Convolution operators on quasianalytic classes of Roumieu type

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Abstract. Extending previous work of Braun, Meise, and Vogt, and of Meyer, we characterize those convolution operators that are surjective on the space $\mathcal{E}_{\{\omega\}}(\mathbb{R})$ of all quasianalytic $\{\omega\}$-ultradifferentiable functions of Roumieu type. We also investigate $\{\omega\}$-ultradifferential operators on $\mathcal{E}_{\{\omega\}}[a,b]$ for compact intervals.

1. Introduction

For a weight function ω let $\mathcal{E}_{\{\omega\}}(\mathbb{R})$ denote the space of all $\{\omega\}$-ultradifferentiable functions of Roumieu type on $\mathbb{R}$. Then each $\mu \in \mathcal{E}'_{\{\omega\}}(\mathbb{R})$ induces a convolution operator $T_\mu : \mathcal{E}_{\{\omega\}}(\mathbb{R}) \to \mathcal{E}_{\{\omega\}}(\mathbb{R})$. If ω is non-quasianalytic, i.e., if $\mathcal{E}_{\{\omega\}}(\mathbb{R})$ contains non-trivial functions with compact support, then Braun, Meise, and Vogt [**7**] characterized those convolution operators T_μ that are surjective on $\mathcal{E}_{\{\omega\}}(\mathbb{R})$. Though the arguments that were used in [**7**] rely heavily on the existence of fundamental solutions for surjective convolution operators, Meyer [**21**] proved a similar result for convolution operators T_μ for which $\mu \in \mathcal{E}'_{\{\omega\}}(\mathbb{R})$ is supported by the origin, even for quasianalytic weight functions ω. In both articles, the proofs are based on properties of the projective limit functor due to Palamodov [**24**] and the sequence space representation for the kernels of slowly decreasing convolution operators T_μ given by Meise [**15**].

In the present paper we show in Theorem 3.10 that the characterization, given in [**7**] also holds for quasiananalytic weight functions ω. More precisely, we prove that for each weight function ω and $\mu \in \mathcal{E}'_{\{\omega\}}(\mathbb{R})$ the convolution operator T_μ is surjective on $\mathcal{E}_{\{\omega\}}(\mathbb{R})$ if and only if the Fourier-Laplace transform $\hat{\mu}$ of μ is $\{\omega\}$-slowly decreasing and the zero set $V(\hat{\mu})$ of $\hat{\mu}$ can be decomposed as $V(\hat{\mu}) = V_0 \cup V_1$ such that

$$\lim_{\substack{|a|\to\infty \\ a\in V_0}} \frac{|\operatorname{Im} a|}{\omega(a)} = 0 \text{ and } \lim \inf_{\substack{|a|\to\infty \\ a\in V_1}} \frac{|\operatorname{Im} a|}{\omega(a)} > 0.$$

The proof uses the better understanding of the slowly decreasing conditions that was achieved by Momm [**22**], Bonet, Galbis, and Meise [**2**], and Bonet, Galbis, and Momm [**3**] together with results about the derived functor of the projective limit functor and about (LF)-spaces, due to Vogt [**29**] and to Wengenroth [**31**]. Applying the Fourier-Laplace transform and methods from Meise [**14**] and [**15**] again together with a recent result of Vogt [**30**] and Bonet and Domanski [**1**], we also show that a convolution operator T_μ acting surjectively on $\mathcal{E}_{\{\omega\}}(\mathbb{R})$ admits a continuous linear right inverse only if $\lim_{|a|\to\infty, a\in V(\hat{\mu})} |\operatorname{Im} a|/\omega(a) = 0$.

We also investigate $\{\omega\}$-ultradifferentiable operators T_μ on $\mathcal{E}_{\{\omega\}}(\mathbb{R})$ and on $\mathcal{E}_{\{\omega\}}[a,b]$ for compact intervals $[a,b]$ with $a < b$ and we show that such an operator is slowly decreasing if and only if $T_{\mu,[a,b]} : \mathcal{E}_{\{\omega\}}[a,b] \to \mathcal{E}_{\{\omega\}}[a,b]$ is surjective for all $a, b \in \mathbb{R}$ with $a < b$. Whenever this condition is satisfied then $\ker T_{\mu,[a,b]}$ is isomorphic to the strong dual of a nuclear power series space of finite type. If in addition $\lim_{|\zeta|\to\infty,\zeta\in V(\hat{\mu})} |\operatorname{Im}\zeta|/\omega(\zeta) = 0$ then the restriction map $\varrho : \ker T_\mu \to \ker T_{\mu,[a,b]}$ is an isomorphism for each $a < b$.

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2. Preliminaries

In this section we introduce the notation that will be used throughout the entire paper.

2.1. Weight functions. A function $\omega : \mathbb{R} \to [0, \infty[$ is called a *weight function* if it is continuous, even, increasing on $[0, \infty[$, and if it satisfies $\omega(0) = 0$ and also the following conditions:

(α) There exists $K \geq 1$ such that $\omega(2t) \leq K\omega(t) + K$.
(β) $\omega(t) = o(t)$ as t tends to infinity.
(γ) $\log(t) = o(\omega(t))$ as t tends to infinity.
(δ) $\varphi : t \mapsto \omega(e^t)$ is convex on $[0, \infty[$.

If a weight function ω satisfies

(Q) $\int_1^\infty \frac{\omega(t)}{t^2} dt = \infty$

then it is called a *quasianalytic weight*. Otherwise it is called *non-quasianalytic*.

A weight function ω satisfies the condition (α_1) if

$$\sup_{\lambda \geq 1} \limsup_{t \to \infty} \frac{\omega(\lambda t)}{\lambda \omega(t)} < \infty.$$

This condition was introduced by Petzsche and Vogt [**25**] and is equivalent to the existence of $C_1 > 0$ such that for each $W \geq 1$ there exists $C_2 > 0$ such that

$$\omega(Wt + W) \leq WC_1\omega(t) + C_2,\ t \geq 0.$$

The *radial extension* $\tilde{\omega}$ of a weight function ω is defined as

$$\tilde{\omega} : \mathbb{C}^n \to [0, \infty[, \quad \tilde{\omega}(z) := \omega(|z|).$$

It will also be denoted by ω in the sequel, by abuse of notation. The *Young conjugate* of the function $\varphi = \varphi_\omega$, which appears in (δ), is defined as

$$\varphi^*(x) := \sup\{xy - \varphi(y) : y > 0\}, \quad x \geq 0.$$

2.2. Example. The following functions are easily seen to be weight functions:

(1) $\omega(t) := |t|(\log(e + |t|))^{-\alpha}, \alpha > 0$.
(2) $\omega(t) := |t|^\alpha, 0 < \alpha < 1$.
(3) $\omega(t) = \max(0, (\log t)^s),\ s > 1$.

2.3. Ultradifferentiable functions defined by weight functions. Let ω be a given weight function. For a compact subset K of $\mathbb{R}^N$ and $m \in \mathbb{N}$ denote by $C^\infty(K)$ the space of all C^∞-Whitney jets on K, define

$$\mathcal{E}^m_{\{\omega\}}(K) := \{f \in C^\infty(K) : \|f\|_{K,m} := \sup_{x \in K} \sup_{\alpha \in \mathbb{N}_0^N} |f^{(\alpha)}(x)| \exp\left(-\frac{1}{m}\varphi^*(m|\alpha|)\right) < \infty\},$$

and let

$$\mathcal{E}_{\{\omega\}}(K) := \operatorname{ind}_{m\to} \mathcal{E}^m_{\{\omega\}}(K)$$

which is a (DFN)-space.

For an open set G in $\mathbb{R}^N$, define the space $\mathcal{E}_{\{\omega\}}(G)$ of all ω-*ultradifferentiable functions of Roumieu type* on G as

$$\mathcal{E}_{\{\omega\}}(G) := \{f \in C^\infty(G) : \text{For each } K \subset G \text{ compact there is } m \in \mathbb{N} \text{ so that } \|f\|_{K,m} < \infty\}.$$

It is endowed with the topology given by the representation

$$\mathcal{E}_{\{\omega\}}(G) = \operatorname{proj}_{\leftarrow K} \mathcal{E}_{\{\omega\}}(K),$$

where K runs over all compact subsets of G.

Note that $\mathcal{E}_{\{\omega\}}(G)$ is a countable projective limit of (DFN)-spaces, which is ultrabornological, reflexive and complete. This follows from Rösner [**26**], Satz 3.25 and Vogt [**30**], Theorem 3.4.

The space $\mathcal{E}_{(\omega)}(G)$ of all *ω-ultradifferentiable functions of Beurling type* on G is defined as

$$\mathcal{E}_{(\omega)}(G) := \{f \in C^\infty(G) : \text{for each } K \subset G \text{ compact and } m \in \mathbb{N}$$
$$p_{K,m}(f) := \sup_{x\in K}\sup_{\alpha\in\mathbb{N}_0^N} |f^{(\alpha)}(x)| \exp\left(-m\varphi^*(\frac{|\alpha|}{m})\right) < \infty\}.$$

It is easy to check that $\mathcal{E}_{(\omega)}(G)$ is a Fréchet space if we endow it with the locally convex topology given by the semi-norms $p_{K,m}$.

If a statement holds in the Beurling and the Roumieu case then we will use the notation $\mathcal{E}_*(G)$. It means that in all cases $*$ can be replaced either by (ω) or by $\{\omega\}$.

2.4. Definition. Let ω be a weight function and G an open convex set in $\mathbb{R}^N$.

(a) We define the space $A_{(\omega)}$ by

$$A_{(\omega)} := \{f \in H(\mathbb{C}) : \exists\, n \in \mathbb{N} : \|f\|_n := \sup_{z\in\mathbb{C}} |f(z)| \exp(-n\omega(z)) < \infty\}$$

and endow it with its natural (LB)-topology. Then $A_{(\omega)}$ is an (DFN)-space. We also define the Fréchet space

$$A_{\{\omega\}} := \{f \in H(\mathbb{C}) : \forall\, n \in \mathbb{N} : \|f\|_n := \sup_{z\in\mathbb{C}} |f(z)| \exp(-\frac{1}{n}\omega(z)) < \infty\}.$$

(b) For each compact set K in G, the *support functional* of K is defined as

$$h_K : \mathbb{R}^N \to \mathbb{R},\ h_K(x) := \sup\{\langle x, y\rangle : y \in K\}.$$

(c) For K as in (b) and $\lambda > 0$ let

$$A(K,\lambda) := \{f \in H(\mathbb{C}^N) : \|f\|_{K,\lambda} := \sup_{z\in\mathbb{C}^N} |f(z)| \exp(-h_K(\operatorname{Im} z) - \lambda\omega(|z|)) < \infty\}$$

and define

$$A_{(\omega)}(\mathbb{C}^N, G) := \operatorname{ind}_{K,n\to} A(K,n)$$
$$A_{\{\omega\}}(\mathbb{C}^N, G) := \operatorname{ind}_{K\to} A(K),\ \text{where } A(K) := \operatorname{proj}_{\leftarrow m} A(K, \frac{1}{m}).$$

It is easy to check that $A(K,\lambda)$ is a Banach space, that $A_{(\omega)}(\mathbb{C}^N, G)$ is an (LB)-space, that $A(K)$ is a Fréchet space, and that $A_{\{\omega\}}(\mathbb{C}^N, G)$ is an (LF)-space.

2.5. The Fourier-Laplace transform. Let ω be a weight function and let G be an open convex set in $\mathbb{R}^N$. For each $u \in \mathcal{E}_*(G)'$ it is easy to check that

$$\widehat{u} : \mathbb{C}^N \to \mathbb{C},\ \widehat{u}(z) := u_x(e^{-i\langle x,z\rangle})$$

is an entire function which belongs to $A_*(\mathbb{C}^N, G)$ and that

$$\mathcal{F} : \mathcal{E}_*'(G) \to A_*(\mathbb{C}^N, G),\ \mathcal{F}(u) := \widehat{u},$$

is linear and continuous.

The following result was proved for $N = 1$ by Meyer [**20**] and for general N in the Roumieu case by Rösner [**26**]. For a unified proof we refer to Heinrich and Meise [**10**], Theorems 3.6 and 3.7.

2.6. Theorem. *For each weight function ω satisfying $\omega(t) = o(t)$ as t tends to infinity and each convex open set $G \subset \mathbb{R}^N$ the Fourier-Laplace transform*

$$\mathcal{F} : \mathcal{E}_*'(G) \to A_*(\mathbb{C}^N, G)$$

is a linear topological isomorphism.

2.7. Convolution operators. For $\mu \in \mathcal{E}_*(\mathbb{R})'$, $\mu \neq 0$, and $\varphi \in \mathcal{E}_*(\mathbb{R})$ we define

$$\check{\mu}(\varphi) := \mu(\check{\varphi}),\ \check{\varphi}(x) := \varphi(-x),\ x \in \mathbb{R}.$$

The convolution operator $T_\mu : \mathcal{E}_{(*}(\mathbb{R}) \to \mathcal{E}_*(\mathbb{R})$ is defined by

$$T_\mu(f) := \check{\mu} * f,\ (\check{\mu} * f)(x) := \check{\mu}(f(x - .)),\ x \in \mathbb{R}.$$

It is a well-defined, linear, continuous operator; see Meyer [**20**] and [**21**]. For $g \in A_*(\mathbb{C}, \mathbb{R})$ we define the multiplication operator $M_g : A_*(\mathbb{C}, \mathbb{R}) \to A_*(\mathbb{C}, \mathbb{R})$ by $M_g(f) = gf$. It is well-known that for $\mu \in \mathcal{E}_*(\mathbb{R})$ we have on $\mathcal{E}_*(\mathbb{R})' : \mathcal{F} \circ T_\mu^t = M_{\hat{\mu}} \circ \mathcal{F}$.

2.8. Definition. Let $X = \operatorname{ind}_{n\to} X_n$ be an (LF)-space.

(a) X is called *sequentially retractive* if for each convergent sequence $(x_j)_{j\in\mathbb{N}}$ in X there exists $n \in \mathbb{N}$ such that $(x_j)_{j\in\mathbb{N}}$ lies in X_n and converges there.

(b) X is called *boundedly stable* if on each set which is bounded in some X_n all but finitely many of the step topologies coincide.

From Wengenroth [**31**], Theorem 6.4 and Corollary 6.7, we recall the following equivalences which we will use in section 3.

2.9. Theorem. *Let $X = \operatorname{ind}_{n\to} X_n$ be an (LF)-space and let $(\|.\|_{n,k})_{k\in\mathbb{N}}$ be a fundamental sequence of semi-norms for X_n. Then the following assertions are equivalent:*

(1) *X is sequentially retractive.*

(2) *There exist absolutely convex zero neighborhoods U_n in X_n for $n \in \mathbb{N}$ such that $U_n \subset U_{n+1}$ and such that for each $n \in \mathbb{N}$ there exists $m \geq n$ such that X and X_m induces the same topology on U_n.*

(3) *X is boundedly stable and satisfies the condition (P_3^*), i.e.,*

$$\forall\, n \in \mathbb{N}\, \exists\, m \geq n\, \forall\, k \geq m\, \exists\, N \in \mathbb{N}\, \forall\, M \in \mathbb{N}\, \exists\, K \in \mathbb{N},\ S > 0\, \forall\, x \in X_n :$$
$$\|x\|_{m,M} \leq S(\|x\|_{k,K} + \|x\|_{n,N}).$$

If X_n is a Fréchet-Montel space for each $n \in \mathbb{N}$ then (1)-(3) are also equivalent to

(4) *X is regular, i.e., for each bounded set B in X there exists $n \in \mathbb{N}$ such that $B \subset X_n$ and is bounded there.*

(5) *X is complete.*

2.10. Corollary. *For each weight function ω and for each convex open set $\Omega \subset \mathbb{R}^N$ the (LF)-space $A_{\{\omega\}}(\mathbb{C}^N, \Omega) = \operatorname{ind}_{n\to} A_{\{\omega\}}(K_n)$ satisfies the equivalent conditions of Theorem 2.9.*

Proof. Since $A_{\{\omega\}}(K_n)$ is a Fréchet-Montel space for each $n \in \mathbb{N}$, it follows that $\operatorname{ind}_{n\to} A_{\{\omega\}}(K_n)$ is boundedly stable. In the proof of Rösner [**26**], Satz 3.25, it is shown that the system $(\|\cdot\|_{n,k})_{n,k\in\mathbb{N}}$, defined by $\|f\|_{n,k} : \sup_{z\in\mathbb{C}} |f(z)| \exp(-n|\operatorname{Im} z| - \frac{1}{k}\omega(z))$ satisfies the condition (P_3^*). Hence condition 2.9 (3) is satisfied and the corollary follows from Theorem 2.9. See also Bonet and Domanski [**1**]. □

2.11. Definition. Let $\alpha = (\alpha_j)_{j\in\mathbb{N}}$ be an increasing, unbounded sequence in $[0, \infty[$. For $R \in \{0, \infty\}$ the power series spaces $\Lambda_R(\alpha)$ are defined as

$$\Lambda_R(\alpha) := \{x = (x_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} :\ \|x\|_r := \sum_{j=1}^{\infty} |x_j| \exp(r\alpha_j) < \infty\ \forall\, r < R\}.$$

$\Lambda_\infty(\alpha)$ is called a power series space of infinite type, while $\Lambda_0(\alpha)$ is said to be of finite type. Note that $\Lambda_R(\alpha)$ is a Fréchet-Schwartz space for each α and each R.

3. Surjectivity

In this section we characterize the surjectivity of the convolution operators $T_\mu : \mathcal{E}_{\{\omega\}}(\mathbb{R}) \to \mathcal{E}_{\{\omega\}}(\mathbb{R})$. We show that some of the equivalences in Braun, Meise, and Vogt [**7**], Theorem 3.8, in combination with Corollary 2.8, that were proved in the non-quasianalytic case also hold in the quasianalytic case. We also extend the characterization which Meyer [**21**] gave for convolution operators T_μ for which $\mu \in \mathcal{E}'_{\{\omega\}}(\mathbb{R})$ is supported by the origin, to arbitrary convolution operators. We begin by recalling several slowly decreasing conditions.

3.1. Definition. Let ω be a weight function.

(a) $F \in A_{\{\omega\}}(\mathbb{C}^N, \mathbb{R}^N)$ is called $\{\omega\}$-slowly decreasing, if for each $m \in \mathbb{N}$ there exists $R > 0$ such that for each $x \in \mathbb{R}^N$ with $|x| \geq R$ there exists $\xi \in \mathbb{C}^N$ satisfying $|x - \xi| \leq \omega(x)/m$ such that $|F(\xi)| \geq \exp(-\omega(\xi)/m)$.

(b) $F \in A_{(\omega)}(\mathbb{C}^N, \mathbb{R}^N)$ is called (ω)-slowly decreasing, if there exists $C > 0$ such that for each $x \in \mathbb{R}$, $|x| \geq C$, there exists $\xi \in \mathbb{C}^N$ such that

$$|x - \xi| \leq C\omega(x) \text{ and } |F(\xi)| \geq \exp(-C|\operatorname{Im}\xi| - C\omega(\xi)).$$

The significance of the $\{\omega\}$-slowly decreasing condition is explained by the following result.

3.2. Proposition. *Let ω be a weight function and let $F \in A_{\{\omega\}}(\mathbb{C}^N, \mathbb{R}^N)$ be given. Then the following assertions are equivalent:*

(a) *F is $\{\omega\}$-slowly decreasing.*
(b) *There exists a weight function σ satisfying $\sigma = o(\omega)$ such that $F \in A_{(\sigma)}(\mathbb{C}^N, \mathbb{R}^N)$ and such that F is (σ)-slowly decreasing.*
(c) *The multiplication operator $M_F : A_{\{\omega\}}(\mathbb{C}^N, \mathbb{R}^N) \to A_{\{\omega\}}(\mathbb{C}^N, \mathbb{R}^N)$, $M_F(g) := Fg$, has closed range.*
(d) *$M_F^{-1} : FA_{\{\omega\}}(\mathbb{C}^N, \mathbb{R}^N) \to A_{\{\omega\}}(\mathbb{C}^N, \mathbb{R}^N)$ is sequentially continuous.*

Proof. (a) $\Rightarrow$ (b): This holds by Bonet, Galbis, and Meise [**2**], Lemma 3.2, since in their proof the non-quasianalyticity of the weight function ω is not needed (see, e.g., Heinrich and Meise [**10**], Corollary 3.8).

(b) $\Rightarrow$ (c): Since every principal ideal in $H(\mathbb{C}^N)$ is closed, it suffices to show that the following assertion holds:

$$\text{If } g \in A_{\{\omega\}}(\mathbb{C}^N, \mathbb{R}^N) \text{ and } g/F \in H(\mathbb{C}^N) \text{ then } g/F \in A_{\{\omega\}}(\mathbb{C}^N, \mathbb{R}^N). \tag{3.1}$$

To prove (3.1), fix $g \in A_{\{\omega\}}(\mathbb{C}^N, \mathbb{R}^N)$ and choose a weight function σ according to (b). Then there exist $A, B > 0$ such that

$$|F(z)| \leq A\exp(B|\operatorname{Im} z| + B\sigma(z)),\ z \in \mathbb{C}^N \tag{3.2}$$

and there exists $\kappa \in \mathbb{N}$ such that for each $p \in \mathbb{N}$ there exists $C_p > 0$ such that

$$|g(z)| \leq C_p \exp(\kappa|\operatorname{Im} z| + \frac{1}{p}\omega(z)),\ z \in \mathbb{C}^N. \tag{3.3}$$

Next note that with $n = 1$ we get from Bonet, Galbis, and Momm [**3**], Proposition 2 (c), that

(3.4) there exist $k, m \in \mathbb{N}$ and $R > 0$ such that for each $z \in \mathbb{C}^N$, $|z| \geq R$, there exists $\zeta \in \mathbb{C}^N$ with $|\zeta - z| \leq |\operatorname{Im} z| + k\sigma(z)$ such that $|F(\zeta)| \geq \exp(-m|\operatorname{Im}\zeta| - m\sigma(\zeta))$.

Now we apply Hörmander [**11**], Lemma 3.2, with $r := |\operatorname{Im} z| + k\sigma(z)$ to get for $|z| \geq R$:

$$\left|\frac{g(z)}{F(z)}\right| \leq \frac{\sup_{|w-z|\leq 4r}|g(w)| \sup_{|w-z|\leq 4r}|F(w)|}{(\sup_{|w-z|\leq r}|F(w)|)^2}.$$

Using the upper estimate (3.2) for F and the lower estimate for $|F(\zeta)|$ it follows that

$$\left|\frac{g(z)}{F(z)}\right| \leq (\sup_{|w-z|\leq 4r}|g(w)|)A\exp((5B|\operatorname{Im} z| + 2m|\operatorname{Im}\zeta| + 4k\sigma(z) + B\sigma(5|z| + 4k\sigma(z)) + 2m\sigma(\zeta)).$$

Obviously, $|\zeta - z| \leq |\operatorname{Im} z| + k\sigma(z)$ implies

$$|\operatorname{Im}\zeta| \leq 2|\operatorname{Im} z| + k\sigma(z) \text{ and } \sigma(\zeta) \leq \sigma(2|z| + k\sigma(z)).$$

Since σ is a weight function, it is easy to check that this implies the existence of $A_1 \geq A$ and $B_1 \geq B$ such that by (3.3) we get for each $p \in \mathbb{N}$

$$\begin{aligned}\left|\frac{g(z)}{F(z)}\right| &\leq (\sup_{|w-z|\leq 4r}|g(w)|)A_1\exp(B_1|\operatorname{Im} z| + B_1\sigma(z)) \\ &\leq A_1 C_p \exp(B_1|\operatorname{Im} z| + (\kappa+4)|\operatorname{Im} z| + B_1\sigma(z) + \frac{1}{p}\omega(5|z| + 4k\sigma(z))).\end{aligned}$$

Since ω is a weight function and since $\sigma = o(\omega)$, it follows from this, that g/F is in $A_{\{\omega\}}(\mathbb{C}^N, \mathbb{R}^N)$. Hence we proved that (3.1) and consequently that (c) holds.

(c) $\Rightarrow$ (d): By Corollary 2.10, the (LF)-space $A_{\{\omega\}}(\mathbb{C}^N, \mathbb{R}^N) = \operatorname{ind}_{n\to} A_n$ is sequentially retractive. The continuous linear map $M_F : A_{\{\omega\}}(\mathbb{C}^N, \mathbb{R}^N) \to A_{\{\omega\}}(\mathbb{C}^N, \mathbb{R}^N)$ has closed range by the present hypothesis. Hence $\operatorname{im}(M_F) \cap A_n = M_F^{-1}(A_n)$ is closed in A_n for each $n \in \mathbb{N}$. This means that $\operatorname{im}(M_F)$

is stepwise closed in the sense of Floret [**9**], Theorem 6.4. By this theorem $M_F^{-1} : FA_{\{\omega\}}(\mathbb{C}^N, \mathbb{R}^N) \to A_{\{\omega\}}(\mathbb{C}^N, \mathbb{R}^N)$ is sequentially continuous. Hence (d) holds.

(d) $\Rightarrow$ (a): Note first that for each $\lambda > 0$ the spaces $A_{\{\omega\}}(\mathbb{C}^N, \mathbb{R}^N)$ and $A_{\{\lambda\omega\}}(\mathbb{C}^N, \mathbb{R}^N)$ are equal. Therefore, we may assume that there exists $t_0 > 0$ such that $\omega(t) \le t/2$ for $t \ge t_0$. Next choose $k \in \mathbb{N}$ so that $F \in A_k$, where $A_k := A(\overline{B(0,k)})$ in the notation of 2.4. To argue by contraposition, we assume that F is not $\{\omega\}$-slowly decreasing. Then there exist $\kappa \in \mathbb{N}$ and an unbounded sequence $(x_j)_{j\in\mathbb{N}}$ in $\mathbb{R}^N$ for which $(|x_j|)_{j\in\mathbb{N}}$ is increasing and for which the following holds for each $j \in \mathbb{N}$

$$(3.5)\qquad |F(\zeta)| \le \exp(-\frac{1}{\kappa}\omega(\zeta)) \text{ for all } \zeta \in \mathbb{C}^N \text{ with } |\zeta - x_j| < \frac{1}{\kappa}\omega(x_j).$$

We claim that this implies the following assertion:

(3.6) There exists a sequence $(g_j)_{j\in\mathbb{N}}$ in A_1 which is unbounded in A_n for each $n \in \mathbb{N}$, while $(M_F(g_j))_{j\in\mathbb{N}}$ is a null-sequence in A_{k+1}.

Obviously, (3.6) implies that $M_F^{-1} : FA_{\{\omega\}}(\mathbb{C}^N, \mathbb{R}^N) \to A_{\{\omega\}}(\mathbb{C}^N, \mathbb{R}^N)$ is not sequentially continuous. Hence (d) implies (a).

To prove (3.6) we argue similarly as in Momm [**22**] (see also [**2**], Proposition 3.4) and define for $j \in \mathbb{N}$ and $R > 0$ the function $h_{j,R} : \mathbb{C}^N \to \mathbb{R}$ by $h_{j,R}(z) := |\operatorname{Im} z|$ for $z \in \mathbb{C}^N \setminus B(x_j, R)$ and for $z \in B(x_j, R)$ by

$$h_{j,R}(z) := \sup\{v(z) : v \text{ is plurisubharmonic on } B(x_j, R) \text{ and for each } \xi \in \partial B(x_j, R) : \limsup_{\zeta\to\xi} v(\zeta) \le |\operatorname{Im} \xi|\}.$$

Then $h_{j,R}$ is continuous and plurisubharmonic on $\mathbb{C}^N$. Next let $K \ge 1$ be the constant from 2.1 (α), choose $p \in \mathbb{N}$, $p \ge 2$, so large that $2K/p \le 1/\kappa$, let $R_j := \omega(x_j)/p$, and define $\varphi_j := h_{j,R_j}$. Since $|x_j| \to \infty$, we may assume that for all $j \in \mathbb{N}$ the following holds:

$$(3.7)\qquad 2 \le \frac{\omega(x_j)}{2p},\ \frac{1}{\omega(x_j)} \le \frac{\omega(x_j)}{8p^2},\ |x_j| \ge t_0 \text{ and hence } \frac{\omega(x_j)}{p} + 1 \le \frac{|x_j|}{2}.$$

Using Hörmander's solution of the $\overline{\partial}$-problem (see Hörmander [**12**], Theorem 4.4.4) it follows as in Momm [**23**], 1.8, that there exists a constant $C_N > 0$ such that for each $j \in \mathbb{N}$ there exists $f_j \in H(\mathbb{C}^N)$ satisfying the following estimates

$$(3.8)\qquad |f_j(x_j)| \ge \exp(\inf_{|w-x_j|\le 1} \varphi_j(w) - C_N \log(1 + |x_j|^2))$$

and

$$(3.9)\qquad |f_j(z)| \le C_N \exp(\sup_{|w-z|\le 1} \varphi_j(w) + C_N \log(1 + |z|^2)),\ z \in \mathbb{C}^N.$$

Next note that for $z \in \mathbb{C}^N \setminus B(x_j, R_j + 1)$ we have

$$(3.10)\qquad \sup_{|w-z|\le 1} \varphi_j(w) = \sup_{|w-z|\le 1} |\operatorname{Im} w| \le |\operatorname{Im} z| + 1.$$

From this estimate and (3.9) we get for each $j \in \mathbb{N}$ and each $m \in \mathbb{N}$

$$\sup_{z\in\mathbb{C}} |f_j(z)| \exp(-|\operatorname{Im} z| - \frac{1}{m}\omega(z)) < \infty.$$

Hence $f_j \in A_1$ for each $j \in \mathbb{N}$. Therefore, also the sequence $(g_j)_{j\in\mathbb{N}}$ defined by

$$g_j := \exp(-\frac{\omega(x_j)}{8p}) f_j,\ j \in \mathbb{N},$$

is in A_1. To show that it is not bounded in A_n for any $n \in \mathbb{N}$, note that the function

$$v_j(z) := \frac{1}{2R_j}(|\operatorname{Im} z|^2 - |\operatorname{Re} z|^2 + R_j^2)$$

is harmonic and satisfies $v_j(z) \le |\operatorname{Im} z|$ for $z \in \partial B(x_j, R_j)$, since $x_j \in \mathbb{R}^N$. By the definition of φ_j, this implies $\varphi_j \ge v_j$ on $B(x_j, R_j)$ and consequently by (3.7)

$$\inf_{|w-x_j|\le 1} \varphi_j(w) \ge \inf_{|w-x_j|\le 1} v_j(w) \ge \frac{1}{2R_j}(-1+R_j^2) = \frac{R_j}{2} - \frac{1}{2R_j} = \frac{\omega(x_j)}{2p} - \frac{2p}{\omega(x_j)} \ge \frac{\omega(x_j)}{4p}.$$

Since $\log(1+t^2) = o(\omega(t))$ for t tending to infinity, there exists $\delta > 0$ such that $\exp(-C_N \log(1+|x_j|^2)) \ge \delta \exp(-\omega(x_j)/32)$ for each $j \in \mathbb{N}$. Therefore, it follows from (3.8) that for each $n \in \mathbb{N}$ and each $m \in \mathbb{N}$ with $m \ge 16p$ we have for each $j \in \mathbb{N}$:

$$\begin{aligned}&\sup_{z\in\mathbb{C}^N} |g_j(z)| \exp(-n|\operatorname{Im} z| - \frac{1}{m}\omega(z))\\ &\qquad \ge \exp((-\frac{1}{8p} - \frac{1}{m} + \frac{1}{4p})\omega(x_j) - \log(1+(x_j)^2)) \ge \delta \exp(\frac{1}{32p}\omega(x_j)).\end{aligned}$$

This shows that $(g_j)_{j\in\mathbb{N}}$ is unbounded in A_n for any $n \in \mathbb{N}$.

To prove that $(M_F(g_j))_{j\in\mathbb{N}}$ is a null-sequence in A_{k+1}, note first that for $z \in \mathbb{C}^N \setminus B(x_j, R_j+1)$ we get from (3.10) and (3.9) that for each $m \in \mathbb{N}$ we have

$$\begin{aligned}|F(z)f_j(z)| &\le \|F\|_{\overline{B(0,k)},1/m} \exp(k|\operatorname{Im} z| + \frac{1}{m}\omega(z)) \exp(|\operatorname{Im} z| + 1)\\ &\le \|F\|_{\overline{B(0,k)},1/m} \exp((k+1)|\operatorname{Im} z| + \frac{1}{m}\omega(z)).\end{aligned} \tag{3.11}$$

To estimate Ff_j in $B(x_j, R_j+1)$, fix $z \in B(x_j, R_j+1)$. Then we have by the maximum principle and (3.7)

$$\sup_{|w-z|\le 1} \varphi_j(w) \le \sup_{|w-x_j|\le R_j+2} \varphi_j(w) \le \sup_{|w-x_j|\le R_j+2} |\operatorname{Im} w| \le R_j + 2 = \frac{\omega(x_j)}{p} + 2 \le \frac{3\omega(x_j)}{2p}$$

and also

$$|\operatorname{Re} z| \ge |x_j| - R_j - 1 = |x_j| - \frac{\omega(x_j)}{p} - 1 \ge \frac{|x_j|}{2}.$$

Since ω satisfies 2.1 (α), the last estimate implies $\omega(x_j) \le \omega(2\operatorname{Re} z) \le K\omega(z) + K$ and consequently

$$\sup_{|w-z|\le 1} \varphi_j(w) \le \frac{3K\omega(z)}{2p} + \frac{3K}{2p}.$$

From this, (3.5), and (3.9) we get the existence of C' such that for each $j \in \mathbb{N}$:

$$\begin{aligned}|F(z)f_j(z)| &\le C_N \exp(-\frac{1}{\kappa}\omega(z) + \frac{3K\omega(z)}{2p} + \frac{3K}{2p} + C_N \log(1+|z|^2)))\\ &\le C' \exp((\frac{2K}{p} - \frac{1}{\kappa})\omega(z)) \le C'.\end{aligned} \tag{3.12}$$

From (3.11) and (3.12) it follows that $(Ff_j)_{j\in\mathbb{N}}$ is bounded in A_{k+1}. Since $(\exp(-\omega(x_j)/8p))_{j\in\mathbb{N}}$ is a null-sequence, we proved that $(M_F(g_j))_{j\in\mathbb{N}}$ is a null-sequence in A_{k+1}. Hence the proof of (3.6) and also the one of the proposition is complete. □

3.3. Corollary. *Let ω be a weight function and let $F \in A_{\{\omega\}}(\mathbb{C}, \mathbb{R})$ be given. Then the conditions (a) - (d) in Proposition 3.2 are equivalent to the following one:*

(e) *There exists a weight function σ satisfying $\sigma = o(\omega)$such that $F \in A_{(\sigma)}(\mathbb{C}, \mathbb{R})$, and there exist $\varepsilon, C, D > 0$ such that for each component S of the set*

$$S(F, \varepsilon, C) := \{z \in \mathbb{C} : |F(z)| < \varepsilon \exp(-C|\operatorname{Im} z| - C\sigma(z))\}$$

the following estimates hold:

$$\sup_{z\in S}(|\operatorname{Im} z| + C\sigma(z)) \le D(1 + \inf_{z\in S}(|\operatorname{Im} z| + \sigma(z))), \quad \sup_{z\in S}\omega(z) \le D(1 + \inf_{z\in S}\omega(z)).$$

Proof. To show that condition 3.2(b) implies the present condition (e), note that by Momm [**22**], Proposition 1, (e) follows from (b), except for the last estimate. This, however, follows from the diameter estimates given in the proof of Meise, Taylor, and Vogt [**17**]. Lemma 2.3.

To show that (e) implies condition 3.2(c) let $V(F) := \{a \in \mathbb{C} : F(a) = 0\}$ and denote for each $a \in V(F)$ by $\operatorname{ord}(F,a)$ the order of vanishing of F at a. Then consider the map

$$\varrho :\ A_{\{\omega\}}(\mathbb{C},\mathbb{R}) \to \prod_{a\in V(F)} \mathbb{C}^{\operatorname{ord}(F,a)},\ \ \varrho(g) :=\ (g(a), g'(a), \dots, g^{(\operatorname{ord}(F,a)-1)}(a))_{a\in V(F)}.$$

It is easy to check that ϱ is linear and continuous. Hence $I_{\mathrm{loc}}(F) := \ker \varrho$ is closed in $A_{\{\omega\}}(\mathbb{C},\mathbb{R})$. Thus, (d) follows if we show that $FA_{\{\omega\}}(\mathbb{C},\mathbb{R}) = \operatorname{im}(M_F) = I_{\mathrm{loc}}(F)$. To do so, note first that obviously we have $\operatorname{im}(M_F) \subset I_{\mathrm{loc}}(F)$. For the converse inclusion fix $g \in I_{\mathrm{loc}}(F)$. Then there exists $k \in \mathbb{N}$ such that for each $m \in \mathbb{N}$ there is $C_m > 0$ such that

$$|g(z)| \le C_m \exp(k|\operatorname{Im} z| + \frac{1}{m}\omega(z)),\ z \in \mathbb{C}.$$

By (e), we can choose σ, ε, C, and D according to (e). Then note that $g \in I_{\mathrm{loc}}(F)$ implies $g/F \in H(\mathbb{C})$. Since $\sigma = o(\omega)$, we get for each $m \in \mathbb{N}$ the existence of C'_m such that for each $z \in \mathbb{C} \setminus S_\sigma(F,\varepsilon,C)$ the following estimate holds

$$\begin{aligned}\left|\frac{g(z)}{F(z)}\right| &\le C_m \exp(k|\operatorname{Im} z| + \frac{1}{m}\omega(z))\frac{1}{\varepsilon}\exp(C|\operatorname{Im} z| + C\sigma(z))\\ &\le C'_m \exp((k+C)|\operatorname{Im} z| + \frac{2}{m}\omega(z)).\end{aligned} \tag{3.13}$$

Now note that from (3.13) and the estimates in (e) it follows by the maximum principle that for each $m \in \mathbb{N}$ there exists C''_m such that for each component s of $S_\sigma(F,\varepsilon,C)$ and each $z \in S$ we get the estimate

$$\begin{aligned}\left|\frac{g(z)}{F(z)}\right| &\le C'_m \exp((k+C)\sup_{\zeta\in S}(|\operatorname{Im}\zeta|) + \frac{2}{m}\sup_{\zeta\in S}\omega(\zeta))\\ &\le C'_m \exp((k+C)D(1+|\operatorname{Im} z| + \sigma(z)) + \frac{2D}{m}(1+\omega(z)))\\ &\le C''_m \exp((k+C)D|\operatorname{Im} z| + \frac{3D}{m}\omega(z)).\end{aligned} \tag{3.14}$$

Obviously, (3.13) and (3.14) imply that $g/F \in A_{\{\omega\}}(\mathbb{C},\mathbb{R})$. Hence $g = F(g/F) \in FA_{\{\omega\}}(\mathbb{C},\mathbb{R})$. □

In order to apply Proposition 3.2 we recall the following sequence spaces from Meise [**15**], 1.4.

3.4. Definition. Let $\alpha = (\alpha_j)_{j\in\mathbb{N}}$ and $\beta = (\beta_j)_{j\in\mathbb{N}}$ be sequences of nonnegative real numbers and let $\mathbb{E} = (E_j)_{j\in\mathbb{N}}$ be a sequence of Banach spaces. For $R > 0$ and $m \in \mathbb{N}$ let

$$K(\mathbb{E},R,m) :=\ \{x = (x_j)_{j\in\mathbb{N}} \in \prod_{j=1}^{\infty} E_j :\ \|x\|_{R,m} :=\ \sup_{j\in\mathbb{N}} \|x_j\|_j \exp(-R\alpha_j - \beta_j/m) < \infty\}$$

and define the Fréchet space $K(\mathbb{E},R,\alpha,\beta)$ and the (LF)-space $K(\mathbb{E},\alpha,\beta)$ by

$$K(\mathbb{E},R,\alpha,\beta) :=\ \operatorname{proj}_{\leftarrow m} K(\mathbb{E},R,m) \text{ and } K(\mathbb{E},\alpha,\beta) :=\ \operatorname{ind}_{k\to} \operatorname{proj}_{\leftarrow m} K(\mathbb{E},k,m).$$

If $E_j = \mathbb{C}$ for each $j \in \mathbb{N}$, then we write $K(\alpha,\beta)$ instead of $K(\mathbb{E},\alpha,\beta)$.

3.5. Remark. If $\lim_{j\to\infty}\beta_j = \infty$ then for each $k \in \mathbb{N}$ the space $\operatorname{proj}_{\leftarrow m} K(k,m)$ is a Fréchet-Schwartz space. Note that by Meise [**15**], Example 1.9 (2), the (LF)-space $K(\alpha,\beta)$ is in fact an (LB)-space, whenever $\liminf_{j\to\infty}\alpha_j/\beta_j > 0$.

Because of Corollary 3.3, we get from Meise [**15**], Theorem 2.6, the following holds (for more details we refer to the proof of Proposition 4.7 below):

3.6. Theorem. *Let ω be a weight function and let $F \in A_{\{\omega\}}(\mathbb{C},\mathbb{R})$ be $\{\omega\}$-slowly decreasing. Then $A_{\{\omega\}}(\mathbb{C},\mathbb{R})/FA_{\{\omega\}}(\mathbb{C},\mathbb{R})$ is either finite dimensional or isomorphic to $K(\alpha,\beta)$, for the sequences α and β defined as*

$$\alpha := (|\operatorname{Im} a_j|)_{j\in\mathbb{N}},\ \beta := (\omega(a_j))_{j\in\mathbb{N}},$$

where $(a_j)_{j\in\mathbb{N}}$ is an enumeration of the points in $V(F)$ with each point repeated as many times as the multiplicity of the zero of F at this point.

From Braun, Meise, and Vogt [**7**], Proposition 3.7, and Vogt [**28**], Theorem 4.3, we recall the following result.

3.7. Proposition. *Let α and β be sequences of nonnegative real numbers such that $\lim_{j\to\infty}\beta_j=\infty$. Then $K(\alpha,\beta)$ is complete if and only if there exists $\delta>0$ such that each limit point of the set $\{\alpha_j/\beta_j : j\in\mathbb{N},\beta_j\neq 0\}$ is contained in $\{0\}\cup[\delta,\infty[$.*

3.8. Lemma. *Let $E=\operatorname{ind}_{n\to}E_n$ be an (LF)-space which is sequentially retractive and for which each E_n is a Fréchet-Schwartz space. Let $S:E\to E$ be a continuous linear operator for which $S(E)\cap E_n$ is closed in E_n for each $n\in\mathbb{N}$. Then the following assertions are equivalent:*

(1) *S is an injective topological homomorphism.*
(2) *$S^t:E'\to E'$ is surjective.*
(3) *The (LF)-space $E/S(E):=\operatorname{ind}_{n\to}E_n/(S(E)\cap E_n)$ is sequentially retractive.*
(4) *$E/S(E)$ is complete.*
(5) *$E/S(E)$ is regular.*

Proof. (1) ⇔ (2): This holds by Floret [**9**], Theorem 6.2.

(1) ⇒ (3): By the present hypothesis, we have the following short algebraically exact sequence of (LF)-spaces with continuous linear maps

$$0\to E\overset{S}{\to}E\overset{q}{\to}E/S(E)\to 0, \tag{3.15}$$

where $S(E)$ carries the topology defined in (3) and where q is the quotient map. Next note that by Wengenroth [**31**], Theorem 6.4, E is sequentially retractive if and only if E is acyclic, a concept explained in [**31**] and Vogt [**29**], Section 1. Hence it follows from (3.15) and [**29**], Theorem 1.5, that $E/S(E)$ is acyclic and consequently sequentially retractive. Thus (3) holds.

(3) ⇒ (1): This implication follows from (3.15) by Vogt [**29**], Theorem 1.4, if we show the following:

$$\text{For each } n\in\mathbb{N} \text{ there is } m\in\mathbb{N} \text{ such that } S^{-1}(E_n)\subset E_m. \tag{3.16}$$

To show this, we define on $S(E)$ the (LF)-topology τ by $(S(E),\tau):=\operatorname{ind}_{n\to}(S(E)\cap E_n)$. Then the map $S:E\to(S(E),\tau)$ is injective and has closed graph. Consequently, it is an injective topological homomorphism. By the continuity of $S^{-1}:(S(E),\tau)\to E$ and Grothendieck's factorization theorem we get for each $n\in\mathbb{N}$ the existence of $m\in\mathbb{N}$ such that

$$S^{-1}(E_n)=S^{-1}(S(E)\cap E_n)\subset E_m.$$

Thus, (3.16) holds and consequently (3) holds.

(3) ⇔ (4) ⇔ (5): This follows from Theorem 2.9. □

3.9. Theorem. *Let ω be a weight function and let $F\in A_{\{\omega\}}(\mathbb{C},\mathbb{R})$ be $\{\omega\}$-slowly decreasing. Then the following conditions are equivalent:*

(1) *$M_F:A_{\{\omega\}}(\mathbb{C},\mathbb{R})\to A_{\{\omega\}}(\mathbb{C},\mathbb{R})$ is an injective topological homomorphism.*
(2) *There exists $\delta>0$ such that each limit point of the set $\{|\operatorname{Im}a|/\omega(a) : a\in V(F),\ \omega(a)\neq 0\}$ is contained in $\{0\}\cup[\delta,\infty[$.*

Proof. Note that $A_{\{\omega\}}(\mathbb{C},\mathbb{R})=\operatorname{ind}_{n\to}A_n$, where each A_n is a Fréchet-Schwartz space. By Corollary 2.10, $A_{\{\omega\}}(\mathbb{C},\mathbb{R})$ is sequentially retractive. Since F is $\{\omega\}$-slowly decreasing, it follows from Proposition 3.2 that M_F hat closed range. Thus, the hypotheses of Lemma 3.8 are fulfilled for $S=M_F$ and $E=A_{\{\omega\}}(\mathbb{C},\mathbb{R})$. Moreover, the open mapping theorem for (LF)-spaces implies that $A_{\{\omega\}}(\mathbb{C},\mathbb{R})/FA_{\{\omega\}}(\mathbb{C},\mathbb{R})$ and $\operatorname{ind}_{n\to}A_n/(A_n\cap FA_{\{\omega\}}(\mathbb{C},\mathbb{R}))$ are topologically equal. Hence Lemma 3.8 implies that condition (1) is equivalent to the completeness of $A_{\{\omega\}}(\mathbb{C},\mathbb{R})/FA_{\{\omega\}}(\mathbb{C},\mathbb{R})$. By Theorem 3.6 the latter space is isomorphic to $K(\gamma,\delta)$. From the definition of the sequences γ and δ in Theorem 3.6 and Proposition 3.7 it now follows that $A_{\{\omega\}}(\mathbb{C},\mathbb{R})/FA_{\{\omega\}}(\mathbb{C},\mathbb{R})$ is complete if and only if condition (2) holds. Hence we proved the equivalence of (1) and (2). □

3.10. Theorem. *Let ω be a weight function and let $\mu\in\mathcal{E}_{\{\omega\}}(\mathbb{R})'$, $\mu\neq 0$, be given. Then the following assertions are equivalent:*

(1) $T_\mu : \mathcal{E}_{\{\omega\}}(\mathbb{R}) \to \mathcal{E}_{\{\omega\}}(\mathbb{R})$ *is surjective.*
(2) *The following two conditions are satisfied:*
   (a) $\hat{\mu}$ *is* $\{\omega\}$*-slowly decreasing.*
   (b) *There exists* $\delta > 0$ *such that each limit point of the set* $\{|\operatorname{Im} a|/\omega(a) : a \in V(\hat{\mu}),\ \omega(a) \neq 0\}$ *is contained in* $\{0\} \cup [\delta, \infty[$.

PROOF. (1) $\Rightarrow$ (2): Since the space $\mathcal{E}_{\{\omega\}}(\mathbb{R})$ is ultrabornological and webbed, the surjectivity of T_μ implies that T_μ is open or equivalently a surjective topological homomorphism. By a result of Grothendieck (see Köthe [**13**], 32, 4.(3)), $T_\mu^t(\mathcal{E}_{\{\omega\}}(\mathbb{R})')$ is weakly closed in $\mathcal{E}_{\{\omega\}}(\mathbb{R})'$ and hence closed. Because of $\mathcal{F} \circ T_\mu^t = M_{\hat{\mu}} \circ \mathcal{F}$, this implies that $M_{\hat{\mu}}$ has closed range. Therefore, $\hat{\mu}$ is $\{\omega\}$-slowly decreasing by Proposition 3.2. Hence condition (a) holds.
Moreover, also the hypotheses of Lemma 3.8 are fulfilled for $E = A_{\{\omega\}}(\mathbb{C}, \mathbb{R})$ and $S = M_{\hat{\mu}}$, since $A_{\{\omega\}}(\mathbb{C}, \mathbb{R})$ is sequentially retractive by Corollary 2.10. From 2.7 we know that

$$\mathcal{F}^t \circ M_{\hat{\mu}}^t = (T_\mu^t)^t \circ \mathcal{F}^t = T_\mu \circ \mathcal{F}^t. \tag{3.17}$$

This shows that $M_{\hat{\mu}}^t$ is surjective. Hence $M_{\hat{\mu}}$ is an injective topological homomorphism, by Lemma 3.8. Consequently, Theorem 3.9 implies that (b) holds.

(2) $\Rightarrow$ (1): By Theorem 3.9 the conditions (a) and (b) imply that $M_{\hat{\mu}} : A_{\{\omega\}}(\mathbb{C}, \mathbb{R}) \to A_{\{\omega\}}(\mathbb{C}, \mathbb{R})$ is an injective topological homomorphism. Hence the Theorem of Hahn-Banach implies that $M_{\hat{\mu}}^t$ is surjective. Since the space $\mathcal{E}_{\{\omega\}}(\mathbb{R})$ is reflexive, we get from (3.17) that T_μ is surjective. □

Of course, one wants to know which surjective convolution operators $\mathcal{E}_{\{\omega\}}(\mathbb{R})$ admit a continuous linear right inverse. We were only able to prove the following necessary condition, which is a characterization in the non-quasianalytic case by Braun, Meise, and Vogt [**7**], Theorem 4.2.

3.11. PROPOSITION. *Let* ω *be a quasianalytic weight function which satisfies the condition* (α_1)*, let* $\mu \in \mathcal{E}'_{\{\omega\}}(\mathbb{R})$*,* $\mu \neq 0$ *be given, and assume that* $T_\mu : \mathcal{E}_{\{\omega\}}(\mathbb{R}) \to \mathcal{E}_{\{\omega\}}(\mathbb{R})$ *is surjective. If* T_μ *admits a continuous linear right inverse, then*

$$\lim_{\substack{a \in V(\hat{\mu}) \\ |a| \to \infty}} \frac{|\operatorname{Im} a|}{\omega(a)} = 0.$$

PROOF. If we assume that the present condition does not hold then we can find a sequence $((a_j)_{j\in\mathbb{N}}$ in $V(\hat{\mu})$ and $\delta > 0$ with $|\operatorname{Im} a_j| \geq \delta\omega(a_j)$ for each $j \in \mathbb{N}$. Proceeding by recurrence, we extract a subsequence of $(a_j)_{j\in\mathbb{N}}$, which we denote in the same way, such that

(i) $|a_{j+1}| \geq 4|a_j|$, and for $n(t) := \operatorname{card}\{j : |a_j| \leq t\}$,
(ii) $n(t)\log t = o(\omega(t))$ as $t \to \infty$.

Applying [**6**], 1.7 and 1.8 (a), we find a weight function $\sigma_0(t)$ such that $n(t)\log t = o(\sigma_0(t))$ and $\sigma_0(t) = o(\omega(t))$ as $t \to \infty$. As in [**7**], 3.11, we define

$$F(z) := \prod_{j=1}^{\infty} \left(1 - \frac{z}{a_j}\right),\ z \in \mathbb{C}.$$

By Rudin [**27**], Theorem 15.6, F is an entire function such that its set of zeros consists of the sequence $(a_j)_j$, and satisfies the following conditions:

(1) There exists $C > 0 :\ |F(z)| \leq C\exp(\sigma_0(z))$, $z \in \mathbb{C}$.
(2) There exists $\varepsilon_0 > 0$ such that $|F(\zeta)| \geq \varepsilon_0 \exp(-\sigma_0(\zeta))$ for all $\zeta \in \mathbb{C} \setminus \bigcup_{j=1}^{\infty} B(a_j, 1)$.
(3) There exist $\varepsilon_0 > 0, K_0 > 0$ such that for all $\zeta \in \mathbb{C}$ with $1 \leq |\zeta - a_j| \leq 2$ for some j:

$$|F(z)| \geq \varepsilon_0 \exp(-K_0\sigma_0(z)), z \in \mathbb{C}.$$

This can be achieved by the arguments given in [**4**], proof of Lemma 3.5, arguments based on Braun, Meise, and Vogt [**7**], 3.11. In particular, F is (σ_0)-slowly decreasing by (ii).

Since each a_j is a zero of $\hat{\mu}(z)$, it follows that $g(z) := \hat{\mu}(z)/F(z)$ is an entire function. Since F is (σ_0)-slowly decreasing, we conclude $g \in A_{(\sigma_0)}(\mathbb{C}, \mathbb{R})$. On the other hand $\sigma_0(t) = o(\omega(t))$, hence $A_{(\sigma_0)}(\mathbb{C}, \mathbb{R}) \subset A_{\{\omega\}}(\mathbb{C}, \mathbb{R})$, and the latter space is an algebra. This yields that $M_g : A_{\{\omega\}}(\mathbb{C}, \mathbb{R}) \to A_{\{\omega\}}(\mathbb{C}, \mathbb{R})$, $M_g(h) := gh$, is a continuous linear operator.

By hypothesis, $M_{\hat{\mu}} : A_{\{\omega\}}(\mathbb{C},\mathbb{R}) \to A_{\{\omega\}}(\mathbb{C},\mathbb{R})$ admits a continuous linear left inverse $L_{\hat{\mu}}$. The operator $L_F := L_{\hat{\mu}} \circ M_g : A_{\{\omega\}}(\mathbb{C},\mathbb{R}) \to A_{\{\omega\}}(\mathbb{C},\mathbb{R})$ is continuous and it is a left inverse of M_F, since $L_F M_F(h) = h$ for each $h \in A_{\{\omega\}}(\mathbb{C},\mathbb{R})$.

We define, for an entire function $f \in H(\mathbb{C})$, $\varrho(f) := (f(a_j))_j \in \mathbb{C}^{\mathbb{N}}$. Proceeding as we did in the proof of [**4**], Lemma 3.8 (a proof based on the method of the proof of Meise [**14**], Theorem 3.7), we conclude from properties (1), (2), and (3) of F that

$$M_F A_{\{\omega\}}(\mathbb{C},\mathbb{R}) = \ker \varrho \cap A_{\{\omega\}}(\mathbb{C},\mathbb{R}),$$

hence this principal ideal is closed, and the quotient $A_{\{\omega\}}(\mathbb{C},\mathbb{R})/M_F A_{\{\omega\}}(\mathbb{C},\mathbb{R})$ coincides with the sequence (LF)-space $G := K(\alpha,\beta)$ for $\alpha := (|\operatorname{Im} a_j|)_{j\in\mathbb{N}}$ and $\beta := (\omega(a_j))_{j\in\mathbb{N}}$. Since $M_F : A_{\{\omega\}}(\mathbb{C},\mathbb{R}) \to A_{\{\omega\}}(\mathbb{C},\mathbb{R})$ has a continuous linear left inverse, we conclude that G is isomorphic to a complemented subspace of $A_{\{\omega\}}(\mathbb{C},\mathbb{R})$.

We now show that the (LF)-space G coincides algebraically and topologically with the (LB)-sequence space

$$E := \{y \in \mathbb{C}^{\mathbb{N}} : \exists m : \|y\|_m := \sup_{j\in\mathbb{N}} |y_j| \exp(-m|\operatorname{Im} a_j|) < \infty\}.$$

Indeed, it is clear that $E \subset G$. On the other hand, if $x \in G$, there is $n \in \mathbb{N}$ such that for $k = 1$ we can find $C_1 > 0$ with

$$|x_j| \le C_1 \exp(n|\operatorname{Im} a_j| + \omega(a_j)) \text{ for each } j \in \mathbb{N}.$$

Since $|\operatorname{Im} a_j| \ge \delta\omega(a_j)$ for each j, we select $m \in \mathbb{N}$, $m > n + \delta^{-1}$, we get

$$|x_j| \le C_1 \exp(m|\operatorname{Im} a_j|) \text{ for each } j, \text{ and } x \in E.$$

By the closed graph theorem $E = G$ also topologically.

This implies that G is isomorphic to the dual of the power series space $\Lambda_\infty((|\operatorname{Im} a_j|)_{j\in\mathbb{N}})$ of infinite type and is complemented in $A_{\{\omega\}}(\mathbb{C},\mathbb{R}) \cong \mathcal{E}_{\{\omega\}}(\mathbb{R})'$. This yields that $\Lambda_\infty((|\operatorname{Im} a_j|)_{j\in\mathbb{N}})$ is isomorphic to a complemented subspace of $\mathcal{E}_{\{\omega\}}(\mathbb{R})$. Since ω satisfies (α_1), this implies by Vogt [**30**] or Bonet and Domanski [**1**], Corollary 2.5, that $\Lambda_\infty((|\operatorname{Im} a_j|)_{j\in\mathbb{N}})$ has property $(\overline{\overline{\Omega}})$. This, however, is a contradiction. □

4. Ultradifferential operators on compact intervals

In this section we show that the surjectivity of $\{\omega\}$-ultradifferential operators on $\mathcal{E}_{\{\omega\}}[a,b]$ is characterized by $\hat{\mu}$ being $\{\omega\}$-slowly decreasing.

4.1. Definition. Let ω be a weight function and assume that for $\mu \in \mathcal{E}'_{\{\omega\}}(\mathbb{R})$ its Fourier-Laplace transform $\hat{\mu}$ is in $A_{\{\omega\}}$. Then the operator T_μ will be called an $\{\omega\}$-*ultradifferential operator* since for each $f \in \mathcal{E}_{\{\omega\}}(\mathbb{R})$ we have

$$T_\mu(f) = \sum_{j=0}^{\infty} i^j \frac{\hat{\mu}^{(j)}(0)}{j!} f^{(j)}.$$

4.2. Definition. For a weight function ω and for $R > 0$ define the Fréchet space $A_{\{\omega,R\}}$ of entire functions by

$$A_{\{\omega,R\}} := \operatorname{proj}_{\leftarrow m} A([-R,R], \frac{1}{m}).$$

We also define the space

$$A_{(\omega,R)} := \operatorname{ind}_{n\to} A([-R,R], n),$$

which is a (DFN)-space.

4.3. Remark. By Rösner [**26**], 2.19, for each weight function ω and each $R > 0$, the Fourier-Laplace transform $\mathcal{F} : \mathcal{E}'_{\{\omega\}}[-R,R] \to A_{\{\omega,R\}}$ is a linear topological isomorphism.

4.4. Proposition. *Let ω be a weight function. For $F \in A_{\{\omega\}}, F \neq 0$, the following conditions are equivalent:*

(1) *F is $\{\omega\}$-slowly decreasing.*
(2) *For each $R > 0$ and each $g \in A_{\{\omega,R\}}$ which satisfies $g/F \in H(\mathbb{C})$, the function g/F is in $A_{\{\omega,R\}}$.*

(3) *For each $R > 0$ the multiplication operator*

$$M_F : A_{\{\omega,R\}} \to A_{\{\omega,R\}},\ M_F(g) := Fg,$$

has closed range.

(4) *For each $R > 0$ the map M_F defined in (3) is an injective topological homomorphism.*

PROOF. (1) ⇒ (2): Note first that a standard application of Braun, Meise, and Taylor [**6**], Lemma 1.7, implies the existence of a weight function σ_1 satisfying $\sigma_1 = o(\omega)$ such that $F \in A_{(\sigma)}$ for each weight function σ which satisfies $\sigma_1 = o(\sigma)$. Since $g \in A_{\{\omega,R\}}$, we can find a weight function σ_2 and $C_2 > 0$ such that $\sigma_2 = o(\omega)$ and such that

$$|g(z)| \le C_2 \exp(R|\operatorname{Im} z| + \sigma_2(z)),\ z \in \mathbb{C}.$$

Next note that because of the hypothesis (1) it follows from Proposition 3.2 that there exists a weight function σ_3 with $\sigma_3 = o(\omega)$ such that $F \in A_{(\sigma_3)}$ and F is (σ_3)-slowly decreasing. Now choose a weight function σ which satisfies $\sigma = o(\omega)$ and $\max(\sigma_1, \sigma_2, \sigma_3) \le \sigma$. Then we have $g \in A_{(\sigma,R)}$, $F \in A_{(\sigma)}$ and that F is (σ)-slowly decreasing. Since $g/F \in H(\mathbb{C})$ by hypothesis, it follows from [**5**], Lemma 4.6, that $g/F \in A_{(\sigma,R)} \subset A_{\{\omega,R\}}$. Hence we showed that (2) holds.

(2) ⇒ (3): Obviously, the inclusion map $J : A_{\{\omega,R\}} \to H(\mathbb{C})$ is linear and continuous and the principal ideal $FH(\mathbb{C})$ is closed in $H(\mathbb{C})$. Hence $J^{-1}(FH(\mathbb{C}))$ is closed in $A_{\{\omega,R\}}$. Because of $J^{-1}(FH(\mathbb{C})) = FA_{\{\omega,R\}} = M_F(A_{\{\omega,R\}})$, this implies that (3) holds.

(3) ⇒ (4): Since M_F is injective and since $A_{\{\omega,R\}}$ is a Fréchet space, this follows from the closed range theorem (see Meise and Vogt [**19**], 26.3).

(4) ⇒ (1): If we show that $M_F^{-1} : FA_{\{\omega\}}(\mathbb{C},\mathbb{R}) \to A_{\{\omega\}}(\mathbb{C},\mathbb{R})$ is sequentially continuous then it follows from Proposition 3.2 (d) that (1) holds. To do so, let $(Fh_j)_{j\in\mathbb{N}}$ be a sequence in $FA_{\{\omega\}}(\mathbb{C},\mathbb{R})$ that satisfies $\lim_{j\to\infty} Fh_j = 0$. By Corollary 2.10, the inductive limit $A_{\{\omega\}}(\mathbb{C},\mathbb{R}) = \operatorname{ind}_{n\to} A_{\{\omega,n\}}$ is sequentially retractive. Hence there exists $n \in \mathbb{N}$ such that $(Fh_j)_{j\in\mathbb{N}}$ is in fact a sequence in $A_{\{\omega,n\}}$ and converges to 0 in this space. Now (2) implies that $(h_j)_{j\in\mathbb{N}}$ converges to zero in $A_{\{\omega,n\}}$ and consequently in $A_{\{\omega\}}(\mathbb{C},\mathbb{R})$. □

4.5. COROLLARY. *Let ω be a weight function and let $T_\mu \neq 0$ be an $\{\omega\}$-ultradifferentiable operator. Then the Fourier-Laplace transform $\hat{\mu}$ of μ is slowly decreasing if and only if for each $a, b \in \mathbb{R}$ with $a < b$ the convolution operator*

$$T_{\mu,[a,b]} : \mathcal{E}_{\{\omega\}}[a,b] \to \mathcal{E}_{\{\omega\}}[a,b]$$

is surjective.

PROOF. Since T_μ commutes with translations, it is enough to prove the corollary for $[a,b] = [-R,R]$ and each $R > 0$. Since $\mathcal{E}_{\{\omega\}}[-R,R]$ is a (DFN)-space the strong dual of which is isomorphic to $A_{\{\omega,R\}}$ via Fourier-Laplace transform (by Remark 4.3) and since $\mathcal{F} \circ T^t_{\mu,[-R,R]} = M_{\hat{\mu}} \circ \mathcal{F}$, the corollary follows from the closed range theorem (see, e.g., Meise and Vogt [**18**], 26.3). □

4.6. LEMMA. *Let ω be a weight function and assume that $F \in A_{\{\omega\}}$ is $\{\omega\}$-slowly decreasing. Then there exists a weight function σ satisfying $\sigma = o(\omega)$ such that $F \in A_{(\sigma)}$. Moreover, there exist ε_0, C_0, and $D > 0$ such that each connected component S of the set*

$$S_\sigma(F, \varepsilon_0, C_0) := \{z \in \mathbb{C} : |F(z)| < \varepsilon_0 \exp(-C_0\sigma(z))\}$$

satisfies

$$\operatorname{diam} S \le D \inf_{z\in S} \sigma(z) + D \text{ and } \sup_{z\in S} \omega(z) \le D \inf_{z\in S} \omega(z) + D.$$

PROOF. By Proposition 3.2 there exists a weight function σ_1 satisfying $\sigma_1 = o(\omega)$ such that $F \in A_{(\sigma_1)}(\mathbb{C},\mathbb{R})$ and F is (σ_1)-slowly decreasing. From Braun, Meise, and Taylor [**6**], Lemma 1.7, we get the existence of a weight function σ_2 satisfying $\sigma_2 = o(\omega)$ and $F \in A_{(\sigma_2)}$. Hence we can choose a weight function σ which satisfies $\max(\sigma_1, \sigma_2) \le \sigma$ and $\sigma = o(\omega)$. Then $F \in A_{(\sigma)}$ and F is (σ)-slowly decreasing. Thus F satisfies the hypotheses of [**5**], Lemma 4.2. Therefore, [**5**], Lemma 4.3, implies the existence of positive numbers ε_0, C_0, and D such that for each component S of $S_\sigma(F, \varepsilon_0, C_0)$ we have $\operatorname{diam} S \le D \inf_{z\in S} \sigma(z) + D$. To show that we also have

$$\sup_{z\in S} \omega(z) \le D \inf_{z\in S} \omega(z) + D \tag{4.1}$$

for each component S of $S_\sigma(F,\varepsilon_0,C_0)$, provided that $D>0$ is large enough, we remark that the following was shown in the proof of [**5**], Lemma 4.3: There exist $m\in\mathbb{N}$ and $R_0\geq 1$ such that for each $z_0\in S_\sigma(F,\varepsilon_0,C_0)$ satisfying $|z_0|\geq R_0$ the connected component S of $S_\sigma(F,\varepsilon_0,C_0)$ which contains z_0 satifies

$$\operatorname{diam} S\leq 4m\sigma(z_0).$$

It is no restriction to assume that R_0 is so large that from 2.1 (α) and (β) and $\sigma=o(\omega)$ we get the existence of L and $K_0\geq 1$ such that

$$\sigma(t)\leq\omega(t)\leq Lt \text{ and } \omega(2t)\leq K_0\omega(t),\ t\geq R_0.$$

Next we fix a component S of $S_\sigma(F,\varepsilon_0,C_0)$ such that $S\cap(\mathbb{C}\setminus B(0,R_0))\neq\emptyset$ and we choose $z_0\in S$ with $|z_0|\geq R_0$ as well as $z_1,z_2\in\overline{S}$ such that

$$\inf_{z\in S}\omega(z)=\omega(z_1) \text{ and } \sup_{z\in S}\omega(z)=\omega(z_2).$$

In the proof of [**5**], Lemma 4.3, it was shown that then $|z_0|\leq 2|z_1|$. By our choices, this implies

$$\begin{aligned}|z_2|&\leq|z_2-z_1|+|z_1|\leq\operatorname{diam}S+|z_1|\leq 4m\sigma(z_0)+|z_1|\\&\leq 4m\omega(2|z_1|)+|z_1|\leq 4mK_0\omega(z_1)+|z_1|\leq(4mLK_0+1)|z_1|.\end{aligned}$$

Since ω satisfies 2.1 (α), this estimate implies the existence of $K_1\geq 1$ such that

$$\sup_{z\in S}\omega(z)=\omega(z_2)\leq\omega((4mLK_0+1)|z_1|)\leq K_1\omega(z_1)=K_1\inf_{z\in S}\omega(z).$$

Since there are only finitely many components S of $S_\sigma(F,\varepsilon_0,C_0)$ which are contained in $B(0,R_0)$, we proved (4.1), provided that we choose $D>0$ large enough. □

4.7. Proposition. *Let ω be a weight function and let $F\in A_{\{\omega\}}$ be $\{\omega\}$-slowly decreasing. For $R>0$ denote by $q_R: A_{\{\omega,R\}}\to A_{\{\omega,R\}}/FA_{\{\omega,R\}}$ and by $q: A_{\{\omega\}}(\mathbb{C},\mathbb{R})\to A_{\{\omega\}}(\mathbb{C},\mathbb{R})/FA_{\{\omega\}}(\mathbb{C},\mathbb{R})$ the corresponding quotient maps. Let $J_R: A_{\{\omega,R\}}\to A_{\{\omega\}}(\mathbb{C},\mathbb{R})$ be the inclusion map. Then for each $R>0$ the map J_R induces a continuous linear injection $j_R: A_{\{\omega,R\}}/FA_{\{\omega,R\}}\to A_{\{\omega\}}(\mathbb{C},\mathbb{R})/FA_{\{\omega\}}(\mathbb{C},\mathbb{R})$ which satisfies $j_R\circ q_R=J_R\circ q$.*

Proof. Fix $R>0$ and note that $FA_{\{\omega,R\}}$ is a closed linear subspace of $A_{\{\omega,R\}}$ by Proposition 4.4, while $FA_{\{\omega\}}(\mathbb{C},\mathbb{R})$ is closed in $A_{\{\omega\}}(\mathbb{C},\mathbb{R})$ by Proposition 2.4. Next note that the result holds trivially if F has only finitely many zeros. Therefore, we assume from now on that $V(F):=\{a\in\mathbb{C}: F(a)=0\}$ is an infinite set. Then we choose a weight function σ and positive numbers ε_0, C_0, and D according to Lemma 4.6 and we label the connected components S of $S_\sigma(F,\varepsilon_0,C_0)$ which satisfy $S\cap V(F)\neq\emptyset$ in such a way that the sequence β, defined by

$$\beta_j:=\sup_{z\in S_j}\omega(z),\ j\in\mathbb{N}.$$

is increasing. Also, we define the sequence α by

$$\alpha_j:=\sup_{z\in S_j}|\operatorname{Im} z|,\ j\in\mathbb{N},$$

Then we define the sequence $\mathbb{E}=(E_j)_{j\in\mathbb{N}}$ by

$$E_j:=\prod_{b\in S_j\cap V(F)}\mathbb{C}^{\operatorname{ord}(F,b)},\ j\in\mathbb{N},$$

and we let

$$\varrho_j:\ H^\infty(S_j)\to E_j,\ \varrho_j(f):=\left(\left(\frac{1}{k!}f^{(k)}(b)\right)_{0\leq k<\operatorname{ord}(F,b)}\right)_{b\in S_j\cap V(F)}.$$

We endow E_j with the quotient norm

$$\|\varrho_j(g)\|:=\ \inf\{\|h\|_{H^\infty(S_j)}:\ \varrho_j(h)=\varrho_j(g)\},\ g\in H^\infty(S_j).$$

Then ϱ_j is linear, continuous, and surjective. If $f\in A_{\{\omega,R\}}$, then for each $m\in\mathbb{N}$ there exists $C_m>0$ such that

$$|f(z)|\leq C_m\exp(R|\operatorname{Im} z|+\frac{1}{m}\omega(z)),\ z\in\mathbb{C}.$$

Obviously, this implies that for each $m \in \mathbb{N}$ and each $j \in \mathbb{N}$ we have

$$\|f|_{S_j}\|_{H^\infty(S_j)} \le C_m \exp(R\alpha_j + \frac{1}{m}\beta_j).$$

Hence the map

$$\varrho^R : A_{\{\omega,R\}} \to K(\mathbb{E}, R, \alpha, \beta),\ \varrho^R(f) := (\varrho_j(f|_{S_j}))_{j\in\mathbb{N}}$$

is well-defined, linear, and continuous. By the definition of the spaces $A_{\{\omega\}}(\mathbb{C},\mathbb{R}) = \mathrm{ind}_{n\to} A_{\{\omega,n\}}$ and $K(\mathbb{E},\alpha,\beta) = \mathrm{ind}_{n\to} K(\mathbb{E}, n, \alpha, \beta)$ also the map

$$\varrho :\ A_{\{\omega\}}(\mathbb{C},\mathbb{R}) \to K(\mathbb{E},\alpha,\beta),\ \varrho(f) := (\varrho_j(f|_{S_j}))_{j\in\mathbb{N}}$$

is well-defined, linear, and continuous.

Next we claim that $\ker \varrho^R = FA_{\{\omega,R\}}$ and $\ker \varrho = FA_{\{\omega\}}(\mathbb{C},\mathbb{R})$. Obviously, $FA_{\{\omega,R\}}$ is contained in $\ker \varrho^R$. To prove the converse inclusion, fix $g \in \ker \varrho^R$. Then g/F is in $H(\mathbb{C})$. By Proposition 4.4 this implies that $g \in FA_{\{\omega,R\}}$. Since $A_{\{\omega\}}(\mathbb{C},\mathbb{R}) = \mathrm{ind}_{n\to} A_{\{\omega,n\}}$, this implies $\ker \varrho = FA_{\{\omega\}}(\mathbb{C},\mathbb{R})$.

To show that ϱ^R is surjective, fix $y = (y_j)_{j\in\mathbb{N}}$ in $K(\mathbb{E}, R, \alpha, \beta)$. By the definition of the norm in E_j, we can choose $\lambda_j \in H^\infty(S_j)$ satisfying

$$\varrho_j(\lambda_j) = y_j \text{ and } \|y_j\|_{H^\infty(S_j)} \le 2\|y_j\|_j,\ j \in \mathbb{N}.$$

Then we define

$$\lambda :\ S_\sigma(F, \varepsilon_0, C_0) \to \mathbb{C},\ \lambda(z) = \lambda_j(z) \text{ if } z \in S_j \text{ and } \lambda(z) = 0 \text{ if } z \in \mathbb{C} \setminus \bigcup_{j=1}^{\infty} S_j$$

and we claim that for each $m \in \mathbb{N}$ there exist $p \in \mathbb{N}$ and $C_m > 0$ such that

$$\sup_{z\in\mathbb{C}} |\lambda(z)| \exp(-R|\operatorname{Im} z| - \frac{1}{m}\omega(z)) \le C_m \|y\|_{R,p}. \tag{4.2}$$

To prove this, fix $m \in \mathbb{N}$ and choose $p \ge 2Dm$. Since $\sigma = o(\omega)$, there exists $C_m > 0$ such that

$$2\exp(RD\sigma(t) + (R+1)D) \le C_m \exp(\frac{D}{p}\omega(t)) \text{ for } t \ge 0.$$

Then we get for each $j \in \mathbb{N}$ and each $z \in S_j$ the following estimate

$$\begin{aligned}
|\lambda_j(z)| &\le 2\|y_j\|_j \le 2\|y\|_{R,p} \exp(R\beta_j + \frac{1}{p}\alpha_j) \\
&\le 2\|y\|_{R,p} \exp(R|\operatorname{Im} z| + R \operatorname{diam} S_j + \frac{1}{p}(D\omega(z) + D)) \\
&\le 2\|y\|_{R,p} \exp(R|\operatorname{Im} z| + RD\sigma(z) + (R+1)D + \frac{D}{p}\omega(z)) \\
&\le C_m \|y\|_{R,p} \exp(R|\operatorname{Im} z| + \frac{1}{m}\omega(z)),
\end{aligned}$$

which implies (4.2).

Next note that by Lemma 4.6 there exists $B > 0$ such that

$$|F(z)| \le B\exp(B\sigma(z)),\ z \in \mathbb{C}.$$

Hence it follows from the proof of [**5**], Lemma 4.7, that there exist $\varepsilon_1, C_1 > 0, \chi \in C^\infty(\mathbb{C})$ and $A_0, B_0 > 0$ such that

(4.3)

$$0 \le \chi \le 1,\ \chi \equiv 1 \text{ on } S_\sigma(F, \varepsilon_1, C_1),\ \operatorname{Supp} \chi \subset S_\sigma(F, \varepsilon_0, C_0), \text{ and } \left|\frac{\partial \chi}{\partial \overline{z}}(z)\right| \le A_0 \exp(B_0\sigma(z)),\ z \in \mathbb{C}.$$

Now define

$$v := -\frac{1}{F}\frac{\partial}{\partial \overline{z}}(\chi\lambda) = -\frac{1}{F}\frac{\partial \chi}{\partial \overline{z}}\lambda$$

and note that v is in $C^\infty(\mathbb{C})$ and vanishes on $S_\sigma(F, \varepsilon_1, C_1)$. Moreover, we get from (4.2) and (4.3) that for each $m \in \mathbb{N}$ there exist $p \in \mathbb{N}$ and $C_m > 0$ such that for each $z \in \mathbb{C}$ we have

$$|v(z)| \le \frac{1}{\varepsilon_1} A_0 C_m \|y\|_{R,p} \exp(R|\operatorname{Im} z| + \frac{1}{m}\omega(z) + (B_0 + C_1)\sigma(z)).$$

Using Lemma 1.7 of Braun, Meise, and Taylor [**6**], we get the existence of a weight function $\tau \geq \sigma$ and of $A_1 > 0$ such that

$$|v(z)| \leq A_1 \exp(R|\operatorname{Im} z| + \tau(z)),\ z \in C.$$

Since τ satisfies condition 2.1 (γ), this estimate implies

$$\int_{\mathbb{C}} (|v(z)| \exp(-R|\operatorname{Im} z| - 2\tau(z)))^2 dz < \infty.$$

By Hörmander [**12**], Theorem 4.4.2, there exists $g \in L^2_{\mathrm{loc}}(\mathbb{C})$ which satisfies $\frac{\partial g}{\partial \bar{z}} = v$ and

$$\int (|g(z)| \exp(-R|\operatorname{Im} z| - 2\tau(z) - \log(1 + |z|^2)))^2 dz < \infty. \tag{4.4}$$

Since v is a C^∞-function on C and since $\frac{\partial}{\partial \bar{z}}$ is elliptic, g belongs to $C^\infty(\mathbb{C})$. By the choice of v, we now get that $f := \chi\lambda + gF \in C^\infty(\mathbb{C})$ and $\frac{\partial f}{\partial \bar{z}} = 0$, i.e., $f \in H(\mathbb{C})$. Now the estimates for λ in (4.2), for g in (4.4), and for F imply a weighted L^2-estimate for f which can be converted by standard arguments to a sup-estimate which shows that f is in fact in $A_{\{\omega,R\}}$. By the definition of f and λ, we get

$$\varrho(f) = (\varrho_j(f|_{S_j}))_{j\in\mathbb{N}} = (\varrho_j(\lambda_j))_{j\in\mathbb{N}} = y.$$

Hence we proved that $\varrho^R : A_{\{\omega,R\}} \to K(\mathbb{E}, R, \alpha, \beta)$ is surjective. Since $K(\mathbb{E}, \alpha, \beta) = \operatorname{ind}_{n\to} K(\mathbb{E}, n, \alpha, \beta)$ and $A_{\{\omega\}}(\mathbb{C}, \mathbb{R}) = \operatorname{ind}_{n\to} A_{\{\omega,R\}}$ we also get that $\varrho : A_{\{\omega\}}(\mathbb{C}, \mathbb{R}) \to K(\mathbb{E}, \alpha, \beta)$ is surjective. Since $\ker \varrho^R = FA_{\{\omega,R\}}$ and $\ker \varrho = FA_{\{\omega\}}(\mathbb{C}, \mathbb{R})$, classical open mapping theorems show that we can identify $A_{\{\omega,R\}}/FA_{\{\omega,R\}}$ with $K(\mathbb{E}, R, \alpha, \beta)$ and $A_{\{\omega\}}(\mathbb{C}, \mathbb{R})/A_{\{\omega\}}(\mathbb{C}, \mathbb{R})$ with $K(\mathbb{E}, \alpha, \beta)$. If we do this ϱ and ϱ^R are the corresponding quotient maps. Now note that by the definition of the maps ϱ^R and ϱ, the following diagram, where $j_R : K(\mathbb{E}, R, \alpha, \beta) \to K(\mathbb{E}, \alpha, \beta)$ denote the inclusion, is commutative

$$\begin{array}{ccc} A_{\{\omega,R\}} & \xrightarrow{\varrho^R} & K(\mathbb{E}, R, \alpha, \beta) \\ \downarrow J_R & & \downarrow j_R \\ A_{\{\omega\}}(\mathbb{C}, \mathbb{R}) & \xrightarrow{\varrho} & K(\mathbb{E}, \alpha, \beta). \end{array}$$

Thus the proof is complete. □

4.8. Remark. Under the hypotheses of Proposition 4.7 we proved that for each $R > 0$ the space $A_{\{\omega,R\}}/FA_{\{\omega,R\}}$ is topologically isomorphic to the Fréchet space $K(\mathbb{E}, R, \alpha, \beta)$, as the proof of 4.7 shows.

4.9. Corollary. *Let ω be a weight function, let F be $\{\omega\}$-slowly decreasing, and assume that $\lim_{|a|\to\infty,\, a\in V(F)} |\operatorname{Im} a|/\omega(a) = 0$. Then for each $R > 0$ the map j_R, defined in Proposition 4.7, $j_R : A_{\{\omega,R\}}/A_{\{\omega,R\}} \to A_{\{\omega\}}(\mathbb{C}, \mathbb{R})/FA_{\{\omega\}}(\mathbb{C}, \mathbb{R})$ is surjective and hence a linear topological isomorphism.*

Proof. From the proof of Proposition 4.7 and the open mapping theorem it follows that we only have to show that $K(\mathbb{E}, \alpha, \beta) \subset K(\mathbb{E}, R, \alpha, \beta)$. In fact we will show that $K(\mathbb{E}, \alpha, \beta) \subset K(\mathbb{E}, 0, \alpha, \beta)$. To do so we fix $y \in K(\mathbb{E}, \alpha, \beta)$. Then there exists $n \in \mathbb{N}$ such that for each $m \in \mathbb{N}$ there exists $C_m > 0$ such that for each $j \in \mathbb{N}$

$$\|y_j\|_j \leq C_m \exp(n\alpha_j + \frac{1}{2m}\beta_j).$$

Next choose a weight function $\sigma = o(\omega)$ so that the assertions of Lemma 4.6 hold and for each $j \in \mathbb{N}$ choose $a_j \in S_j$. (If $V(F)$ is finite, there is nothing to prove). Then we get from Lemma 4.6

$$\alpha_j = \sup_{z\in S_j} |\operatorname{Im} z| \leq |\operatorname{Im} a_j| + \operatorname{diam} S_j \leq |\operatorname{Im} a_j| + D\sigma(a_j) + D.$$

Since $\lim_{|a|\to\infty,\, a\in V(F)} |\operatorname{Im} a|/\omega(a) = 0$, for each $m \in \mathbb{N}$ there exists $D_m > 0$ such that

$$|\operatorname{Im} a| \leq \frac{1}{4mn}\omega(a) + D_m,\ a \in V(F)$$

and we can choose $K_m > 0$ such that

$$D\sigma(t) + D \leq \frac{1}{4mn}\omega(t) + K_m,\ t \geq 0.$$

Then we get

$$n\alpha_j + \frac{1}{2m}\beta_j \le \frac{1}{4m}\omega(a_j) + \frac{1}{4n}\omega(a_j) + nD_m + K_m \le \frac{1}{2m}\beta_j + nD_m + K_m$$

and hence

$$\|y_j\|_j \le C_m \exp(nD_m + K_m)\exp(\frac{1}{m}\beta_j),\ j \in \mathbb{N}.$$

This shows that y is in fact in $K(\mathbb{E}, 0, \alpha, \beta)$. □

4.10. PROPOSITION. *Let ω be a weight function and let $T_\mu \neq 0$ be an $\{\omega\}$-ultradifferentiable operator. If the Fourier-Laplace transform $\hat{\mu}$ of μ is slowly decreasing, then for each $a, b \in \mathbb{R}$ with $a < b$ the following assertions hold:*

(1) *$\ker T_{\mu,[a,b]}$ is isomorphic to $\Lambda_0(\gamma)'_b$, where $\gamma = (\omega(a_j))_{j\in\mathbb{N}}$ and where $(a_j)_{j\in\mathbb{N}}$ counts the zeros of $\hat{\mu}$ with multiplicities in such a way that $(\omega(a_j))_{j\in\mathbb{N}}$ is increasing.*
(2) *If $\lim_{|z|\to\infty, z\in V(\hat{\mu})} |\mathrm{Im}(z)|/\omega(z) = 0$ then the map $\varrho_{[a,b]} : \ker T_\mu \to \ker T_{\mu,[a,b]}$, $\varrho_{[a,b]}(f) := f|_{[a,b]}$, is an isomorphism.*

PROOF. Since T_μ commutes with translations, it suffices to consider intervals of the form $[-R, R]$ for $R > 0$. By Proposition 4.7 the short sequence

$$0 \to A_{\{\omega,R\}} \overset{M_{\hat{\mu}}}{\to} A_{\{\omega,R\}} \overset{q_R}{\to} A_{\{\omega,R\}}/\hat{\mu}A_{\{\omega,R\}} \to 0$$

of Fréchet-Schwartz spaces and continuous linear maps is exact. Hence its dual sequence is exact, too, by Meise and Vogt [**18**], Proposition 26.24. Since the spaces $\mathcal{E}_{\{\omega\}}[-R, R]$ are reflexive, it follows from Remark 4.3 and $\hat{\mu}A_{\{\omega,R\}} = imM_{\hat{\mu}} = (\ker M^t_\mu)^\perp$ that up to Fourier-Laplace transform the dual sequence can be identified with

$$0 \to \ker T_{\mu,[-R,R]} \to \mathcal{E}_{\{\omega\}}[-R, R] \overset{T_{\mu,[-R,R]}}{\to} \mathcal{E}_{\{\omega\}}[-R, R] \to 0.$$

Hence we get from Remark 4.8 that $\ker T_{\mu,[-R,R]}$ is isomorphic to $(A_{\{\omega,R\}}/\hat{\mu}A_{\{\omega,R\}})' \cong (K(\mathbb{E}, R, \alpha, \beta))'$. Now note that $K(\mathbb{E}, R, \alpha, \beta)$ is a nuclear Fréchet space which is isomorphic to $K(\mathbb{E}, 0, \alpha, \beta) = \Lambda_0(\mathbb{E}, \beta)$ by an obvious diagonal transform. Now (1) follows from Meise [**14**], Proposition 1.4, by the definition of the sequence $\mathbb{E}$ and the diameter estimates for the sets S_j in the proof of Proposition 4.7.

To prove (2), note that by the arguments in Meise [**15**], 3.4, we have $(\ker T_\mu)' \cong \mathcal{E}'_{\{\omega\}}(\mathbb{R})/(\ker T_\mu)^\perp \cong A_{\{\omega\}}(\mathbb{C}, \mathbb{R})/\hat{\mu}A_{\{\omega\}}(\mathbb{C}, \mathbb{R})$ via Fourier-Laplace transform. Hence for each $R > 0$ we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & \ker T_\mu & \to & \mathcal{E}_{\{\omega\}}(\mathbb{R}) & \overset{T_\mu}{\to} & \mathcal{E}_{\{\omega\}}(\mathbb{R}) & \to & 0 \\ & \downarrow \varrho_{[-R,R]} & & & \downarrow \varrho_{[-R,R]} & & & \downarrow \varrho_{[-R,R]} & \\ 0 & \to & \ker T_{\mu,[-R,R]} & \to & \mathcal{E}_{\{\omega\}}[-R, R] & \overset{T_{\mu,[-R,R]}}{\to} & \mathcal{E}_{\{\omega\}}[-R, R] & \to & 0\ . \end{array}$$

If we dualize it and apply the Fourier-Laplace transform, the dual map of $\varrho_{[-R,R]} : \ker T_\mu \to \ker T_{\mu,[-R,R]}$ corresponds to the map $j_R : A_{\{\omega,R\}}/\hat{\mu}A_{\{\omega,R\}} \to A_{\{\omega\}}(\mathbb{C}, \mathbb{R})/\hat{\mu}A_{\{\omega\}}(\mathbb{C}, \mathbb{R})$, defined in Proposition 4.7. As we showed in the proof of 4.7, j_R becomes the inclusion of $K(\mathbb{E}, R, \alpha, \beta)$ in $K(\mathbb{E}, \alpha, \beta)$, if we identify the corresponding quotient spaces with these vector-valued sequence spaces. Since $\lim_{|z|\to\infty, z\in V(\hat{\mu})} |\mathrm{Im}\, z|/\omega(z) = 0$ holds by hypothesis, it follows easily that

$$K(\mathbb{E}, R, \alpha, \beta) = K(\mathbb{E}, 0, \alpha, \beta) = K(\mathbb{E}, \alpha, \beta)$$

as sets but also as locally convex spaces. Therefore, j_R is a linear topological isomorphism. Next note that $\ker T_{\mu,[-R,R]}$ is reflexive as closed subspace of a (DFS)-space. To see that also $\ker T_\mu$ is reflexive, we argue as follows: By Theorem 3.10, the present hypotheses imply that $T_\mu : \mathcal{E}_{\{\omega\}}(\mathbb{R}) \to \mathcal{E}_{\{\omega\}}(\mathbb{R})$ is surjective. Since $\mathrm{Proj}^1\, \mathcal{E}_{\{\omega\}}(\mathbb{R}) = 0$ by Meyer [**21**], Theorem 3.7, (or Rösner [**26**], Satz 3.25) it follows from the long exact sequence theorem (see Wengenroth [**31**], Corollary 3.1.5) that $\mathrm{Proj}^1 \ker T_\mu = 0$. Hence $\ker T_\mu$ is ultrabornological by Wengenroth [**31**], Theorem 3.3.4. Therefore, the semi-reflexive space $\ker T_\mu$ is reflexive. Hence $\varrho_{[-R,R]} : \ker T_\mu \to \ker T_{\mu,[-R,R]}$ is an isomorphism, too. □

4.11. REMARK. If ω is non-quasianalytic and T_μ is a convolution operator on $\mathcal{E}_{\{\omega\}}(\mathbb{R})$ which is surjective, then Theorem 4.2 in Braun, Meise, and Vogt [**7**] shows that T_μ admits a continuous linear right inverse if and only if $\lim_{|a|\to\infty, a\in V(\hat{\mu})} |\mathrm{Im}\, a|/\omega(a) = 0$. In the quasianalytic case, so far we only have the necessity of this condition by Proposition 3.11. For $\{\omega\}$-ultradifferential operators, the sufficiency of this condition will follow from Proposition 4.10 as soon as one knows that for some $R > 0$ the operator $T_{\mu,[-R,R]}$ admits a continuous linear right inverse. Because then one can apply the formal arguments that were used in [**5**], Corollary 4.11, in the Beurling case and which were first applied by Domanski and Vogt [**8**], Theorem 4.7, in the real-analytic case. However, it is still open, whether $T_{\mu,[-R,R]}$ admits a continuous linear right inverse. The main difficulty is that the linear topological structure of $\mathcal{E}_{\{\omega\}}[-R, R]$ or equivalently of $A_{\{\omega,R\}}$ is not known.

Problem: Is $A_{\{\omega,R\}}$ isomorphic to a power series space of finite type?

Remark. It follows easily from Meise and Taylor [**16**], Lemma 1.10, that $A_{\{\omega,R\}}$ has the property ($\underline{\text{DN}}$). If ω is non-quasianalytic then [**16**], Corollary 6.4, in connection with [**18**], Proposition 29.18, shows that $A_{\{\omega,R\}}$ is isomorphic to a power series space of finite type.

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