GROTHENDIECK SPACES WITH THE DUNFORD–PETTIS PROPERTY

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Abstract. Banach spaces which are Grothendieck spaces with the Dunford–Pettis property (briefly, GDP) are classical. A systematic treatment of GDP–Fréchet spaces occurs in [12]. This investigation is continued here for locally convex Hausdorff spaces. The product and (most) inductive limits of GDP–space are again GDP–spaces. Also, every complete injective space is a GDP–space. For $p \in \{0\} \cup [1, \infty)$ it is shown that the classical co–echelon spaces $k_p(V)$ and $K_p(\overline{V})$ are GDP–spaces if and only if they are Montel. On the other hand, $K_\infty(\overline{V})$ is always a GDP–space and $k_\infty(V)$ is a GDP–space whenever its (Fréchet) predual, i.e., the Köthe echelon space $\lambda_1(A)$, is distinguished.

1. Introduction.

Grothendieck spaces with the Dunford–Pettis property (briefly, GDP) play a prominent role in the theory of Banach spaces and vector measures; see Ch.VI of [17], especially the Notes and Remarks, and [18]. Known examples include L^∞, $H^\infty(\mathbb{D})$, injective Banach spaces (e.g. ℓ^∞) and certain $C(K)$ spaces. D. Dean showed in [14] that a GDP–space does not admit any Schauder decomposition; see also [26, Corollary 8]. This has serious consequences for spectral measures in such spaces, [31].

For non–normable spaces the situation changes dramatically. Every Fréchet Montel space X is a GDP–space, [12, Remark 2.2]. Other than Montel spaces, the only known non–normable Fréchet space which is a GDP–space is the Köthe echelon space $\lambda_\infty(A)$, for an arbitrary Köthe matrix A, [12, Proposition 3.1]. Moreover, such spaces often admit Schauder decompositions, even unconditional ones in the presence of the density condition, [12, Proposition 4.4].

Our aim here is to continue and expand on the investigation begun in [12]. We exhibit large classes of locally convex Hausdorff spaces (briefly, lcHs) which are GDP–spaces. Many of these admit Schauder decompositions, some even unconditional ones. The methods also exhibit new classes of Fréchet GDP–spaces which are neither Montel nor isomorphic to any space of the kind $\lambda_\infty(A)$. Consequences for spectral measures are also presented. In the final section we characterize those co–echelon spaces $k_p(V)$, for $p \in \{0\} \cup [1, \infty)$, which are GDP–spaces. For $p \neq \infty$,

Key words and phrases. Grothendieck space, Dunford–Pettis property, co–echelon space, Schauder decomposition.

Mathematics Subject Classification 2000: Primary 46A20, 46A45, 47B37; Secondary 46A04, 46A08, 46A13.

* Support of the Alexander von Humboldt Foundation is gratefully acknowledged.

+ Support of the Alexander von Humboldt Foundation and MEC and FEDER Project MTM 2007-62643 (Spain) are gratefully acknowledged.

this is the case precisely when $k_p(V)$ is Montel but, for $p = \infty$, the situation is different.

2. Preliminaries.

If X is a lcHs and Γ_X is a system of continuous seminorms determining the topology of X, then the strong operator topology τ_s in the space $\mathcal{L}(X)$ of all continuous linear operators from X into itself (from X into another lcHs Y we write $\mathcal{L}(X,Y)$) is determined by the family of seminorms

$$q_x(S) := q(Sx), \quad S \in \mathcal{L}(X),$$

for each $x \in X$ and $q \in \Gamma_X$ (in which case we write $\mathcal{L}_s(X)$). Denote by $\mathcal{B}(X)$ the collection of all bounded subsets of X. The topology τ_b of uniform convergence on bounded sets is defined in $\mathcal{L}(X)$ via the seminorms

$$q_B(S) := \sup_{x\in B} q(Sx), \quad S \in \mathcal{L}(X),$$

for each $B \in \mathcal{B}(X)$ and $q \in \Gamma_X$ (in which case we write $\mathcal{L}_b(X)$). For X a Banach space, τ_b is the operator norm topology in $\mathcal{L}(X)$. If Γ_X is countable and X is complete, then X is called a Fréchet space.

By X_σ we denote X equipped with its weak topology $\sigma(X,X')$, where X' is the topological dual space of X. The strong topology in X (resp. X') is denoted by $\beta(X,X')$ (resp. $\beta(X',X)$) and we write X_β (resp. X'_β); see [23, §21.2] for the definition. The strong dual space $(X'_\beta)'_\beta$ of X'_β is denoted simply by X''. By X'_σ we denote X' equipped with its weak–star topology $\sigma(X',X)$. Given $T \in \mathcal{L}(X)$, its *dual operator* $T^t\colon X' \to X'$ is defined by $\langle x, T^t x'\rangle = \langle Tx, x'\rangle$ for all $x \in X$, $x' \in X'$. It is known that $T^t \in \mathcal{L}(X'_\sigma)$ and $T^t \in \mathcal{L}(X'_\beta)$, [24, p.134].

The following two known facts are included for ease of reading.

Lemma 2.1. *Let X, Y be lcHs' with Y quasi–barrelled. Then the linear map $\Phi\colon \mathcal{L}_b(X,Y) \to \mathcal{L}_b(Y'_\beta, X'_\beta)$ defined by $\Phi(T) := T^t$, for $T \in \mathcal{L}_b(X,Y)$, is continuous.*

In particular, if X is quasi–barrelled and a sequence $\{T_n\}_{n=1}^\infty \subseteq \mathcal{L}(X)$ satisfies τ_b-$\lim_{n\to\infty} T_n = T$ in $\mathcal{L}_b(X)$, then also τ_b-$\lim_{n\to\infty} T_n^t = T^t$ in $\mathcal{L}_b(X'_\beta)$.

Proof. A basis of 0–neighbourhoods in $\mathcal{L}_b(X,Y)$ consists of all sets of the form $W(B,U) := \{T \in \mathcal{L}(X,Y) : T(B) \subseteq U\}$ as B runs through $\mathcal{B}(X)$ and U runs through the collection $\mathcal{U}_0(Y)$ of all 0–neighbourhoods in Y.

Let $C \in \mathcal{B}(Y'_\beta)$ and $V \in \mathcal{U}_0(X'_\beta)$ be given. Since Y is quasi–barrelled, C is equicontinuous, [23, p.368]. So, there exist $U \in \mathcal{U}_0(Y)$ with $C \subseteq U^\circ$ (the polar of U) and, by definition of the topology of X'_β, a set $D \in \mathcal{B}(X)$ such that $D^\circ \subseteq V$. To complete the proof, we check that $\Phi(W(D,U)) \subseteq W(C,V)$. Fix $T \in W(D,U) \subseteq \mathcal{L}(X,Y)$, in which case $T(D) \subseteq U$. It suffices to show that $T^t(U^\circ) \subseteq D^\circ$ since this implies that $T^t(C) \subseteq V$. So, fix $y' \in U^\circ$ and $d \in D$. Then $|\langle d, T^t y'\rangle| = |\langle Td, y'\rangle| \le 1$ as $Td \in U$ and $y' \in U^\circ$. Accordingly, $T^t y' \in D^\circ$ for all $y' \in U^\circ$. □

Lemma 2.2. *Let A be a subset of a lcHs X such that, for every $U \in \mathcal{U}_0(X)$, there exists a precompact set $B \subseteq X$ (depending on U) with $A \subseteq B + U$. Then A is precompact.*

Proof. Fix U and B as in the statement of the lemma with $A \subseteq B + \frac{1}{2}U$. Since B is precompact, select $x_1, \ldots, x_k$ from B such that $B \subseteq \cup_{j=1}^k (x_j + \frac{1}{2}U)$. It follows that $A \subseteq \cup_{j=1}^k (x_j + U)$. Hence, A is precompact. □

A sequence $(P_n)_{n=1}^\infty \subseteq \mathcal{L}(X)$ is a *Schauder decomposition* of X if it satisfies:
(S1) $P_n P_m = P_{\min\{m,n\}}$ for all $m, n \in \mathbb{N}$,
(S2) $P_n \to I$ in $\mathcal{L}_s(X)$ as $n \to \infty$, and
(S3) $P_n \neq P_m$ whenever $n \neq m$.
By setting $Q_1 := P_1$ and $Q_n := P_n - P_{n-1}$ for $n \geq 2$ we arrive at a sequence of pairwise orthogonal projections (i.e. $Q_n Q_m = 0$ if $n \neq m$) satisfying $\sum_{n=1}^\infty Q_n = I$, with the series converging in $\mathcal{L}_s(X)$. If the series is unconditionally convergent in $\mathcal{L}_s(X)$, then $\{P_n\}_{n=1}^\infty$ is called an *unconditional Schauder decomposition*, [28]. Such decompositions are intimately associated with (non–trivial) spectral measures; see (the proof of) [12, Proposition 4.3] and [28, Lemma 5 and Theorem 6]. If X is barrelled, then (S2) implies that $\{P_n\}_{n=1}^\infty$ is an equicontinuous subset of $\mathcal{L}(X)$. According to (S1) each P_n and Q_n, for $n \in \mathbb{N}$, is a projection and $Q_n \to 0$ in $\mathcal{L}_s(X)$ as $n \to \infty$. Condition (S3) ensures that $Q_n \neq 0$ for each $n \in \mathbb{N}$.

Let $\{P_n\}_{n=1}^\infty \subseteq \mathcal{L}(X)$ be a Schauder decomposition of X. Then the dual projections $\{P_n^t\}_{n=1}^\infty \subseteq \mathcal{L}(X'_\sigma)$ always form a Schauder decomposition of X'_σ, [22, p.378]. If, in addition, $\{P_n^t\}_{n=1}^\infty \subseteq \mathcal{L}(X'_\beta)$ is a Schauder decomposition of X'_β, then the original sequence $\{P_n\}_{n=1}^\infty$ is called *shrinking*, [22, p.379]. Since (S1) and (S3) clearly hold for $\{P_n^t\}_{n=1}^\infty$, this means precisely that $P_n^t \to I$ in $\mathcal{L}_s(X'_\beta)$; see (S2).

3. GDP–spaces.

A lcHs X is called a *Grothendieck space* if every sequence in X' which is convergent in X'_σ is also convergent for $\sigma(X', X'')$. Clearly every reflexive lcHs is a Grothendieck space. A lcHs X is said to have the *Dunford–Pettis property* (briefly, DP) if every element of $\mathcal{L}(X, Y)$, for Y any quasicomplete lcHs, which transforms elements of $\mathcal{B}(X)$ into relatively $\sigma(Y, Y')$–compact subsets of Y, also transforms $\sigma(X, X')$–compact subsets of X into relatively compact subsets of Y, [21, p.633–634]. Actually, it suffices if Y runs through the class of all Banach spaces, [12, p.79]. A reflexive lcHs satisfies the DP–property if and only if it is Montel, [21, p.634]. According to [21, pp.633–634], a lcHs X has the DP–property if and only if every absolutely convex, $\sigma(X, X')$–compact subset of X (denote all such sets by Σ) is precompact for the topology $\tau_{\Sigma'}$ of uniform convergence on the absolutely convex, equicontinuous, $\sigma(X', X'')$–compact subsets of X' (denote all such sets by Σ'). Clearly the topology $\tau_{\Sigma'}$ is finer than $\sigma(X, X')$.

Examples of GDP–spaces, beyond those given in [12] for certain kinds of non–normable Fréchet spaces, are given via the next result.

Proposition 3.1. *(i) Every complemented subspace of a GDP–space is a GDP–space.*

(ii) An arbitrary product of lcHs' is a GDP–space if and only if each factor is a GDP–space.

Proof. (i) Concerning the DP–property, see [21, p.635]. The proof of the Grothendieck property for Fréchet spaces, as given in [12, Lemma 2.1(iv)], is valid in a general lcHs.

(ii) For the DP–property of a product space $X = \prod_{\alpha\in A} X_\alpha$, with each X_α a GDP–space, we refer to [21, p.635]. Concerning the Grothendieck property, let $\{u^{(k)}\}_{k=1}^\infty \subseteq X'$ be a $\sigma(X', X)$–null sequence. By [23, (2) p.284], $X' = \oplus_{\alpha\in A} X'_\alpha$ in the canonical way. Moreover, [23, (4) p.286], implies (for the respective Mackey topologies) that $(X', \mu(X', X)) = \oplus_{\alpha\in A}(X'_\alpha, \mu(X'_\alpha, X_\alpha))$. Now, $\{u^{(k)}\}_{k=1}^\infty$ is bounded in $(X', \mu(X', X))$ because the topologies $\sigma(X', X)$ and $\mu(X', X)$ have the same bounded sets. We can then apply [23, (4) p.213] to conclude that there exists a finite sum $\oplus_{j=1}^n X'_{\alpha(j)}$ such that $\{u^{(k)}\}_{k=1}^\infty$ is bounded in $\oplus_{j=1}^n (X'_{\alpha(j)}, \mu(X'_{\alpha(j)}, X_{\alpha(j)}))$. In particular, if $u^{(k)} = (u_\alpha^{(k)})_{\alpha\in A}$, then the coordinates $u_\alpha^{(k)} = 0$ for all $\alpha \notin \{\alpha(j)\}_{j=1}^n$ and $k \in \mathbb{N}$. Since each $X_{\alpha(j)}$ is a Grothendieck space, we have $u_{\alpha(j)}^{(k)} \to 0$ in $(X'_{\alpha(j)}, \sigma(X'_{\alpha(j)}, X''_{\alpha(j)}))$ as $k \to \infty$. It is the routine to conclude that $u^{(k)} \to 0$ in $(X', \sigma(X', X''))$ as $k \to \infty$.

Conversely, since each factor in a product space is a complemented subspace, it follows from part (i) that each factor is a GDP–space whenever the product is a GDP–space. □

Let us present an immediate application. Recall that a lcHs X is called *injective* if, whenever a lcHs Y contains a closed subspace isomorphic to X, then this subspace is complemented in Y. For Banach spaces the following fact is known, [25, p.121]

Corollary 3.2. *Every injective complete lcHs X is a GDP–space.*

Proof. As a complete lcHs, X is isomorphic to a closed subspace of a product $\prod_\alpha Y_\alpha$ of Banach spaces $\{Y_\alpha\}_\alpha$, [23, p.208]. On the other hand, each Y_α is isomorphic to a closed subspace of $\ell^\infty(I_\alpha)$ for some index set I_α. So, X is isomorphic to a closed subspace of $\prod_\alpha \ell^\infty(I_\alpha)$ and hence, being injective, X is isomorphic to a complemented subspace of $\prod_\alpha \ell^\infty(I_\alpha)$. But, $\prod_\alpha \ell^\infty(I_\alpha)$ is a GDP–space by Proposition 3.1(ii) and the fact that each Banach space $\ell^\infty(I_\alpha)$ is a GDP–space, [26]. Hence, X is a GDP–space by Proposition 3.1(i). □

For examples of (non–normable) injective lcHs' we refer to [19], [20], for example, and the references therein.

By taking any infinite sequence $\{X_n\}_{n=1}^\infty$ of Banach GDP–spaces (e.g. the classical ones listed in Section 1) and forming the product $\prod_{n=1}^\infty X_n$, one can exhibit many Fréchet GDP–spaces which are neither Montel nor Köthe echelon spaces. The dual X'_β of a GDP–space need not be a GDP–space, e.g. $(\ell^\infty)'$ contains a complemented copy of ℓ^1, which is not a GDP–space.

We now turn to an extension of the Brace–Grothendieck characterization of the DP–property. For Banach spaces we refer to [17, p.177], [21, pp.635–636], and for Fréchet spaces to [7, p.397].

A subset A of a lcHs X is called *relatively sequentially $\sigma(X, X')$–compact* if every sequence in A contains a subsequence which is convergent in X_σ. Such sets belong to $\mathcal{B}(X)$, [23, §24;(1)], after recalling that every sequentially compact set in any lcHs is also countably compact, [23, p.310].

Proposition 3.3. *Let X be a quasicomplete lcHs.*

(i) If X is barrelled and has the DP–property, then for every $\sigma(X,X')$–null sequence $\{x_k\}_{k=1}^\infty \subseteq X$ and every $\sigma(X',X'')$–null sequence $\{x'_k\}_{k=1}^\infty \subseteq X'$ we have $\lim_{k\to\infty}\langle x_k, x'_k\rangle = 0$.

(ii) Let both X and X'_β have the property that their relatively weakly compact subsets are relatively sequentially weakly compact. Suppose that $\lim_{k\to\infty}\langle x_k, x'_k\rangle = 0$ whenever $\{x_k\}_{k=1}^\infty \subseteq X$ is a $\sigma(X,X')$–null sequence and $\{x'_k\}_{k=1}^\infty \subseteq X'$ is a $\sigma(X',X'')$–null sequence. Then X has the DP–property.

It is routine to check (but, in practice quite useful) that the condition $x_k \to 0$ for $\sigma(X,X')$ and $x'_k \to 0$ for $\sigma(X',X'')$ in part (i) can be replaced with $x_k \to 0$ for $\sigma(X,X')$ and $\{x'_k\}_{k=1}^\infty$ is $\sigma(X',X'')$–convergent in X'. For Banach spaces this was noted in [21, (c') p.636].

Proof. (i) Fix null sequences $\{x_k\}_{k=1}^\infty \subseteq X_\sigma$ and $\{x'_k\}_{k=1}^\infty \subseteq X'$ for $\sigma(X',X'')$. Since X is barrelled, the lcHs $(X',\sigma(X',X''))$ is quasicomplete, [23, (3) p. 297], and the closed absolutely convex hull B of $\{0\}\cup\{x'_k\}_{k=1}^\infty$ is equicontinuous and $\sigma(X',X'')$–compact by Krein' Theorem, [23, (4) p. 325]. Since X is quasicomplete, the closed absolutely convex hull A of $\{0\}\cup\{x_k\}_{k=1}^\infty$ is $\sigma(X,X')$–compact, again by Krein's theorem. Via the discussion prior to Proposition 3.1, A is precompact for the topology $\tau_{\Sigma'}$. By Grothendieck's Theorem applied to $u := I$ from X to X, [21, Theorem 9.2.1], the topology $\sigma(X,X')$ is finer on A than $\tau_{\Sigma'}$. In particular, $x_k \to 0$ for $\tau_{\Sigma'}$ as $k\to\infty$ and hence, uniformly on B. This yields that $\lim_{k\to\infty}\langle x_k, x'_k\rangle = 0$.

(ii) Under the given hypotheses, to conclude that X has the DP–property it is enough to show that $x_k \to 0$ for $\tau_{\Sigma'}$ as $k\to\infty$ whenever $\{x_k\}_{k=1}^\infty \subseteq X$ is a $\sigma(X,X')$–null sequence. To this effect, we show that the stated condition implies that every $A\in\Sigma$ is relatively $\tau_{\Sigma'}$–countably compact and hence, is $\tau_{\Sigma'}$–precompact. So, take any sequence $\{x_k\}_{k=1}^\infty \subseteq A$. Since A is $\sigma(X,X')$–compact, it is sequentially $\sigma(X,X')$–compact (by hypothesis), and so we can select a subsequence $\{x_{k(j)}\}_{j=1}^\infty$ converging to some $x_0\in X$ for $\sigma(X,X')$. Then $(x_{k(j)} - x_0)\to 0$ for $\sigma(X,X')$ as $j\to\infty$ and hence, by the stated condition, also for $\tau_{\Sigma'}$ as $j\to\infty$. So, $x_{k(j)}\to x_0$ for $\tau_{\Sigma'}$ as $j\to\infty$ and hence, A is $\tau_{\Sigma'}$–countably compact.

To complete the proof we proceed by contradiction, i.e. assume there is a $\sigma(X,X')$–null sequence $\{x_k\}_{k=1}^\infty \subseteq X$ which does not converge to 0 for $\tau_{\Sigma'}$. Then there exist $\varepsilon > 0$, a sequence $k(1) < k(2) < \dots$ and a sequence $\{x'_j\}_{j=1}^\infty$ contained in some set $B\in\Sigma'$ such that $|\langle x_{k(j)}, x'_j\rangle| > \varepsilon$ for each $j\in\mathbb{N}$. By hypothesis (since $(X'_\beta)' = X''$), B is sequentially $\sigma(X',X'')$–compact and hence, there is a subsequence $\{x'_{j(i)}\}_{i=1}^\infty$ of $\{x'_j\}_{j=1}^\infty$ and $x'\in X'$ such that $x'_{j(i)}\to x'$ for $\sigma(X',X'')$ as $i\to\infty$. Since $x_{k(j(i))}\to 0$ for $\sigma(X,X')$ as $i\to\infty$ and $x'_{j(i)}\to x'$ for $\sigma(X',X'')$ as $i\to\infty$, it follows from the assumed hypotheses (in the form of the comment prior the proof) that $\langle x_{k(j(i))}, x'_{j(i)}\rangle \to 0$ as $i\to\infty$, which is a contradiction. □

Remark 3.4. (i) The following requirements on a lcHs X ensure that X is quasicomplete and that both X and X'_β satisfy the hypotheses of part (ii) in Proposition 3.3.

(a) X is a Fréchet space. The space X'_β is then a complete (DF)–space and the claim follows from [13, Theorem 1.1 and Example 1.2].

(b) X is a complete (DF)–space, in which case X'_β is a Fréchet space. Again the claim follows from [13, Theorem 1.1 and Example 1.2].

(c) $X = \operatorname{ind}_n X_n$ is a complete (LF)–space. According to [23, p.368] the space is barrelled and [13, Example 1.2(A)] implies that the relatively $\sigma(X, X')$–compact sets are relatively sequentially $\sigma(X, X')$–compact. Concerning X'_β, since every bounded set in X is contained and bounded in one of the component spaces X_n, [23, (5) p.225], it follows that $X'_\beta = \operatorname{proj}_n (X_n)'_\beta$ and so X'_β is a subspace of the countable product $\prod_{n\in\mathbb{N}}(X_n)'_\beta$. The desired conclusion for X'_β then follows from Proposition 5, Proposition 6 and Theorem 11 of [13].

(ii) Suppose that X is a complete (DF)–space; see [29, p.248] for definition. Such a space is necessarily $\aleph_0$–barrelled, [29, Observation 8.2.2. (c)]; see [29, p.236] for the definition of $\aleph_0$–quasibarrelled and $\aleph_0$–barrelled. We claim that X has the DP–property if and only if $\lim_{k\to\infty}\langle x_k, x'_k\rangle = 0$ for every $\sigma(X, X')$–null sequence $\{x_k\}_{k=1}^\infty \subseteq X$ and every $\sigma(X', X'')$–null sequence $\{x'_k\}_{k=1}^\infty \subseteq X'$. Indeed, that the validity of the stated condition on pairs of null sequences implies the DP–property follows from part (i)-(b) above and Proposition 3.3(ii). Conversely, suppose that X has the DP–property. We cannot apply Proposition 3.3(i) directly as X need not be barrelled. Nevertheless, suppose that $\{x_k\}_{k=1}^\infty \subseteq X$ is $\sigma(X, X')$–null and $\{x'_k\}_{k=1}^\infty \subseteq X'$ is $\sigma(X', X'')$–null. Then the absolutely convex hull of $\{0\} \cup \{x'_k\}_{k=1}^\infty$ is both $\sigma(X', X'')$–compact by Krein's Theorem (applied in the Fréchet space X'_β) and equicontinuous because X is $\aleph_0$–barrelled. The proof that $\lim_{k\to\infty}\langle x_k, x'_k\rangle = 0$ can then be completed as in the proof of Proposition 3.3(i).

The following technical result will be useful in the sequel.

Lemma 3.5. *Let X be a barrelled lcHs which is a Grothendieck space and $\{T_j\}_{j=1}^\infty \subseteq \mathcal{L}(X)$ be a sequence of pairwise commuting operators satisfying*

$$\lim_{j\to\infty} T_j = 0 \quad \text{in } \mathcal{L}_s(X) \tag{3.1}$$

and

$$\lim_{j\to\infty} (I - T_k)T_j = 0 \quad \text{in } \mathcal{L}_b(X), \text{ for each } k \in \mathbb{N}. \tag{3.2}$$

Then the following assertions are valid.

(i) For each bounded sequence $\{x_j\}_{j=1}^\infty \subseteq X$ we have

$$\lim_{j\to\infty} T_j x_j = 0 \quad \text{in } X_\sigma.$$

(ii) For each $\sigma(X', X)$–bounded sequence $\{x'_j\}_{j=1}^\infty \subseteq X'$ we have

$$\lim_{j\to\infty} T_j^t x'_j = 0 \quad \text{in } (X', \sigma(X', X'')).$$

Proof. (ii) Let $\{x'_j\}_{j=1}^\infty \subseteq X'$ be a $\sigma(X', X)$–bounded sequence. Since X is barrelled, the set $B := \{x'_j\}_{j=1}^\infty \subseteq X'$ is equicontinuous. For fixed $x \in X$, it follows from (3.1) that $\sup_{z'\in B} |\langle T_j x, z'\rangle| \to 0$ as $j \to \infty$, that is, $|\langle x, T_j^t x'_j\rangle| \to 0$ as $j \to \infty$. This shows that $\lim_{j\to\infty} T_j^t x'_j = 0$ in $(X', \sigma(X', X))$. Since X is a Grothendieck space, we can conclude that $\lim_{j\to\infty} T_j^t x'_j = 0$ in $(X', \sigma(X', X''))$.

(i) Set $S_j := I - T_j$, for $j \in \mathbb{N}$, in which case $S_j^t \in \mathcal{L}(X'_\beta)$. By part (ii) applied to $\{S_j\}_{j=1}^\infty$ we have (in $(X', \sigma(X', X''))$) that

$$\lim_{j\to\infty} S_j^t u' = u', \quad u' \in X'.$$

Define a linear subspace $H \subseteq X'$ by

$$H := \{u' \in X' : \ u' = \lim_{j\to\infty} S_j^t u' \ \text{ in } X'_\beta\}.$$

To show that H is closed in X'_β, fix a net $\{x'_\alpha\}_{\alpha\in A} \subseteq H$ such that $\lim_{\alpha\in A} x'_\alpha = x'$ in X'_β. Fix $B \in \mathcal{B}(X)$. The barrelledness of X and the fact that $\{T_j\}_{j=1}^\infty$ is bounded in $\mathcal{L}_s(X)$ (see (3.1)) imply that $\{S_j\}_{j=1}^\infty$ is equicontinuous in $\mathcal{L}(X)$. Accordingly, $C := B \cup \cup_{j=1}^\infty S_j(B) \in \mathcal{B}(X)$. Since $x'_\alpha \to x'$ in X'_β, it follows that there exists $\alpha(0) \in A$ such that

$$\sup_{x\in C} |\langle x, (x'_\alpha - x')\rangle| \le \frac{1}{3}, \quad \alpha \ge \alpha(0). \tag{3.3}$$

For each $\alpha \ge \alpha(0)$ and $z \in B$ we have, for all $j \in \mathbb{N}$, that

$$\begin{aligned}
&|\langle z, x'\rangle - \langle z, S_j^t x'\rangle| \\
&\quad \le |\langle z, x'\rangle - \langle z, x'_\alpha\rangle| + |\langle z, x'_\alpha\rangle - \langle z, S_j^t x'_\alpha\rangle| + |\langle z, S_j^t(x'_\alpha - x')\rangle| \\
&\quad = |\langle z, (x' - x'_\alpha)\rangle| + |\langle z, (x'_\alpha - S_j^t x'_\alpha)\rangle| + |\langle S_j z, (x'_\alpha - x')\rangle| \\
&\quad \le \frac{2}{3} + |\langle z, (x'_\alpha - S_j^t x'_\alpha)\rangle|,
\end{aligned}$$

where (3.3) is applied twice to deduce the final inequality. In particular, for $\alpha := \alpha(0)$ we have, for all $j \in \mathbb{N}$, that

$$|\langle z, x'\rangle - \langle z, S_j^t x'\rangle| \le \frac{2}{3} + |\langle z, (x'_{\alpha(0)} - S_j^t x'_{\alpha(0)})\rangle|, \quad z \in B.$$

Since $x'_{\alpha(0)} \in H$, there exists $j(0) \in \mathbb{N}$ such that

$$\sup_{z\in B} |\langle z, (x'_{\alpha(0)} - S_j^t x'_{\alpha(0)})\rangle| \le \frac{1}{3}, \quad j \ge j(0).$$

This shows that $\{x' - S_j^t x'\}_{j=j(0)}^\infty \subseteq B^\circ$ (the polar of B). It follows that $S_j^t x' \to x'$ in X'_β as $j \to \infty$, i.e., $x' \in H$.

Next we show that $\cup_{k=1}^\infty S_k^t(X') \subseteq H$. So, fix any $k \in \mathbb{N}$. By (3.2) we have

$$\lim_{j\to\infty} T_j S_k = \lim_{j\to\infty} S_k T_j = 0 \ \text{ in } \mathcal{L}_b(X).$$

In particular, for each $x' \in X'$, we have $\lim_{j\to\infty} x' \circ (T_j S_k) = 0$ in X'_β or, equivalently, that $\lim_{j\to\infty} S_k^t T_j^t x' = 0$ in X'_β. It follows, for any $x' \in X'$, that

$$\lim_{j\to\infty} S_j^t(S_k^t x') = \lim_{j\to\infty} S_k^t(I - T_j^t)x' = S_k^t x' - \lim_{j\to\infty} S_k^t T_j^t x' = S_k^t x'$$

with the limits taken in X'_β. This shows that $S_k^t x' \in H$, for each $x' \in X'$, i.e. $S_k^t(X') \subseteq H$.

For each $x' \in X'$, it follows from (ii) that

$$x' = \lim_{j\to\infty} S_j^t x'$$

in $(X', \sigma(X', X''))$, with $\{S_j^t x'\}_{j=1}^\infty \subseteq H$ because of $\cup_{k=1}^\infty S_k^t(X') \subseteq H$. Accordingly, H is dense in $(X', \sigma(X', X''))$. On the other hand, H is closed in X'_β and both $\sigma(X', X'')$ and $\beta(X', X)$ are topologies for the dual pairing (X', X''), which implies that $H = X'$. In particular, in X'_β we have $x' = \lim_{j\to\infty} S_j^t x'$, for each $x' \in X'$, that is, $\lim_{j\to\infty} T_j^t x' = 0$, for each $x' \in X'$.

To complete the proof, let $\{x_j\}_{j=1}^\infty$ be any bounded sequence in X and set $D := \{x_j\}_{j=1}^\infty \in \mathcal{B}(X)$. Fix $x' \in X'$. Since $\lim_{j\to\infty} T_j^t x' = 0$ in X'_β we get

$$\lim_{j\to\infty} \sup_{z\in D} |\langle z, T_j^t x'\rangle| = 0$$

and hence, in particular, that $0 = \lim_{j\to\infty} |\langle x_j, T_j^t x'\rangle| = \lim_{j\to\infty} |\langle T_j x_j, x'\rangle|$. This shows that $\lim_{j\to\infty} T_j x_j = 0$ in X_σ, as required. □

Remark 3.6. Lemma 3.5 is an extension of Proposition 4.1 in [12]. Indeed, let $\{P_n\}_{n=1}^\infty \subseteq \mathcal{L}(X)$ be any Schauder decomposition. Set $T_j := (I - P_j)$, for $j \in \mathbb{N}$, in which case $T_j \to 0$ in $\mathcal{L}_s(X)$ as $j \to \infty$, i.e., (3.1) holds. Moreover, for $k \in \mathbb{N}$ fixed we have

$$(I - T_k)T_j = P_k(I - T_j) = P_k - P_k P_j = 0, \quad j \geq k,$$

and so (3.2) also holds.

The proof of Lemma 3.5 is based on methods introduced by H.P. Lotz, [25, §3].

The following notion is due to J.C. Díaz and M.A. Miñarro, [15, p.194]. A Schauder decomposition $\{P_n\}_{n=1}^\infty$ in a lcHs X is said to have *property* (M) if $P_n \to I$ in $\mathcal{L}_b(X)$ as $n \to \infty$. Since every non-zero projection P in a Banach space satisfies $\|P\| \geq 1$, it is clear that no Schauder decomposition in any Banach space can have property (M). For non-normable spaces the situation is quite different. For instance, if X is a Fréchet Montel space (resp. Fréchet GDP–space, which is a larger class of spaces; see [12]), then *every* Schauder decomposition in X has property (M); see [15] (resp. [12, Proposition 4.2]). The next result significantly extends these classes of spaces. First a useful observation (see [12, Remark 2.2] for Fréchet spaces).

Remark 3.7. Every Montel lcHs X is a GDP–space. Indeed, since X is reflexive, [23, (1) p.369], it is surely a Grothendieck space. That every Montel space has the DP–property is known, [21, Example 9.4.2]. Actually, it was already noted above, [21, Example 9.4.2], that a lcHs X is Montel if and only if it is semireflexive and has the DP–property.

Proposition 3.8. *Let X be any quasicomplete, barrelled lcHs and $\{P_n\}_{n=1}^\infty \subseteq \mathcal{L}(X)$ be a Schauder decomposition of X.*

(i) If X is a GDP–space, then $\{P_n\}_{n=1}^\infty$ has property (M).

(ii) If X is a GDP–space, then $\{P_n^t\}_{n=1}^\infty \subseteq \mathcal{L}(X'_\beta)$ is a Schauder decomposition of X'_β with property (M). In particular, $\{P_n\}_{n=1}^\infty$ is a shrinking Schauder decomposition of X.

(iii) Suppose that $\{P_n\}_{n=1}^\infty$ has property (M) and that each complemented subspace $Q_n(X)$ of X, where $Q_n := P_n - P_{n-1}$ with $P_0 := 0$ for $n \in \mathbb{N}$, is a Grothendieck space (resp. has the DP–property, resp. is Montel). Then X is also a Grothendieck space (resp. has the DP–property, resp. is Montel).

Proof. (i) We proceed with a contradiction argument as in the proof of Proposition 4.2 in [12]. Setting $T_n := I - P_n$, for $n \in \mathbb{N}$, it follows from that proof that there exist $p \in \Gamma_X$ and $B \in \mathcal{B}(X)$, an $\varepsilon > 0$, and an increasing sequence $n(k) \nearrow \infty$ in $\mathbb{N}$ together with sequences $\{x'_k\}_{k=1}^\infty \subseteq X'$ and $\{x_k\}_{k=1}^\infty \subseteq B$ such that $|\langle x, x'_k\rangle| \leq p(x)$, for all $x \in X$, and

$$|\langle T_{n(k)}x_k, T^t_{n(k)}x'_k\rangle| > \varepsilon, \quad k \in \mathbb{N}. \tag{3.4}$$

We check the hypotheses of Lemma 3.5. Clearly, the sequence $\{T_{n(k)}\}_{k=1}^\infty$ is pairwise commuting. Since $P_n \to I$ in $\mathcal{L}_s(X)$ as $n \to \infty$, it is clear that (3.1) is satisfied. The condition (3.2) follows from Remark 3.6. So, Lemma 3.5 does indeed apply and hence, $\lim_{k\to\infty} T_{n(k)}x_k = 0$ in X_σ (since $\{x_{n(k)}\}_{k=1}^\infty \subseteq B$ is a bounded sequence). Moreover, the equicontinuity of $\{x'_k\}_{k=1}^\infty \subseteq X'$ implies that $\{x'_k\}_{k=1}^\infty$ is $\sigma(X', X)$–bounded and hence, part (ii) of Lemma 3.5 implies that $\lim_{k\to\infty} T^t_{n(k)}x'_k = 0$ in $(X', \sigma(X', X''))$. According to Proposition 3.3(i), the DP–property of X implies that $\lim_{k\to\infty}\langle T_{n(k)}x_k, T^t_{n(k)}x'_k\rangle = 0$ which contradicts (3.4). So, $P_n \to I$ in $\mathcal{L}_b(X)$ as $n \to \infty$.

(ii) According to [24, p.134] we have $\{P^t_n\}_{n=1}^\infty \subseteq \mathcal{L}(X'_\beta)$. Part (i) ensures that $P_n \to I$ in $\mathcal{L}_b(X)$ and hence, by Lemma 2.1, $P^t_n \to I$ in $\mathcal{L}_b(X'_\beta)$. So, $\{P^t_n\}_{n=1}^\infty$ is a Schauder decomposition of X'_β with property (M). In particular, $P^t_n x' \to x'$ in X'_β for each $x' \in X'$, i.e., $\{P_n\}_{n=1}^\infty$ is a shrinking Schauder decomposition of X.

(iii) Suppose first that each $Q_n(X)$, for $n \in \mathbb{N}$, is a Grothendieck space. Fix $x'' \in X''$. We claim that

$$\lim_{n\to\infty}(I - P^{tt}_n)x'' = 0 \ \text{ in } (X'_\beta)'_\beta. \tag{3.5}$$

To see this, fix $B \in \mathcal{B}(X'_\beta)$. Then

$$W := \{S \in \mathcal{L}(X'_\beta) : \ |\langle Sv', x''\rangle| \leq 1, \ \forall v' \in B\}$$

is a 0–neighbourhood in $\mathcal{L}_b(X'_\beta)$. Since $(I - P^t_n) \to 0$ in $\mathcal{L}_b(X'_\beta)$ as $n \to \infty$ (see Lemma 2.1), there is $m \in \mathbb{N}$ such that $(I - P^t_n) \in W$ for all $n \geq m$. That is, for all $v' \in B$ and $n \geq m$ we have

$$|\langle v', (I - P^{tt}_n)x''\rangle| = |\langle (I - P^t_n)v', x''\rangle| \leq 1.$$

This shows that $(I - P^{tt}_n)x'' \in B^\circ$ for $n \geq m$, i.e., $\lim_{n\to\infty}(I - P^{tt}_n)x'' = 0$ in $(X'_\beta)'_\beta$, as claimed.

Let $\{x'_k\}_{k=1}^\infty \subseteq X'$ be any sequence such that $x'_k \to 0$ in $(X', \sigma(X', X))$ as $k \to \infty$. Since X is barrelled, the sequence $\{x'_k\}_{k=1}^\infty$ is bounded in X'_β. So, (3.5) implies that for every $\varepsilon > 0$ there is $m \in \mathbb{N}$ with

$$|\langle x'_k, (I - P^{tt}_n)x''\rangle| < \varepsilon, \quad k \in \mathbb{N}, \ n \geq m.$$

It follows, for each $k \in \mathbb{N}$, that

$$|\langle x'_k, x''\rangle| \leq |\langle x'_k, (I - P^{tt}_m)x''\rangle| + |\langle x'_k, P^{tt}_m x''\rangle| \leq \varepsilon + |\langle x'_k, P^{tt}_m x''\rangle|.$$

Setting $E_m := P_m(X)$, we have that the restriction $P^t_m : (E_m)'_\beta \to X'_\beta$ is continuous and hence, $x'' \circ P^t_m : (E_m)'_\beta \to \mathbb{C}$ is continuous, i.e., $P^{tt}_m x'' \in ((E_m)'_\beta)'_\beta$. But, E_m is a Grothendieck space (c.f. proof of (i) of Proposition 3.1) and the restrictions $x'_k|_{E_m} \to 0$ in $(E'_m, \sigma(E'_m, E_m))$ as $k \to \infty$. Accordingly, $x'_k|_{E_m} \to 0$ in $(E'_m, \sigma(E'_m, E''_m))$ as $k \to \infty$. This implies that $\limsup_k |\langle x'_k, x''\rangle| \leq \varepsilon$. Since

$\varepsilon > 0$ is arbitrary, we have $\langle x'_k, x'' \rangle \to 0$ as $k \to \infty$. Hence, X is a Grothendieck space.

Next assume that $Q_n(X)$ has the DP–property for each $n \in \mathbb{N}$. Fix any operator $T \in \mathcal{L}(X,Y)$, with Y a Banach space, which maps bounded sets in X into $\sigma(Y,Y')$–compact sets in Y. Let $B \subseteq X$ be any $\sigma(X,X')$–compact set. Given $\varepsilon > 0$ there exists a 0–neighbourhood $U \subseteq X$ with $T(U) \subseteq \varepsilon B_Y$, where B_Y is the closed unit ball of Y. Since $\lim_{n\to\infty}(I - P_n) = 0$ in $\mathcal{L}_b(X)$, there is $n(0) \in \mathbb{N}$ such that $(I - P_{n(0)})(B) \subseteq U$. Then

$$T(B) \subseteq T(P_{n(0)}(B)) + T((I - P_{n(0)})(B)) \subseteq T(P_{n(0)}(B)) + \varepsilon B_Y. \tag{3.6}$$

Clearly $P_{n(0)}(B)$ is weakly compact in $E_{n(0)} := P_{n(0)}(X)$. Moreover, $E_{n(0)}$ has the DP–property since it is the direct sum (hence, also product) of finitely many spaces $Q_n(X)$, for $1 \leq n \leq n(0)$, each one with the DP–property, [21, p.635]. Hence, $T(P_{n(0)}(B))$ is relatively compact in Y and it follows from (3.6) and Lemma 2.2 that $T(B)$ is precompact (hence, relatively compact) in Y. It follows that X has the DP–property.

Finally, assume that each $Q_n(X)$, for $n \in \mathbb{N}$, is Montel. Fix $B \in \mathcal{B}(X)$. Since X is quasicomplete, it suffices to show that B is precompact. Let U be a 0–neighbourhood in X. Since $P_n \to I$ in $\mathcal{L}_b(X)$, there is $m \in \mathbb{N}$ such that $(I - P_m)(B) \subseteq U$. On the other hand, $P_m(B) \subseteq \sum_{j=1}^m Q_j(B)$. Since $Q_j(B)$ is bounded in the Montel space $Q_j(X)$, it follows that $Q_j(B)$ is relatively compact in $Q_j(X)$, for each $1 \leq j \leq m$ and hence, also in X. It follows that $P_m(B)$ is relatively compact. Since $B \subseteq P_m(B) + U$, it follows from Lemma 2.2 that B is precompact. □

Remark 3.9. (i) For X a Fréchet space, that part of (iii) in Proposition 3.8 which states if each $Q_n(X)$, $n \in \mathbb{N}$, is Montel, then X is Montel, is known, [15, Proposition 4].

(ii) Suppose that X is quasicomplete, barrelled GDP–space which admits a Schauder decomposition $\{P_n\}_{n=1}^\infty$ (necessarily with property (M) by Proposition 3.8(i)) such that each space $Q_n(X)$, with $Q_n := P_n - P_{n-1}$ for $n \geq 1$, is finite dimensional. Then X is a Montel space; see Proposition 3.8(iii). In particular, every quasicomplete barrelled GDP–space with a Schauder basis is Montel.

Let $X = \operatorname{ind}_n X_n$ be a countable inductive limit of lcHs' with canonical inclusions $j_n \colon X_n \to X$, for each $n \in \mathbb{N}$. Recall that X is *quasi–regular* if for every $B \in \mathcal{B}(X)$ there exist $m \in \mathbb{N}$ and $C \in \mathcal{B}(X_m)$ such that $B \subseteq \overline{C}$, where the closure $\overline{C}$ of C is taken in X, [16]. If every bounded subset of X is contained and bounded in a step X_m, for some $m \in \mathbb{N}$, then X is called *regular*, [29]. Every (LB)–space $X = \operatorname{ind}_n X_n$ is quasi–regular; this follows from [29, Corollary 8.3.19]. If X is quasi–regular, then $X'_\beta = \operatorname{proj}_n (X_n)'_\beta$, where the projective limit is formed with respect to the linking maps $j_n^t \colon X' \to X'_n$; see, for example, [23, §22.6 and 22.7], [16]. In particular, if X is a quasi–regular (LF)–space, then every relatively weakly compact subset of X'_β is relatively sequentially weakly compact, [13, Propositions 5 and 6, Theorem 11]. The following result is a reformulation of [8, Proposition 22].

Lemma 3.10. *Let X be a $\aleph_0$–barrelled lcHs such that every relatively weakly compact subset of X'_β is relatively sequentially weakly compact. Then X is a*

Grothendieck space if and only if every operator $T \in \mathcal{L}(X, c_0)$ maps bounded subsets of X to relatively weakly compact subsets of c_0.

We can now establish a useful fact concerning inductive limits.

Proposition 3.11. *Let $X = \operatorname{ind}_n X_n$ be a quasi–regular (LF)–space such that each Fréchet space X_n, for $n \in \mathbb{N}$, is a Grothendieck space. Then X is also a Grothendieck space.*

Proof. Let $T \in \mathcal{L}(X, c_0)$ and $B \in \mathcal{B}(X)$. Since X is quasi–regular, there exist $m \in \mathbb{N}$ and $C \in \mathcal{B}(X_m)$ such that $B \subseteq \overline{C}$, with the closure of C formed in X. The restriction $S := T|_{X_m}$ belongs to $\mathcal{L}(X_m, c_0)$. Since X_m is a Grothendieck space, the set $T(C) = S(C)$ is relatively weakly compact in c_0. Moreover, $T(B) \subseteq T(\overline{C}) \subseteq \overline{T(C)}$ and so $T(B)$ is also relatively weakly compact in c_0. Observing that Lemma 3.10 can be applied to the (barrelled) quasi–regular (LF)–space X, since $X'_\beta = \operatorname{proj}_n (X_n)'_\beta$ satisfies the required hypothesis, we can conclude that X is a Grothendieck space. □

For the DP–property of inductive limits we have the following result.

Proposition 3.12. *Let $X = \operatorname{ind}_n X_n$ be an (LF)–space which satisfies:*

(3.7) Every weakly compact subset of X is contained and weakly compact in some step X_m.

If each Fréchet space X_n, for $n \in \mathbb{N}$, has the DP–property, then also X has the DP–property.

Proof. Let Y be any Banach space and $T \in \mathcal{L}(X, Y)$ transform bounded subsets of X into relatively weakly compact subsets of Y. Fix a weakly compact set $A \subseteq X$. By (3.7) there exists $m \in \mathbb{N}$ such that $A \subseteq X_m$ and A is weakly compact in X_m. Since the restriction $T|_{X_m}$ of T to X_m maps bounded sets of X_m to relatively weakly compact subsets of Y and X_m has the DP–property, it follows that $T(A)$ is relatively weakly compact in Y. Accordingly, X has the DP–property. □

Remark 3.13. (i) Every (LF)–space $X = \operatorname{ind}_n X_n$ satisfying (3.7) is necessarily regular. To see this, let $\{x_k\}_{k=1}^\infty \subseteq X$ be a $\sigma(X, X')$–null sequence. Then it suffices to show that $\{x_k\}_{k=1}^\infty$ is contained and $\sigma(X_m, X'_m)$–null in some step X_m; the conclusion will then follow from [30, Theorem 1]. But, since $A := \{0\} \cup \{x_k\}_{k=1}^\infty$ is weakly compact in X, (3.7) ensures that there is $m \in \mathbb{N}$ such that $A \subseteq X_m$ and A is $\sigma(X_m, X'_m)$–compact. As the topology $\sigma(X_m, X'_m)$ restricted to A is finer that $\sigma(X, X')$ restricted to A and the latter topology is Hausdorff, the two topologies coincide on A. Hence, $\lim_{k\to\infty} x_k = 0$ for $\sigma(X_m, X'_m)$.

(ii) An (LF)–space $X = \operatorname{ind}_n X_n$ is said to satisfy (Retakh's) *condition* (M_0) if there exists an increasing sequence $\{U_n\}_{n=1}^\infty$ of absolutely convex 0–neighbourhoods U_n in X_n such that for each $n \in \mathbb{N}$ there exists $m(n) > n$ with the property that the topologies $\sigma(X, X')$ and $\sigma(X_{m(n)}, X'_{m(n)})$ coincide on U_n. A relevant reference for condition (M_0) is [34]. An (LF)–space satisfying condition (M_0) need not be quasi–regular, [16]. On the other hand, if a regular (LF)–space satisfies condition (M_0), then it necessarily possesses the property (3.7). To see this, let $A \subseteq X$ be $\sigma(X, X')$–compact. By regularity of X there exists $n \in \mathbb{N}$ such that $A \subseteq X_n$ and $A \in \mathcal{B}(X_n)$. According to condition (M_0) there exists $m(n) > n$ and a 0–neighbourhood W_n in X_n such that the topologies $\sigma(X, X')$

and $\sigma(X_{m(n)}, X'_{m(n)})$ agree on W_n. Select $\lambda > 0$ such that $\lambda A \subseteq W_n$ and note that λA is $\sigma(X, X')$–compact as A is. Accordingly, λA (and, hence also A) is $\sigma(X_{m(n)}, X'_{m(n)})$–compact, i.e., (3.7) is valid.

There is a condition which is more restrictive than (M_0) but, has the advantage in practice that it is easier to verify. Namely, an (LF)–space $X = \operatorname{ind}_n X_n$ satisfies (Retakh's) *condition* (M) if there exists an increasing sequence $\{U_n\}_{n=1}^{\infty}$ of absolutely convex 0–neighbourhoods U_n in X_n such that for each $n \in \mathbb{N}$ there exists $m(n) > n$ with the property that X and $X_{m(n)}$ induce the same topology on U_n. It is known that an (LF)–space $X = \operatorname{ind}_n X_n$ satisfies condition (M) if and only if it is *sequentially retractive*, i.e., every convergent sequence in X is convergent in some step X_m, [36].

(iii) A regular co–echelon space $k_\infty(V) = \operatorname{ind}_n \ell^\infty(v_n)$ of order infinity satisfies condition (M_0) if and only if it satisfies condition (M), [5], which in turn is equivalent to the defining sequence $V = (v_n)$ being regularly decreasing. Co–echelon spaces of the form $k_\infty(V)$ will be treated in more detail later. □

Since every (LF)–space satisfying (3.7) is regular (c.f. Remark 3.13(i)), the following fact is a consequence of Propositions 3.11 and 3.12.

Proposition 3.14. *Let $X = \operatorname{ind}_n X_n$ be an (LF)–space satisfying property (3.6). If all the Fréchet spaces X_n, for $n \in \mathbb{N}$, are GDP–spaces, then X is also a GDP–space.*

An immediate consequence is the following result.

Corollary 3.15. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of Fréchet spaces. Then the lc–direct sum $X = \oplus_{n=1}^{\infty} X_n$ is a GDP–space if and only if each X_n, for $n \in \mathbb{N}$, is a GDP–space.*

Proof. Each space X_n is complemented in X, for $n \in \mathbb{N}$. Moreover, $X = \operatorname{ind}_n \oplus_{j=1}^{n} X_j$ is a strict (LF)–space and hence, has property (3.6). Indeed, if $A \subseteq X$ is weakly compact, then $A \in \mathcal{B}(X)$ and hence, $A \subseteq \oplus_{j=1}^{m} X_j$ for some $m \in \mathbb{N}$. Since X induces the given topology on $\oplus_{j=1}^{m} X_j$, it follows that A is weakly compact in $\oplus_{j=1}^{m} X_j$. □

Let us discuss another application of Proposition 3.14. Let Ω denote either $\mathbb{D} := \{z \in \mathbb{C} : \ |z| > 1\}$ or $\mathbb{C}$ and let $a = 1$ or $a = \infty$, respectively. A radial weight is a continuous, non–increasing function $v\colon \Omega \to (0, \infty)$ such that $v(z) = v(|z|)$ for each $z \in \Omega$ and $\lim_{r \to a} r^m v(r) = 0$ for all $m \geq 0$ (for $\Omega = \mathbb{D}$ this condition is $\lim_{r \to 1} v(r) = 0$). Given such a weight, the space of holomorphic functions

$$\mathcal{H}_v(\Omega) := \{f \in \mathcal{H}(\Omega) : \ \|f\|_v := \sup_{z \in \Omega} v(z)|f(z)| < \infty\}$$

is a Banach space when endowed with the norm $\|\cdot\|_v$. W. Lusky showed that $\mathcal{H}_v(\Omega)$ is isomorphic to either ℓ^∞ or to the Hardy space $H^\infty(\mathbb{D})$, [27]. In particular, $\mathcal{H}_v(\Omega)$ is always a GDP–space.

Let $V := \{v_n\}_{n=1}^{\infty}$ be any decreasing sequence of strictly positive radial weights on Ω and consider the weighted inductive limit $V\mathcal{H}(\Omega) := \operatorname{ind}_n \mathcal{H}_{v_n}(\Omega)$. It follows from results in [2] that $V\mathcal{H}(\Omega)$ is a complete (LB)–space. The sequence V is called *regularly decreasing* if for each $n \in \mathbb{N}$ there is $m(n) \geq n$ such that for all $\varepsilon > 0$ and $k \geq m(n)$ there is a $\delta > 0$ such that $v_k \geq \delta v_n$ pointwise on Ω whenever

$v_{m(n)} \geq \varepsilon v_n$ pointwise on Ω. In this case $V\mathcal{H}(\Omega)$ is a sequentially retractive (LB)–space, [2, Theorem 2.3], and hence, it satisfies (3.7); see Remark 3.13(ii). As a consequence of Proposition 3.14 we have the following result.

Corollary 3.16. *Let Ω denote either $\mathbb{D}$ or $\mathbb{C}$ and $V = \{v_n\}_{n=1}^{\infty}$ be a decreasing sequence of radial weights on Ω which is regularly decreasing. Then the weighted (LB)–space $V\mathcal{H}(\Omega) = \mathrm{ind}_n\, \mathcal{H}_{v_n}(\Omega)$ of holomorphic functions is a GDP–space.*

Recall that a *spectral measure* in a lcHs X is a multiplicative map $P\colon \Sigma \to \mathcal{L}(X)$, defined on a σ–algebra Σ of subsets of a non–empty set Ω, which satisfies $P(\Omega) = I$ and is σ–additive in $\mathcal{L}_s(X)$. If, in addition, P is σ–additive in $\mathcal{L}_b(X)$, then P is called *boundedly σ–additive*. It is known that every spectral measure in a Fréchet GDP–space (hence, in every Fréchet Montel space) is necessarily boundedly σ–additive; see Proposition 4.3 of [12]. An examination of the proof given in [12] shows that it can be adapted (by using Propositions 3.1(i) and 3.8(i) above at the appropriate stage) to yield the following extension.

Proposition 3.17. *Let X be any quasicomplete, barrelled lcHs which is a GDP–space. Then every spectral measure in X is necessarily boundedly σ–additive.*

For examples of spectral measures in classical spaces, some of which are boundedly σ–additive and others which are not, we refer to [9], [10], [11], [12], [32], [33], for example.

The following result is an extension of Proposition 4.2 in [1], where it is formulated for Fréchet spaces. However, an examination of the proof shows that the metrizability of X is not necessary, in that the topology of X need *not* be given by a sequence of continuous seminorms and the use of [35, Proposition 2.3] applies in general spaces.

Proposition 3.18. *Let X be a quasicomplete, barrelled lcHs. Suppose that there exists a spectral measure in X which fails to be boundedly σ–additive. Then X admits an unconditional Schauder decomposition without property (M).*

For explicit examples of spaces which admit spectral measures which *fail* to be boundedly σ–additive we refer to Remark 4.3 in [1].

Unconditional Schauder decompositions are of particular interest in non–normable GDP–spaces (in GDP–Banach spaces they do not exist). It was shown in [12, Proposition 3.1] that every Fréchet Köthe echelon space $\lambda_\infty(A)$ of infinite order is a GDP–space and, under certain conditions on the Köthe matrix A, that $\lambda_\infty(A)$ admits unconditional Schauder decompositons, [12, Proposition 4.4]. Further examples can now be exhibited. For example, let $X = \prod_{n=1}^{\infty} X_n$ be any countable product of Fréchet GDP–spaces X_n, for $n \in \mathbb{N}$, in which case X is also a Fréchet GDP–space. Define a continuous projection $Q_j \in \mathcal{L}(X)$ by $Q_j x := (0, \ldots, 0, x_j, 0, \ldots)$ for $x = (x_n) \in X$, where x_j is in position j, for each $j \in \mathbb{N}$, and set $P_n := \sum_{j=1}^{n} Q_j$ for $n \in \mathbb{N}$. Then $\{P_n\}_{n=1}^{\infty}$ is a Schauder decomposition of X. Moreover, for a fixed $x \in X$, it is routine to verify that $\sum_{n=1}^{\infty} Q_n$ is unconditionally convergent to x in X_σ and hence, by the Orlicz–Pettis theorem, also in X. Accordingly, $\{P_n\}_{n=1}^{\infty}$ is an unconditional Schauder decomposition of X. Since X always contains a complemented copy of the Fréchet sequence space $\omega = \prod_{n=1}^{\infty} \mathbb{C}$, other unconditional Schauder decompositions also exist in X. Or, consider the lc–direct sum $X = \oplus_{n=1}^{\infty} X_n$ of Fréchet GDP–spaces. Again the coordinate projections generate an unconditional Schauder decomposition of the (non–metrizable) GDP–space X.

4. Co–echelon spaces.

In the final section we make a detailed investigation of co–echelon spaces from the viewpoint of GDP–spaces. For Köthe echelon spaces such an analysis was undertaken in [12].

In this section, I will always denote a fixed countable index set and $A = (a_n)_{n\in\mathbb{N}}$ an increasing sequence of functions $a_n \colon I \to (0,\infty)$, which is called a *Köthe matrix* on I. Corresponding to each $p \in \{0\} \cup [1,\infty]$ we associate the spaces

$$\lambda_p(A) := \left\{ x = (x(i))_{i\in I} \in \mathbb{C}^I : \, q_n^{(p)}(x) = \left(\sum_{i\in I} (a_n(i)|x(i)|)^p \right)^{1/p} < \infty, \forall n \in \mathbb{N} \right\}$$

$$\lambda_\infty(A) := \left\{ x = (x(i))_{i\in I} \in \mathbb{C}^I : \; q_n^{(\infty)}(x) = \sup_{i\in I} a_n(i)|x(i)| < \infty, \forall n \in \mathbb{N} \right\}$$

$$\lambda_0(A) := \left\{ x = (x(i))_{i\in I} \in \mathbb{C}^I : \; a_n x \in c_0(I), \forall n \in \mathbb{N} \right\},$$

with the last space being endowed with the topology induced by $\lambda_\infty(A)$. The spaces $\lambda_p(A)$ are called (Köthe) *echelon spaces* of order p; they are Fréchet spaces relative to the sequence of seminorms $\{q_n^{(p)}\}_{n=1}^\infty$ for $p \in \{0\} \cup [1,\infty]$.

For a Köthe matrix $A = (a_n)_{n=1}^\infty$, let $V = (v_n)_{n=1}^\infty$ with $v_n := 1/a_n$ for $n \in \mathbb{N}$, and set

$$k_p(V) =: \operatorname*{ind}_n \ell^p(v_n), \; p \in [1,\infty], \text{ and } \; k_0(V) := \operatorname*{ind}_n c_0(v_n),$$

where $\ell^p(v_n) \subseteq \mathbb{C}^I$ and $c_0(v_n) \subseteq \mathbb{C}^I$ are the usual (weighted) Banach spaces, for $n \in \mathbb{N}$. So, $k_p(V)$ is the increasing union $\cup_{n=1}^\infty \ell^p(v_n)$ (resp. $\cup_{n=1}^\infty c_0(v_n)$) endowed with the strongest lc–topology under which the natural injection of each of the Banach spaces $\ell^p(v_n)$ (resp. $c_0(v_n)$), for $n \in \mathbb{N}$, is continuous. The spaces $k_p(V)$ are called *co–echelon spaces* of order p. The natural map $k_0(V) \to k_\infty(V)$ is clearly continuous but, it is even a topological isomorphism into $k_\infty(V)$. For a systematic treatment of echelon and co–echelon spaces see [3].

Given any decreasing sequence $V = (v_n)_{n=1}^\infty$ of strictly positive functions on I (or for the corresponding Köthe matrix $A = (a_n)_{n=1}^\infty$ with $a_n := 1/v_n$) we introduce

$$\overline{V} := \left\{ \overline{v} = (\overline{v}(i))_{i\in I} \in [0,\infty)^I : \; \sup_{i\in I} \frac{\overline{v}(i)}{v_n(i)} = \sup_{i\in I} a_n(i)\overline{v}(i) < \infty, \; \forall n \in \mathbb{N} \right\}.$$

Since I is countable, the system $\overline{V}$ always contains strictly positive functions. Next, associated with $\overline{V}$ is the family of spaces

$$K_p(\overline{V}) := \operatorname*{proj}_{\overline{v}\in\overline{V}} \ell^p(\overline{v}), \; p \in [1,\infty], \text{ and } \; K_0(\overline{V}) := \operatorname*{proj}_{\overline{v}\in\overline{V}} c_0(\overline{v}).$$

These spaces are equipped with the complete lc–topology given by the collection of seminorms

$$q_{\overline{v}}^{(p)}(x) := \left(\sum_{i\in I} (\overline{v}(i)|x(i)|)^p \right)^{1/p}, \; 1 \le p < \infty, \;\text{ and }\; q_{\overline{v}}^{(\infty)}(x) := \sup_{i\in I} \overline{v}(i)|x(i)|,$$

for each $\overline{v} \in \overline{V}$. For $1 \leq p < \infty$ it is known that $k_p(V)$ equals to $K_p(\overline{V})$ as vector spaces and also topologically. In particular, the inductive limit topology is given by the system of seminorms $\{q_{\overline{v}}^{(p)} : \overline{v} \in \overline{V}\}$ and $k_p(V)$ is always complete. Moreover, $K_0(\overline{V})$ is the completion of $k_0(V)$ and the inductive limit topology of $k_0(V)$ is given by the system of seminorms $\{q_{\overline{v}}^{(\infty)} : \overline{v} \in \overline{V}\}$. However, it can happen that $k_0(V)$ is a proper subspace of $K_0(\overline{V})$. Finally, $k_\infty(V)$ and $K_\infty(\overline{V})$ are equal as vector spaces and the two spaces have the same bounded sets. Moreover, $k_\infty(V)$ is the bornological space associated with $K_\infty(\overline{V})$ but, in general, the inductive limit topology is genuinely stronger than the topology of $K_\infty(\overline{V})$.

Concerning duality we have $(\lambda_p(A))'_\beta = K_q(\overline{V})$ and $(k_p(V))'_\beta = \lambda_q(A)$, where $p \in \{0\} \cup [1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$ (and $q = \infty$ if $p = 1$; $q = 1$ if $p = 0$). Also, for $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ or $p = 0$ and $q = 1$, we have $(\lambda_p(A))'_\beta = k_q(V)$. In case $1 < p < \infty$, the spaces $\lambda_p(A)$ and $k_p(V)$ are reflexive. The space $\lambda_0(A)$ is distinguished and satisfies $((\lambda_0(A))'_\beta)'_\beta = (k_1(V))'_\beta = \lambda_\infty(A)$. Furthemore, $K_0(\overline{V})$ is a barrelled (DF)–space with $(K_0(\overline{V}))'_\beta = (k_0(V))'_\beta = \lambda_1(A)$. Hence, there is the biduality $((k_0(V))'_\beta)'_\beta = ((K_0(\overline{V}))'_\beta)'_\beta = K_\infty(\overline{V})$. The inductive dual $(\lambda_1(A))'_i = k_\infty(V)$ and this space is complete. We point out that $k_\infty(V) = (\lambda_1(A))'_\beta$ if and only if $K_\infty(\overline{V}) = k_\infty(V)$ if and only if $\lambda_1(A)$ is distinguished. For all the above facts on echelon and co–echelon spaces we refer to [3], [6].

Proposition 4.1. *Let $V = (v_n)_{n=1}^\infty$ be any decreasing sequence of strictly positive functions defined on a countable index set I and $p \in [1, \infty)$. The following assertions are equivalent.*

(i) *$k_p(V) = \operatorname{ind}_n \ell^p(v_n)$ is a GDP–space.*

(ii) *$k_p(V)$ is a Montel space.*

(iii) *For every infinite set $I_0 \subseteq I$ and every $n \in \mathbb{N}$ there exists $m(n) > n$ such that*

$$\inf_{i \in I_0} \frac{v_{m(n)}(i)}{v_n(i)} = 0.$$

Proof. (ii)⇔(iii) is well known, [3, Theorem 4.7].

(ii)⇒(i); see Remark 3.7.

(i)⇒(iii). Suppose that (iii) fails. Then there is an infinite set $I_0 \subseteq I$ and $n \in \mathbb{N}$ such that for each $m > n$ there exists $\varepsilon_m > 0$ satisfying $v_m(i) \geq \varepsilon_m v_n(i)$ for all $i \in I_0$. Consider the (complemented) sectional subspace

$$X_0 := \{x = (x(i))_{i \in I} \in k_p(V) : x(i) = 0 \ \forall i \notin I_0\}.$$

If $x \in X_0$, then $x \in \ell^p(v_m)$ for some $m > n$. It follows from the previous inequalities that $\|x\|_n \leq \varepsilon_m^{-1} \|x\|_m$, where $\| \cdot \|_r$ is the norm of $\ell^p(v_r)$ for each $r \in \mathbb{N}$. Hence, $X_0 \subseteq \ell^p(v_n)$ and so we can endow X_0 with the norm $\| \cdot \|$ induced by $\ell^p(v_n)$, i.e., $\|x\| = \left(\sum_{i \in I_0} (v_n(i)|x(i)|)^p\right)^{1/p}$. The injection $j_0 \colon X_0 \to k_p(V)$ is continuous because the injection $\tilde{j}_0 \colon X_0 \to \ell^p(v_n)$ is continuous. Moreover, the projection $P \colon k_p(V) \to X_0$ defined by $Px := x\chi_{I_0}$, for $x \in k_p(V)$, is continuous. To see this we need to show that the restriction $P \colon \ell^p(v_m) \to X_0$ is continuous for each $m > n$. But, this is precisely the inequality $\|Px\| = \|x\chi_{I_0}\|_n \leq \varepsilon_m^{-1} \|x\|_m$ indicated above. Accordingly, $(X_0, \| \cdot \|)$ is a Banach space which is isomorphic to a complemented subspace of $k_p(V)$ and also isomorphic to ℓ^p. Since ℓ^p, for

$1 \le p < \infty$, is not a GDP–space neither is $k_p(V)$; see Proposition 3.1(i). So, (i) fails. □

For the notion of $V = (v_n)_{n=1}^\infty$ being regularly decreasing we refer to Section 3.

Proposition 4.2. *Let $V = (v_n)_{n=1}^\infty$ be any decreasing sequence of strictly positive functions defined on a countable index set I which is regularly decreasing. The following assertions are equivalent.*

(i) $k_0(V) = \operatorname{ind}_n c_0(v_n)$ is a GDP–space.
(ii) $k_0(V)$ is a Montel space.
(iii) For every infinite set $I_0 \subseteq I$ and every $n \in \mathbb{N}$ there exists $m(n) > n$ such that

$$\inf_{i \in I_0} \frac{v_{m(n)}(i)}{v_n(i)} = 0.$$

Proof. Since $k_0(V)$ is complete and $k_0(V) \simeq K_0(\overline{V})$, [3, Lemma 3.6], the equivalences (ii)⇔(iii) are known; see (1)⇔(3) in [3, Theorem 4.7].

(ii)⇒(i); see Remark 3.7.

Finally, if (iii) fails to hold, then we can repeat the argument in the proof of (i)⇒(iii) in Propositon 4.1 above to conclude that $k_0(V)$ contains a complemented copy of c_0. Since the Banach space c_0 is not a GDP–space neither is $k_0(V)$; see Proposition 3.1(i). □

Proposition 4.3. *Let $V = (v_n)_{n=1}^\infty$ be any decreasing sequence of strictly positive functions defined on a countable index set I which is regularly decreasing. Then $k_\infty(V) = \operatorname{ind}_n \ell^\infty(v_n)$ is a GDP–space.*

Proof. Each Banach space $\ell^\infty(v_n)$ is isomorphic to ℓ^∞ and hence, is a GDP–space. Since V is regularly decreasing, $k_\infty(V) = \operatorname{ind}_n \ell^\infty(v_n)$ is sequentially retractive, [2, Theorem 2.3]. In particular, $k_\infty(V)$ satisfies (3.6); see parts (ii) and (iii) of Remark 3.13. So, Proposition 3.12 ensures that $k_\infty(V)$ has the DP–property. Since $k_\infty(V)$ is an (LB)–space, the discussion prior to Lemma 3.10 implies that $k_\infty(V)$ is quasi–regular. Then Proposition 3.11 implies that $k_\infty(V)$ is a Grothendieck space. □

Observe that the proof of $k_\infty(V)$ being a Grothendieck space does *not* require V to be regularly decreasing and hence, holds for arbitrary V.

Concerning the GDP–property of $K_\infty(\overline{V})$ it is possible to remove the requirement of regularly decreasing. To achieve this we require a technical result.

Lemma 4.4. *Let $V = (v_n)_{n=1}^\infty$ be any decreasing sequence of strictly positive functions defined on a countable index set I. For each $\overline{v} \in \overline{V}$ there exists an increasing sequence $\{I_k\}_{k=1}^\infty$ of subsets of I such that*

(i) for every $k \in \mathbb{N}$ and every $n > k$ there exists $\alpha_{n,k} > 0$ satisfying

$$v_k(i) \le \alpha_{n,k} v_n(i), \quad i \in I_k,$$

(ii) for every $m \in \mathbb{N}$ and every $\varepsilon > 0$ there exists $k = k(m, \varepsilon)$ satisfying

$$\overline{v}(i) \le \varepsilon v_m(i), \quad i \in I \setminus I_k.$$

Proof. Fix $\overline{v} \in \overline{V}$. By definition of $\overline{V}$ we can select, for each $n \in \mathbb{N}$, a constant $M_n \geq 1$ with $\overline{v} \leq M_n v_n$ pointwise on I. Set $\beta_k := 2^k M_k \geq 2^k$, for each $k \in \mathbb{N}$, and define (possibly some empty) subsets of I by

$$J_1 := \cap_{n=2}^{\infty}\{i \in I : \ v_{n-1}(i) \leq \beta_n v_n(i)\}$$

and

$$J_k := (\cap_{s>k}\{i \in I : \ v_{s-1}(i) \leq \beta_s v_s(i)\}) \cap \{i \in I : \ v_{k-1}(i) > \beta_k v_k(i)\},$$

for $k \geq 2$. Then the sets $\{J_k\}_{k=1}^{\infty}$ are pairwise disjoint and, from their definition it follows, for each $k \in \mathbb{N}$, that

$$v_k(i) \leq (\beta_{k+1} \ldots \beta_l) v_l(i), \quad i \in J_k, \ \forall l > k. \tag{4.1}$$

Set $I_k := \cup_{s=1}^{k} J_s$, for $k \in \mathbb{N}$. If $i \in I_k$ and $n > k$ and $1 \leq s \leq k$, then it follows from (4.1) that

$$v_k(i) \leq v_s(i) \leq (\beta_{s+1} \ldots \beta_n) v_n(i) \leq \alpha_{n,k} v_l(i),$$

where $\alpha_{n,k} := (\beta_{k+1} \ldots \beta_n)$. Then (i) is clear.

Now, fix $i \notin \cup_{k=1}^{\infty} J_k$. Since $i \notin J_1$ there is $n(1) \in \mathbb{N}$ with

$$v_{n(1)-1}(i) > \beta_{n(1)} v_{n(1)}(i). \tag{4.2}$$

As $i \notin J_{n(1)}$ and (4.2) holds, there is $n(2) > n(1)$ with

$$v_{n(2)-1}(i) > \beta_{n(2)} v_{n(2)}(i).$$

Proceeding by induction, we can select $n(1) < n(2) < \ldots < n(k) < \ldots$ such that, for each $k \in \mathbb{N}$, we have

$$v_{n(k)-1}(i) > \beta_{n(k)} v_{n(k)}(i).$$

In particular, as V is decreasing, it follows that

$$v_1(i) > \beta_{n(k)} v_{n(k)}(i), \quad k \in \mathbb{N}.$$

So, for each $k \in \mathbb{N}$, we have

$$2^{n(k)} \overline{v}(i) \leq 2^{n(k)} M_{n(k)} v_{n(k)}(i) = \beta_{n(k)} v_{n(k)}(i) < v_1(i).$$

That is, $\overline{v}(i) < 2^{-n(k)} v_1(i)$ for each $k \in \mathbb{N}$ which implies that $\overline{v}(i) = 0$. Accordingly, we have established that

$$\overline{v}(i) = 0, \quad i \notin \cup_{r=1}^{\infty} J_r.$$

We can now complete the proof of (ii). Fix $m \in \mathbb{N}$ and $\varepsilon > 0$ and select $k \in \mathbb{N}$ satisfying $k > m$ and $2^k > \varepsilon^{-1}$. Then, for $i \notin I_k$, either $i \notin \cup_{r=1}^{\infty} J_r$, in which case $\overline{v}(i) = 0$ from above, or $i \in J_l$ for some $l > k$. In this latter case we have

$$\begin{aligned} \overline{v}(i) &\leq M_l v_l(i) < M_l \beta_l^{-1} v_{l-1}(i) \quad (\text{as } i \in J_l) \\ &= 2^{-l} v_{l-1}(i) \leq 2^{-l} v_m(i) \quad (\text{as } m < k < l) \\ &\leq 2^{-k} v_m(i) < \varepsilon v_m(i) \quad (\text{as } k < l). \end{aligned}$$

This establishes (ii) and the proof is thereby complete. □

It should be pointed out that Lemma 4.4 implies Lemma 2.4 of [12]; see also the discussion prior to this result in [12].

For an absolutely convex bounded subset B of a lcHs X, there is always an associated normed space X_B, [23, Section 20.11]. We recall that a sequence $\{B_n\}_{n=1}^\infty$ of sets $B_n \in \mathcal{B}(X)$ is called *fundamental* if, for every $B \in \mathcal{B}(X)$ there exists $n \in \mathbb{N}$ such that $B \subseteq B_n$. Associated with such a sequence is the bornological space $X^\times := \operatorname{ind}_n X_{B_n}$. As vector spaces $X = X^\times$ and the topology of $X^\times$ is finer than that of X. Nevertheless, X and $X^\times$ have the same bounded sets. For the definition of $X^\times$ and many of its properties we refer to [29, Section 6.2].

Lemma 4.5. *Let X be a complete (DF)–space with a fundamental sequence of bounded sets $\{B_n\}_{n=1}^\infty$. If the associated bornological space $X^\times := \operatorname{ind}_n X_{B_n}$, which is an (LB)–space, is a Grothendieck space, then X is also a Grothendieck space.*

Proof. Let $T \in \mathcal{L}(X, c_0)$. Since $X^\times$ has a finer topology than X, we also have $T \in \mathcal{L}(X^\times, c_0)$. As $X^\times$ is an (LB)–space, it is barrelled and a (DF)–space and so Lemma 3.10 can be applied in $X^\times$. So, $X^\times$ being a Grothendieck space, it follows from Lemma 3.10 that $T \in \mathcal{L}(X^\times, c_0)$ maps bounded sets of $X^\times$ into relatively weakly compact subsets of c_0. But, $\mathcal{B}(X) = \mathcal{B}(X^\times)$ and so $T \in \mathcal{L}(X, c_0)$ maps bounded sets of X into relatively weakly compact subsets of c_0. By Lemma 3.10 applied in X, we conclude that X is a Grothendieck space. □

Proposition 4.6. *Let $V = (v_n)_{n=1}^\infty$ be any decreasing sequence of strictly positive functions on a countable index set I. Then $K_\infty(\overline{V})$ is a GDP–space.*

Proof. It is known that $K_\infty(\overline{V})^\times = k_\infty(V)$, [6, Theorem 15(c)], with $k_\infty(V) = \operatorname{ind}_n \ell^\infty(v_n)$ an (LB)–space. By Proposition 3.11, $k_\infty(V)$ is a Grothendieck space and so, by Lemma 4.5, $K_\infty(\overline{V})$ is also a Grothendieck space.

Concerning the DP–property, it was noted at the beginning of this section that $K_\infty(\overline{V}) = (\lambda_1(A))'_\beta$, that is, $K_\infty(\overline{V})$ is a complete (DF)–space. So, it suffices to show that if $x^j \to 0$ weakly in $K_\infty(\overline{V})$ and $u^j \to 0$ for $\sigma((K_\infty(\overline{V}))', (K_\infty(\overline{V}))'')$, then $\lim_{j\to\infty}\langle x^j, u^j\rangle = 0$; see Remark 3.4(i)(b) and Remark 3.4(ii). To establish this, first observe that $\{x^j\}_{j=1}^\infty \in \mathcal{B}(K_\infty(\overline{V}))$. Since $\mathcal{B}(K_\infty(\overline{V})) = \mathcal{B}(k_\infty(V))$ with $k_\infty(V) = \operatorname{ind}_n \ell^\infty(v_n)$ an (LB)–space, there exist $m \in \mathbb{N}$ and $C > 0$ such that

$$\sup_{j\in\mathbb{N}} \sup_{i\in I} v_m(i)|x^j(i)| \le C. \tag{4.3}$$

Since $\{u^j\}_{j=1}^\infty$ is bounded for $\sigma((K_\infty(\overline{V}))', (K_\infty(\overline{V}))'')$ and $K_\infty(\overline{V})$ is a complete (DF)–space (hence, $\aleph_0$–barrelled), it follows that $\{u^j\}_{j=1}^\infty$ is equicontinuous. Accordingly, choose $\overline{v} \in \overline{V}$ such that $\{u^j\}_{j=1}^\infty \subseteq W^\circ$, where

$$W := \{x \in K_\infty(\overline{V}) : \sup_{i\in I} \overline{v}(i)|x(i)| \le 1\}.$$

For this particular $\overline{v}$ we select the subsets $\{I_k\}_{k=1}^\infty$ of I according to Lemma 4.4. By (i) of that lemma, for each $k \in \mathbb{N}$, the sectional subspace

$$X_k := \{x \in K_\infty(\overline{V}) : x(i) = 0 \ \forall i \notin I_k\}$$

is a Banach space isomorphic to ℓ^∞.

Fix $\varepsilon > 0$. For that $m \in \mathbb{N}$ and $C > 0$ as given in (4.3), we select $k(0) = k(m, \varepsilon/2C) \in \mathbb{N}$ via (ii) of Lemma 4.4 to get

$$\overline{v}(i) \leq \frac{\varepsilon}{2C} v_m(i), \quad i \notin I_{k(0)}.$$

For each $i \notin I_{k(0)}$ and $j \in \mathbb{N}$, we then have

$$\overline{v}(i)|x^j(i)| \leq \frac{\varepsilon}{2C} v_m(i)|x^j(i)| \leq \frac{\varepsilon}{2}.$$

That is, $\{x^j \chi_{I \setminus I_{k(0)}}\}_{j=1}^{\infty} \subseteq \frac{\varepsilon}{2} W$. Since $\{u^j\}_{j=1}^{\infty} \subseteq W^{\circ}$, this implies that

$$\sup_{j \in \mathbb{N}} \left| \langle x^j \chi_{I \setminus I_{k(0)}}, u^j \rangle \right| \leq \frac{\varepsilon}{2}. \tag{4.4}$$

Now, $\{x^j \chi_{I_{k(0)}}\}_{j=1}^{\infty}$ is a $\sigma(X_{k(0)}, X'_{k(0)})$–null sequence in $X_{k(0)}$ and the restrictions $\{u^j|_{X_{k(0)}}\}_{j=1}^{\infty}$ form a $\sigma(X'_{k(0)}, X''_{k(0)})$–null sequence in $X'_{k(0)}$. Since $X_{k(0)}$ is isomorphic to ℓ^{∞}, it has the DP–property and so there is $j(0) \in \mathbb{N}$ such that

$$\sup_{j \geq j(0)} |\langle x^j \chi_{I_{k(0)}}, u^j \rangle| \leq \frac{\varepsilon}{2}. \tag{4.5}$$

For each $j \geq j(0)$ we can conclude from (4.4) and (4.5) that

$$|\langle x^j, u^j \rangle| \leq |\langle x^j \chi_{I_{k(0)}}, u^j \rangle| + |\langle x^j \chi_{I \setminus I_{k(0)}}, u^j \rangle| \leq \varepsilon.$$

This shows that $\lim_{j \to \infty} \langle x^j, u^j \rangle = 0$ and completes the proof. □

For the definition of a sequence $V = (v_n)_{n=1}^{\infty}$ (as above) satisfying *condition* (D) we refer to [4], [6]. It can be shown directly that if V is regularly decreasing, then it satisfies condition (D) but, not conversely. So, our final result is an extension of Proposition 4.3.

Corollary 4.7. *Let $V = (v_n)_{n=1}^{\infty}$ be any decreasing sequence of strictly positive functions on a countable index set I such that V satisfies condition (D). Then $k_{\infty}(V) = \mathrm{ind}_n\, \ell^{\infty}(v_n)$ is a GDP–space.*

Proof. Condition (D) implies that $k_{\infty}(V)$ and $K_{\infty}(\overline{V})$ coincide as vector spaces and also topologically, [6, Corollary 8 and Theorem 8]. Then apply Proposition 4.6. □

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