NON COMPLETE MACKEY TOPOLOGIES ON BANACH SPACES

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ABSTRACT. Answering in the negative a question of W. Arendt and M. Kunze, we construct Banach spaces $X$ and a norm closed weak*-dense subspace $Y$ of the dual space $X'$ of $X$ such that the space $X$ endowed with the Mackey $\mu(X,Y)$ of the dual pair $\langle X,Y \rangle$ is not complete.

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The following problem aroused in a natural way in connection with the study of Pettis integrability with respect to norming subspaces developed in his Ph.D. thesis by Markus Kunze [5] and was asked to the authors by Kunze himself and his thesis advisor W. Arendt.

Problem. Suppose that $(X, \|\cdot\|)$ is a Banach space and $Y$ is a subspace of its topological dual $X'$ which is norm closed and weak*-dense. Is there a complete topology of the dual pair $\langle X,Y \rangle$ in $X$?

We use freely the notation for locally convex spaces (shortly, lcs) as in [4, 6, 7]. In particular, we denote, respectively, by $\sigma(X,Y)$ and $\mu(X,Y)$ the weak and the Mackey topology in $X$ associated to the dual pair $\langle X,Y \rangle$. For a Banach space $X$ with topological dual $X'$, the weak*-topology is $\sigma(X',X)$. By the Bourbaki Robertson lemma [4, §18.4.4], there is a complete topology in $X$ of the dual pair $\langle X,Y \rangle$ if and only if the space $(X, \mu(X,Y))$ is complete. Therefore, the original question is equivalent to the following

Problem A: Let $(X, \|\cdot\|)$ be a Banach space. Is $(X, \mu(X,Y))$ complete for every norm closed weak*-dense subspace $Y$ of the dual space $X'$?

Let $(X, \|\cdot\|)$ be a normed space. A subspace $Y$ of $X'$ is said to be norming if the function $p$ of $X$ given by $p(x) = \sup \{|x'(x)| : x' \in Y \cap B_{X'}\}$ is a norm equivalent to $\|\cdot\|$. Notice that Problem A is not affected by changing the given norm of $X$ by any equivalent one. Thus if we want to study Problem A for some norming $Y \subset X'$ we can and will always assume that $Y$ is indeed 1-norming, i.e., $\|x\| = \sup \{|x'(x)| : x' \in Y \cap B_{X'}\}$.

We start by noting that, in the in the conditions of Problem A, if $(X, \mu(X,Y))$ is quasi-complete (in particular complete) then Krein-Smulian’s theorem, see [4, §24.5.(4)], implies that for every $\sigma(X,Y)$-compact subset $H$ of $X$ its $\sigma(X,Y)$-closed absolutely convex hull $M := \text{aco}\sigma(X,Y)$ is $\sigma(X,Y)$-compact. There are several papers dealing with the validity of Krein-Smulian theorem for topologies
Weaker than the weak topology; see for instance [1, 2] where it is proved that for every Banach space \( X \) not containing \( \ell^1([0, 1]) \) and every 1-norming subspace \( Y \subset X' \), if \( H \) is a norm bounded \( \sigma(X, Y) \)-compact subset of \( X \) then \( \text{aco}H^{\sigma(X,Y)} \) is \( \sigma(X, Y) \)-compact. It was proved in [3] that the hypothesis \( \ell^1([0, 1]) \not\subset X \) is needed in the latter.

We start with the following very useful observation:

**Proposition 1.** Let \((X, \|\cdot\|)\) be a Banach space and let \( Y \) be a 1-norming subspace of \( X' \). If \((X, \mu(X,Y))\) is quasi-complete, then every \( \sigma(X, Y) \) compact of \( X \) is norm bounded.

**Proof.** Let \( H \subset X \) be \( \sigma(X, Y) \)-compact. As noted before, Krein-Smulyan’s theorem, [4, §24.5.(4)], implies that the \( \sigma(X, Y) \)-closed absolutely convex hull \( M := \text{aco}H^{\sigma(X,Y)} \) is \( \sigma(X, Y) \)-compact. Therefore, \( M \) is an absolutely convex, bounded and complete subset of the locally convex space \((X, \sigma(X, Y))\). Now we can apply [4, §20.11.(4)] to obtain that \( M \) is a Banach disc, i.e., \( X_M := \bigcup_{n \in \mathbb{N}} nM \) is a Banach space with the norm

\[
\|x\|_M := \inf\{\lambda \geq 0 : x \in \lambda M\}, \quad x \in X_M.
\]

Since \( M \) is bounded in \((X, \sigma(X, Y))\), the inclusion \( J : X_M \to (X, \sigma(X, Y)) \) is continuous, hence \( J : X_M \to (X, \|\cdot\|) \) has closed graph, hence it is continuous by the closed graph theorem. In particular, the image of the closed unit ball \( M \) in \( X_M \) is bounded in \((X, \|\cdot\|)\), and the proof is complete. \( \square \)

As an immediate consequence of the above we have the following:

**Example A.** Let \( X = C([0, 1]) \) be with its sup norm and take

\[ Y := \text{span} \{ \delta_x : x \in [0, 1] \} \subset X'. \]

Then \((X, \mu(X,Y))\) is not quasi-complete.

**Proof.** Notice that \( \sigma(X, Y) \) coincides with the topology \( \tau_p \) of pointwise convergence on \( C([0, 1]) \). Since there are sequences \( \tau_p \)-convergent to zero which are not norm bounded, \((X, \mu(X,Y))\) cannot be quasi-complete when bearing in mind Proposition 1. \( \square \)

The subspace \( Y \) of \( X' \) in Example A is weak*-dense in \( X' \) but not closed. It is in fact easy to give even simpler examples: Take \( X = c_0 \), \( Y = \varphi \), the space of sequences with finitely many non-zero coordinates, which is weak*-dense in \( X' = \ell_1 \). In this case \( \mu(X,Y) = \sigma(X, Y) \), since every absolutely convex \( \sigma(Y, X) \)-compact subset of \( Y \) is finite dimensional by Baire category theorem. In this case \((X, \sigma(X,Y))\) is even not sequentially complete.

The following example, taken from Lemma 11 in [3], provides the negative solution to Problem A.

**Example B.** Take \( X = (\ell^1([0, 1]), \|\cdot\|_1) \) and consider the space \( Y = C([0, 1]) \) of continuous functions on \([0, 1]\) as a norming subspace of the dual \( X' = \ell^\infty([0, 1]) \). Then \((X, \mu(X,Y))\) is not quasi-complete.
Let $\phi : X \to M([0, 1])$ be the canonical basis of $\ell^1([0, 1])$. The set $H$ is clearly $\sigma(X, Y)$-compact but we will prove that $\text{aco}H_{\sigma(X,Y)}$ is not $\sigma(X, Y)$-compact, and therefore $(X, \mu(X, Y))$ cannot be quasi-complete. Indeed, we proceed by contradiction and assume that $W := \text{aco}H_{\sigma(X,Y)}$ is $\sigma(X, Y)$-compact.

We write $M([0,1]) = (C([0,1]), \|\cdot\|_\infty)'$ to denote the space of Radon measures in $[0,1]$ endowed with its variation norm. The map

$$\phi : X \to M([0, 1])$$

given by $\phi((\xi_x)_{x \in [0,1]}) = \sum_{x \in [0,1]} \xi_x \delta_x$ is $\sigma(X, Y)$-w*-continuous. We notice that:

1. $\phi(W) \subset \phi(\ell^1([0,1]))$;
2. $\phi(W)$ is an absolutely convex w*-compact subset of $M([0,1])$;
3. $\{\delta_x : x \in [0,1]\} \subset \phi(W)$.

From the above we obtain that

$$B_{M([0,1])} = \text{aco}\{\delta_x : x \in [0,1]\}^{w^*} \subset \phi(W) \subset \phi(\ell^1([0,1]),$$

which is a contradiction because there are Radon measures on $[0,1]$ which are not of the form $\sum_{x \in [0,1]} \xi_x \delta_x$. The proof is complete. \hfill \Box

**Proposition 2.** If $X$ is a Banach space such that $\ell^1([0,1]) \subset X$, then there is a subspace $Y \subset X'$ norm closed and norming such that $(X, \mu(X, Y))$ is not quasi-complete.

**Proof.** In the proof of [3, Proposition 3] it is constructed a norming subspace $E \subset X'$ and $H \subset X$ norm bounded $\sigma(X, E)$-compact such that $\text{aco}H_{\sigma(X,E)}$ is not $\sigma(X, E)$-compact. If we take $Y = E \subset X'$, norm closure, then $\sigma(X, E)$ and $\sigma(X, Y)$ coincide on norm bounded sets of $X$. Thus $H \subset X$ is $\sigma(X, Y)$-compact with $\text{aco}H_{\sigma(X,E)}$ not $\sigma(X, E)$-compact and therefore $(X, \mu(X, Y))$ cannot be quasi-complete. \hfill \Box

We conclude this note with a few comments about the relation of the questions considered here with Mazur property. We say that a lcs $(E, T)$ is Mazur if every sequentially $T$-continuous form defined on $E$ is $T$-continuous. We quote the following result:

**Theorem 3.** [7, Theorem 9.9.14] Let $(X, Y)$ be a dual pair. If $(X, \sigma(X,Y))$ is Mazur and $(X, \mu(X,Y))$ is complete, then $(Y, \mu(Y, X))$ is complete.

**Proposition 4.** Let $X$ be a Banach space, $Y \subset X'$ proper subspace and $w^*$-dense. Assume that:

1. the norm bounded $\sigma(X,Y)$-compact subsets of $X$ are weakly compact.
2. $(X, \sigma(X, Y))$ is Mazur.

Then $(X, \mu(X,Y))$ is not complete.

**Proof.** Assume that $(X, \mu(X, Y))$ is complete. Then Proposition 1 implies that every $\sigma(X,Y)$-compact subset of $X$ is norm bounded. Therefore the family of $\sigma(X,Y)$-compact subset coincide with the family of weakly compact sets. So the Mackey topology $\mu(Y, X)$ in $Y$ associated to the pair $(X, Y)$ is the topology induced in $Y$ by the Mackey topology $\mu(X', X)$ in $X'$ associated to the dual pair.
If we use now Theorem 3 we obtain that $Y$ is $\mu(Y, X)$ is complete, what implies that $Y \subset X'$ is $\mu(X', X)$ closed. Thus:

$$Y = \overline{\mu(Y', X)} = \overline{\mu(X', X)} = X',$$

that is a contradiction with the fact that $Y$ is a proper subspace of $X'$.

We observe that hypothesis (1) in the above Proposition is satisfied for Banach spaces without copies of $\ell^1([0, 1])$ whenever $Y$ contains a boundary for the norm, see [1, 2].

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