Dynamics of linear operators on non-metrizable vector spaces

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(joint work with L. Frerick, A. Peris and J. Wengenroth)

TOPOLOGICAL TRANSITIVITY

Definition 1. *Let X be a topological space, $T : X \to X$ a continuous map, and $x \in X$. The* ***orbit*** *of x under T is*

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*T is said to be **topologically transitive** if, for every pair $U, V \subseteq X$ of non-empty open sets, there exists $n \in \mathbb{N}$ such that*

$$T^n(U) \cap V \neq \emptyset.$$

In other words, there is $x \in U$ whose orbit intersects V.

Example:

$$T : [0,1] \longrightarrow [0,1], \quad T(x) = 4x(1-x) \quad \text{is transitive}$$

If X is a Baire space, 2AN, without isolated points, then

T is transitive $\Leftrightarrow$ T admits a dense orbit i.e.,

$$\exists x \in X : \quad \overline{Orb(x,T)} = X.$$

$(G_n)_n$ basis of open subsets of X

$$H := \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} (T^m)^{-1}(G_n) \quad \text{dense in } X$$

T is called **chaotic** in the sense of Devaney if it is topologically transitive and the set of periodic points of T is dense in X.

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From now on $T : X \to X$ will be a (linear and continuous) operator on a separable locally convex space X.

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Kitai, 1982: No finite dimensional space E admits hypercyclic operators.

Theorem 1. *If T is transitive, then the transpose operator*

$$T^t : E' \to E'$$

has no eigenvalues.

CLASSICAL AND RECENT RESULTS

Birkhoff, 1929: The translation operator

$$T_a : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C}), \quad (T_a f)(z) := f(z+a)$$

on the Fréchet space $H(\mathbb{C})$ of entire functions is hypercyclic if $a \neq 0$.

MacLane, 1952: The derivative operator $Df = f'$ is hypercyclic on $\mathcal{H}(\mathbb{C})$.

Godefroy, Shapiro, 1991: Let T be an operator on $\mathcal{H}(\mathbb{C}^N)$ which commutes with translations (or, equivalently, with the operators of partial differentiation). If T is not a scalar multiple of the identity, then T is chaotic.

• On $C^{\infty}(\mathbb{R}^{\mathbb{N}})$ the translation operators are hypercyclic (**Duyos-Ruíz, 1983**) and every partial differential operator with constant coefficients, which is not a multiple of the identity, is hypercyclic (**Godefroy, Shapiro, 1991**).

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Theorem 2. *(J.H. Shapiro) Let $G \subset \mathbb{C}$ be an open connected set, and let $P(D)$ be a non-constant polynomial. The following conditions are equivalent:*

(1) $P(D)$ is chaotic on $H(G)$

(2) $P(D)$ is hypercyclic on $H(G)$

(3) G is simply connected.

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(3) G is simply connected.

Rolewicz, 1969: Let $T := \lambda B : \ell^p \to \ell^p$, $1 \leq p < \infty$, $|\lambda| > 1$,

$$T(x_1, x_2, \dots) = (\lambda x_2, \lambda x_3, \dots)$$

be the *backward shift*. Then T is hypercyclic (and chaotic).

• Every separable infinite dimensional Banach space admits hypercyclic operators (Ansari, 1997/Bernal 1999). This results solves a problem of Rolewicz.

• The theorem is also true for Fréchet spaces (Bonet, Peris, 1998).

• However, there are infinite dimensional separable Banach spaces which admit no chaotic operator. This was proved by Bonet, F. Martínez-Giménez, Peris, 2001.

The construction depends on the hereditarily indecomposable Banach spaces of Gowers and Maurey.

The space $\varphi := \oplus_{i\in\mathbb{N}}\mathbb{C}$ of scalar sequences with finitely many coordinates different from 0 endowed with the finest locally convex topology does not admit hypercyclic operators, but it admits transitive operators.

$T : \varphi \longrightarrow \varphi$ continuous hypercyclic, x hypercyclic vector

$\Rightarrow$ $\{T^n x \mid n = 1, 2 \ldots\}$ dense

$\Rightarrow$ span $\{T^n x \mid n \in \mathbb{N}\}$ closed and dense

$$\Rightarrow \quad x = \sum_{n=1}^{m} \alpha_i \, T^n x$$

$$\Rightarrow \quad \exists s : \; T^s x \in \mathrm{span}(x, Tx, \ldots, T^{s-1}x) =: H$$

$$\Rightarrow \quad T^n x \in H \quad \forall n$$

$\Rightarrow$ $\varphi = H$ contradiction.

CONVOLUTION OPERATORS

Let Ω be an open subset of $\mathbb{R}^N$.

- Given a compact set $K \subset \mathbb{R}^N$, we denote by $\mathcal{D}(K)$ the Fréchet space of all C^∞-functions with support contained in K.

- The space $\mathcal{D}(\Omega)$ of test functions is the (strict) inductive limit of the system $(\mathcal{D}(K), K \subset \Omega$ compact).

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- The dual $\mathcal{D}'(\Omega)$ of $\mathcal{D}(\Omega)$ is the space of distributions of Laurent Schwartz.

- The space of C^∞ functions on Ω is denoted by $\mathcal{E}(\Omega)$. Its dual $\mathcal{E}'(\Omega)$ is the space of distributions with compact support.

- $\mathcal{E}'_*(\Omega)$ and $\mathcal{D}'_*(\Omega)$ denote the corresponding spaces of ultradistributions of Beurling or Roumieu type.

Theorem 3. *Let G be an open subset of $\mathbb{R}^d$. Let $\mu \in \mathcal{E}'_*(\mathbb{R}^N)$ satisfy $G - supp\mu \subset G$. If μ is not a scalar multiple of the Dirac measure, then T_μ is hypercyclic and chaotic on $\mathcal{E}_*(G)$. If the class is non-quasianalytic, the convolution operator is also hypercyclic and chaotic on $\mathcal{D}'_*(G)$.*

Every linear partial differential operator $P(D)$ with constant coefficients satisfies the assumptions of Theorem 3.

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> Every linear partial differential operator $P(D)$ with constant coefficients satisfies the assumptions of Theorem 3.

The proof depends on:

- The theorem of Godefroy and Shapiro on operators on $H(\mathbb{C}^N)$.
- Density theorems of $H(\mathbb{C}^N)$ in $\mathcal{E}_*(\Omega)$. Heinrich and Meise for quasyanalytic functions (2005).
- The following result:

Theorem 4. *Let T be a continuous and linear operator on a locally convex space E. Let F be a locally convex space which is continuously and densely contained in E. If the restriction $T|F$ of T defines a continuous and linear operator on F which is hypercyclic (resp. chaotic), then T is also hypercyclic (resp. chaotic).*

• This type of reduction argument had already been used by Peris and I to show that certain hypercyclic operators exist on certain LB-spaces.

• The proof depends on the existence of a dense subspace which is a Fréchet space for a stronger topology, hence the conclusion follows from Baire theorem and a comparison principle.

• This kind of reduction to the Fréchet case **fails** if one considers spaces of test functions such as $\mathcal{D}$, which is a strict inductive limit of Fréchet spaces, due to Grothendieck factorization theorem.

THE SPACE OF TEST FUNCTIONS

We want to show that $\mathcal{D}(\Omega)$ admits a hypercyclic operator T.

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To do this we will use the fact that

$$\boxed{\mathcal{D}(\Omega) \cong \oplus_{i\in\mathbb{N}} s}$$

(Valdivia / Vogt 1982)

where

$$\boxed{s := \{x = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \|x\|_t := \sum_{n=1}^{\infty} |x_n| n^t < \infty \text{ for every } t \in \mathbb{N}\}}$$

is the Fréchet space of all rapidly decreasing sequences.

Theorem 5. *There are hypercyclic operators* T *on* $X := \oplus_{i\in\mathbb{N}}\ell^1$. *More precisely, there exist* $x \in X$ *whose* T*-orbit is sequentially dense in* X.

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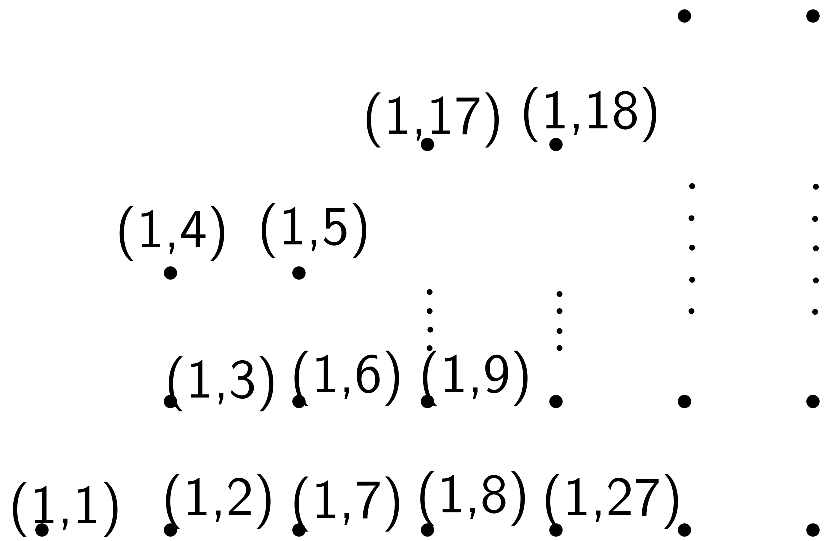
Sketch of proof:

We represent the elements $x \in X$ as $x = (x_{i,n})_{i,n\in\mathbb{N}}$, where $(x_{i,n})_{n\in\mathbb{N}} \in \ell^1$ for each $i \in \mathbb{N}$. We then construct a "snake" backward shift operator T on X:

$$Te_{1,1} = 0.$$

$$Te_{i,j} = \lambda e_{\sigma(i,j)} \text{ if } (i,j) \neq (1,1).$$

where $\lambda \in \mathbb{C}$, $|\lambda| > 1$ is fixed, and $\sigma : \mathbb{N}\times\mathbb{N} \setminus \{(1,1)\} \to \mathbb{N}\times\mathbb{N}$ is certain bijection.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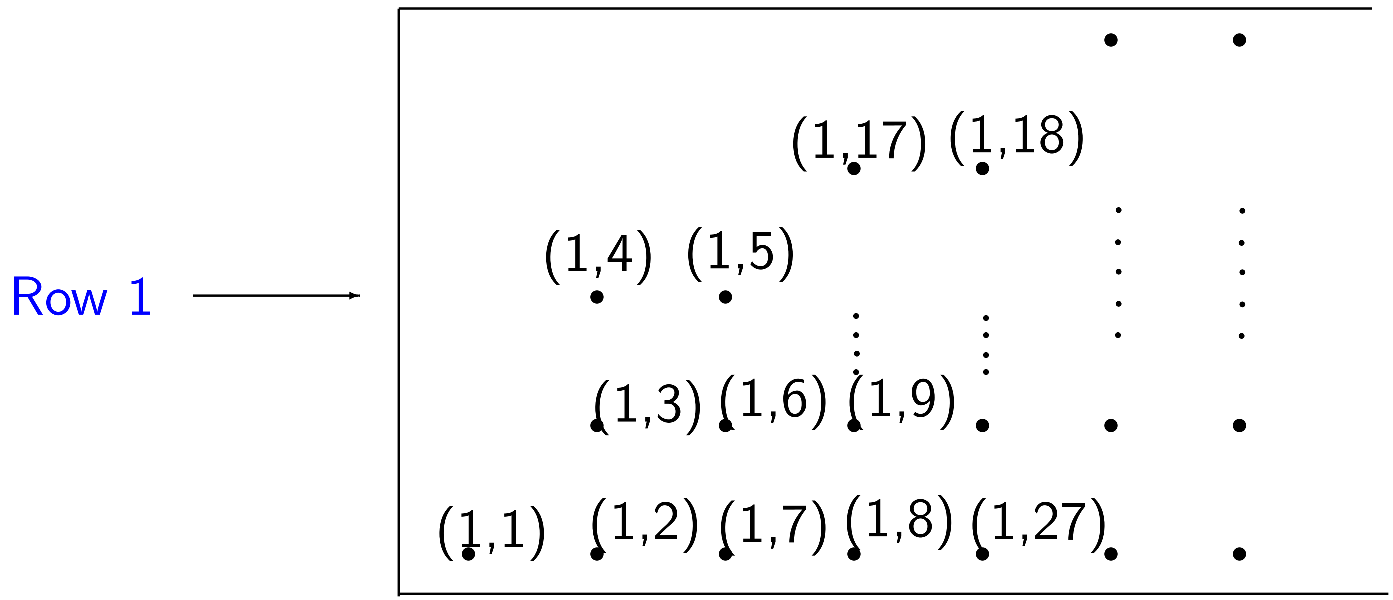
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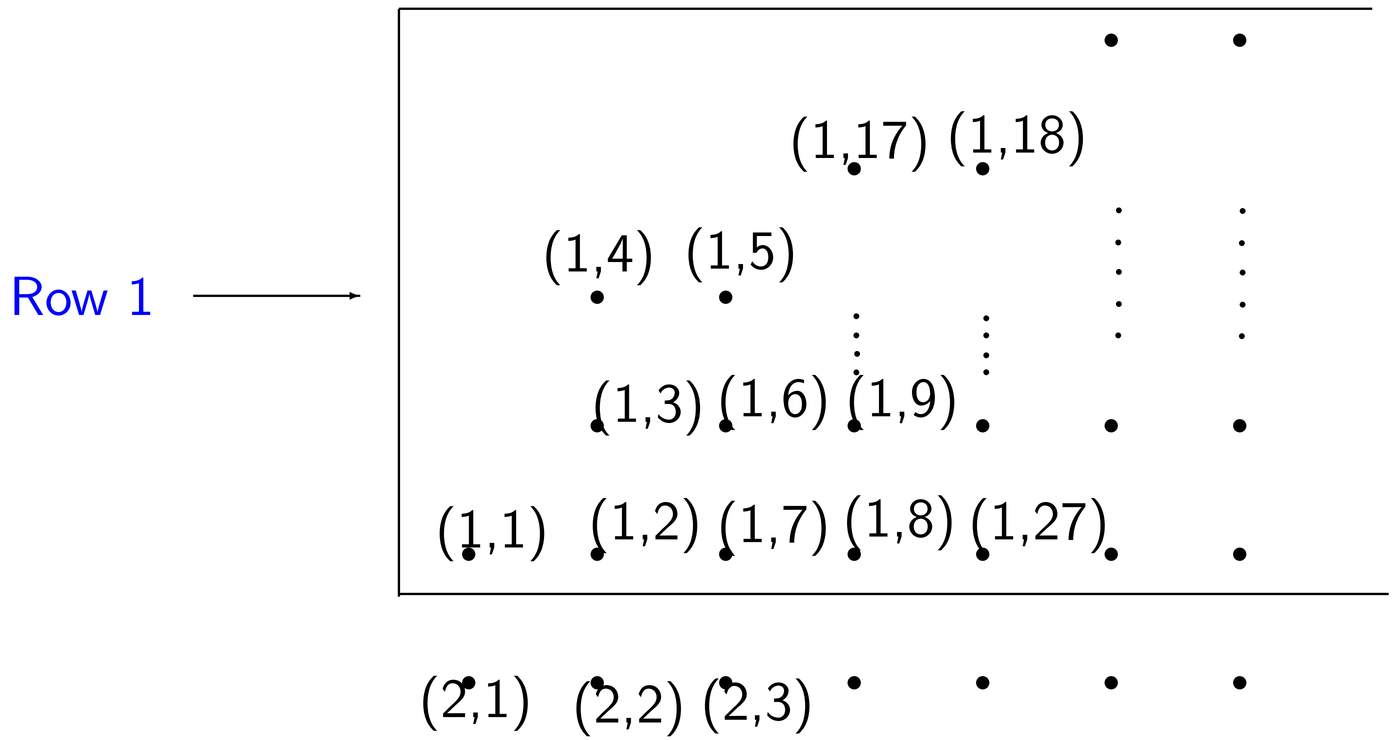
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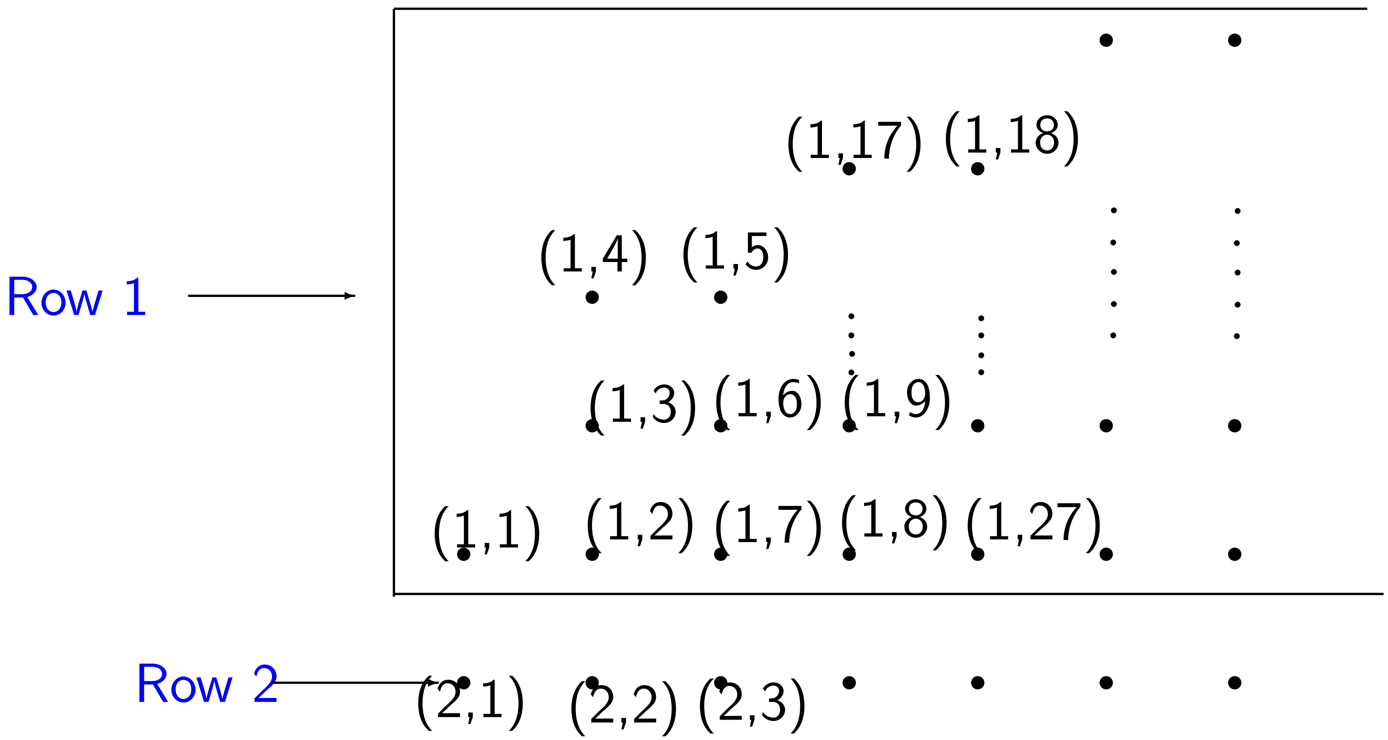
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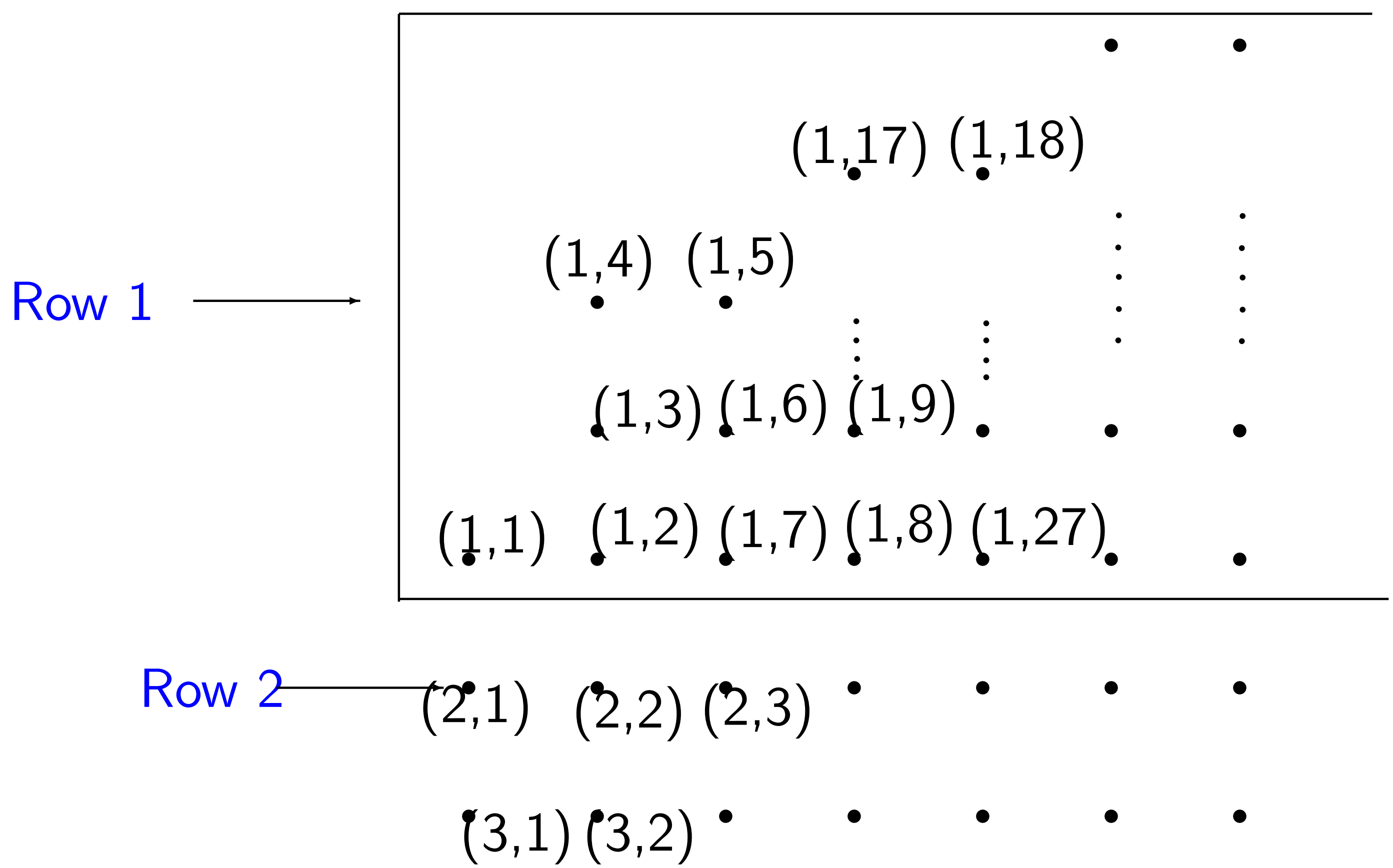
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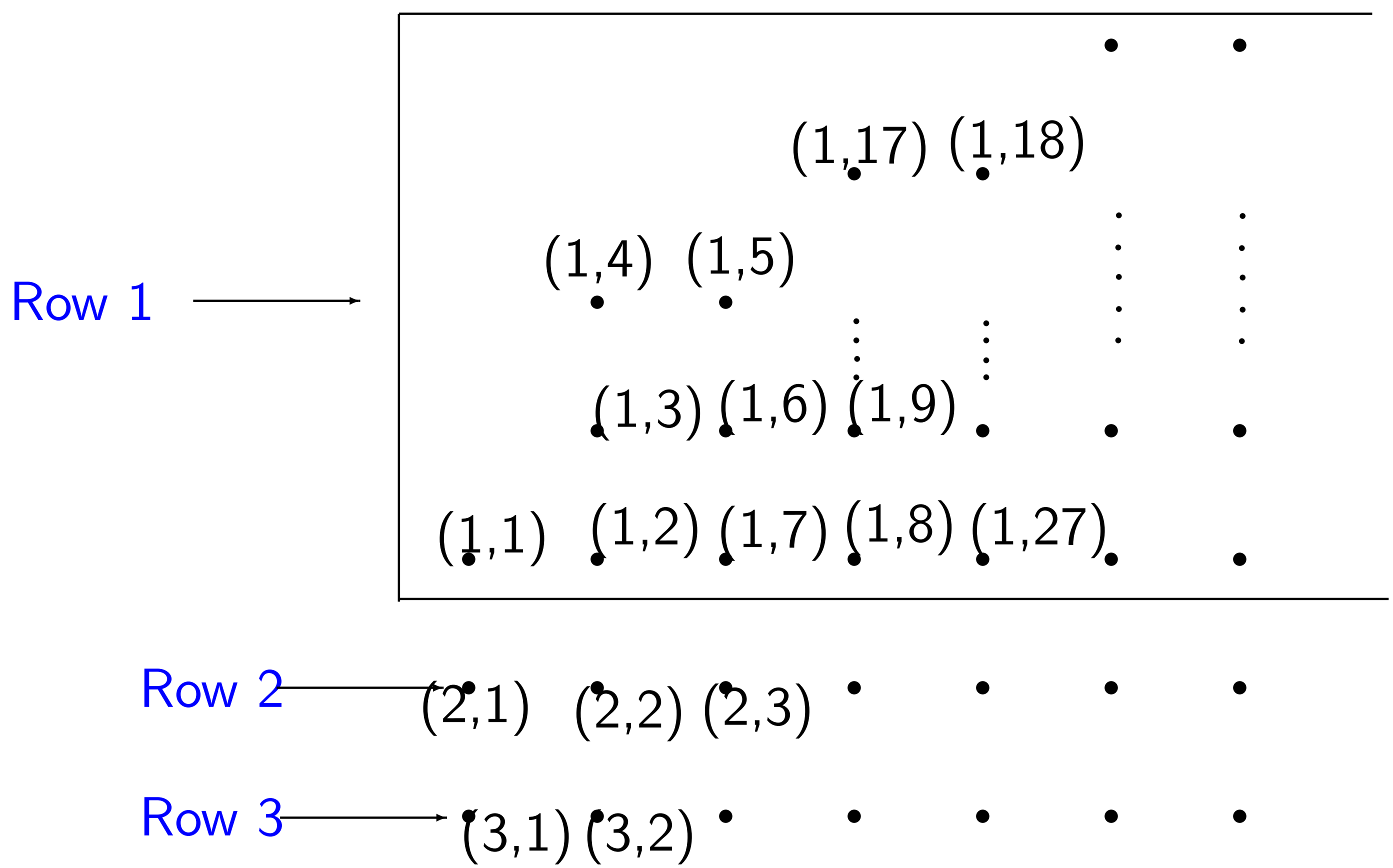

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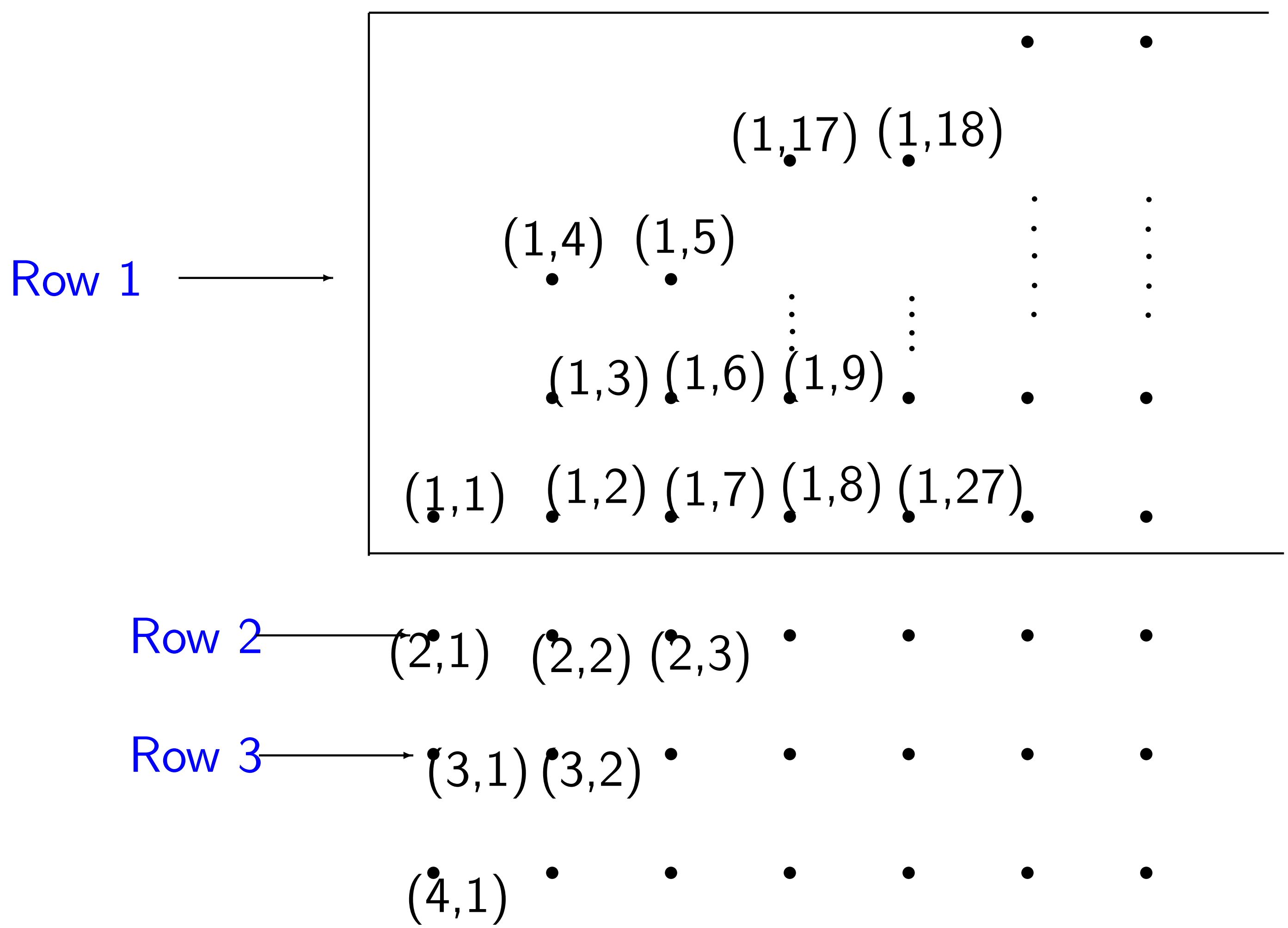

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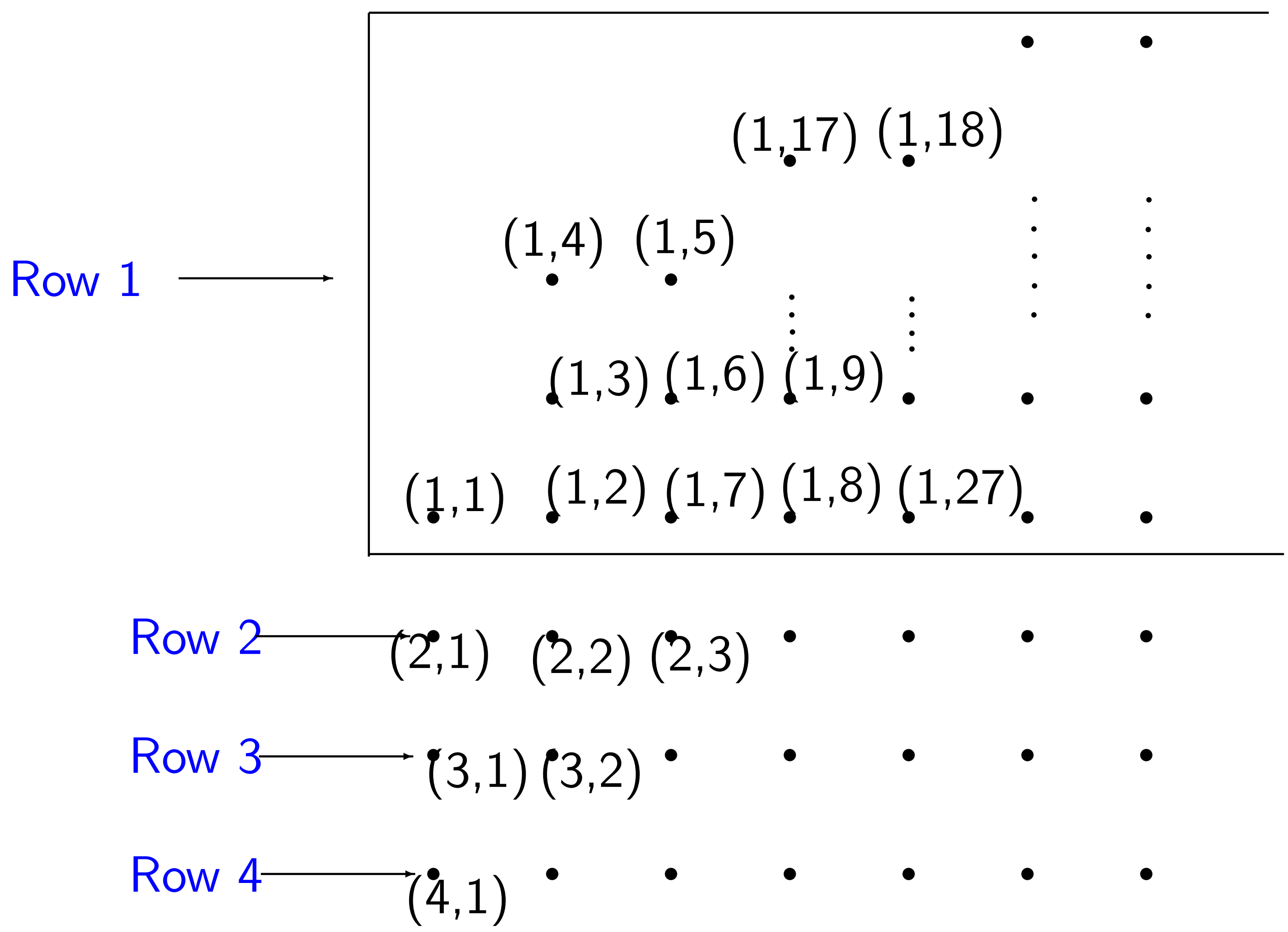
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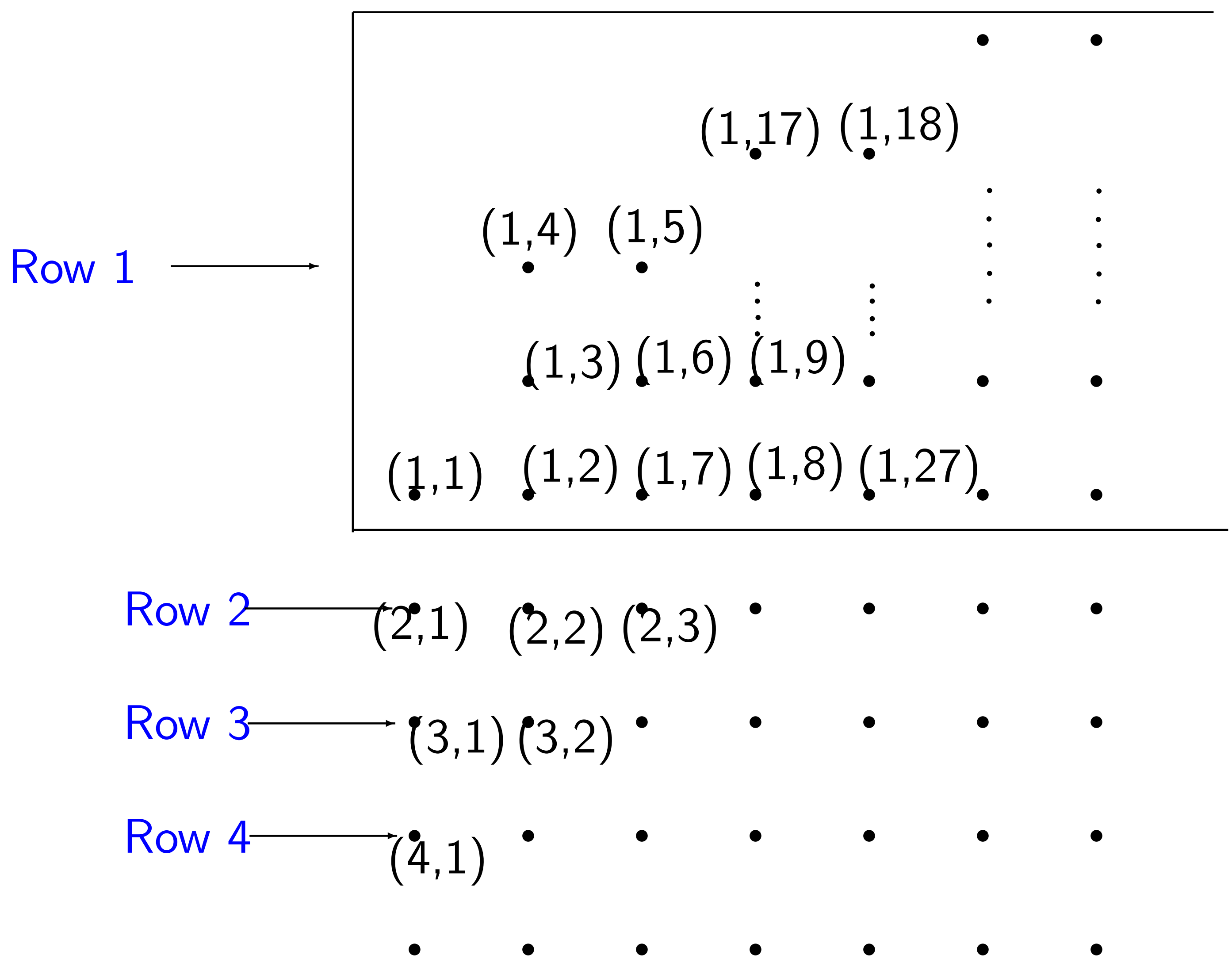

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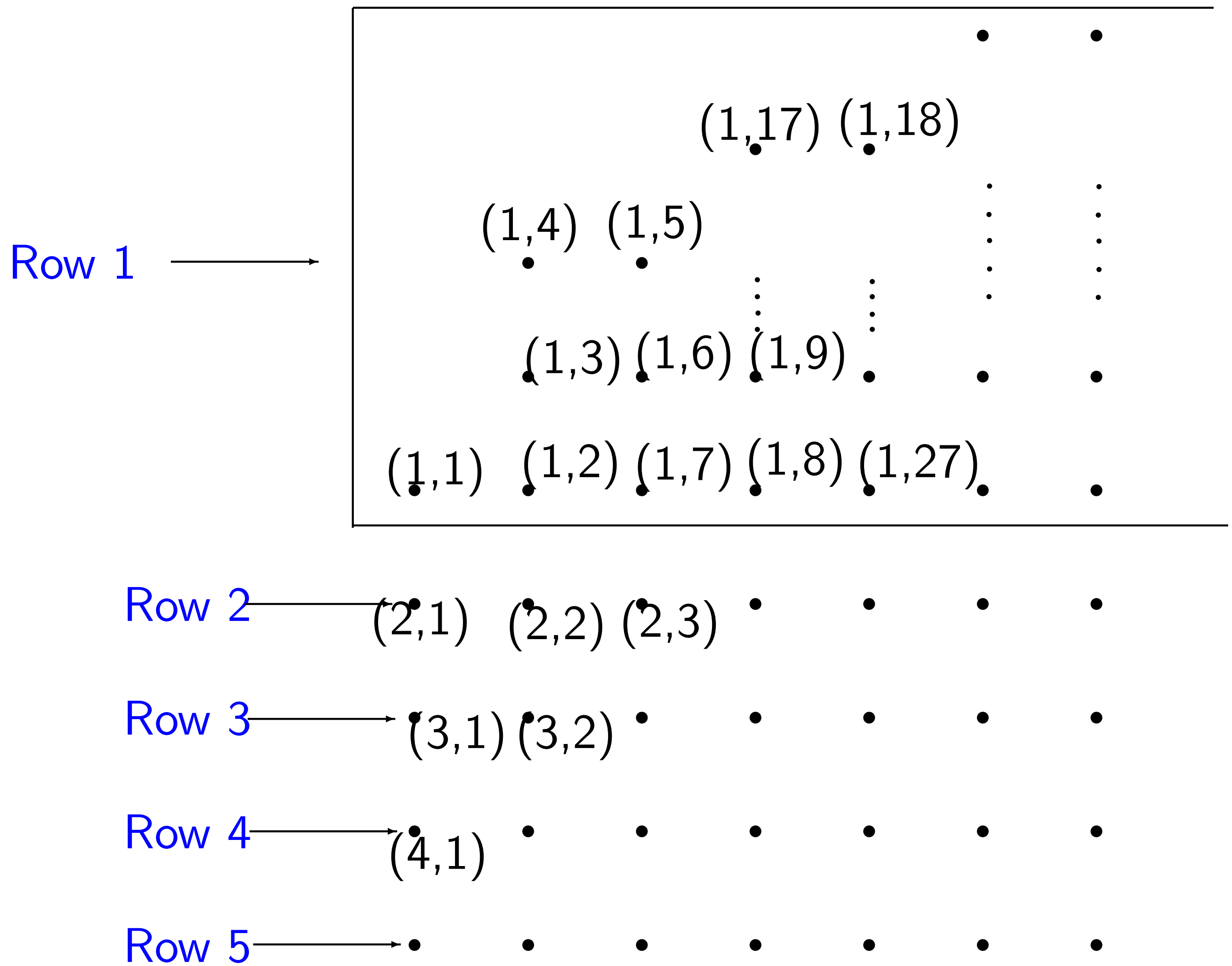
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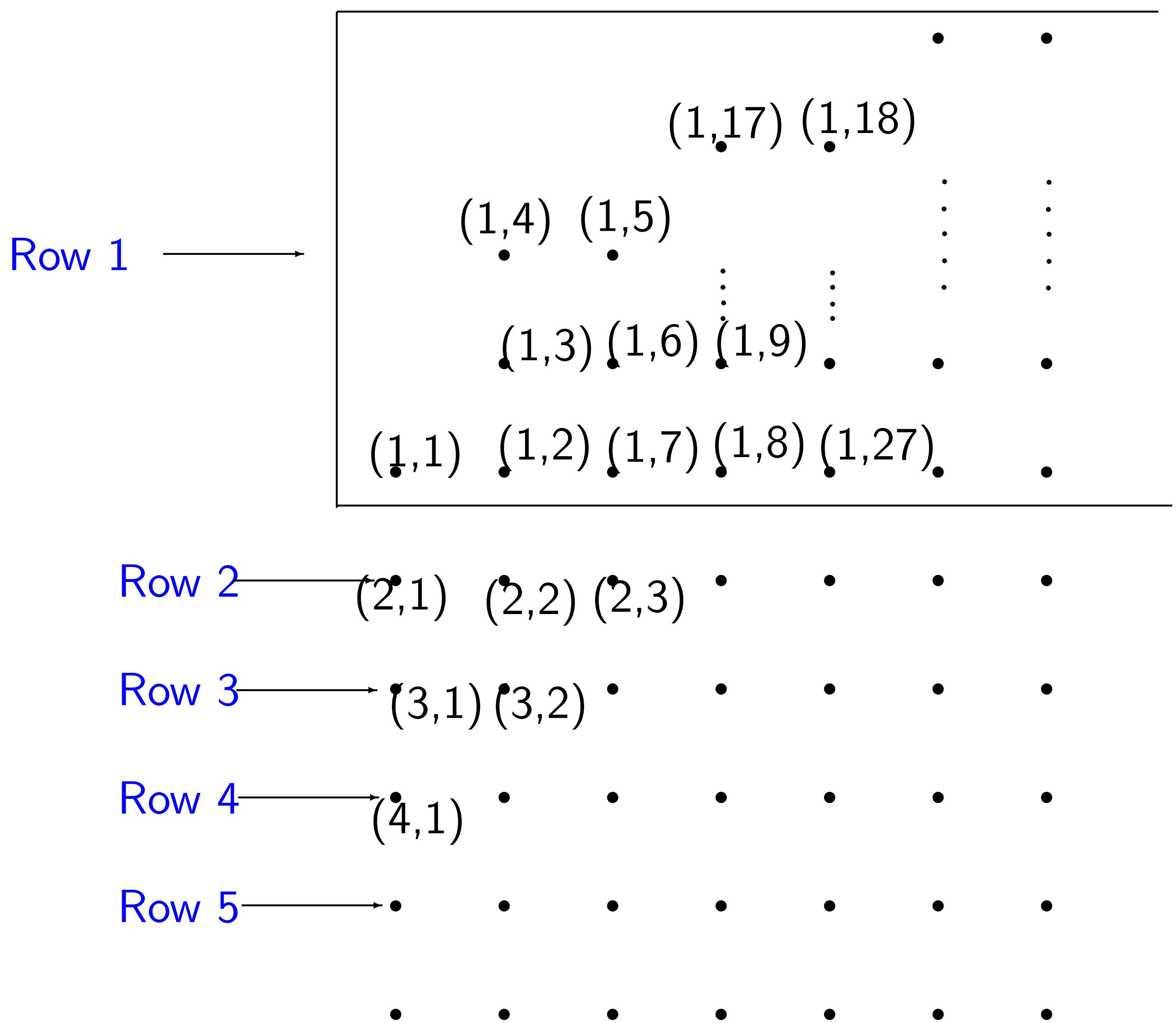
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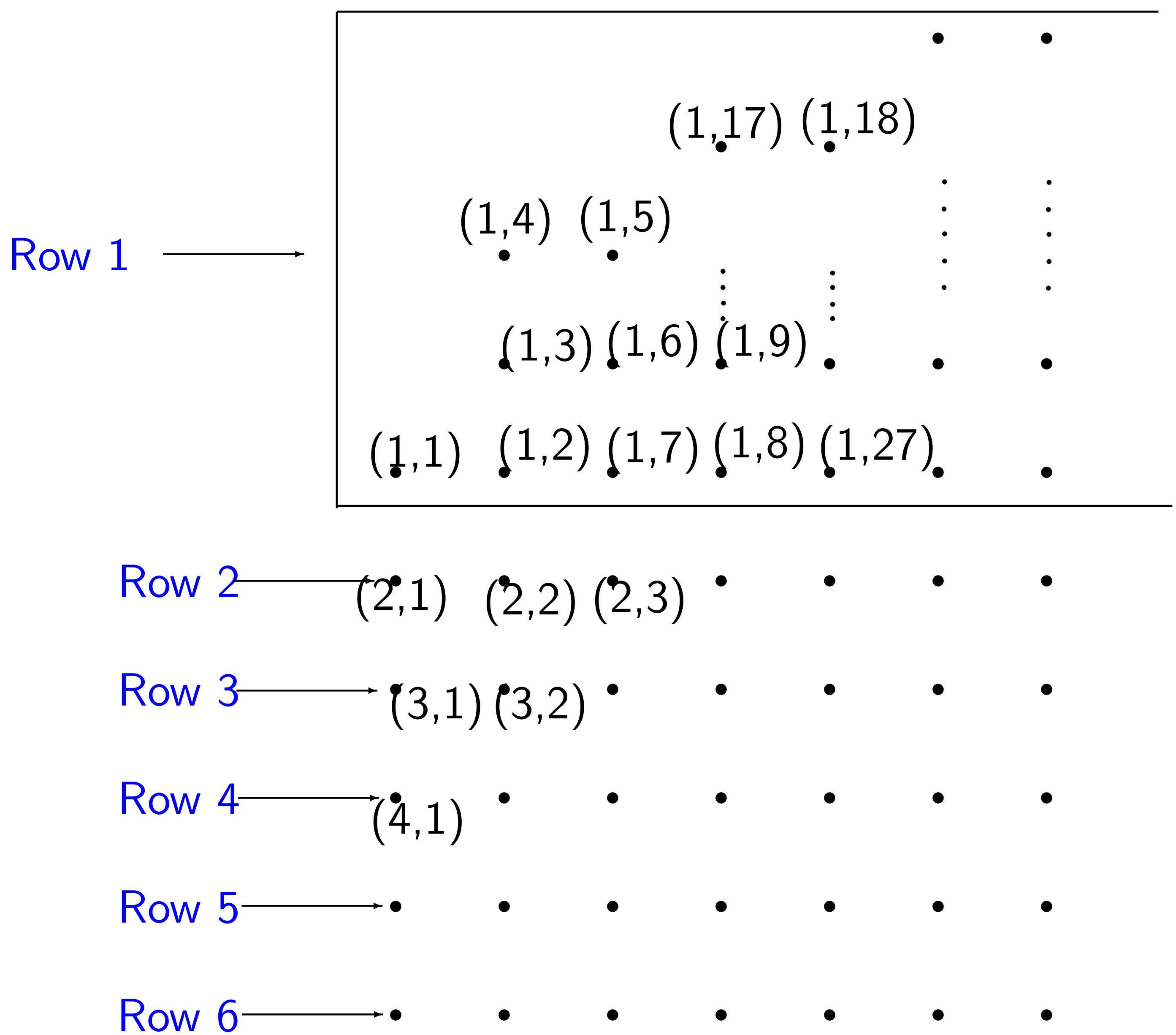
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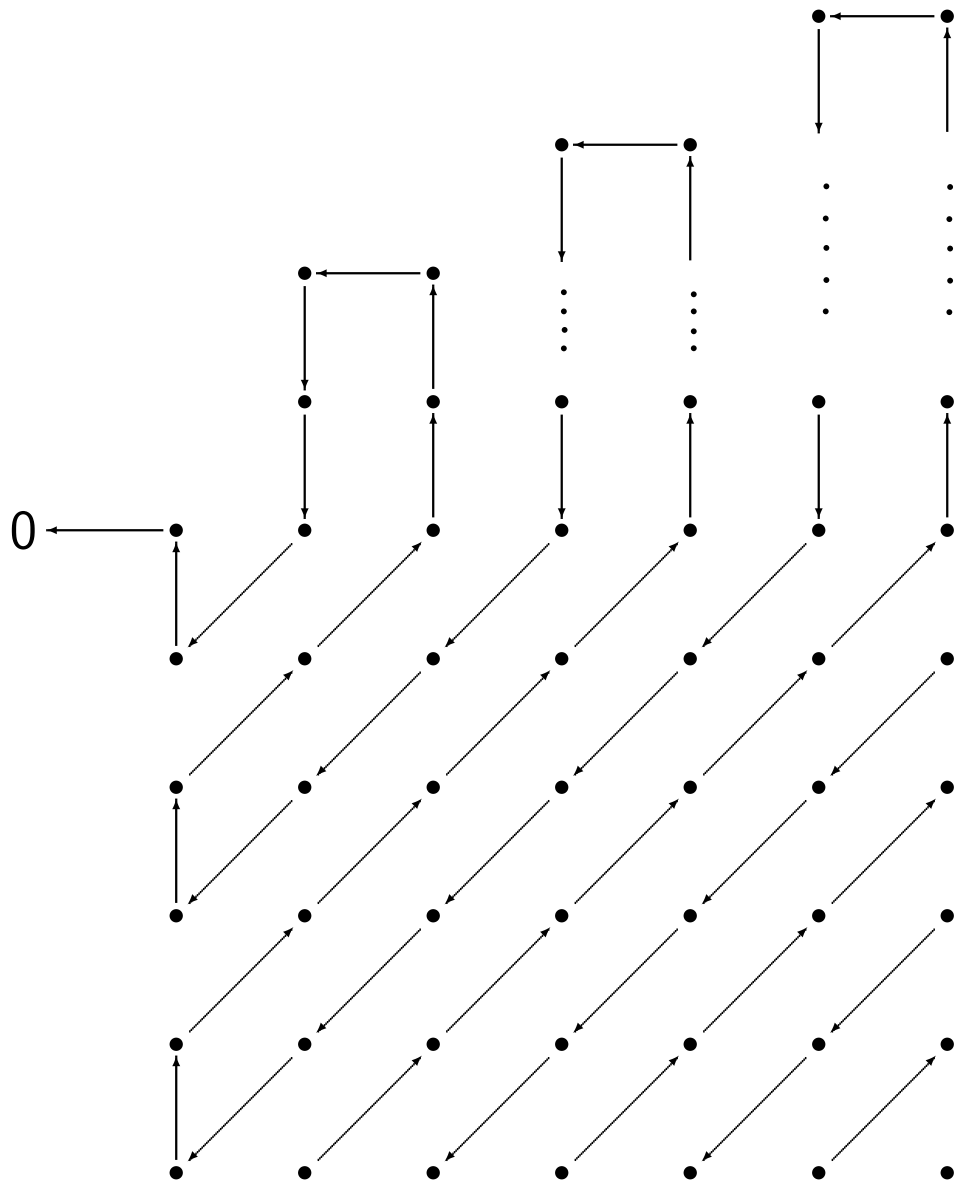
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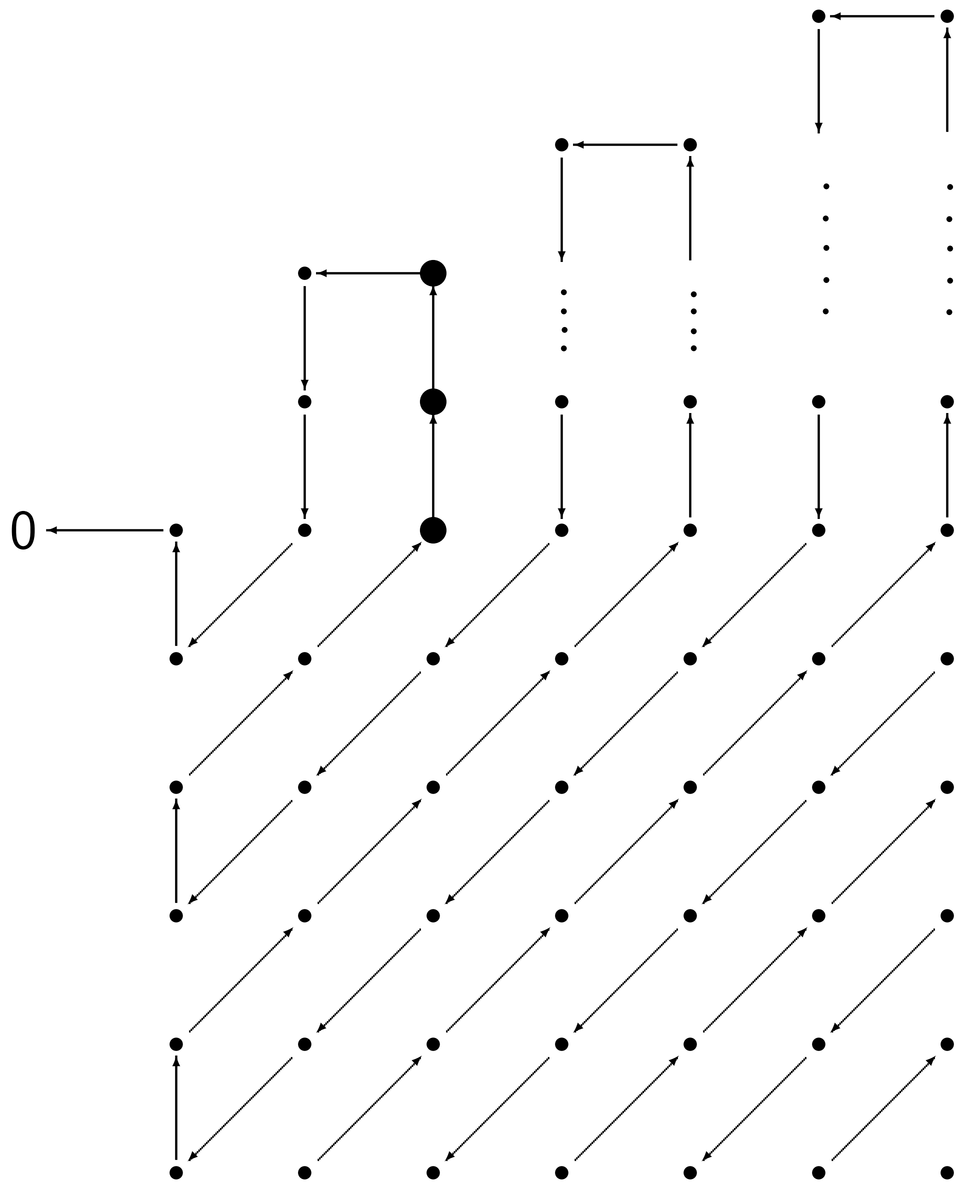
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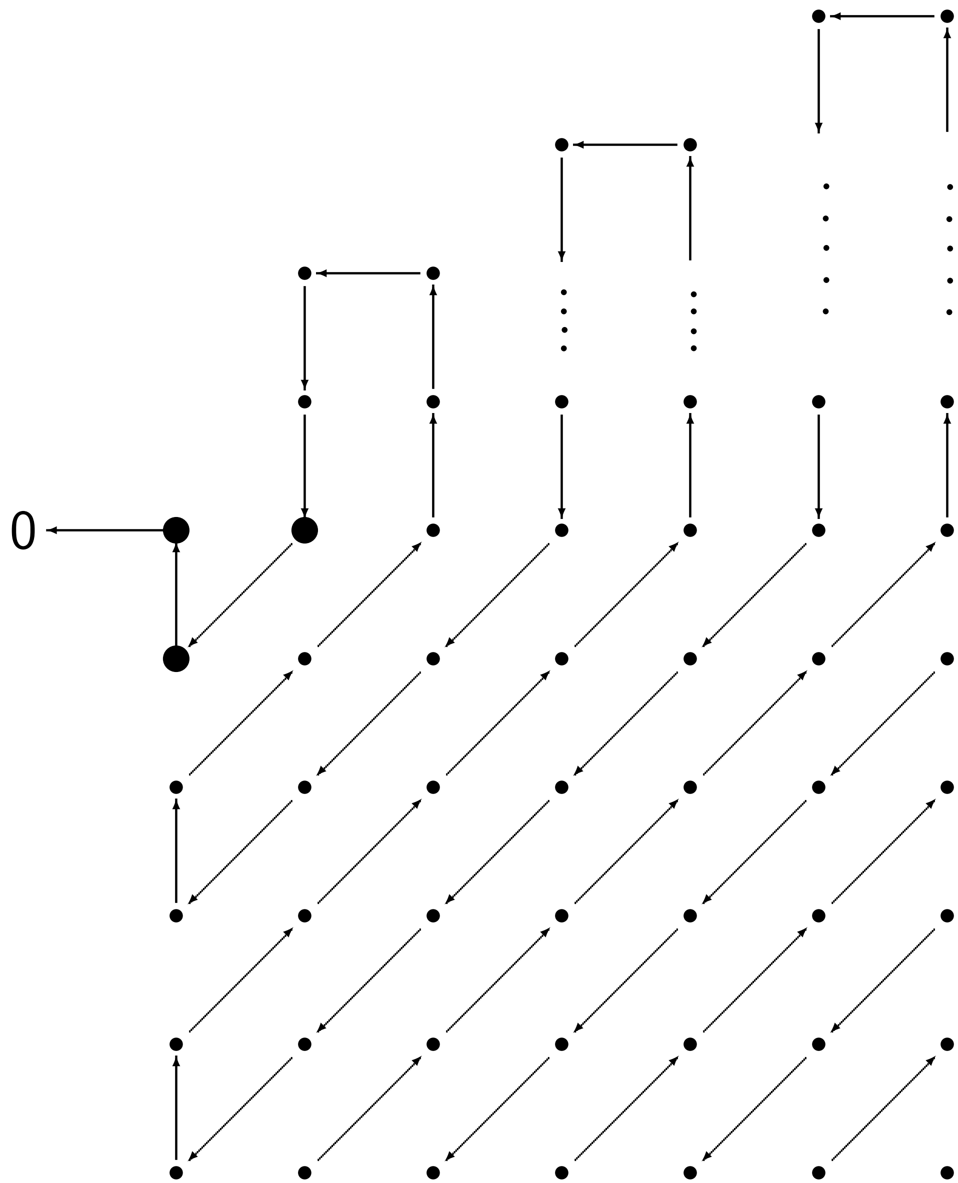
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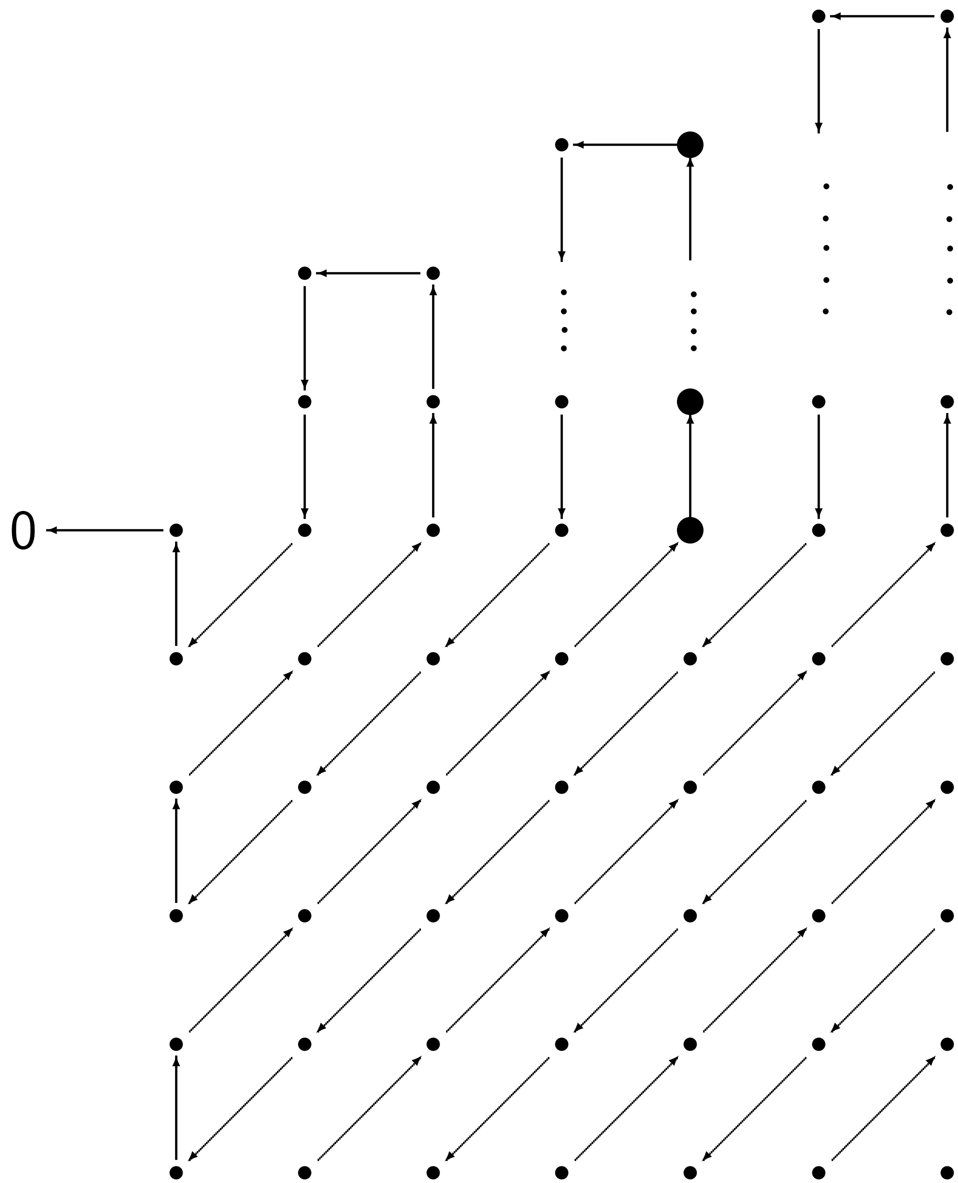
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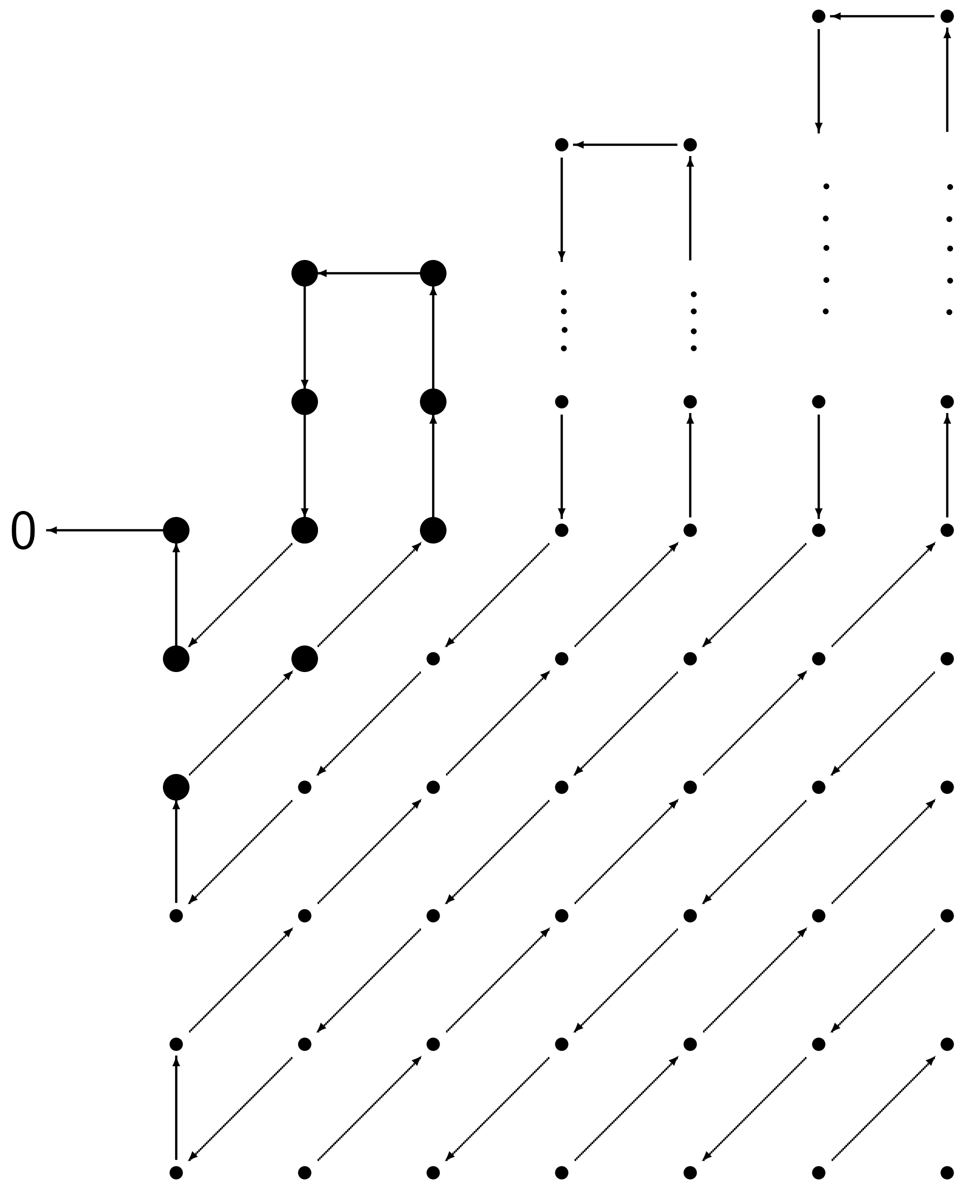
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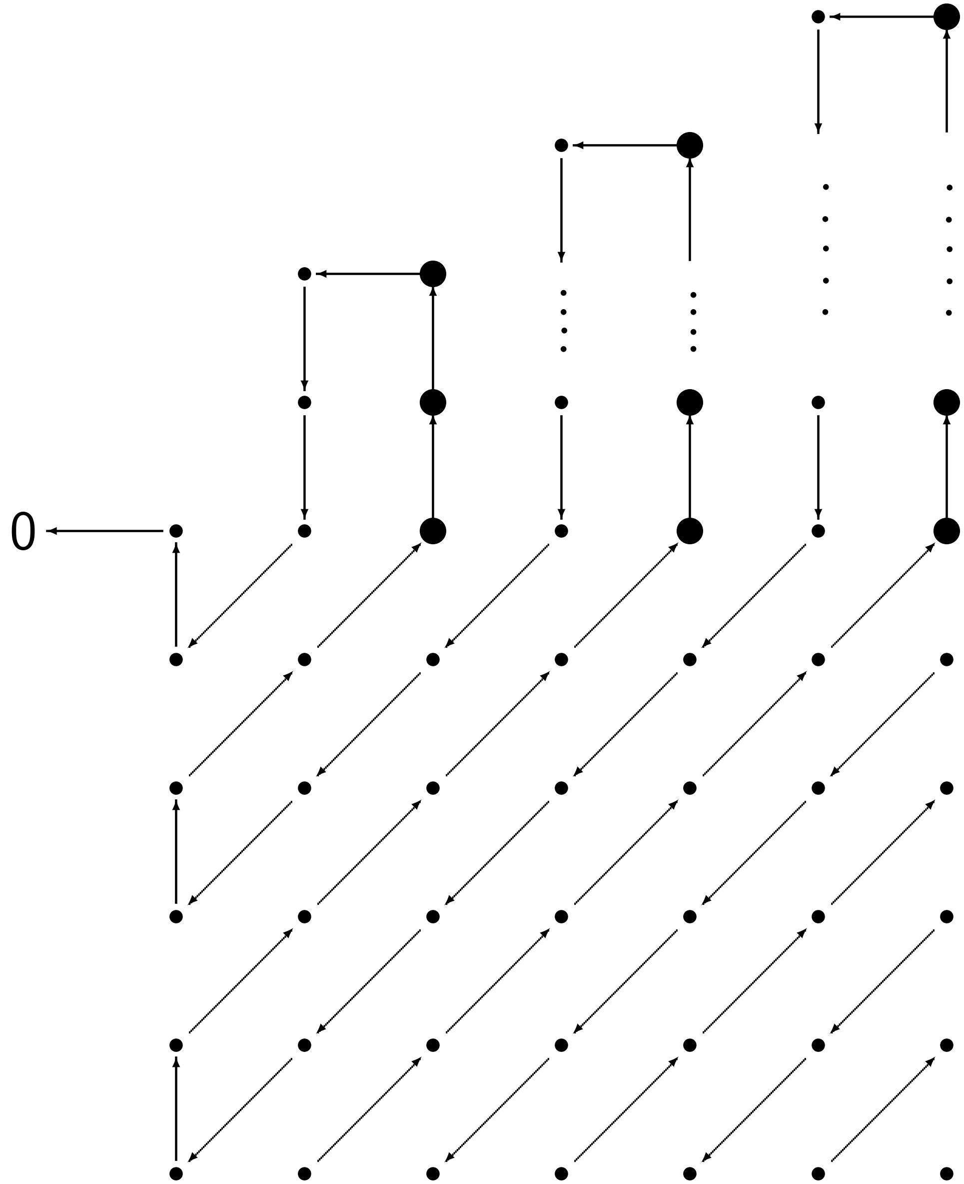
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To construct a hypercyclic vector x, let $\{v_k = (a^k_{i,j})_{(i,j)\in\mathbb{N}^2} : k \geq 1\}$ be a sequentially dense subset of $\oplus_{i\in\mathbb{N}}\ell^1$ such that

$$\sup_{(i,j)\in\mathbb{N}^2} |a^k_{i,j}| \leq k, \quad \text{and} \quad a^k_{i,j} = 0 \ \text{ if } \ i+j > k+2.$$

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There are suitable strictly increasing sequences of positive integers $(\ell(k))_k$, $(m(k))_k$ and $(n(k))_k$, and (unique) scalars x_j, $m(k) \leq j \leq n(k)$, such that

$$T^{\ell(k)}(\sum_{j=m(k)}^{n(k)} \frac{x_j}{\lambda^{\ell(k)}} e_{1,j}) = v_k, \ k \geq 1, \qquad |x_j| \leq k, \quad k \geq 1.$$

We then define

$$\boxed{x := \sum_{k=1}^{\infty} \sum_{j=m(k)}^{n(k)} \frac{x_j}{\lambda^{\ell(k)}} e_{1,j}}$$

which satisfies that $\{T^{\ell(k)}x \ : \ k \in \mathbb{N}\}$ is sequentially dense in X. ■

Is this operator T valid for $\mathcal{D}(\Omega) = \bigoplus_{i \in \mathbb{N}} s$?

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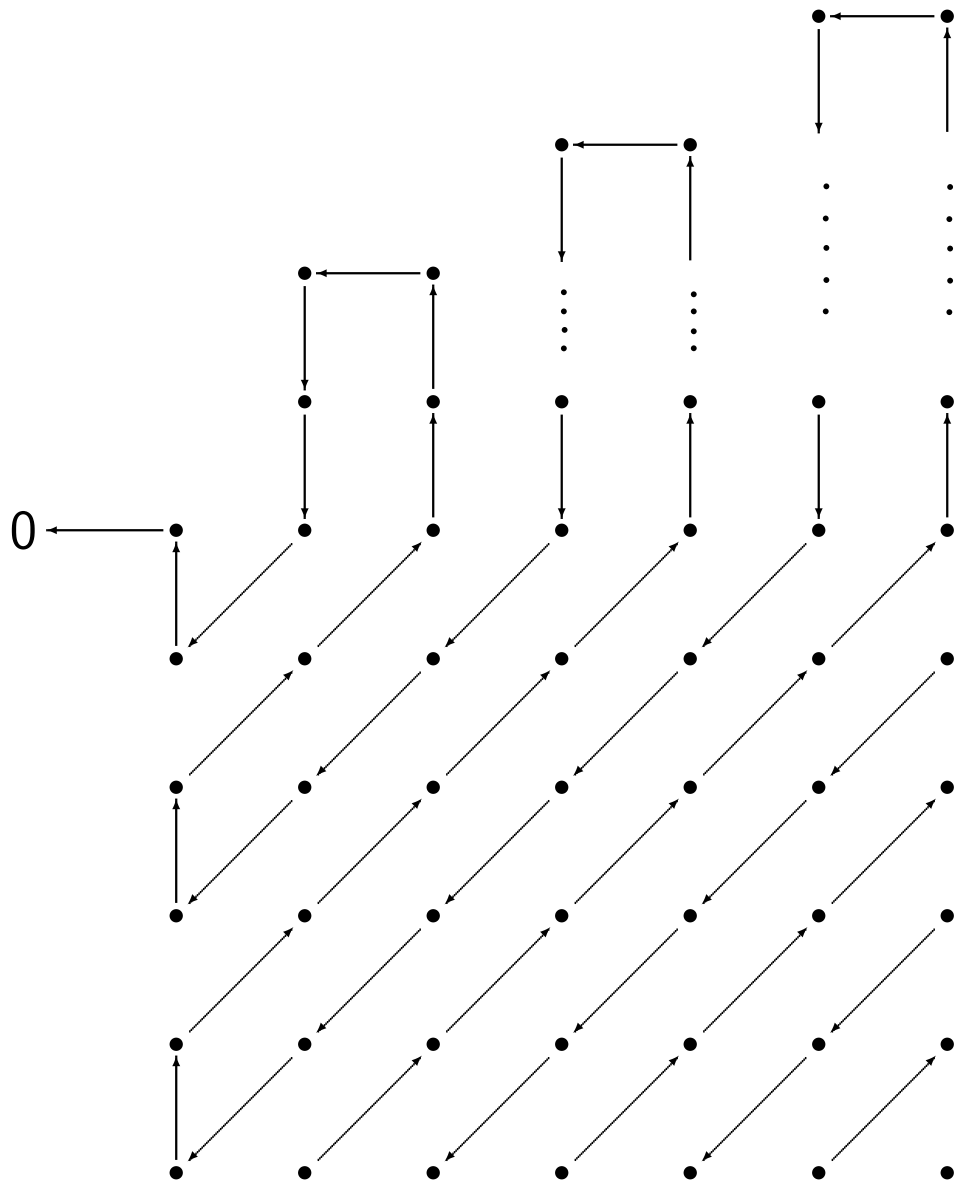
Let us recall that

$$s := \{x = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \|x\|_t := \sum_{n=1}^{\infty} |x_n| n^t < \infty \text{ for every } t \in \mathbb{N}\}$$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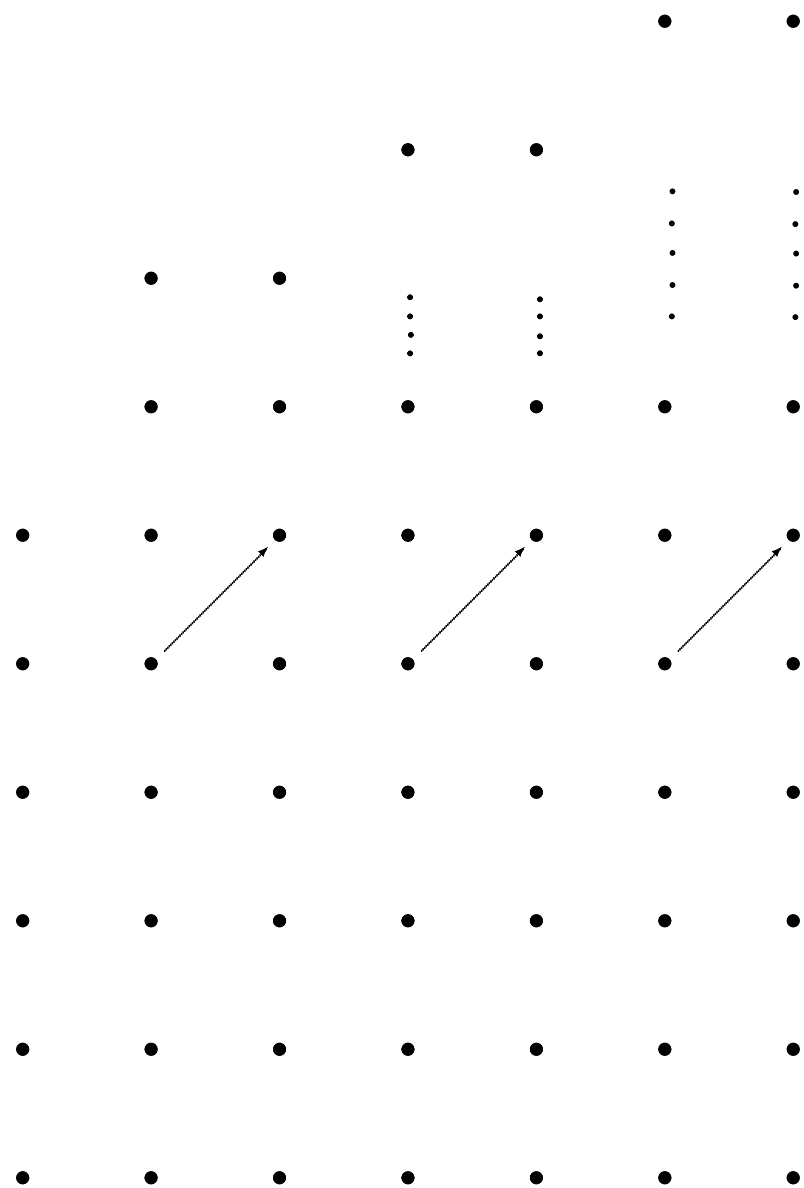

Let us consider the set of indices

$$J := \{j \in \mathbb{N} \ : \ Te_{2,j} = \lambda e_{1,p(j)} \text{ for some } p(j) \in \mathbb{N}\}.$$

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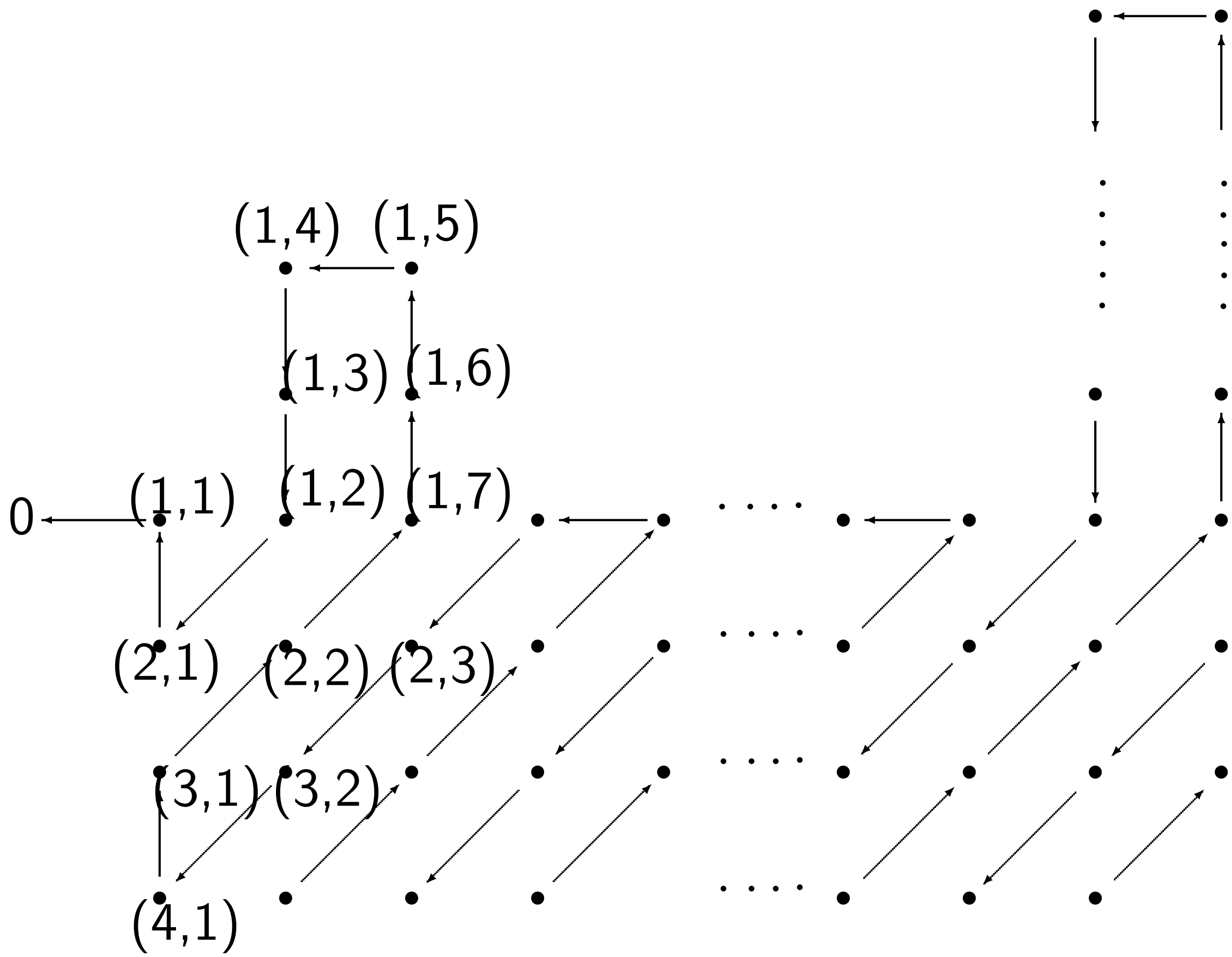
We modify the "snake-shift" operator:



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Under this modification, the new snake-shift operator T, is so that the set of indices

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Theorem 6. *There are hypercyclic operators T on $X := \mathcal{D}(\Omega)$ and vectors $x \in X$ whose orbit with respect to T is sequentially dense in X.*