Factorization of weakly compact operators between Banach spaces and Fréchet or (LB)-spaces

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Abstract

In this note we show that weakly compact operators from a Banach space X into a complete (LB)-space E need not factorize through a reflexive Banach space. If E is a Fréchet space, then weakly compact operators from a Banach space X into E factorize through a reflexive Banach space. The factorization of operators from a Fréchet or a complete (LB)-space into a Banach space mapping bounded sets into relatively weakly compact sets is also investigated.

1 Introduction and preliminaries

A linear operator $T \in L(X, Y)$ between Banach spaces is weakly compact if it maps the closed unit ball of X into a weakly relatively compact subset of Y. There are two possible extensions of this concept when the continuous linear operator $T \in L(F, E)$ is defined between locally convex spaces F and E. As in [5], we say that T is *reflexive* if it maps bounded sets into weakly relatively compact sets, and it is called *weakly compact* (as in [10, 42.2]) if there is a 0-neighborhood U in F such that $T(U)$ is relatively weakly compact in E. It can be easily seen that if $T \in L(F, E)$ is weakly compact, then T is reflexive. Although the converse is true if F is a Banach space, in general this is false, as the identity $T : E \to E$ on an infinite dimensional Fréchet Montel space E shows. One can take, for example, the space E of entire functions on the complex plane endowed with the compact open topology. On the other hand, van Dulst [22] showed that if F is a (DF)-space and E is a Fréchet space and T is reflexive, then T is weakly compact (see also [9, Corollary 6.3.8]). We refer the reader to [14] or [11] for (DF)-spaces. Grothendieck [6, Cor. 1 of Thm 11] and [7, IV,4.3,Cor. 1 of Thm 2] proved that if F is a quasinormable locally convex space (cf. [2] or [11]), E is a Banach space and T is reflexive, then T is weakly compact. This result can be seen e.g. in Junek [9, 6.3.4 and 6.3.5]. Extensions of these results for sets of operators can be seen in [17].

Davis, Figiel, Johnson and Pełczyński [4] proved the following beautiful and important result: Every weakly compact operator between Banach spaces factorizes through a reflexive

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Banach space. J.C. Díaz and Domański [5] investigated the factorization of reflexive operators between Fréchet spaces through reflexive Fréchet spaces. In connection with our research on weakly compact operators between C^*-algebras and locally convex spaces in [3], we became interested in the factorization of weakly compact and reflexive operators between Banach spaces and Fréchet or complete (LB)-spaces. In [3] we continued work by Brooks, Saitô and Wright, showing that weakly compact operators $T : A \to E$ from a C^*-algebra A into a complete locally convex space E constitute the natural non-commutative version of vector measures with values in E. See also [12], [13], [19], [25] and [26]. A recent expository article on this topic is [24].

We use standard notation for functional analysis and locally convex spaces [7, 8, 10, 11, 14]. The closed unit ball of a Banach space Y will be denoted by Y_1. A Fréchet space is a complete metrizable locally convex space. We refer the reader to [2, 10, 11] for the theory of Fréchet and (DF)-spaces. For a locally convex space $E = (E, \tau)$, E' stands for the topological dual of E and we denote by $\sigma(E, E')$ and $\beta(E, E')$ the weak and strong topologies on E respectively. The family of all absolutely convex 0-neighborhoods of a locally convex space E is denoted by $\mathcal{U}_0(E)$, the family of all absolutely convex bounded subsets of E by $\mathcal{B}(E)$, and the family of all continuous seminorms on E by $cs(E)$. If E is a locally convex space and $q \in cs(E)$, E_q is the Banach space which appears as the completion of $(E/Kerq, \hat{q})$, $\hat{q}(x + Kerq) = q(x)$, $x \in E$. We denote by $\pi_q : E \to E_q$, $\pi_q(x) = x + Kerq$ and by $\pi_{p,q} : E_q \to E_p, p \leq q$, the canonical maps. If $B \in \mathcal{B}(E)$, the normed space generated by B is $E_B := (\text{span}B, p_B)$, p_B being the Minkowski functional of B. If $B \in \mathcal{B}(E)$, then $E_B \hookrightarrow E$ continuously. If E is sequentially complete, then E_B is a Banach space for every $B \in \mathcal{B}(E)$ which is closed. If X is a Banach space, X_1 stands for the closed unit ball of X.

An (LB)-space $E := \text{ind}_n E_n$ is a Hausdorff countable inductive limit of Banach spaces. Every (LB)-space is a (DF)-space and every (DF)-space is quasinormable (see [14, 8.3.37]). An (LB) space E is called *regular* if every bounded subset in E is contained and bounded in a step E_m. An (LB)-space is complete if and only if it is quasicomplete. Every complete (LB)-space is regular.

2 Results

Proposition 2.1 *Let $E := \text{ind}_n E_n$ be a complete (LB)-space. The following conditions are equivalent:*

(1) Every weakly compact operator T from an arbitrary Banach space X into E factorizes through a reflexive Banach space.

(2) Every weakly compact subset of E is contained and weakly compact in some step E_m.

Proof. We assume first that (1) is satisfied and fix a weakly compact subset K of E. Since E is complete, the closed absolutely convex hull B of K is also weakly compact by Krein's theorem [10, 24.5.(4')]. Accordingly, the canonical inclusion $T : E_B \to E$ is weakly compact. By condition (1), there are a reflexive Banach space Y and continuous linear operators $T_1 \in L(E_B, Y)$ and $T_2 \in L(Y, E)$ such that $T = T_2 \circ T_1$. The continuity of T_1 yields $\lambda > 0$ such that $T_1(B) \subset \lambda Y_1$. Since $T_2 : Y \to \text{ind}_n E_n$ is continuous, we can apply Grothendieck's factorization theorem [11, Theorem 24.33] to find m such that $T_2(Y) \subset E_m$ and $T_2 : Y \to E_m$ is continuous. The unit ball Y_1 of the reflexive Banach space Y is $\sigma(Y, Y')$-compact, hence

$T_2(Y_1)$ is $\sigma(E_m, E'_m)$-compact in E_m. Now, $T(B) = B = T_2T_1(B) \subset T_2(\lambda Y_1) = \lambda T_2(Y_1)$. Therefore B, and hence K, is $\sigma(E_m, E'_m)$-compact in E_m and condition (2) is proved.

Conversely, we assume that condition (2) holds and take a weakly compact operator $T : X \to E$ from a Banach space X into E. The image $T(X_1)$ of the unit ball of X is $\sigma(E, E')$-compact in E. We can apply condition (2) to find m such that the closure K of $T(X_1)$ is $\sigma(E_m, E'_m)$-compact in E_m. This implies that $T(X) \subset E_m$ and that $T : X \to E_m$ is weakly compact between the Banach spaces X and E_m. By the theorem of Davis, Figiel, Johnson and Pełczyński [4], $T : X \to E_m$ factorizes through a reflexive Banach space Y. This implies that the operator $T : X \to E$ also factorizes through the reflexive Banach space Y.

□

Theorem 2.2 *There are a Banach space X, a complete (LB)-space $E := \text{ind}_n E_n$ and a weakly compact operator $T \in L(X, E)$ which does not factorize through a reflexive Banach space.*

Proof. By Valdivia [20, Chapter1,9.4.(11)], a complete (LB)-space E satisfies condition (2) in Theorem 2.1 if and only if it satisfies Retakh's condition (M_0): there exists an increasing sequence $\{U_n\}_{n=1}^{\infty}$ of absolutely convex 0–neighbourhoods U_n in E_n such that for each $n \in \mathbb{N}$ there exists $m(n) > n$ with the property that the topologies $\sigma(E, E')$ and $\sigma(E_{m(n)}, E'_{m(n)})$ coincide on U_n. See also [23], [15] and [16]. Bierstedt and Bonet [1] showed that there exist complete co–echelon spaces $E = \text{ind}_n \, \ell^\infty(v_n)$ of order infinity which do not satisfy condition (M_0). In fact it is enough to take E as the strong dual of a distinguished non quasinormable Köthe echelon space $\lambda_1(A)$ of order 1; see e.g. [2]. We can apply Theorem 2.1 to find a Banach space X and a weakly compact operator $T : X \to E$ which does not factorize through a reflexive Banach space.

□

The following result is well-known to specialists. For the convenience of the reader, we give a brief proof.

Proposition 2.3 *Every weakly compact operator $T : X \to E$ from a Banach space X into a Fréchet space E factorizes through a reflexive Banach space.*

Proof. The closure B of $T(X_1)$ in E is a weakly compact set. By a result of Grothendieck (see e.g. [9, Corollary 6.4.5] or [17, Section 1 Lemma (e)]), there is $C \in \mathcal{B}(E)$ such that B is weakly compact in E_C. The map $T : X \to E_C$ is well defined and weakly compact, hence it factorizes through a reflexive Banach space Y, by the theorem of Davis, Figiel, Johnson and Pełczyński [4]. Since the inclusion from E_C into E is continuous, the original map $T : X \to E$ factorizes through Y, too. □

We now consider the factorization of reflexive maps from a locally convex space F into a Banach space X through a reflexive Banach space Y. Deep results concerning the factorization of reflexive maps between Fréchet spaces through a reflexive Fréchet space can be seen in Díaz, Domanski [5].

Proposition 2.4 *The following conditions are equivalent for a reflexive operator T from a locally convex space F into a Banach space X:*

(1) T factorizes through a reflexive Banach space.

(2) T is weakly compact.

Proof. We assume that a reflexive operator $T \in L(F, X)$ satisfies condition (1). There are a reflexive Banach space Y and continuous linear operators $T_1 \in L(F, Y)$ and $T_2 \in L(Y, X)$ such that $T = T_2 \circ T_1$. Find $U \in \mathcal{U}_0(E)$ such that $T_1(U) \subset Y_1$. Since Y is reflexive, $T_2(Y_1)$ is $\sigma(X, X')$-compact in X. Therefore $T(U) \subset T_2(Y_1)$ is also $\sigma(X, X')$-compact in X, and T is weakly compact.

Conversely, suppose that $T \in L(F, X)$ is weakly compact and find $U \in \mathcal{U}_0(E)$ such that $T(U)$ is $\sigma(X, X')$-compact in X. Let q be the Minkowski functional of U. Then $T = S \circ \pi_q$, with $S : F_q \to X$ the unique continuous extension of $S(x + Ker q) := T(x)$. The closed unit ball of F_q is $B_q = \overline{\pi_q(U)}$, the closure taken in F_q. Therefore $S(B_q) \subset \overline{T(U)}$, the closure taken in X. This implies that S is weakly compact between the Banach spaces. By the theorem of Davis, Figiel, Johnson and Pełczyński [4], S factorizes through a reflexive Banach space Y; hence T also factorizes through Y. □

Corollary 2.5 *Every reflexive operator from a quasinormable locally convex space into a Banach space factorizes through a reflexive Banach space. In particular, reflexive operators from a (DF)-space into a Banach space factorize through a reflexive Banach space.*

Proof. This is a consequence of Proposition 2.4 and Grothendieck [6, Cor. 1 of Thm 11] and [7, IV,4.3, Cor. 1 of Thm 2]. □

There is another class of locally convex spaces F such that every Banach valued reflexive operator factorizes through a reflexive Banach space. A locally convex space F is called *infra-Schwartz* (cf. [9, 7.1.3]) if for every continuous seminorm p on F there is a continuous seminorm $q \geq p$ such that the canonical map $\pi_{p,q} : F_q \to F_p$ is weakly compact. Complete infra-Schwartz spaces are projective limits of spectra of reflexive Banach spaces [9, 7.5.3] and every reflexive quasinormable locally convex space is infra-Schwartz [9, 7.5.2]. There are infra-Schwartz Fréchet spaces which are not quasinormable and quasinormable Fréchet spaces which are not infra-Schwartz. It follows from the definition that *every reflexive operator from an infra-Schwartz space into a Banach space factorizes through a reflexive Banach space.* Infra-Schwartz Fréchet spaces were investigated by Floret; see [14, section 8.5]. Valdivia [21] proved that a Fréchet space F is infra-Schwartz if and only if it is totally reflexive, i.e. every separated quotient of F is reflexive. Reflexive non totally reflexive Fréchet spaces exist. They are used in our last result.

Proposition 2.6 *There exist a Fréchet Montel space F and a continuous surjection $T : F \to X$ onto a Banach space X which does not factorize through a reflexive Banach space.*

Proof. Köthe and Grothendieck constructed Fréchet Montel spaces F which have a quotient isomorphic to ℓ_1 or c_0; see [10, 31.7] and [11, Example 27.21]. Denote by $T : F \to X$ the quotient map. Since every bounded set in F is relatively compact, T is reflexive. If there were a reflexive Banach space Y and continuous linear operators $T_1 \in L(F, Y)$ and $T_2 \in L(Y, X)$ such that $T = T_2 \circ T_1$, then T_2 would be a continuous surjection from Y into X. By the open mapping theorem, this would imply that the non-reflexive Banach space X is a quotient of the Banach space Y. A contradiction. □

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