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A problem on the structure of Fréchet spaces

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Abstract. The following open problem is stated: Is there a non-normable Fréchet space E such that every continuous linear operator T on E has the form $T = \lambda I + S$, where S maps a 0-neighbourhood of E into a bounded set? A few remarks and the relation of this question with other still open problems on operators between Fréchet spaces are mentioned.

Un problema sobre la estructura de espacios de Fréchet

Resumen. Se plantea el siguiente problema: ¿Existe un espacio de Fréchet no normable E tal que todo operador lineal y continuo T en E tiene la forma $T = \lambda I + S$, donde S manda un entorno de E en un conjunto acotado? Se mencionan algunas observaciones y la relación de esta cuestión con otros problemas aún abiertos acerca de operadores en espacios de Fréchet.

Dedicated to Prof. Manuel Valdivia on the occasion of his 80th birthday

1. Introduction

The purpose of this short expository article is to state an open problem about the linear structure of non-normable Fréchet spaces, and to relate this question with some other open problems about continuous linear operators on Fréchet spaces. Our notation for functional analysis is standard; see e.g. [30]. All the vector spaces are considered over the field $\mathbb{C}$ of complex numbers.

After the seminal papers of Gowers and Maurey [19, 20] on hereditarily indecomposable Banach spaces, several important open problems about the linear structure of infinite dimensional Banach spaces have been solved in the last fifteen years; see the excellent survey by Maurey [29]. Here are some of them:

(A) (Banach, 1932) Is every infinite dimensional Banach space X isomorphic to its closed hyperplanes?

(B) (Homogeneous Banach space problem, Banach 1932) If an infinite dimensional Banach space X is isomorphic to every infinite dimensional subspace of itself, does it follow that X is isomorphic to ℓ_2?

(C) (Bessaga, Pełczyński, 1958) Does every infinite dimensional Banach space contain an infinite unconditional basic sequence?

(D) (Lindenstrauss, 1970) Is it possible to decompose every infinite dimensional Banach space as a topological sum of two infinite dimensional subspaces?

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(E) (Scalar-plus-Compact Problem, Lindenstrauss, 1976) Does there exist an infinite dimensional Banach space on which every continuous linear operator is a compact perturbation of a scalar multiple of the identity?

As Maurey says in [29], "the results and examples obtained in this direction represent a significant progress in our understanding of infinite dimensional Banach spaces". Let us review some of these results. Gowers and Maurey's example [19] was the first infinite dimensional Banach space X_{GM} which is *hereditarily indecomposable* (in short: HI) in the sense that if a closed subspace Z of X_{GM} is the direct sum $Z = X \oplus Y$ of two closed subspaces, then either X or Y must be finite dimensional. Question (D) was then solved by proving much more than asked. Clearly, a HI space cannot contain an infinite unconditional basic sequence, since the closed span of that sequence would be decomposable into two closed subspaces generated by corresponding unconditional block basis. Thus the space X_{GM} gave also a negative solution to Problem (C).

Every bounded operator T on a HI-space X has the form $T = \lambda I + S$ where $\lambda \in \mathbb{C}$, I is the identity and S is a strictly singular operator. An operator is *strictly singular* if it cannot be restricted to an infinite dimensional subspace on which it is an isomorphic embedding. In particular, every operator on a HI space must be strictly singular or Fredholm with index zero. As a consequence, X is not isomorphic to any proper closed subspace of itself, since any isomorphism from X into a proper subspace would have to be a Fredholm operator with non-zero index, therefore strictly singular, a contradiction. See e.g. [26, Section 3.7]. In particular, X_{GM} solved Problem (A) in the negative.

The homogeneous Banach space Problem (B) has a positive solution, which follows from a result by Komorowski and Tomczak [24] and Gowers' beautiful dichotomy theorem [18]: *An arbitrary infinite dimensional Banach space X either contains an infinite unconditional sequence or contains a HI subspace.*

The example of Gowers and Maurey X_{GM} is separable. Argyros, López-Abad and Todorcevic [4] presented an example of a non-separable reflexive Banach space which contains no unconditional basic sequence. For a better understanding of the difficulty of this construction, recall that every non-separable reflexive Banach space admits non-trivial projections although, by Gowers' dichotomy theorem, this space is saturated with HI space. The construction of [4] also gives an uncountable family of reflexive HI separable Banach spaces which are totally incomparable.

The scalar-plus-compact Problem (E) has been solved very recently by Argyros and Haydon [3]. Androulakis and Schlumprecht [1] had exhibited strictly singular, non-compact operators defined on the whole space X_{GM}; thus showing that X_{GM} could not be a counterexample to Problem (E). Argyros and Haydon construct a Banach space X_{AH} such that every continuous linear operator T on X_{AH} is of the form $T = \lambda I + K, \lambda \in \mathbb{C}, K$ compact. The space X_{AH} is HI and its dual is isomorphic to ℓ_1. This result is relevant in connection with other problems in Banach space theory. As a consequence of a famous theorem of Lomonosov on invariant subspaces (see e.g. [25, 42.9.(6)]), the space X_{AH} is the first example of a Banach space such that every continuous linear operator on it has a non-trivial invariant subspace. Moreover, the space $\mathcal{K}(X_{AH})$ of compact operators on X_{AH} is a proper complemented subspace of the space $\mathcal{L}(X_{AH})$ of all continuous linear operators.

It seems to be a still open problem whether there is a Banach space X such that every continuous linear operator T on X is of the form $T = \lambda I + N$, where N is a nuclear operator.

Let us point out right away that *there are no hereditarily indecomposable non-normable Fréchet spaces.* Indeed, this follows from classical results due to Bessaga, Pełczyński and Rolewicz [7, 8]: *Let E be a non-normable Fréchet space. If E admits a continuous norm, then E contains a subspace which is isomorphic to a nuclear Köthe echelon space $\lambda_1(A)$. If E does not have a continuous norm, then it contains a copy of the countable product ω of copies of the scalar field. In any case, every non-normable Fréchet space E contains a subspace H which can be written as $H = F \oplus G$ of infinite dimensional nuclear subspaces F and G.* Miñarro [31] complemented this result with the following theorem: *Every non-normable, non-nuclear Fréchet space E contains a closed subspace $Y = F \oplus G$ where F and G are infinite dimensional and one of them is not nuclear.* However, nuclear Fréchet spaces with "few" operators do exist: Dubinski and Vogt [16, Lemma 2.1] give examples of nuclear power series spaces E such that every continuous linear operator T on E has the form $T = D + K$, where D is a diagonal operator and K maps a 0-neighbourhood of E into a relatively compact set. In Section 2 we state the main open problem discussed in this paper. A Fréchet space solving this problem would have very "few" operators in a non-normable sense and would satisfy, in particular, that every infinite dimensional complemented subspace must be a Banach space.

2. Statement of the Problem. Remarks

First we recall some notation. For a locally convex space E, which we assume to be Hausdorff, E' stands for its topological dual. The set of continuous seminorms on the space E is denoted by $cs(E)$. We denote by $\beta(E, F)$ the strong topology and by $\sigma(E, F)$ the weak topology on E with respect to a dual pair $\langle E, F\rangle$. The strong dual $(E', \beta(E', E))$ of E is also denoted by E'_b. If E is a locally convex space, then $\mathcal{L}(E)$ denotes the vector space of all continuous linear maps from E to E. The composition of an operator T with itself n times is denoted by T^n. Given $T \in \mathcal{L}(E)$ we denote by $T^t \in \mathcal{L}(E')$ its transpose defined by $T^t(u) = u \circ T \in E'$ for each $u \in E'$. A locally convex space is called *normable* if it is isomorphic to a normed space, or equivalently if it has a bounded 0-neighbourhood. A complete, metrizable, locally convex space is called a Fréchet space, see [30].

A linear operator T on a locally convex space E is called *bounded (resp. compact)* if there is a 0-neighbourhood U in E such that $T(U)$ is a bounded (resp. relatively compact) subset of E. Clearly every compact operator is bounded and every bounded operator is continuous. A continuous linear operator T on E is called *Montel* if it maps bounded sets into relatively compact subsets of E. Every bounded operator on a Fréchet Montel space is compact. Examples and results concerning these classes of operators in the frame of Fréchet spaces can be seen in [12]. The following problem is well-known in the theory of Fréchet spaces and has been discussed by the author with L. Frerick and D. Vogt among others.

Problem 1 *Is there a non-normable Fréchet space E such that every continuous linear operator T on E has the form $T = \lambda I + S$, where S is a bounded operator?*

Proposition 1 *(i) If E is a non-normable Fréchet space such that every continuous linear operator T on E has the form $T = \lambda I + S$ with S bounded, then every projection P on E satisfies that either $P(E)$ or $(I - P)(E)$ is normable. In particular every Fréchet Montel space with this property would have only finite dimensional or finite codimensional complemented subspaces.*

(ii) If a Fréchet space contains an infinite dimensional Montel subspace with an unconditional basis, then there is a continuous linear operator T on E which is not of the form $T = \lambda I + S$ with S bounded.

(iii) If E is a non-normable Fréchet space such that every continuous linear operator T on E has the form $T = \lambda I + S$ with S bounded, then E admits a continuous norm.

PROOF. (i) Let P be a projection on E. There are $\lambda \in \mathbb{C}$ and a bounded operator S on E such that $P = \lambda I + S$. If $\lambda = 1$, then $I - P = -S$ is bounded in E, hence the identity on $(I - P)(E)$ is a bounded operator and $(I - P)(E)$ is normable. If $\lambda \neq 1$, then $P - \lambda I = S$ is bounded. If $y \in P(E)$, then $(1 - \lambda)y = Sy$, and the identity on $P(E)$ is a bounded operator. Consequently $P(E)$ is normable.

(ii) Suppose that $E = F \oplus G$ with F an infinite dimensional Montel subspace with an unconditional basis. Decomposing the basis of F in two infinite blocks, we conclude that $F = F_1 \oplus F_2$, with F_1 and F_2 infinite dimensional Montel subspaces. Now $E = F_1 \oplus (F_2 \oplus G)$ and neither F_1 nor $F_2 \oplus G$ is normable. We can apply (i) to conclude that there is a continuous linear operator T on E which is not of the form $T = \lambda I + S$ with S bounded.

(iii) If E does not admit a continuous norm, by a classical result of Bessaga and Pełczyński [7], there is a complemented subspace of E isomorphic to a countable product ω of copies of $\mathbb{C}$. We can apply (ii) to conclude that there is a continuous linear operator T on E which is not of the form $T = \lambda I + S$ with S bounded. A contradiction. ■

The proof of the following result depends on a remarkable construction of hyperplanes due to Valdivia [32]. Since every continuous linear operator with finite dimensional range is bounded, it gives a positive answer to Problem 1 for non-complete metrizable locally convex spaces.

Theorem 1 ***(Bonet, Frerick, Peris, Wengenroth [11])*** *Every separable infinite dimensional Fréchet space E contains a dense hyperplane Y such that every continuous linear operator T on Y has the form $T = \lambda I + F$, where $\lambda \in \mathbb{C}$ and F has finite dimensional range.*

3. Related problems on hypercyclic operators

A continuous and linear map $T \in \mathcal{L}(E)$ on a locally convex space E is called *hypercyclic* if there exists a vector $x \in E$ (which is called hypercyclic vector) such that its orbit $O(T, x) := \{T^n x : n = 0, 1, 2, ...\}$ is dense in E. The operator T is called *transitive* if for each pair of non-empty open subsets U, V of E there is $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$. A continuous linear operator on a separable Fréchet space is transitive if and only if it is hypercyclic. The operator is called *chaotic* (in the sense of Devaney) if it is hypercyclic and the set of periodic points of T is dense in E. Although the first examples of hypercyclic operators were given in the first half of the last century, much research has been done concerning hypercyclic and chaotic operators during the last years especially after the unpublished but well-known PhD thesis of C. Kitai, the article by Godefroy and Shapiro [17] and the excellent surveys by Grosse-Erdmann [21, 22]. The books of Bayart and Matheron [5] and Grosse-Erdmann and Peris [23] give a complete picture.

Ansari [2] and Bernal [6] proved independently that every separable infinite dimensional Banach space admits a hypercyclic operator. This result was extended to separable infinite dimensional Fréchet spaces by Bonet and Peris [14]. On the other hand, Bonet, Martínez-Giménez and Peris [13] showed that there is no continuous chaotic operator on the separable Banach space X_{GM} constructed by Gowers and Maurey. See also [5, Theorem 6.36]. This was the first time that the relevance of HI spaces in hypercyclicity was observed. We have the following result for especial types of operators on Fréchet spaces.

Theorem 2 *(1)* ***(Bonet, Peris [14, Prop. 8])*** *No compact operator on a Fréchet space is hypercyclic.*

(2) ***(Martínez-Giménez, Peris [28, Prop. 6.1])*** *No perturbation of a non-zero multiple of the identity by a compact operator on a Fréchet space is chaotic.*

(3) ***(Bonet, Peris [14, Example 9])*** *There is a bounded hypercyclic operator defined on the countable product of copies of the space $\ell_p, 1 \leq p < \infty$.*

The present formulation of Theorem 2 requires small changes in the proof of [28, Prop. 6.1]. Parts (1) and (2) above are valid for arbitrary Hausdorff locally convex spaces. Theorem 2 (1) and (2) imply that the separable Banach space X_{AH} constructed by Argyros and Haydon admits no chaotic operator.

The following two problems were mentioned by the author in [10].

Problem 2 *Does every infinite dimensional Fréchet Montel space admit a chaotic operator?*

If the solution to Problem 1 yields an infinite dimensional Fréchet Montel space E such that every continuous linear operator T on E has the form $T = \lambda I + S$, where S is a bounded operator, then S is compact, since E is Montel. We can apply Theorem 2 (1) and (2) to conclude that E admits no chaotic operator. Accordingly a positive solution of Problem 1 with a Fréchet Montel example, would give a negative solution of problem 2.

The best positive result related to Problem 2 has been recently obtained by de la Rosa, Frerick and Peris [15].

Theorem 3 ***(de la Rosa, Frerick, Grivaux and Peris [15]).*** *Every Fréchet space with an unconditional basis, in particular every nuclear Fréchet space with basis, admits a chaotic operator.*

Problem 3 *Does every infinite dimensional non-normable separable Fréchet space support a hypercyclic operator T such that λT is hypercyclic for all $\lambda \neq 0$? The most important case is for infinite dimensional Fréchet Montel spaces.*

This is clearly a question on non-normable Fréchet spaces, since every operator on a Banach space with norm less or equal than 1 cannot be hypercyclic. The work of Godefroy and Shapiro [17] contains several examples of nuclear Fréchet spaces which admit an operator T such that λT is hypercyclic for al $\lambda \neq 0$.

By a deep result of León-Saavedra and Müller [27], whenever T is a hypercyclic operator, the rotation λT of T, where λ is any complex number of modulus 1, is also hypercyclic and they have the same hypercyclic vectors. Accordingly, to solve Problem 3 it is enough to show that every non-normable Fréchet space supports a continuous linear operator T such that λT is hypercyclic for each $\lambda > 0$.

The following result shows that if there is a a non-normable Fréchet space E_0 such that every operator $T \in \mathcal{L}(E_0)$ is of the form $T = \lambda I + S$, with S bounded, i.e. the solution of Problem 1 is positive, then no hypercyclic operator T on E_0 satisfies that μT is hypercyclic for al $\mu \neq 0$; thus solving Problem 3 in the negative.

Proposition 2 *If S is a bounded operator on a Fréchet space E and $\lambda \in \mathbb{C}$, then there is $\mu > 0$ such that $\mu(\lambda I + S)$ is not hypercyclic on E.*

PROOF. Let $(U_k)_k$ be a basis of open absolutely convex neighbourhoods in E such that $\overline{U_{k+1}} \subset U_k$ for each $k \in \mathbb{N}$ and $U_1 \neq E$. Since S is bounded, there is $m > 1$ such that $S(U_m)$ is bounded in E. For each $k \geq m$ there is $M_k > 0$ such that $S(U_m) \subset M_k U_k$. Therefore $(\lambda I + S)(U_k) \subset (|\lambda| + M_k)U_k$ for each $k \geq m$. Set $T := \lambda I + S$ and $\mu := (|\lambda| + M_{m+1})^{-1}$. We have $(\mu T)(U_{m+1}) \subset U_{m+1}$. The sets $U := U_{m+1}$ and $V := E \setminus \overline{U_m}$ are non-empty and open in E and $(\mu T)^n(U) \cap V = \emptyset$ for each $n \in \mathbb{N}$. This implies that μT is not transitive, thus it is not hypercyclic. ■

The following observation is an easy extension of this result.

Remark 1 *If an operator T on a locally convex space E satisfies that there are $M > 0$ and an absolutely convex closed 0-neighbourhood U different from E with $T(U) \subset MU$, then $(1/M)T$ is not transitive on E. It is enough to observe that the interior V of U and the open set $E \setminus U$ satisfy $((1/M)T)^n(U) \cap V = \emptyset$ for each $n \in \mathbb{N}$.*

It is not clear whether every non-normable Fréchet space E admits an operator T such that $T(U)$ is not absorbed by U for every absolutely convex open 0-neighbourhood U different from E. Recall that Theorem 2 (3) implies that there are hypercyclic operators R on non-normable Fréchet spaces F which are bounded. In particular they satisfy that $R(U_k)$ is absorbed by U_k, $k \in \mathbb{N}$, for a basis of open absolutely convex neighbourhoods $(U_k)_k$ in F.

An operator T on a locally convex space E is called *power bounded* if $(T^n)_n$ is equicontinuous in $\mathcal{L}(E)$.

Proposition 3 *If the operator T on a locally convex space E is power bounded, then there is an absolutely convex closed 0-neighbourhood U in E different from E such that $T^n(U) \subset U$ for each $n \in \mathbb{N}$.*

PROOF. Let W be an absolutely convex closed 0-neighbourhood in E different from E. Since $(T^n)_n$ is equicontinuous, there is an absolutely convex 0-neighbourhood V in E such that $T^n(V) \subset W$ for each $n = 0, 1, 2, ...$ Clearly $V \subset \bigcap_{n=0}^{\infty} (T^n)^{-1}(W)$. Denote by U the set of $\bigcap_{n=0}^{\infty} (T^n)^{-1}(W)$, which is an absolutely convex closed 0-neighbourhood in E satisfying $V \subset U \subset W$, hence U is different from E. It is easy to see that $T^n(U) \subset U$ for each $n \in \mathbb{N}$. ■

4. A related problem on topologizable operators

The following two classes of operators were defined and studied by Żelazko in [33] in the context of operator algebras on locally convex spaces.

Definition 1 An operator $T \in L(E)$ on a locally convex space E is called *topologizable* if for every continuous seminorm $p \in cs(E)$ there is a continuous seminorm $q \in cs(E)$ such that for every $n \in \mathbb{N}$ there is $M_n > 0$ such that

$$p(T^n x) \leq M_n q(x)$$

for each $x \in E$.

Definition 2 An operator $T \in L(E)$ on a locally convex space E is called *m-topologizable* if for every continuous seminorm $p \in cs(E)$ there are a continuous seminorm $q \in cs(E)$ and $C \geq 1$ such that for every $n \in \mathbb{N}$ such that

$$p(T^n x) \leq C^n q(x)$$

for each $x \in E$.

Observe that in the definitions above it is essential that the seminorm q only depends on the seminorm p and not on the iteration k. By Żelazko [33], Theorems 5 and 9, an operator $T \in L(E)$ is topologizable (resp. m-topologizable) if and only if T belongs to some operator algebra $A \subset L(E)$ on E (resp. T belongs to an operator algebra of the form $L_\Gamma(E)$); see [33] for notation.

Clearly every m-topologizable operator is topologizable. If the locally convex space E is normable, then every operator $T \in L(E)$ is m-topologizable. The translation operator is mentioned in Żelazko [33], Examples 8, as a natural operator which is not topologizable in many cases. On the other hand, the differentiation operator on the Fréchet space $H(\mathbb{C})$ of entire functions endowed with the compact open topology is a topologizable operator [9]. Accordingly, there are hypercyclic operators which are topologizable and hypercyclic operators which are not topologizable. Every bounded operator on a locally convex space is even m-topologizable, [9, prop. 7].

Theorem 4 ***(Bonet [9]).***

(i) If E is a Fréchet space without a continuous norm, then there is $T \in \mathcal{L}(E)$ which is not topologizable.

(ii) Every power series space of infinite type E admits an operator $T \in \mathcal{L}(E)$ which is not topologizable.

The following problem remains open.

Problem 4 ***(Bonet [9]).*** *Does every infinite dimensional Fréchet space with a continuous norm admit an operator which is not topologizable?*

If the solution to Problem 1 is positive, then there is a non-normable Fréchet space E_0, with a continuous norm by Proposition 1 (iii), such that every operator $T \in \mathcal{L}(E_0)$ is of the form $T = \lambda I + S$, with S bounded. In this case, every operator on E_0 would be m-topologizable by [33, Prop. 12], thus solving in the negative Problem 4.

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