \},
\end{aligned}
\tag{2.9}
$$

and for each L and λ denote $z_{L,\lambda} := (1 - 2^{-L})e^{i2\pi\lambda 2^{-L}} \in D_{L,\lambda}$.

Remark 2.4. For positive real valued functions F and G (defined for example on $\mathbb{D}$ or on some other domain) we denote $F \cong G$, if there exist absolute, positive constants $C > c > 0$ such that

$$
cF \leq G \leq CF \tag{2.10}
$$

on the whole domain of definition. The following simple estimate will be needed:

$$
1 - |z| + |\arg z| \cong |1 - z|, \tag{2.11}
$$

where $z \in \mathbb{D}$ with $\arg z \in [-\pi, \pi]$. Namely, it is obviously enough to prove the claim on a domain where, say, $|1 - z| \leq 1/4$. (For other z both sides of (2.11) are bounded from above and below by positive constants.) But for such $z =: re^{i\theta}$ we have $|1 - z| \cong |1 - r\cos\theta| + |\sin\theta| \cong |1 - r| + |\theta|$, which is the statement.

3. Characterization of continuity and compactness for positive symbols.

In the following main results of our paper we characterize the continuity and compactness of T_φ with positive φ in terms of the growth properties of the Berezin transform of φ. For general, not necessarily positive φ, we show that these growth criterions are necessary conditions for continuity and compactness, respectively.

Theorem 3.1. *Assume $\varphi \geq 0$. The Toeplitz operator T_φ is continuous $H_V^\infty \to H_V^\infty$ if and only if there exist $k \in \mathbb{N}$ and $C > 0$ such that the Berezin transform $\tilde{\varphi}$ satisfies*

$$
\tilde{\varphi}(z) \leq Cw(|z|)^k = C\Big(1 + |\log(1 - |z|)|\Big)^k, \qquad z \in \mathbb{D}. \tag{3.1}
$$

Theorem 3.2. *Assume $\varphi \geq 0$. The operator $T_\varphi : H_V^\infty \to H_V^\infty$ is compact, if and only there exist $k \in \mathbb{N}$ such that for every $q \in \mathbb{N}$ there exists $C_q > 0$ with*

$$
\int\limits_{\mathbb{D}} \frac{\varphi(\zeta)(1 - |z|^2)^2}{|1 - z\zeta|^4} w(\zeta\bar{z})^q dA(\zeta) \leq C_q w(|z|)^k, \qquad z \in \mathbb{D} \tag{3.2}
$$

Proof of Theorems 3.1 and 3.2. I. *Sufficiency.* We prove the sufficiency of the condition (3.1) in Theorem 3.1 (we call this the case (A); of course, $k \in \mathbb{N}$ will be fixed all the time). During the course of the proof we describe the changes needed to prove the sufficiency of (3.2) in Theorem 3.2; this is called the case (B).

We first claim that in the case (A)

$$
\int\limits_{D_{L,\lambda}} \varphi \, dA \leq C_k L^k 2^{-2L} \tag{3.3}
$$

for all L and λ. In the case (B) and the assumption (3.2) we get

$$
\int\limits_{D_{L,\lambda}} \varphi \, dA \leq C_q L^{-q+k} 2^{-2L} \qquad \text{for all } q. \tag{3.4}
$$

In the case (A), fixing L and λ we use the positivity of the symbol, the definition of the Berezin transform and the assumption (3.1) to obtain

$$\begin{aligned} &\int_{D_{L,\lambda}} \frac{\varphi(\zeta)(1-|z_{L,\lambda}|^2)^2}{|1-z_{L,\lambda}\bar{\zeta}|^4} dA(\zeta) \\ &\leq \int_{\mathbb{D}} \frac{\varphi(\zeta)(1-|z_{L,\lambda}|^2)^2}{|1-z_{L,\lambda}\bar{\zeta}|^4} dA(\zeta) \leq C|\log(1-|z_{L,\lambda}|)|^k. \end{aligned} \tag{3.5}$$

We have $1-|z_{L,\lambda}| \cong 2^{-L}$ and also $|1-z_{L,\lambda}\bar{\zeta}| \geq C2^{-L}$ for $\zeta \in D_{L,\lambda}$. The inequality (3.3) follows already from this, since $\varphi \geq 0$.

In the case (B) we in addition make the observation that

$$|\log(1-\zeta\bar{z}_{L,\lambda})| \cong -\log|1-\zeta\bar{z}_{L,\lambda}| + |\arg(1-\zeta\bar{z}_{L,\lambda})| \cong L \tag{3.6}$$

for $\zeta \in D_{L,\lambda}$, since the modulus of $1-\zeta\bar{z}_{L,\lambda}$ is of order 2^{-L} and the argument of it is only between $-\pi$ and π. Otherwise (3.4) follows in the same way as in (3.5).

In the case (A), for the continuity of T_φ it is sufficient to show that for an arbitrary $n \in \mathbb{N}$ one can find a constant $C_{n,k} > 0$ such that $T_\varphi(B_n) \subset C_{n,k}B_{n+k+1}$ (see Remark 2.2, (1)). For the compactness, case (B), we show that for every $n \in \mathbb{N}$ there exists $c_n > 0$ such that $T_\varphi(B_n) \subset c_n B_{2+k}$ (Remark 2.2, (2)).

To prove these facts, we let $n \in \mathbb{N}$ and $f \in B_n$ be arbitrary and evaluate the expression

$$|T_\varphi f(z)| \leq \int_{\mathbb{D}} \frac{\varphi(\zeta)|f(\zeta)|}{|1-z\bar{\zeta}|^2} dA(\zeta), \tag{3.7}$$

$z \in \mathbb{D}$. We may assume that $z > 0$ (by the rotation symmetry) and that z actually of the form $z = z_{N,\nu} =: z_N$ for some $N \in \mathbb{N}$ and $\nu = 2^N$ (since a positive z anyway belongs to some set $D_{N,\nu}$ with $\nu = 2^N$; we have then $|1-z_N\bar{\zeta}| \cong |1-z\bar{\zeta}|$ for $\zeta \in \mathbb{D}$). We get

$$|T_\varphi f(z_N)| \leq \sum_{L=1}^{\infty} \sum_{\lambda \in I_L} \int_{D_{L,\lambda}} \frac{\varphi(\zeta)|f(\zeta)|}{|1-z_N\bar{\zeta}|^2} dA(\zeta). \tag{3.8}$$

For $\zeta \in D_{L,\lambda}$ one has $1-|z_N\bar{\zeta}| \cong 1-(1-2^{-N})(1-2^{-L}) \cong 2^{-N}+2^{-L}$ and, since z_N is real, we obtain from (2.11) of the Remark 2.4,

$$|1-z_N\bar{\zeta}| \cong 2^{-N}+2^{-L}+|\lambda|2^{-L}. \tag{3.9}$$

Moreover, since $f \in B_n$, we have

$$|f(\zeta)| \leq C_n L^n. \tag{3.10}$$

On account these estimates and (3.3), we find that in the case (A) that (3.8) can be bounded by a constant times

$$\begin{aligned} &\sum_{L,\lambda} \frac{C_n L^n}{(2^{-N}+2^{-L}+|\lambda|2^{-L})^2} \int_{D_{L,\lambda}} \varphi(\zeta) dA(\zeta) \\ &\leq \sum_{L,\lambda} \frac{C_n' L^{n+k} 2^{-2L}}{(2^{-N}+2^{-L}+|\lambda|2^{-L})^2}. \end{aligned} \tag{3.11}$$

In case (B) we get an estimate where CL^{n+k} is replaced by $C_q L^{n-q+k}$. We are thus left with combinatorial estimates. In the case (A) (3.11) will be shown to have a bound

$$CN^{n+k+1}. \tag{3.12}$$

This proves the continuity of T_φ, since this shows

$$|T_\varphi f(z_N)| \le CN^{n+k+1} \le C' |\log(1-|z_N|)|^{n+k+1},$$

i.e., T_φ maps B_n into $C_n B_{n+k+1}$ (see the discussion about z and z_N above). In the case (B) we aim to the bound

$$|T_\varphi f(z_N)| \le C_q n! N^{n-q+1+k} \tag{3.13}$$

for every $q \le n$. Then we choose $q = n-1$, and obtain

$$|T_\varphi f(z_N)| \le CN^{n-q+1+k} \le C' |\log(1-|z_N|)|^{2+k},$$

and thus T_φ maps B_n into $c_n B_{2+k}$.

We need to estimate

$$\sum_{\lambda \in I_L} \frac{2^{-2L}}{(2^{-N} + 2^{-L} + |\lambda| 2^{-L})^2} = \sum_{\lambda \in I_L} \frac{1}{(1 + 2^{L-N} + |\lambda|)^2}. \tag{3.14}$$

Comparing with the integral (with $\alpha = 2^{L-N}$)

$$\int_1^{2^L} \frac{1}{(1+\alpha+x)^2} dx \le \frac{C}{1+\alpha} \tag{3.15}$$

we find that the right hand side of (3.14) is bounded by $C/(1+2^{L-N})$. From this we find that (3.11) has in the case (A) the bound

$$\sum_L \frac{C_n L^{n+k}}{1+2^{L-N}}. \tag{3.16}$$

In the case (B) the bound is the same with L^{n+k} replaced by L^{n-q+k}.

The inequality

$$\sum_{L \ge N} \frac{L^{n+k}}{1+2^{L-N}} \le C(n+k)! N^{n+k+1} \tag{3.17}$$

can easily be derived e.g. by integrating by parts the expression $\int_N^\infty x^{n-k} e^{-x} dx$. On the other hand,

$$\sum_{L \le N} \frac{L^{n+k}}{1+2^{L-N}} \le \sum_{L \le N} L^{n+k} \le N^{n+k+1}. \tag{3.18}$$

This proves the case (A). Obviously, in the case (B) we also get the desired estimate $N^{n-q+1+k}$.

II. *Necessity for Theorem 3.1.* We assume T_φ is continuous and find $k \in \mathbb{N}$ such that T_φ maps B_1 into CB_k (see the remark after (2.8) and Remark 2.2, (1)). Considering for a moment the number $z \in \mathbb{D}$ as a parameter, we define the analytic function

$$K_z(\zeta) := \frac{1}{(1-\zeta\bar{z})^2} \quad , \quad \zeta \in \mathbb{D}. \tag{3.19}$$

Trivially the estimate

$$\|(1-|z|^2)^2 K_z\|_\infty \le 4 \tag{3.20}$$

holds for every z. Hence $T_\varphi((1-|z|^2)^2 K_z) \in 4CB_k$ for every z, and

$$|T_\varphi((1-|z|^2)^2 K_z)(\omega)| \le 4Cw(|\omega|)^k \tag{3.21}$$

for all $\omega \in \mathbb{D}$. But taking $\omega = z$, this means that

$$\begin{aligned}
|\tilde{\varphi}(z)| &= \Big| \int\limits_{\mathbb{D}} \frac{\varphi(\zeta)(1-|z|^2)^2}{|1-z\bar{\zeta}|^4} dA(\zeta) \Big| \\
&= \Big| \int\limits_{\mathbb{D}} \frac{\varphi(\zeta)}{(1-z\bar{\zeta})^2} \frac{(1-|z|^2)^2}{(1-\zeta\bar{z})^2} dA(\zeta) \Big| \\
&= |T_\varphi\big((1-|z|^2)^2 K_z\big)(z)| \\
&\le Cw(|z|)^k. \qquad (3.22)
\end{aligned}$$

This is the condition (3.1).

III. *Necessity for Theorem 3.2.* Our assumption is that there exists a $k \in \mathbb{N}$ such that for every $q \in \mathbb{N}$, $T(B_q) \subset C_q B_k$ for some constant $C_q > 0$. We define for this q and each $z \in \mathbb{D}$ the analytic function

$$\Lambda_{q,z}(\zeta) := \Big(\log \frac{1-\zeta\bar{z}}{e^{10q}} \Big)^q \quad , \quad \zeta \in \mathbb{D}. \tag{3.23}$$

Combining (3.20) and (3.25) of Lemma 3.3 (below) we find that the function $(1-|z|^2)^2 K_z \Lambda_{q,z}$ belongs to the set $C_q B_q$ for all z. Hence, it is mapped into the set CB_k by T_φ, i.e.

$$\begin{aligned}
Cw(|z|)^k &\ge |T_\varphi\big((1-|z|^2)^2 K_z \Lambda_{q,z}\big)(z)| \\
&= \Big| \int\limits_{\mathbb{D}} \frac{\varphi(\zeta)}{(1-z\bar{\zeta})^2} \frac{(1-|z|^2)^2 \Lambda_{q,z}(\zeta)}{(1-\zeta\bar{z})^2} dA(\zeta) \Big| \\
&= \Big| \int\limits_{\mathbb{D}} \frac{\varphi(\zeta)(1-|z|^2)^2 \Lambda_{q,z}(\zeta)}{|1-\zeta\bar{z}|^4} dA(\zeta) \Big|. \qquad (3.24)
\end{aligned}$$

Since all the factors of the integrand are positive, except the function $\Lambda_{q,z}$, we can conclude from (3.26) of Lemma 3.3 that (3.24) is bounded from below by a constant times

$$\Big| \int\limits_{\mathbb{D}} \frac{\varphi(\zeta)(1-|z|^2)^2 w(\zeta\bar{z})^q}{|1-\zeta\bar{z}|^4} dA(\zeta) \Big|. \qquad \square$$

To complete the proof above, we still have to show the following lemma.

Lemma 3.3. *For each $q \in \mathbb{N}$ there exists $C_q > 0$ such that the function $\Lambda_{q,z}$ of* (3.23) *satisfies*

$$|\Lambda_{q,z}(\zeta)| \le C_q w(|\zeta|)^q \tag{3.25}$$

and

$$(-1)^q \mathrm{Re}\Lambda_{q,z}(\zeta) \ge \frac{1}{C_q} w(\zeta\bar{z})^q \tag{3.26}$$

for all z, $\zeta \in \mathbb{D}$.

Proof. We may assume that $|z| \geq 1 - e^{-10q}$; otherwise $\Lambda_{q,z}$ is bounded from below and above by positive constants.

To prove (3.25), we have

$$|\mathcal{I}| := \big|\mathrm{Im}\big(\log(e^{-10q}(1-\zeta\bar{z}))\big)\big| = \big|\arg\big(e^{-10q}(1-\zeta\bar{z})\big)\big| \leq \pi, \tag{3.27}$$

since $\zeta\bar{z} \in \mathbb{D}$. Hence, denoting

$$\mathcal{R} := -\mathrm{Re}\big(\log(e^{-10q}(1-\zeta\bar{z}))\big) = -\log|e^{-10q}(1-\zeta\bar{z})| > 0 \tag{3.28}$$

we can estimate

$$\begin{aligned} |\Lambda_{q,z}(\zeta)| &\leq C_q(1+\mathcal{R})^q = C_q\big(1 - \log|e^{-10q}(1-\zeta\bar{z})|\big)^q \\ &\leq C_q'\big(1 + |\log(1-|\zeta|)|\big)^q, \end{aligned} \tag{3.29}$$

since $C \geq |1-\zeta\bar{z}| \geq 1 - |\zeta|$. This proves (3.25).

Now we deal with (3.26). To estimate the real part of $\Lambda_{q,z}$ we first observe that $\mathcal{R} \geq 10q$, since $|e^{-10q}(1-\zeta\bar{z})| \leq 3e^{-10q}$. By (3.27) we thus have $\mathcal{R} \geq 2q|\mathcal{I}|$. But for the number $\mathcal{R} + i\mathcal{I}$ one thus has (by elementary geometry in the complex plane, based on the fact that the modulus of the argument of $\mathcal{R} + i\mathcal{I}$ is small, i.e., about $1/(4\pi q)$)

$$\mathrm{Re}(\mathcal{R} + i\mathcal{I})^q \geq \frac{1}{2}|\mathcal{R} + i\mathcal{I}|^q \tag{3.30}$$

and, furthermore,

$$\begin{aligned} \mathrm{Re}(\mathcal{R} + i\mathcal{I})^q &\geq c_q(1 + |\mathcal{R} + i\mathcal{I}|)^q = c_q(1 + |\log(e^{-10q}(1-\zeta\bar{z})|)^q \\ &\geq C_q(1 + |\log(1-\zeta\bar{z})|)^q = C_q w(\bar{z}\zeta)^q. \end{aligned} \tag{3.31}$$

This is our statement (3.26). □

Remark 3.4. If φ is not positive, condition (3.1) is still necessary for the continuity. *Sufficient conditions for continuity and compactness are obtained by replacing* φ *by* $|\varphi|$ *in* (3.1) *and* (3.2), *respectively.* These claims will not be given separate proofs. A careful inspection of our proof above will reveal their validity.

4. Suffient conditions for general symbols.

In the case of the reflexive Bergman spaces A^p it is clear that a Toeplitz operator is bounded, if its symbol φ is bounded on $\mathbb{D}$. This follows from the boundedness of the Bergman projection $P : L^p(dA) \to A^p$ for $1 < p < \infty$. In case of H_V^∞ there is an analogous, quite straightforward suffient condition based on a pointwise estimate of φ. It is however of some interest that many unbounded symbols satisfy this condition. Moreover, there is an similar condition for the compactness of T_φ. The analogy of these pointwise conditions with those in Theorems 3.1 and 3.2 should be observed.

Proposition 4.1. *If there exists a* $k \in \mathbb{N}$ *such that*

$$|\varphi(z)| \leq Cw(|z|)^k \qquad \textit{for all } z \in \mathbb{D}, \tag{4.1}$$

then $T_\varphi : H_V^\infty \to H_V^\infty$ *is continuous.*

Proof. If $v \in V$, then the weight $\omega := w^k v$ still belongs to V, by Proposition 2.1, (1). The assumption (4.1) moreover implies that

$$\|f\varphi\|_v \leq C\|f\|_\omega \tag{4.2}$$

for all $f \in H_V^\infty$, i.e., the multiplication operator M_φ is continuous $H_V^\infty \to L_V^\infty$. By Proposition 2.1, (2), the Bergman projection is continuous $L_V^\infty \to H_V^\infty$, hence, $T_\varphi = PM_\varphi$ is continuous on H_V^∞. □

Theorem 4.2. *Assume that for all $q \in \mathbb{N}$ there exist $C_q \in \mathbb{N}$ such that*

$$|\varphi(z)| \le C_q w(|z|)^{-q} \qquad \text{for all } z \in \mathbb{D}. \tag{4.3}$$

Then $T_\varphi : H_V^\infty \to H_V^\infty$ is compact.

Proof. We first show that under the condition (4.3) the operator $M_\varphi : H_V^\infty \to L_V^\infty$ is bounded, i.e. it maps a neighbourhood U of 0 into a bounded set. We first define

$$v(z) := \inf_{q\in\mathbb{N}} C_q w(|z|)^{-q}. \tag{4.4}$$

Clearly, $v \in V$; see the definition of the weight family V in (2.7). Let U be a neighbourhood of 0 which is the closed unit ball of $\|\cdot\|_v$, i.e. $U := \{f \in H_v^\infty \mid \|f\|_v \le 1\}$. Every $f \in U$ satisfies $|f(z)| \le 1/v(z)$, hence, using (4.3), we obtain

$$|M_\varphi f(z)| = |\varphi(z)||f(z)| \le |\varphi(z)| \sup_{q\in\mathbb{N}} C_q^{-1} w(|z|)^q \le 1. \tag{4.5}$$

Hence, $M_\varphi f \in B_1^L \subset L_V^\infty$. Proposition 2.1, (3), shows that P maps B_1^L into $B_2 \subset H_V^\infty$, which is a precompact subset of H_V^∞. Hence, $T_\varphi = PM_\varphi$ is compact. □

It is not so difficult to see that there are positive symbols which do not satisfy (4.1) but nevertheless determine continuous and even compact Toeplitz operators on H_V^∞. We actually have the following result.

Proposition 4.3. *For any $\varphi \in L^1(\mathbb{D})$ such that the support of φ is a compact subset of $\mathbb{D}$, the operator $T_\varphi : H_V^\infty \to H_V^\infty$ is compact.*

Proof. Let $0 < r < 1$ be such that $\mathrm{supp}(\varphi)$ is contained in the closed disc $D(0,r)$ (with center 0 and radius r). There exists a constant $C > 0$ such that $|1 - z\bar\zeta| = |1 - \zeta\bar z| \ge C$ for all $\zeta \in D(0,r)$ and $z \in \mathbb{D}$. Hence, for every q we have $w(\zeta\bar z)^q \le C_q$ for such ζ and z, and we can estimate in (3.2)

$$\int_{\mathbb{D}} \frac{\varphi(\zeta)(1-|z|^2)^2}{|1-z\bar\zeta|^4} w(\zeta\bar z)^q dA(\zeta) = \int_{D(0,r)} \frac{\varphi(\zeta)(1-|z|^2)^2}{|1-z\bar\zeta|^4} w(\zeta\bar z)^q dA(\zeta)$$

$$\le C_q \int_{D(0,r)} \varphi(\zeta) dA(\zeta)(1-|z|^2)^2 \le C_q'. \tag{4.6}$$

The result follows from Theorem 3.2. □

Obviously, all functions φ which satisfy the assumption of Proposition 4.3, and which in addition are essentially unbounded, violate the condition (4.1).

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