The dual of the space of holomorphic functions on locally closed convex sets

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NOTATION AND STATEMENT OF THE PROBLEM

A subset Q of $\mathbb{C}^N$ is called **locally closed** if for each $z \in Q$ there is a closed neighbourhood U of z in $\mathbb{C}^N$ such that $Q \cap U$ is closed.

- Every open subset and every closed subset of $\mathbb{C}^N$ is locally closed.

- Every convex open set in $\mathbb{R}^N$ is locally closed too.

For a convex set $Q \subset \mathbb{C}^N$ the symbols $\mathrm{int}_r Q$ denote the relative interior and $\partial_r Q$ the relative boundary of Q *with respect to the affine hull of* Q.

For example, if $0 \in Q$, the affine hull of Q is the the real linear span of Q.

We write $\omega := Q \cap \partial_r Q$.

Proposition 1. *The following assertions are equivalent for a convex subset Q of $\mathbb{C}^N$:*

Q *is locally closed.*

$$\Updownarrow$$

Q *admits a countable fundamental sequence* $(Q_n)_{n\in\mathbb{N}}$ *of compact subsets*

$$\Updownarrow$$

Q *is the union of the relative interior* $\mathrm{int}_r Q$ *of* Q *and a subset* ω *of* $\partial_r Q$ *which is open in* $\partial_r Q$

A locally closed convex set Q is called ($\mathbb{C}$-)**strictly convex at the relative boundary of** ω if the intersection of Q with each supporting (complex) hyperplane to the closure $\overline{Q}$ of Q is compact.

• If the interior of Q is empty,

Q is strictly convex at the relative boundary of ω	$\Leftrightarrow$	Q is compact

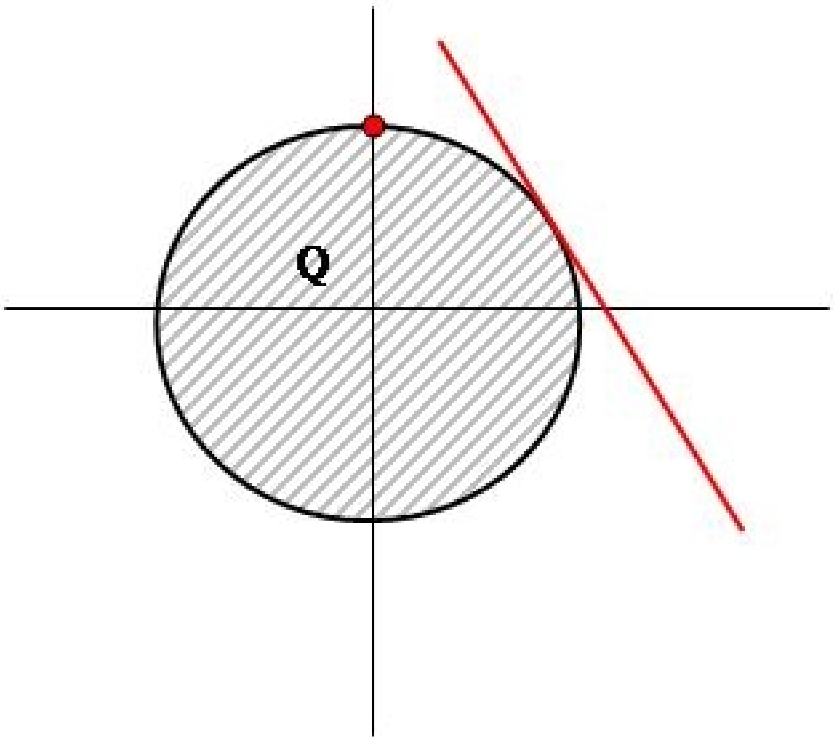
• If the interior of Q is not empty,

Q is ($\mathbb{C}$-)strictly convex at the relative boundary of ω	$\Leftrightarrow$	each line segment (of which the $\mathbb{C}$-linear affine hull belongs to some supporting hyperplane of $\overline{Q}$) of $\omega = Q \cap \partial_r Q$ is relatively compact in ω

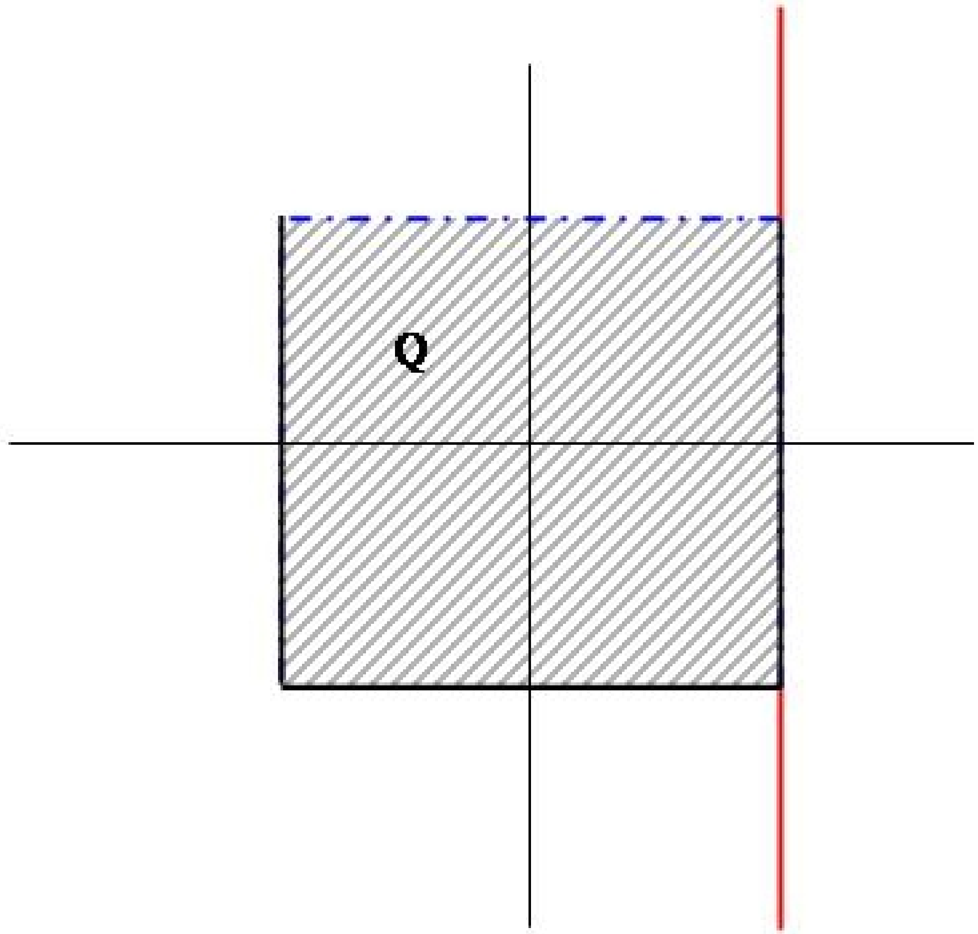
Proposition 2. *A locally closed convex set Q is strictly convex at the relative boundary of ω if and only if Q has a neighbourhood basis of convex domains.*

For example Q is strictly convex at the relative boundary of ω if Q is open or compact.

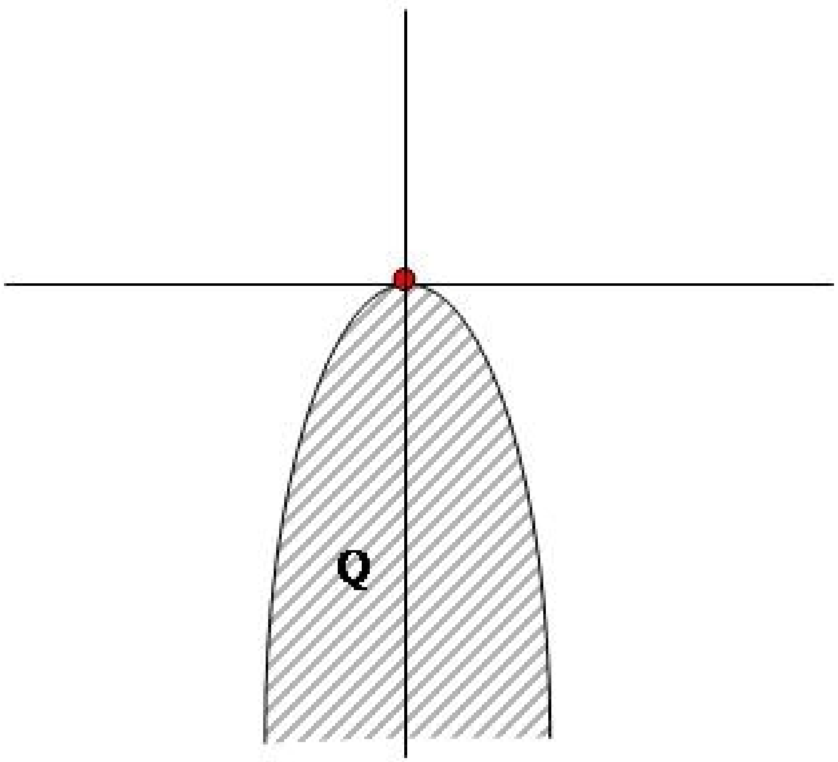
Strictly convex



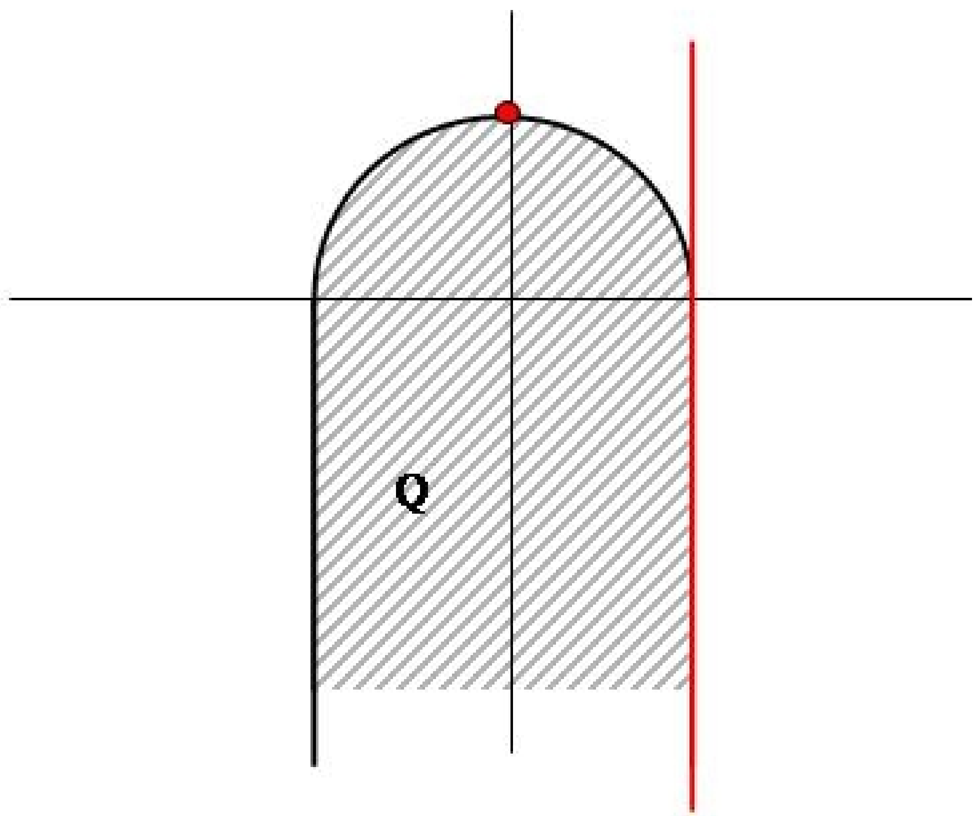
Not strictly convex



Strictly convex



Not strictly convex



Let $Q \subset \mathbb{C}^N$ be a convex locally closed set such that

$$\overline{Q} = \{x \in \mathbb{C}^N \mid \quad x_1 \geq f(x_2, ..., x_{2N}), \quad (x_2, ..., x_{2N}) \in \mathbb{R}^{2N-1}\}$$

for a convex function $f : \mathbb{R}^{2N-1} \longrightarrow \mathbb{R}$.

- If the function f is strictly convex $\Rightarrow$ $\left\{ \begin{array}{l} Q \text{ is strictly convex at the} \\ \text{relative boundary of } \omega \end{array} \right.$

- If Q is closed and there is a unbounded interval in $\mathbb{R}^{2N-1}$ on which f is affine $\Rightarrow$ $\left\{ \begin{array}{l} Q \text{ is not strictly convex} \\ \text{at the relative boundary of } \omega \end{array} \right.$

We denote by $\mathbf{H}(\mathbf{Q})$ the vector space of all functions which are holomorphic on some open neighbourhood of the locally closed convex set Q.

Let $(Q_n)_{n\in\mathbb{N}}$ be an increasing fundamental sequence of compact convex sets in Q. Since the algebraic equality $H(Q) = \cap_{n\in\mathbb{N}} H(Q_n)$ holds, we endow $H(Q)$ with the projective topology of

$$\boxed{H(Q) := \mathrm{proj}_n H(Q_n)}$$

This topology does not depend of the choice of the fundamental system $(Q_n)_{n\in\mathbb{N}}$. The space $H(Q)$ is a (PLN)-space.

- If Q is an open convex subset of $\mathbb{R}^N$, then

$$H(Q) = A(Q)$$

where $A(Q)$ denotes the space of all real analytic functions on Q.

- 1966, **Martineau** investigated the spaces $H(Q)$ of analytic functions on a locally closed convex set Q in $\mathbb{C}^N$ and convolution operators on these spaces in 1967.

- 1990's, **Napalkov**, **Udakov**, **Korobeinik** and **Maltsev**,

- 2000, **Melikhov** and **Momm**.

  The Laplace transform

$$\mathcal{F} : H(Q)'_b \longrightarrow VH(\mathbb{C}^N) := \operatorname*{ind}_{n\rightarrow} \operatorname*{proj}_{\leftarrow k} \; H(v_{n,k}, \mathbb{C}^N)$$

  is a linear topological isomorphism.

The Laplace transform: $\mathcal{F}(\varphi)(z) := \varphi(\exp\langle \cdot, z\rangle), \; z \in \mathbb{C}^N$,

$$VH(\mathbb{C}^N) := \operatorname{ind}_{n\rightarrow} \operatorname{proj}_{\leftarrow k} \; H(v_{n,k}, \mathbb{C}^N)$$

$VH(\mathbb{C}^N)$ is a weighted inductive limit of Fréchet spaces

The steps $H(v, \mathbb{C}^N)$ are defined, for a positive weight v on $\mathbb{C}^N$, as

$$H(v, \mathbb{C}^N) := \{f \in H(\mathbb{C}^N) | \sup_{z \in \mathbb{C}^N} v(z)|f(z)| < \infty\} \quad \text{(Banach space)}$$

and $v_{n,k}(z) := \exp(-H_n(z) - |z|/k), \quad n, k \in \mathbb{N}, \quad z \in \mathbb{C}^N$.

For each set $D \subset \mathbb{C}^N$ we denote by H_D the **support function of** D:

$$H_D(z) := \sup_{w \in D} \operatorname{Re}\langle z, w\rangle, \quad z \in \mathbb{C}^N \quad \text{with } \langle z, w\rangle := \sum_{j=1}^{N} z_j w_j.$$

For each $n \in \mathbb{N}$ let $\boxed{H_n := H_{Q_n}}$ be the support function of the convex compact set Q_n.

Bierstedt, Meise, and Summers (1982)

System $\overline{V}$ of all those weights $\overline{v} : \mathbb{C}^N \to]0, \infty[$ which are continuous and have the property that

for each n there are $\alpha_n > 0$ and $k = k(n)$ such that $\overline{v} \leq \alpha_n v_{n,k}$ on $\mathbb{C}^N$.

The projective hull of the weighted inductive limit is defined by

$$H\overline{V}(\mathbb{C}^N) := \{f \in H(\mathbb{C}^N) \mid ||f||_{\overline{v}} := \sup_{z \in \mathbb{C}^N} \overline{v}(z)|f(z)| < \infty \text{ for all } \overline{v} \in \overline{V}\},$$

endowed with the Hausdorff locally convex topology defined by the system of seminorms

$$\{||.||_{\overline{v}} \mid \overline{v} \in \overline{V}\}$$

The projective hull is a complete locally convex space and

$$VH(\mathbb{C}^N) \subset H\overline{V}(\mathbb{C}^N) \quad \text{with continuous inclusion.}$$

PROBLEM

Characterize in terms of the locally closed convex set Q when the inclusion

$$VH(\mathbb{C}^N) \subset H\overline{V}(\mathbb{C}^N)$$

is a topological isomorphism into and when it is surjective.

Spaces of holomorphic functions

Theorem 3. *Let $Q \subset \mathbb{C}^N$ be a convex locally closed set. If Q is strictly convex at the relative boundary of ω, then*

$$VH(\mathbb{C}^N) = H\overline{V}(\mathbb{C}^N) \qquad \textit{(algebraically and topologically)}$$

- **Maltsev** (1994): Permit us to construct, if $Q \subset \mathbb{C}$ is locally closed and not strictly convex at the relative boundary of ω, an entire function $P(z)$ of order at most one and zero type such that the linear differential operator $P(D)$ associated with $P(z)$ is not surjective on $H(Q)$. A reduction argument for $N > 1$ yields

Proposition 4. *Suppose that $Q \subset \mathbb{C}^N$ is convex, locally closed and has a neighbourhood basis of domains of holomorphy.*

$$\left.\begin{array}{l}\textit{If each nonzero differential operator}\\ \quad P(D) : H(Q) \longrightarrow H(Q)\\ \textit{is surjective}\end{array}\right\} \Rightarrow \begin{array}{l}Q \textit{ is strictly convex at}\\ \textit{the relative boundary of } \omega\end{array}$$

Theorem 5. *Let $Q \subset \mathbb{C}^N$ be a convex locally closed set. Suppose that Q has a neighbourhood basis of domains of holomorphy. If $VH(\mathbb{C}^N)$ is a topological subspace of $H\overline{V}(\mathbb{C}^N)$, then Q is strictly convex at the relative boundary of ω.*

Proof. Suppose that $VH(\mathbb{C}^N)$ is a topological subspace of $H\overline{V}(\mathbb{C}^N)$. With a division argument one shows that, for each nonzero entire function of order at most one and zero type P, the multiplication operator $M_P = P(D)^t$: $VH(\mathbb{C}^N) \to VH(\mathbb{C}^N)$ is an injective topological homomorphism. Since the space $H(Q)$ is reflexive, an application of Hahn-Banach theorem gives that

$$P(D) : H(Q) \longrightarrow H(Q)$$

is surjective for each such P. By Proposition 4, Q is strictly convex at the relative boundary of ω.

Corollary 6. *Let Q be a convex subset of $\mathbb{R}^N$ which is locally closed.*

$$\begin{array}{c} VH(\mathbb{C}^N) \text{ is a topological subspace of} \\ \text{its projective hull } H\overline{V}(\mathbb{C}^N) \end{array} \Leftrightarrow \quad Q \text{ is compact.}$$

If a convex and locally closed set $Q \subset \mathbb{C}^N$ is $\mathbb{C}$-strictly convex at the relative boundary of ω then Q has a neighbourhood basis of domains of holomorphy.

In fact, by Martineau 1966, if Q is $\mathbb{C}$-strictly convex at the relative boundary of ω, then Q has a neighbourhood basis of linearly convex open sets, hence a basis of domains of holomorphy.

An open convex set in $\mathbb{C}^N$ is linearly convex if its complement is a union of complex hyperplanes.

If Q is a convex and locally closed subset of $\mathbb{R}^N$, then also Q has a neighbourhood basis of domains of holomorphy by a lemma of Cartan.

Spaces of continuous functions

The weighted (LF)-space of continuous functions $VC(\mathbb{C}^N)$ and its projective hull $C\overline{V}(\mathbb{C}^N)$ associated with the sequence $V = (v_{n,k})_{n,k\in\mathbb{N}}$ of Section 2 are defined by replacing entire functions by continuous ones.

Theorem 7. (Bierstedt, Meise, Summers, 1982) *For every locally closed convex set $Q \subset \mathbb{C}^N$ the weighted (LF)-space $VC(\mathbb{C}^N)$ is a topological subspace of its projective hull $C\overline{V}(\mathbb{C}^N)$.*

Theorem 8. *Let $Q \subset \mathbb{C}^N$ is convex and locally closed. The following are equivalent:*

(i) The algebraic equality $VC(\mathbb{C}^N) = C\overline{V}(\mathbb{C}^N)$ holds.

(i)' $VC(\mathbb{C}^N) = C\overline{V}(\mathbb{C}^N)$ (algebraically and topologically)

(ii) Q is strictly convex at the relative boundary of ω.

A necessary and sufficient condition for the algebraic equality $VH(\mathbb{C}) = H\overline{V}(\mathbb{C})$ in the case of a bounded convex locally closed set Q in $\mathbb{C}$ was obtained in 2003.

Theorem 9. *Let Q be a bounded convex locally closed subset of $\mathbb{C}^N$.*

(i) Assume that the following conditions $(*)$ holds:

There is a supporting hyperplane Π to $\overline{Q}$ such that $\Pi \cap Q \neq \emptyset$ and there exists a $z_0 \in (\Pi \cap \overline{Q}) \setminus Q$ which is a smooth point of ∂Q,

then $VH(\mathbb{C}^N) \neq H\overline{V}(\mathbb{C}^N)$.

(ii) $VH(\mathbb{C}) \neq H\overline{V}(\mathbb{C})$ if and only if the condition $(*)$ holds.

The algebraic identity $VH(\mathbb{C}^N) = H\overline{V}(\mathbb{C}^N)$ also holds in case Q is a convex open subset of $\mathbb{R}^N$ as it was proved in 2004. This is the case of the Fourier Laplace transform of the space of analytic functionals.