Dynamics of linear operators on spaces of ultradistributions

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DISCRETE DYNAMICAL SYSTEMS

Let X be a topological space, $T : X \to X$ a continuous map, and $x \in X$. The **orbit** of x by T is

$$Orb(x, T) := \{x, Tx, T^2x, \dots\}.$$

T is called **topologically transitive** if for every pair of non-empty open subsets $U, V \subseteq X$ there is $n \in \mathbb{N}$ such that

$$T^n(U) \cap V \neq \emptyset.$$

In other words, if there is $x \in U$ whose orbit intersects V.

• If X is separable, Baire, and has no isolated points, then

T is transitive $\Leftrightarrow$ T admits a dense orbit, i.e., $\exists x \in X$:

$$\overline{Orb(x, T)} = X.$$

T is called **chaotic** in the sense of Devaney if it is topologically transitive and the set of periodic points of T is dense in X.

GENERAL CONTEXT

$T : E \to E$ is a continuous and linear operator on a complex separable locally convex space E. For example T is a convolution operator or a partial differential operator on the space E of smooth functions.

T is **hypercyclic** if there is $x \in E$ such that $\overline{Orb(x,T)} = E$. In this case x is called a hypercyclic vector for T.

- Read, 1988:
  There are operators T on ℓ^1 such that every non-zero vector is hypercyclic for T. This means that T admits no non-trivial invariant closed subset.

- No finite dimensional space E admits hypercyclic operators (Kitai, 1982).

Theorem 1 If T is transitive, then the transpose operator

$$T^t : E' \to E'$$

has no eigenvalues.

CLASSICAL RESULTS

Birkhoff, 1929:

The translation operator $T_a : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$, $(T_a f)(z) := f(z+a)$, on the Fréchet space $H(\mathbb{C})$ of entire functions is hypercyclic if $a \neq 0$.

MacLane, 1952:

The derivative operator $Df = f'$ is hypercyclic on $\mathcal{H}(\mathbb{C})$.

Godefroy, Shapiro, 1991:

Let T be an operator on $\mathcal{H}(\mathbb{C}^N)$ which commutes with translations (or, equivalently, with the operators of partial differentiation). If T is not a scalar multiple of the identity, then T is chaotic.

Rolewicz, 1969:

Let $T := \lambda B : l^p \to l^p$, $1 \leq p < \infty$, $|\lambda| > 1$,

$$T(x_1, x_2, \dots) = (\lambda x_2, \lambda x_3, \dots)$$

be the backward shift. Then T is hypercyclic (and chaotic).

Every separable infinite dimensional Banach space admits hypercyclic operators (Ansari, 1997/Bernal 1999). This results solves a problem of Rolewicz.

The theorem is also true for Fréchet spaces (Bonet, Peris, 1998).

However, there are infinite dimensional separable Banach spaces which admit no chaotic operator. This was proved by Bonet, F. Martínez-Giménez, Peris, 2001.

Theorem 2 Let $G \subset \mathbb{C}$ be an open connected set, and let $P(D)$ be a non-constant polynomial. The following conditions are equivalent:

(1) $P(D)$ is chaotic on $H(G)$

(2) $P(D)$ is hypercyclic on $H(G)$

(3) G is simply connected.

WEIGHT FUNCTION

A **weight function** is an increasing continuous function $\omega : [0, \infty[\longrightarrow [0, \infty[$ with the following properties:

(α) $\omega(2t) \leq L(\omega(t) + 1)$ for some $L \geq 1$ for all $t \geq 0$,

(β) $\omega(t) \leq Lt + L$ for some $L \geq 1$ for all $t \geq 0$,

(γ) $\log(t) = o(\omega(t))$ as t tends to ∞,

(δ) $\varphi : t \to \omega(e^t)$ is convex.

• The radial extension of ω is $\omega(z) = \omega(|z|)$ where $|z| = \sum_{j=1}^{N} |z_j|$.

• The function $\varphi^* : [0, \infty[\to \mathbb{R}$ defined by $\varphi^*(s) := \sup\limits_{t \geq 0}\{st - \varphi(t)\}$ is called the **Young conjugate** of φ.

• If $\displaystyle\int_1^\infty \frac{\omega(t)}{t^2}\, dt < \infty$ the weight ω is called **non-quasianalytic**. Otherwise, it is called **quasianalytic**

EXAMPLES

The following functions are weight functions (possibly after a change on the interval $[-\delta, \delta]$, for suitable $\delta > 0$).

(1) $\omega(t) := |t|^{\alpha}$, $0 < \alpha < 1$.

(2) $\omega(t) := \log(1 + |t|)^{\beta}$, $\beta > 1$.

(3) $\omega(t) := |t| \, (\log(1 + |t|))^{-\beta}$, $\beta > 0$.

(4) $\omega(t) := |t|$.

The weights in (1) permit us to define the Gevrey classes.

The weight function in (3) is quasianalytic for $\beta \in]0, 1]$ and non-quasianalytic for $\beta > 1$.

The weight function in (4) is also quasianalytic.

FUNCTIONS AND ULTRADISTRIBUTIONS

Let ω be a weight function and let $K \subset \mathbb{R}^N$ be a compact set. For $\lambda > 0$ and $f \in \mathcal{C}^\infty(K)$ we let

$$\| f \|_{K,\lambda} := \sup_{x \in K} \sup_{\alpha \in \mathbb{N}_0^N} |f^{(\alpha)}(x)| \exp\left(-\lambda \varphi^*\left(\frac{|\alpha|}{\lambda}\right)\right).$$

$$\mathcal{E}_{(\omega)}(G) = \{f \in \mathcal{C}^\infty : \| f \|_{K,\lambda} < \infty \text{ for all compact } K \subset G \text{ and for all } \lambda > 0\}.$$

This is the space of **ultradifferentiable functions of Beurling type**

• If ω is non-quasianalitic, we define

$$\mathcal{D}_{(\omega)}(K) := \{f \in \mathcal{D}(K) : \| f \|_{K,\lambda} < \infty \quad \text{for every } \lambda > 0\},$$

For a fundamental sequence $(\mathrm{K}_j)_{j \in \mathbb{N}}$ of compact subsets of G we let

$$\mathcal{D}_{(\omega)}(G) := \operatorname*{ind}_{j \to} \mathcal{D}_{(\omega)}(\mathrm{K_j}).$$

The elements of $\mathcal{D}'_{(\omega)}(G)$ are called **ultradistributions of Beurling type on** G.

$\mathcal{E}_{(\omega)}(G)$ is Fréchet space and $\mathcal{D}_{(\omega)}(G)$ is a strict (LF)-space.

• Let ω be a given weight function. For a compact subset K of $\mathbb{R}^N$ and $m \in \mathbb{N}$ denote by $C^\infty(K)$ the space of all C^∞-Whitney jets on K and define

$$\mathcal{E}^m_{\{\omega\}}(K) := \{f \in C^\infty(K) : \|f\|_{K,m} < \infty\}.$$

where

$$\|f\|_{K,m} := \sup_{x\in K} \sup_{\alpha\in\mathbb{N}_0^N} \left|f^{(\alpha)}(x)\right| \exp\left(-\frac{1}{m}\phi^*(m\,|\alpha|)\right).$$

• For an open set G in $\mathbb{R}^N$, the space of all ω-***ultradifferentiable functions of Roumieu type* on** G is defined as

$$\mathcal{E}_{\{\omega\}}(G) := \{\quad f \in C^\infty(G) : \text{For each } K \subset G \text{ compact there is } m \in \mathbb{N} \text{ so that } \|f\|_{K,m} < \infty\}.$$

It is endowed with the topology given by the representation

$$\mathcal{E}_{\{\omega\}}(G) = \operatorname*{proj}_{\leftarrow K} \ \operatorname*{ind}_{m\rightarrow} \mathcal{E}^m_{\{\omega\}}(K),$$

where K runs over all compact subsets of G. $\mathcal{E}_{\{\omega\}}(G)$ is a countable projective limit of (DFN)-spaces which is complete.

It is easy to define $\mathcal{D}_{\{\omega\}}(G)$ and $\mathcal{D}'_{\{\omega\}}(G)$ if ω is non-quasyanalytic.

• If ω is the weight function $\omega(t) = |t|$, then the space $\mathcal{E}_{\{\omega\}}(G)$ coincides with the space $\mathcal{A}(G)$ of all real analytic functions on G.

CONVOLUTION OPERATORS

• Suppose $\mu \in \mathcal{E}_*(\mathbb{R}^N)'$ with $G - supp\mu \subset G$. The ultradistribution μ induces a *convolution operator*,

$$T_\mu : \mathcal{E}_*(G) \to \mathcal{E}_*(G), \quad T_\mu(\phi) := \mu * \phi : x \mapsto \mu(\phi(x - \cdot)),$$

which is linear and continuous.

• The adjoint operator T_μ^t of T_μ is given by

$$T_\mu^t : \mathcal{E}_*(G)' \to \mathcal{E}_*(G)', \quad T_\mu^t(\nu) := \check{\mu} * \nu,$$

where

$$(\check{\mu} * \nu)(\phi) = \nu(\mu * \phi),$$

while $\check{\mu}(\phi) := \mu(\check{\phi})$ and $\check{\phi}(x) = \phi(-x)$ for $\phi \in \mathcal{E}_*(\mathbb{R}^N)$, $x \in \mathbb{R}^N$.

• In case ω is non-quasianalytic, the ultradistribution $\mu \in \mathcal{E}_*(\mathbb{R}^N)'$ defines also a convolution operator

$$T_\mu : \mathcal{D}'_*(G) \to \mathcal{D}'_*(G)$$

Theorem 3 Let G be an open subset of $\mathbb{R}^d$. Let $\mu \in \mathcal{E}_*(\mathbb{R}^N)'$ satisfy $G - supp\mu \subset G$. If μ is not a scalar multiple of the Dirac measure, then T_μ is hypercyclic and chaotic on $\mathcal{E}_*(G)$. If the weight ω is non-quasianalytic, the convolution operator is also hypercyclic and chaotic on $\mathcal{D}'_*(G)$.

The proof of this result depends on the theorem of Godefroy and Shapiro, and the following results.

Theorem 4 (Heinrich, Meise) Let ω be a weight function and let G be an open subset of $\mathbb{R}^d$. Then:

(1) $H(\mathbb{C}^N)$ is dense in $\mathcal{E}_{\{\omega\}}(G)$.

(2) $H(\mathbb{C}^N)$ is dense in $\mathcal{E}_{(\omega)}(G)$ if $\omega(t) = o(t)$ as t goes to infinity.

Theorem 5 Let T be a continuous and linear operator on a locally convex space E. Let F be a locally convex space which is continuously and densely contained in E. If the restriction $T|F$ of T defines a continuous and linear operator on F which is hypercyclic (resp. chaotic), then T is also hypercyclic (resp. chaotic).

THE SPACE OF TEST FUNCTIONS

Here we report on joint work with Frerick, Peris and Wengenroth.

Let Ω an open subset of $\mathbb{R}^N$. The space $\mathcal{D}(\Omega)$ of test functions is the strict inductive limit of the Fréchet spaces $\mathcal{D}(K), K \subset \Omega$ compact.

In fact $\mathcal{D}(\Omega)$ is isomorphic to $\oplus_{i\in\mathbb{N}} s$ (Valdivia, 1982), where

$$s := \{x = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \|x\|_t := \sum_{n=1}^{\infty} |x_n| n^t < \infty \text{ para todo } t \in \mathbb{N}\}$$

is Fréchet space of rapidly decreasing sequences.

The space $\mathcal{D}(\Omega)$ has a structure which is "similar" to the one of

$$\varphi := \oplus_{i\in\mathbb{N}} \mathbb{C}.$$

Theorem 6

(1) (B., Peris, 1998/ Grosse-Erdmann, 1999):

There are no hypercyclic operators on φ.

(2) (B, Frerick, Peris and Wengenroth):

There are transitive operators on φ.

Theorem 7 There are hypercyclic operators on $E := \oplus_{i\in\mathbb{N}} l^1$.

Idea of the proof:

Representing the elements $x \in E$ as

$$x = (x_{i,n})_{i,n\in\mathbb{N}}, \quad \text{with } (x_{i,n})_{n\in\mathbb{N}} \in \ell^1 \quad \text{for each } \ i \in \mathbb{N},$$

we construct a shift operator T on E:

$$Te_{1,1} = 0.$$

$$Te_{i,j} = \lambda e_{\sigma(i,j)} \quad \text{si } (i,j) \neq (1,1).$$

where $\lambda \in \mathbb{C}$ is fixed and has modulus strictly greater than 1, and

$$\sigma : \mathbb{N} \times \mathbb{N} \setminus \{(1,1)\} \rightarrow \mathbb{N} \times \mathbb{N}$$

is certain bijection.

To construct the hypercyclic vector x, let

$$\{v_k = (a^k_{i,j})_{(i,j)\in\mathbb{N}^2} : k \geq 1\}$$

be a sequentially dense subset of $\oplus_{i\in\mathbb{N}} l^1$ such that

$$\begin{aligned} \sup_{(i,j)\in\mathbb{N}^2} |a^k_{i,j}| &\leq k, \quad \text{y} \\ a^k_{i,j} &= 0 \ \text{ si } \ i+j > k+2. \end{aligned}$$

There are strictly increasing sequences of natural numbers $(l(k))_k$, $(m(k))_k$ y $(n(k))_k$ and unique scalars x_j, $m(k) \leq j \leq n(k)$, such that

$$T^{l(k)}\Big(\sum_{j=m(k)}^{n(k)} \frac{x_j}{\lambda^{l(k)}} e_{1,j}\Big) = v_k,\ k \geq 1,$$

where

$$|x_j| \leq k, \quad k \geq 1.$$

We define

$$x := \sum_{k=1}^{\infty} \sum_{j=m(k)}^{n(k)} \frac{x_j}{\lambda^{l(k)}} e_{1,j},$$

obtaining a vector such that

$$\{T^{l(k)}x \ : \ k \in \mathbb{N}\}$$

is sequentially dense in E. ■

Is this operator good enough for $\mathcal{D}(\Omega) = \oplus_{i\in\mathbb{N}} s$?

Recall that

$$s := \{x = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \|x\|_t := \sum_{n=1}^{\infty} |x_n| n^t < \infty \text{ para todo } t \in \mathbb{N}\}.$$

Consider the set of indices

$$J := \{j \in \mathbb{N} \ : \ Te_{2,j} = \lambda e_{1,p(j)} \text{ para cierto } p(j) \in \mathbb{N}\}.$$

$$\lim_{j\in J} \frac{p(j)}{j^t} = \infty$$

for each $t \in \mathbb{N}$.

Therefore the operator T is not well-defined on $\mathcal{D}(\Omega)$.

We modify it "separating the humps":

The new shift operator T satisfies that the set

$$J := \{j \in \mathbb{N} \; : \; Te_{2,j} = \lambda e_{1,p(j)} \text{ para cierto } p(j) \in \mathbb{N}\}$$

satisfies

$$p(j) = o(j^2),$$

which gives a continuous operator.

Theorem 8 On $E := \mathcal{D}(\Omega)$ there are hypercyclic operators.