Splitting of short exact sequences of PLS-spaces

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PLS-Spaces

X is a PLS-space $\Leftrightarrow$

$$X = \mathrm{proj}_{N\in\mathbb{N}} \; X_N = \mathrm{proj}_{N\in\mathbb{N}} \; \mathrm{ind}_{n\in\mathbb{N}} \; X_{N,n},$$

X_N DFS-space, $(X_{N,n}, \|\cdot\|_{N,n})$ Banach space.

Examples:

- Fréchet Schwartz spaces and their duals
- $\mathscr{A}(\Omega)$ and $\mathscr{D}'(\Omega)$
- Beurling classes of ultradistributions $\mathscr{D}'_{(\omega)}(\Omega)$
- Roumieu classes of ultradifferentiable functions $\mathscr{E}_{\{\omega\}}(\Omega)$
- Köthe type sequence PLS-spaces

Splitting problem

$\mathrm{Ext}^1(Z, X) = 0$

Every short exact sequence of PLS-spaces:

$$0 \longrightarrow X \xrightarrow{\;j\;} Y \xrightarrow{\;q\;} Z \longrightarrow 0$$

splits, i.e., q has a right linear continuous inverse.

Exact sequence

Every map in the diagram is

- continuous,
- open onto its image
- surjective onto the kernel of the next map.

Main result

Theorem

Let X be an ultrabornological PLS-space and E be a nuclear Fréchet space then:

1) $\mathrm{Ext}^1(E, X) = 0 \quad \Leftrightarrow \quad (E, X) \in (H)$:
$\forall\ N\ \exists\ M\ \forall\ K\ \exists\ n, \nu\ \forall\ m, \mu\ \exists\ k, \kappa, S$

$$\|y\|^*_{M,m}\|x\|_\mu \leq S\left(\|y\|^*_{N,n}\|x\|_\nu + \|y\|^*_{K,k}\|x\|_\kappa\right)$$

for $\forall\ y \in X'_N, x \in E$.

2) $\mathrm{Ext}^1(E', X) = 0 \quad \Leftrightarrow \quad (E', X) \in (G)$:
$\forall\ N, \nu\ \exists\ M, \mu\ \forall\ K, \kappa\ \exists\ n\ \forall\ m\ \exists\ k, S$

$$\|y\|^*_{M,m}\ \|x\|^*_\mu \leq S\left(\|y\|^*_{N,n}\ \|x\|^*_\nu + \|y\|^*_{K,k}\ \|x\|^*_\kappa\right)$$

for $\forall\ y \in X'_N, x \in E'_\nu$.

$$X = \mathrm{proj}_{N\in\mathbb{N}}\ \mathrm{ind}_{n\in\mathbb{N}}\ X_{N,n}, \quad (X_{N,n}, \|\cdot\|_{N,n}), \quad E = (E, (\|\cdot\|_\nu)_{\nu\in\mathbb{N}})$$

Power series spaces

$\alpha = (\alpha_j) \nearrow \infty$, $r = 0$ or $r = \infty$

$$\Lambda_r^p(\alpha) \quad := \quad \{x = (x_j)_{j \in \mathbb{N}} : \ \forall\ t < r \quad \|x\|_t < \infty\}$$

$$p = 1, \qquad \|x\|_t := \sum_j |x_j| e^{t\alpha_j}$$

$$p = \infty, \qquad \|x\|_t := \sup_j |x_j| e^{t\alpha_j}$$

Examples:

$$H(\mathbb{D}^d) \simeq \Lambda_0(j^{1/d}), \qquad H(\mathbb{C}^d) \simeq \Lambda_\infty(j^{1/d}), \qquad C^\infty[0,1] \simeq \Lambda_\infty(\log j)$$

Necessary condition

Let X be an ultrabornological PLS-space and α be stable.
If $\mathrm{Ext}^1(Y, X) = 0$ then X satisfies

$$\forall\, N\ \exists\, M\ \forall\, K\ \exists\, n\ \forall\, m\ \boxed{\ \ldots\ldots\ }\ \forall\, y \in X'_N$$

$$\|y\|^*_{M,m} \leq C\left(r^\eta\, \|y\|^*_{K,k} + \frac{1}{r}\|y\|^*_{N,n} \right), \quad \text{for some} 0 < r_0 < \infty.$$

- $Y = \Lambda^1_\infty(\alpha) \quad \Rightarrow \quad (P\Omega)$: $\boxed{\exists\, \eta\ \exists\, k, C\ \forall\, r > r_0}$
- $Y = \Lambda^1_0(\alpha) \quad \Rightarrow \quad (P\overline{\overline{\Omega}})$: $\boxed{\forall\, \eta\ \exists\, k, C\ \forall\, r > r_0}$
- $Y = \Lambda^\infty_r{}'(\alpha) \quad \Rightarrow \quad (PA)$: $\boxed{\forall\, \eta\ \exists\, k, C\ \forall\, r < r_0}$

Additionally, $(P\underline{A})$: $\boxed{\exists\, \eta\ \exists\, k, C\ \forall\, r < r_0}$

Sufficient condition

(DN)-(Ω)-type Splitting Theorem

Let X be a PLS-space and E a nuclear Fréchet space.

$$\left.\begin{array}{ll} E \in (DN), & X \in (P\Omega) \\ E \in (\underline{DN}), & X \in (P\overline{\overline{\Omega}}) \end{array}\right\} \quad \Rightarrow \quad \mathrm{Ext}^1(E, X) = 0.$$

$$\left.\begin{array}{ll} E \in (\Omega), & X \in (PA) \\ E \in (\overline{\overline{\Omega}}), & X \in (P\underline{A}) \end{array}\right\} \quad \Rightarrow \quad \mathrm{Ext}^1(E', X) = 0.$$

Proof for $(P\Omega)$ and (DN):

$X \in (P\Omega)$: $\forall\, N\ \exists\, M\ \forall\, K\ \exists\, n\ \forall\, m\ \exists\, \eta\ \exists\, k, C\ \forall\, r > r_0$

$$\forall\, y \in X'_N : \quad \|y\|^*_{M,m} \ \leq\ C \left(r^\eta \ \|y\|^*_{K,k} \ +\ \frac{1}{r}\|y\|^*_{N,n} \right),$$

$E \in (DN)$: $\exists\, \nu\ \forall\, \mu, \eta\ \exists\, \kappa, D$:

$$\|x\|_\mu^{(\eta+1)} \ \leq\ D\ \|x\|_\nu^\eta\ \|x\|_\kappa$$

$$r := \frac{\|x\|_\mu}{\|x\|_\nu} \geq 1, \qquad r^\eta = \frac{\|x\|_\mu^\eta}{\|x\|_\nu^\eta} \leq D\frac{\|x\|_\kappa}{\|x\|_\mu}$$

$$\|y\|^*_{M,m} \leq C\left(D\frac{\|x\|_\kappa}{\|x\|_\mu}\|y\|^*_{K,k} + \frac{\|x\|_\nu}{\|x\|_\mu}\|y\|^*_{N,n}\right),$$

A characterization

Let X be an ultrabornological PLS-space and α be stable.

- $\mathrm{Ext}^1(\Lambda^1_\infty(\alpha), X) = 0 \quad \Leftrightarrow \quad X \in (P\Omega)$;
- $\mathrm{Ext}^1(\Lambda^1_0(\alpha), X) = 0 \quad \Leftrightarrow \quad X \in (P\overline{\overline{\Omega}})$;
- $\mathrm{Ext}^1(\Lambda^\infty_r{}'(\alpha), X) = 0 \quad \Leftrightarrow \quad X \in (PA)$.

Fréchet Schwartz spaces and their duals with P-properties

Property	Fréchet	DFS-space
$(P\Omega)$	(Ω)	always
$(P\overline{\overline{\Omega}})$	$(\overline{\overline{\Omega}})$	always
$(P\underline{A})$	always	dual $(\underline{DN}) = (\underline{A})$
(PA)	always	dual $(DN) = (A)$

(Ω): $\forall\, \nu\ \exists\, \mu\ \forall\, \kappa\ \exists\, \theta, C: \quad \|\cdot\|_\mu^* \leq C \|\cdot\|_\nu^{*\,\theta} \|\cdot\|_\kappa^{*\,1-\theta}$

$(\overline{\overline{\Omega}})$: $\forall\, \nu\ \exists\, \mu\ \forall\, \kappa, \theta\ \exists\, C: \quad \|\cdot\|_\mu^* \leq C \|\cdot\|_\nu^{*\,\theta} \|\cdot\|_\kappa^{*\,1-\theta}$

$(\underline{DN})$: $\exists\, \nu\ \forall\, \mu\ \exists\, \kappa, \theta, C: \quad \|\cdot\|_\mu \leq C \|\cdot\|_\nu^{\theta} \|\cdot\|_\kappa^{1-\theta}$

(DN) : $\exists\, \nu\ \forall\, \mu, \theta\ \exists\, \kappa, C: \quad \|\cdot\|_\mu \leq C \|\cdot\|_\nu^{\theta} \|\cdot\|_\kappa^{1-\theta}$

- $H(U) \in (\underline{DN})$ for connected Stein mnf. U
- $H(U) \in (DN)$ for U with strong Liouville prop.
- $H(U) \in (\Omega)$ for all Stein manifold U
- $\Lambda_\infty \in (DN)$, $\Lambda_0 \in (\underline{DN})$, $\Lambda_r \in (\Omega)$ and $\notin (\overline{\overline{\Omega}})$

Spaces with P-properties

- ω non-quasianalytic:
  $\mathscr{D}'(\Omega), \mathscr{D}'_{(\omega)}(\Omega) \in (P\overline{\overline{\Omega}})+(PA)$,
  $\mathscr{E}_{\{\omega\}}(\Omega) \in (P\overline{\overline{\Omega}})+(P\underline{A})$ and $\notin (PA)$;
- ω quasianalytic, Ω convex:
  $\mathscr{E}_{\{\omega\}}(\Omega) \in (P\underline{A})$ and for ω subadditive $\mathscr{E}_{\{\omega\}}(\Omega) \in (P\overline{\overline{\Omega}})$;
- $\mathscr{A}(\Omega) \ \in (P\overline{\overline{\Omega}})+(P\underline{A})$ and $\notin (PA)$;
- ker $T_\mu \in (P\Omega)+(PA)$, if $T_\mu : \mathscr{D}'_{(\omega)}(\mathbb{R}) \to \mathscr{D}'_{(\omega)}(\mathbb{R})$ surjective;
- ker $T_\mu \in (P\Omega)+(P\underline{A})$, if T_μ surjective and
  $T_\mu : \mathscr{E}_{\{\omega\}}(\mathbb{R}) \to \mathscr{E}_{\{\omega\}}(\mathbb{R})$ or $T_\mu : \mathscr{E}_{\{\omega\}}(]-1,1[) \to \mathscr{E}_{\{\omega\}}(]-1,1[)$.

Theorem (Vogt)

Let E be an FN-space.

$$\mathrm{Ext}^1(E, \mathscr{A}(\Omega)\ ) = 0 \quad \Longleftrightarrow \quad E \in (\underline{DN})$$

$$\mathrm{Ext}^1(E', \mathscr{A}(\Omega)\ ) = 0 \quad \Longleftrightarrow \quad E \in (\overline{\overline{\Omega}})$$

Applications to parameter dependence

Linear P.D.O. with constant coefficients:

$$P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$$

$f_\lambda \in \mathscr{D}'(\Omega)$ depends on $\lambda \in U$ holomorphically, smoothly,...

Does there exist $u_\lambda \in \mathscr{D}'(\Omega)$ depending on $\lambda \in U$ holomorphically, smoothly,...:

$$P(D)u_\lambda = f_\lambda, \qquad \forall \lambda \in U?$$

Are the following operators surjective?

$$\begin{aligned} P(D) &: H(U, \mathscr{D}'(\Omega)) \to H(U, \mathscr{D}'(\Omega)) \\ P(D) \otimes \mathrm{id} &: \mathscr{D}'(\Omega) \hat{\otimes}_\varepsilon H(U) \to \mathscr{D}'(\Omega) \hat{\otimes}_\varepsilon H(U) \\ P(D) &: \mathscr{D}'(\Omega, H(U)) \to \mathscr{D}'(\Omega, H(U)) \end{aligned}$$

$$H(U, \mathscr{D}'(\Omega)) \simeq \mathscr{D}'(\Omega) \hat{\otimes}_\varepsilon H(U) \simeq \mathscr{D}'(\Omega, H(U))$$

A functional analytic approach

$$H(U,\mathscr{D}'(\Omega)) \leftrightarrow H(U)\hat{\otimes}_\varepsilon \mathscr{D}'(\Omega) = L(H(U)',\mathscr{D}'(\Omega))$$

$$H(U,\mathscr{D}'(\Omega)) \ni u \longleftrightarrow T \in L(H(U)',\mathscr{D}'(\Omega))$$

$$T(\delta_\lambda) := u(\lambda), \quad P(D)(T) := P(D)\circ T$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker P(D) & \longrightarrow & \mathscr{D}' & \xrightarrow{P(D)} & \mathscr{D}' & \longrightarrow & 0 \\ & & & & & & \uparrow T & & \\ & & & & & & H(U)' & & \end{array}$$

The map

$$P(D) \ : \ L(H(U)',\mathscr{D}'(\Omega)) \longrightarrow L(H(U)',\mathscr{D}'(\Omega))$$

is surjective if and only if every T lifts.

Theorem

Let E be a Fréchet Schwartz space and $R : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ be surjective.

$$R \otimes \mathrm{id}_{\,E} \;:\; \mathscr{D}'(\Omega)\hat{\otimes}_\varepsilon E \longrightarrow \mathscr{D}'(\Omega)\hat{\otimes}_\varepsilon E \simeq \mathscr{D}'(\Omega, E)$$

$$R \otimes \mathrm{id}_{\,E'} \;:\; \mathscr{D}'(\Omega)\hat{\otimes}_\varepsilon E' \longrightarrow \mathscr{D}'(\Omega)\hat{\otimes}_\varepsilon E' \simeq \mathscr{D}'(\Omega, E')$$

Then

$$\mathrm{Ext}^1(E, \ker\, R) = 0 \quad \Leftrightarrow \quad R \otimes \mathrm{id}_{\,E'} \text{ is surjective,}$$

$$\mathrm{Ext}^1(E', \ker\, R) = 0 \quad \Rightarrow \quad R \otimes \mathrm{id}_{\,E} \text{ is surjective}$$

If additionally $E \in (\Omega)$ then

$$\mathrm{Ext}^1(E', \ker\, R) = 0 \quad \Leftrightarrow \quad R \otimes \mathrm{id}_{\,E} \text{ is surjective.}$$

Examples of spaces with (Ω): $H(U)$, $C^\infty(U)$, $\mathscr{S}$, $\Lambda_r(\alpha)$.

Examples

1. Example:

$$P(D): \; H(\mathbb{C}, \mathscr{D}'(\Omega)) \longrightarrow H(\mathbb{C}, \mathscr{D}'(\Omega))$$

$$P(D) \otimes \mathrm{id} \,:\, \mathscr{D}'(\Omega)\hat{\otimes}_\varepsilon H(\mathbb{C}) \longrightarrow \mathscr{D}'(\Omega)\hat{\otimes}_\varepsilon H(\mathbb{C})$$

For Ω convex the map

$$P(D) \otimes \mathrm{id} \,:\, \mathscr{D}'(\Omega)\hat{\otimes}_\varepsilon H(\mathbb{C}) \longrightarrow \mathscr{D}'(\Omega)\hat{\otimes}_\varepsilon H(\mathbb{C})$$

is surjective.

$$\mathrm{Ext}^1(H(\mathbb{C})', \ker\, P(D)) = 0 \quad \Longrightarrow \quad \ker\, P(D) \in (PA)$$

2. Example:

$$P(D) : \mathscr{D}'(\Omega, \mathscr{D}'(\mathbb{R})) \longrightarrow \mathscr{D}'(\Omega, \mathscr{D}'(\mathbb{R}))$$

$$\mathscr{D}'(\Omega, \mathscr{D}'(\mathbb{R})) \simeq \mathscr{D}'(\Omega \times \mathbb{R})$$

If Ω is convex then $P(D)$ is surjective.

$$P(D) \otimes \text{id} : \mathscr{D}'(\Omega)\hat{\otimes}_\varepsilon s' \longrightarrow \mathscr{D}'(\Omega)\hat{\otimes}_\varepsilon s' \text{ is onto.}$$

$$\text{Ext}^1(s, \ker P(D)) = 0 \quad \Longrightarrow \quad \ker P(D) \in (P\Omega)$$

Corollary

If Ω convex then $\ker P(D) \subseteq \mathscr{D}'(\Omega)$ has (PA) and $(P\Omega)$, i.e., it has dual interpolation estimate for small θ:

$$\forall\ N\ \exists\ M\ \forall\ K\ \exists\ n\ \forall\ m, \theta < \theta_0\ \exists\ k, C\ \forall\ y \in X'_N :$$

$$\|y\|^*_{M,m} \ \leq\ C \left(\|y\|^{*\ (1-\theta)}_{K,k} \cdot \|y\|^{*\ \theta}_{N,n} \right).$$

- A generalization of the result known for hypoelliptic operators and due to Petsche, Vogt and Wiechert.

The Main Consequence:

If Ω is convex then the following operators are surjective:

(a) $P(D) : H(U, \mathscr{D}'(\Omega)) \longrightarrow H(U, \mathscr{D}'(\Omega))$ for every Stein manifold U.

(b) $P(D) : C^\infty(U, \mathscr{D}'(\Omega)) \longrightarrow C^\infty(U, \mathscr{D}'(\Omega))$ for every smooth manifold U.

(c) $P(D) : \mathscr{D}'(\Omega, E) \longrightarrow \mathscr{D}'(\Omega, E)$ for every FN space $E \in (\Omega)$ (for instance, $E \simeq H(U), C^\infty(U), \mathscr{S}, \Lambda_r(\alpha)$).

(d) $P(D) : \mathscr{D}'(\Omega, E') \longrightarrow \mathscr{D}'(\Omega, E')$ for every FN space $E \in (DN)$ (for instance, $E' \simeq \mathscr{S}', H(\{0\}), \Lambda'_\infty(\alpha), \mathscr{D}'(U)$).

Vogt: For elliptic $P(D)$ the converse to (d) holds! No surjectivity:

$$P(D) \text{ elliptic, } E = H(\overline{\mathbb{D}^d}) \text{ or } \Lambda_0(\alpha)'!$$

Main Theorem Again:

Theorem

Let X be an ultrabornological PLS-space and E be a nuclear Fréchet space then:

1) $\text{Ext}^1(E,X)=0 \quad \Leftrightarrow \quad (E,X)\in (H)$:
$\forall\ N\ \exists\ M\ \forall\ K\ \exists\ n,\nu\ \forall\ m,\mu\ \exists\ k,\kappa,S$

$$\|y\|^*_{M,m}\|x\|_\mu \leq S\left(\|y\|^*_{N,n}\|x\|_\nu + \|y\|^*_{K,k}\|x\|_\kappa\right)$$

for $\forall\ y\in X'_N, x\in E$.

2) $\text{Ext}^1(E',X)=0 \quad \Leftrightarrow \quad (E',X)\in (G)$:
$\forall\ N,\nu\ \exists\ M,\mu\ \forall\ K,\kappa\ \exists\ n\ \forall\ m\ \exists\ k,S$

$$\|y\|^*_{M,m}\ \|x\|^*_\mu \leq S\left(\|y\|^*_{N,n}\ \|x\|^*_\nu + \|y\|^*_{K,k}\ \|x\|^*_\kappa\right)$$

for $\forall\ y\in X'_N, x\in E'_\nu$.

$$X=\text{proj}_{N\in\mathbb{N}}\ \text{ind}_{n\in\mathbb{N}}\ X_{N,n}, \quad (X_{N,n},\|\cdot\|_{N,n}), \quad E=(E,(\|\cdot\|_\nu)_{\nu\in\mathbb{N}})$$

Sketch of the proof of Main Thm.

Reduction to $\mathrm{Proj}^1 = 0$ case 1):

Let $W = \mathrm{proj}_{N\in\mathbb{N}} W_N$. Fundamental resolution of the spectrum $\mathscr{W}$:

$$0 \longrightarrow W \longrightarrow \prod_{N\in\mathbb{N}} W_N \xrightarrow{\Sigma} \prod_{N\in\mathbb{N}} W_N$$

$$\Sigma\left((w_N)_{N\in\mathbb{N}}\right) := \left(i_N^{N+1} w_{N+1} - w_N\right)_{N\in\mathbb{N}}$$

$$\mathrm{Proj}^1 \mathscr{W} = 0 \quad \Leftrightarrow \quad \Sigma \text{ is surjective.}$$

$$\begin{array}{ccccccccc}
0 \longrightarrow & X & \longrightarrow & \prod_{N\in\mathbb{N}} X_N & \xrightarrow{\sigma} & \prod_{N\in\mathbb{N}} X_N & \longrightarrow 0 \\
& \uparrow \mathrm{id} & & \uparrow S & & \uparrow T & \\
0 \longrightarrow & X & \xrightarrow{j} & Y & \xrightarrow{q} & E & \longrightarrow 0
\end{array}$$

Σ is surjective for spectrum $(L(E, X_N))$ iff every T lifts with respect to σ.

$$\mathrm{Ext}^1(E, X) = 0 \quad \Leftrightarrow \quad \mathrm{Proj}^1 \left(L(E, X_N)\right)_{N\in\mathbb{N}} = 0$$

Sketch of the proof of Main Thm.

Reduction to $\mathrm{Proj}^1 = 0$ case 2):

$\mathrm{Ext}^1(E', X) = 0 \Leftrightarrow$ for arbitrary complete LFS-space Y every short exact sequence splits:

$$0 \longrightarrow E \xrightarrow{j} Y \xrightarrow{q} X' \longrightarrow 0.$$

$$\begin{array}{ccccccccc} 0 \longrightarrow & E & \longrightarrow & \prod_{N\in\mathbb{N}} E_N & \xrightarrow{\sigma} & \prod_{N\in\mathbb{N}} E_N & \longrightarrow 0 \\ & \uparrow \mathrm{id} & & \uparrow S & & \uparrow T & \\ 0 \longrightarrow & E & \xrightarrow{j} & Y & \xrightarrow{q} & X' & \longrightarrow 0 \end{array}$$

$$\mathrm{Ext}^1(E', X) = 0 \quad \Leftrightarrow \quad \mathrm{Proj}^1 \; (L(X', E_N))_{N\in\mathbb{N}} = 0$$

Lemma (Vogt):

$$\mathrm{Proj}^1 \; (L(X', E_N))_{N\in\mathbb{N}} \simeq \mathrm{Proj}^1 \; (L(X'_N, E_N))_{N\in\mathbb{N}}$$

$$\mathrm{Ext}^1(E,X)=0 \quad\Leftrightarrow\quad \mathrm{Proj}^1\ (L(E,X_N))_{N\in\mathbb{N}}=0$$
$$\mathrm{Ext}^1(E',X)=0 \quad\Leftrightarrow\quad \mathrm{Proj}^1\ (L(X_N',E_N))_{N\in\mathbb{N}}=0$$

Theorem (Braun-Vogt, Frerick-Wengenroth, Langenbruch)

Let $W=\mathrm{proj}_{N\in\mathbb{N}}\ \mathrm{ind}_{n\in\mathbb{N}}\ W_{N,n}$ be a projective limit of LB-spaces, $B_{N,n}$ the unit ball in $W_{N,n}$, then:

$$\forall\ N\ \exists\ M\ \forall\ K\ \exists\ n\ \forall\ m,\varepsilon>0\ \exists\ k,S<\infty$$

$$B_{M,m}\subseteq SB_{K,k}+\varepsilon B_{N,n}$$

$$\Rightarrow \mathrm{Proj}^1\ W=0\Rightarrow$$

$$\forall\ N\ \exists\ M\ \forall\ K\ \exists\ n\ \forall\ m\ \exists\ k,S<\infty$$

$$B_{M,m}\subseteq S\left(B_{K,k}+B_{N,n}\right).$$

Modifications of (G) and (H) to (G_ε) and (H_ε)

E quasinormable:

$\forall\ \nu\ \exists\ \mu\ \forall\ \kappa, \varepsilon > 0\ \exists\ D\ \forall\ x \in E'$

$$\|x\|_\mu^* \ \leq\ \varepsilon\ \|x\|_\nu^*\ +\ D\ \|x\|_\kappa^*$$

X ultrabornological:

$\forall\ N\ \exists\ M\ \forall\ K\ \exists\ n\ \forall\ m, \varepsilon > 0\ \exists\ k, C\ \forall\ y \in X_N'$

$$\|y\|_{M,m}^* \ \leq\ \varepsilon\ \|y\|_{N,n}^*\ +\ C\ \|y\|_{K,k}^*$$

PLS-type power series spaces

$$\alpha = (\alpha_j) > 0, \quad \beta = (\beta_j) > 0, \quad \alpha_j + \beta_j \to \infty$$

$$r_N \nearrow r, \qquad s_n \nearrow s$$

$$\Lambda_{r,s}(\alpha, \beta) := \{x : \ \forall\ N \ \exists\ n \ \ \|x\|_{N,n} < \infty\}$$

$$\|x\|_{N,n} := \sup_j |x(j)| \exp\left(r_N \alpha_j - s_n \beta_j\right)$$

Examples:

$T_\mu : X \to X$ surjective convolution operator, (z_j) zeros of $\hat{\mu}$, $\alpha_j = |\operatorname{Im} z_j|$, $\beta_j = \omega(z_j)$ then:

- if $X = \mathscr{D}'(\mathbb{R})$ then $\ker\ T_\mu \simeq \Lambda_{\infty,\infty}(\alpha, \beta)$;
- if $X = \mathscr{E}_{\{\omega\}}(\mathbb{R})$ then $\ker\ T_\mu \simeq \Lambda_{\infty,0}(\alpha, \beta) \simeq \Lambda_\infty(\gamma) \oplus \Lambda_0'(\delta)$;
- if $X = \mathscr{E}_{\{\omega\}}(]-1, 1[)$ then $\ker\ T_\mu \simeq \Lambda_{0,0}(\alpha, \beta) \simeq \Lambda_0(\gamma) \oplus \Lambda_0'(\delta)$.

P-properties of PLS-type power series spaces

- $\Lambda_{r,s}(\alpha,\beta) \in (PA) \quad \Leftrightarrow \quad s = \infty$ or Fréchet;
- $\Lambda_{r,s}(\alpha,\beta) \in (P\Omega)$ or $(P\underline{A}) \quad \Leftrightarrow \quad s = \infty$ or Fréchet$\times$LS-space;
- $\Lambda_{r,s}(\alpha,\beta) \in (P\overline{\overline{\Omega}}) \quad \Leftrightarrow$ LS-space.

Open problems

1. Main Theorem with a nuclearity assumption for X only.

2. $(P\Omega)$ for $\ker P(D)$ if

$$P(D) : \mathscr{D}'_{(\omega)}(\Omega) \to \mathscr{D}'_{(\omega)}(\Omega), \quad \Omega \text{ non-convex},$$

   or, equivalently, show that $\Omega \times \mathbb{R}$ is P-convex for singular support if Ω is P-convex for singular support.

3. The same problem for (PA) instead of $(P\Omega)$.

4. Does $\ker P(D) \in (P\overline{\overline{\Omega}}) + (P\underline{A})$ imply that $P(D)$ on $\mathscr{E}_{\{\omega\}}$ has a right continuous linear inverse?

5. The same problem for $P(D)$ on $\mathscr{D}'_{(\omega)}$ and $\ker P(D) \in (P\overline{\overline{\Omega}}) + (PA)$.