Convolution operators on quasianalytic classes that admit a continuous linear right inverse

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Weight function

A function $\omega : \mathbb{R} \to [0, \infty[$ is called a *weight function* if it is continuous, even, increasing on $[0, \infty[$, satisfies $\omega(0) = 0$, and also the following conditions:

(α) There exists $K \geq 1$ such that $\omega(2t) \leq K\omega(t) + K$ for all $t \geq 0$.

(β) $\omega(t) = o(t)$ as t tends to infinity.

(γ) $\log(t) = o(\omega(t))$ as t tends to infinity.

(δ) $\varphi : t \mapsto \omega(e^t)$ is convex on $[0, \infty[$.

quasianalytic weight function

(Q) $$\int_1^\infty \frac{\omega(t)}{t^2} dt = \infty$$

Otherwise it is called *non-quasianalytic*.

- **radial extension of** ω : $\tilde{\omega} : \mathbb{C}^n \to [0, \infty[, \quad \tilde{\omega}(z) := \omega(|z|)$.

Example. The function $\omega(t) := |t|(\log(e + |t|))^{-\alpha}, \alpha > 0$ is a weight function which is quasianalytic if and only if $0 < \alpha \leq 1$.

The Spaces.

Let ω be a given weight function, let K be a compact and G be an open subset of $\mathbb{R}^N$.

ω-ultradifferentiable functions of Beurling type on G

$$\mathcal{E}_{(\omega)}(G) = \{f \in C^\infty(G) : \text{ for each } K \subset G \text{ compact and } m \in \mathbb{N}$$
$$p_{K,m}(f) := \sup_{x \in K} \sup_{\alpha \in \mathbb{N}_0^N} |f^{(\alpha)}(x)| \exp\left(-m\varphi^*\left(\frac{|\alpha|}{m}\right)\right) < \infty\}.$$

$$\varphi^*(x) := \sup\{xy - \varphi(y) : y > 0\}, \quad x \geq 0.$$

- $\mathcal{E}_{(\omega)}(G)$ is a Fréchet space if we endow it with the locally convex topology given by the semi-norms $p_{K,m}$.

$$\mathcal{E}_{(\omega)}(K) := \{f \in C^\infty(K) : p_{K,m}(f) < \infty \quad \forall\, m \in \mathbb{N}\}.$$

The Convolution Operators

For $\mu \in \mathcal{E}_{(\omega)}(\mathbb{R})'$, $\mu \neq 0$, and $\varphi \in \mathcal{E}_{(\omega)}(\mathbb{R})$ we define

$$\check{\mu}(\varphi) := \mu(\check{\varphi}), \quad \check{\varphi}(x) := \varphi(-x), \ x \in \mathbb{R}.$$

The **convolution operator**

$$\begin{array}{llll} T_\mu : & \mathcal{E}_{(\omega)}(\mathbb{R}) & \longrightarrow & \mathcal{E}_{(\omega)}(\mathbb{R}) \\ & f & \rightsquigarrow & T_\mu(f) := \check{\mu} * f, \\ & & & (\check{\mu} * f)(x) := \check{\mu}(f(x - .)), \ x \in \mathbb{R}. \end{array}$$

It is a well-defined, linear, continuous operator.

Problem

When does operator T_μ admit a continuous linear right inverse?

Non quasianalytic classes

The problem was solved by **Meise** and **Vogt** in 1987

- Their proof used non quasianalyticity in an essential way.
- However, the basic tool here and in their paper is the Fourier Laplace transform

$$\mathcal{F}(u)(z) = \widehat{u}(z) := u_x(e^{-i\langle x,z\rangle}).$$

$\mathcal{F} : \mathcal{E}_{(\omega)}(\mathbb{R})' \longrightarrow A_{(\omega)}$ is a linear topological isomorphism.

- Moreover, $\mathcal{F} \circ T_\mu^t = M_{\hat{\mu}} \circ \mathcal{F}$ for each $\mu \in \mathcal{E}_{(\omega)}(\mathbb{R})'$. here $M_{\hat{\mu}}$ is the multiplication operator.

$$A_{(\omega)} = \{f \in H(\mathbb{C}) : \exists\, n \in \mathbb{N} : \|f\|_n := \sup_{z\in\mathbb{C}} |f(z)| \exp(-n(|\operatorname{Im} z| + \omega(z))) < \infty\}$$

Surjective convolution operators

Theorem 1. (Momm, 1992)

For each weight function ω the following conditions are equivalent for $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R})$, $\mu \neq 0$:

(1) $T_\mu : \mathcal{E}_{(\omega)}(\mathbb{R}) \to \mathcal{E}_{(\omega)}(\mathbb{R})$ is surjective.

(2) The principal ideal $\hat{\mu} A_{(\omega)}$ is closed in $A_{(\omega)}$.

(3) $\hat{\mu}$ is **(ω)-slowly decreasing in the sense of Ehrenpreis**, i.e., there exist $k > 0$, $x_0 > 0$, such that for each $x \in \mathbb{R}$ with $|x| \geq x_0$ there exists $t \in \mathbb{R}$ with $|t - x| \leq k\omega(x)$ such that

$$|\hat{\mu}(t)| \geq \exp(-k\omega(t)).$$

(4) $\hat{\mu}$ is (ω)-**slowly decreasing**, i.e., there exists $C > 0$ such that for each $x \in \mathbb{R}, |x| \geq C$ there exists exists $\xi \in \mathbb{C}$ such that

$$|x - \xi| \leq C\omega(x) \text{ and } |\hat{\mu}(\xi)| \geq \exp(-C|\operatorname{Im}\xi| - C\omega(\xi)).$$

Necessary condition for the existence of right inverse

Proposition 2.

Let ω be a weight function which satisfies (α_1) and let $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R})$ be given. If the convolution operator

$$T_\mu : \mathcal{E}_{(\omega)}(\mathbb{R}) \to \mathcal{E}_{(\omega)}(\mathbb{R})$$

admits a continuous linear right inverse then the following two conditions are satisfied:

(a) $\hat{\mu}$ is (ω)-slowly decreasing.

(b) There exists $C > 0$ such that

$$|\operatorname{Im} a| \leq C(1 + \omega(a)), \; a \in \mathbb{C}, \; \hat{\mu}(a) = 0.$$

A weight function ω satisfies the condition (α_1) if

$$\sup_{\lambda \geq 1} \operatorname{lím\,sup}_{t \to \infty} \frac{\omega(\lambda t)}{\lambda \omega(t)} < \infty.$$

This condition was introduced by **Petzsche** and **Vogt** in 1984.

(α_1) $\Leftrightarrow$

$\exists C_1 > 0$ such that for each $W \geq 1$there exists $C_2 > 0$ such that

$$\omega(Wt + W) \leq WC_1\omega(t) + C_2, \ t \geq 0.$$

Idea of the proof of Proposition 2.

- The surjectivity of T_μ follows trivially. Assume that condition (b) does not hold.
- We can choose a sequence $(a_j)_{j\in\mathbb{N}}$ of complex numbers, such that $\hat{\mu}(a_j) = 0$ and a weight function σ with $\omega = o(\sigma)$, which also satisfies the condition (α_1), with $\sigma(a_j) = O(|\operatorname{Im} a_j|)$ as $j \to \infty$, and such that

$$F(z) := \prod_{j=1}^{\infty}(1 - \frac{z}{a_j}),\ z \in \mathbb{C},$$

is an entire function with

$$\sup_{z\in\mathbb{C}} |F(z)| \exp(-n\sigma_0(z)) < \infty$$

for some n and some weight $\sigma_0 = o(\omega)$.

- Moreover, $M_f : A_{(\omega)} \to A_{(\omega)}$ has closed range, and there is $g \in A_{(\omega)}$ such that $\hat{\mu} = gF$.
- Since T_μ admits a continuous linear right inverse, $M_{\hat{\mu}}$ has a continuous linear left inverse $L_{\hat{\mu}}$. This implies that M_F has this property too, since

$$L_{\hat{\mu}} M_g M_F(h) = L_{\hat{\mu}}(gFh) = L_{\hat{\mu}}(\hat{\mu}h) = h.$$

- To continue, we denote by $A_{\{\sigma\}}$ the space of all the entire functions g satisfying that there is n such that for each m we have

$$\sup_{z \in \mathbb{C}} |g(z)| \exp(-n|\operatorname{Im} z| - \frac{1}{m}\sigma(z)) < \infty.$$

Observe that $A_{(\omega)} \subset A_{\{\sigma\}}$.

- Define $\varrho : H(\mathbb{C}) \to \mathbb{C}^{\mathbb{N}}$, $\varrho(f) := (f(a_j))_{j\in\mathbb{N}}$. By the properties of the sequence $(a_j)_j$ and results due to **Meise** in 1985 and 1989, we have the diagram

$$\begin{array}{ccccccccc} 0 & \to & A_{(\omega)} & \xrightarrow{M_F} & A_{(\omega)} & \xrightarrow{\varrho_1} & E & \to & 0 \\ & & \cap & & \cap & & & & \\ 0 & \to & A_{\{\sigma\}} & \xrightarrow{M_F} & A_{\{\sigma\}} & \xrightarrow{\varrho_2} & G & \to & 0, \end{array}$$

where ϱ_1 and ϱ_2 are the restrictions of ϱ, and E and G are sequence spaces, which coincide algebraically and topologically with the power series space $\Lambda_\infty((|\operatorname{Im} a_j|)_{j\in\mathbb{N}})'$ and are isomorphic to the corresponding quotients.

- Since ϱ_1 admits a continuous linear right inverse, so does ϱ_2.
- Therefore $\Lambda_\infty((|\mathrm{Im}\, a_j|)_{j\in\mathbb{N}})'$ is isomorphic to a complemented subspace of $A_{\{\sigma\}}$, hence $\Lambda_\infty((|\mathrm{Im}\, a_j|)_{j\in\mathbb{N}})$ is isomorphic to a quotient of the space $\mathcal{E}_{\{\sigma\}}(\mathbb{R}) = A'_{\{\sigma\}}$ of Roumieu type.
- By a result of **Vogt** 2002 (see also **Bonet**, **Domanski** 2006), this cannot be the case if condition (α_1) holds. This contradiction completes the proof.

Remark

For $\omega(t) = t$ we get $\mathcal{E}_{(\omega)}(\mathbb{R}) = H(\mathbb{C})$. **Meise** and **Taylor** proved that each convolution operator on $H(\mathbb{C})$ admits a continuous linear right inverse. This is the reason why we assume $\omega(t) = o(t)$ as t tends to infinity.

Sufficient condition for the existence of right inverse

To prove a partial converse of Proposition 2 we need the following condition on the weight.

(DN)-weight function

For each $C > 1$ there exist $R_0 > 0$ and $0 < \delta < 1$ such that for each $R \geq R_0$

$$\omega^{-1}(CR)\omega^{-1}(\delta R) \leq (\omega^{-1}(R))^2 .$$

- (DN)-weight functions were introduced by **Meise** and **Taylor** in 1987.
- It is equivalent to the property (DN) of **Vogt** for the Fréchet space A'_ω.

$$A_\omega = \{f \in H(\mathbb{C}) \mid \exists n \in \mathbb{N} \ : \ \sup_{z \in \mathbb{C}} |f(z)| \exp(-n\omega(z)) < \infty\}$$

Let ω be a weight function for which there exists $A > 0$ such that

$$2\omega(t) \leq \omega(At) + A, \ t \geq 0.$$

Then ω is a (DN)-weight function.

- The following functions are quasianalytic (DN)-weight functions which also satisfy (α_1):

  (1) $\omega(t) := \dfrac{|t|}{(\log(e+|t|))^\alpha}$, $0 < \alpha \leq 1$.

  (2) $\omega(t) := \dfrac{|t|}{(\log(e+\log(e+|t|)))}$.

Proposition 3

Let ω be a (DN)-weight function. Then for $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R})$ the convolution operator

$$T_\mu : \mathcal{E}_{(\omega)}(\mathbb{R}) \to \mathcal{E}_{(\omega)}(\mathbb{R})$$

admits a continuous linear right inverse if the following conditions hold.

(a) $\hat{\mu}$ is (ω)-slowly decreasing.

(b) There exists $C > 0$ such that

$$|\operatorname{Im} a| \leq C(1 + \omega(a)), \ a \in \mathbb{C}, \ \hat{\mu}(a) = 0.$$

Idea of the Proof of Proposition 3.

- By duality and the Fourier Laplace transform, it suffices to show that the short exact sequence

$$0 \longrightarrow A_{(\omega)} \xrightarrow{M_{\hat{\mu}}} A_{(\omega)} \xrightarrow{\varrho} A_{(\omega)}/\hat{\mu}A_{(\omega)} \longrightarrow 0$$

  splits.
- It is no restriction to assume that $\hat{\mu}$ has infinitely many zeros. In this case, **Meise** proved in 1985 that the quotient $A_{(\omega)}/\hat{\mu}A_{(\omega)}$ is isomorphic to a sequence space Λ.

- In fact, after guessing what Λ is, one defines a linear map $\Phi : A_{(\omega)} \longrightarrow \Lambda$ with $\ker(\Phi) = \hat{\mu}A_{(\omega)}$.

- Several steps are needed to show that Φ is surjective. All but one can be done in a continuous linear way.

- Since ω is a (DN)-weight function, we can use a result due to **Meise** and **Taylor** on the splitting of certain $\overline{\partial}$-complex, and show that Φ, and hence ϱ, admits a continuous linear right inverse.

We restrict the attention now to (ω)-ultradifferential operators, and we see that the situation changes.

Proposition 4

Let ω be a (DN)-weight function. Assume that for $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R})$ its Fourier-Laplace transform $\hat{\mu}$ is (ω)-slowly decreasing and satisfies $|\hat{\mu}(z)| \leq C \exp(C\omega(z))$ for some $C > 0$ and all $z \in \mathbb{C}$. Then for each $a, b \in \mathbb{R}$ with $a < b$ the sequence

$$0 \longrightarrow \ker T_{\mu,[a,b]} \longrightarrow \mathcal{E}_{(\omega)}[a,b] \xrightarrow{T_{\mu,[a,b]}} \mathcal{E}_{(\omega)}[a,b] \longrightarrow 0$$

is exact and splits.

Remark

Let ω be a quasianalytic (DN)-weight function, which satisfies the condition (α_1). Then there exist (ω)-ultradifferential operators T_μ which admit a continuous linear right inverse on $\mathcal{E}_{(\omega)}[a,b]$, but which do not admit a continuous linear right inverse on $\mathcal{E}_{(\omega)}(\mathbb{R})$.

Ingredients of the proof of Proposition 4.

(1) $\mathcal{E}_{(\omega)}[a,b]$ has (DN), since ω is a (DN)-weight.

(2) $\ker T_{\mu,[a,b]}$ is isomorphic to a power series space of infinite type.

(3) $T_{\mu,[a,b]}$ is surjective.

Hence the result follows from the splitting theorem of **Vogt** and **Wagner**.

We obtain the following result which is due in the real analytic case to **Langenbruch** 1994 and to **Domanski** and **Vogt** 2001.

Theorem 5.

Let ω be a quasianalytic (DN)-weight function which satisfies the condition (α_1). Assume that for $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R})$ its Fourier-Laplace transform $\hat{\mu}$ is (ω)-slowly decreasing and satisfies

$$|\hat{\mu}(z)| \leq C \exp(C\omega(z))$$

for some $C > 0$ and all $z \in \mathbb{C}$. Then the following assertions are equivalent:

(1) $T_\mu : \mathcal{E}_{(\omega)}(\mathbb{R}) \to \mathcal{E}_{(\omega)}(\mathbb{R})$ admits a continuous linear right inverse.

(2) There exists $C > 0$ such that $|\mathrm{Im}\, a| \leq C(\omega(a) + 1)$ for each $a \in V(\hat{\mu})$.

(3) For each/some $a, b \in \mathbb{R}$ with $a < b$ and each $f \in \ker T_{\mu,[a,b]}$ there exists $g \in \ker T_\mu$ such that $f = g|_{[a,b]}$.

Let ω be a given weight function. For a compact subset K of $\mathbb{R}^N$ and $m \in \mathbb{N}$

$$\|f\|_{K,m} := \sup_{x\in K} \sup_{\alpha\in\mathbb{N}_0^N} |f^{(\alpha)}(x)| \exp\left(-\frac{1}{m}\varphi^*(m|\alpha|)\right).$$

For an open set G in $\mathbb{R}^N$,

ω-ultradifferentiable functions of Roumieu type on G

$$\mathcal{E}_{\{\omega\}}(G) = \{f \in C^\infty(G) : \ \forall\, K \subset G \text{ compact } \ \exists m \in \mathbb{N} \ \ \|f\|_{K,m} < \infty\}$$

The Fourier Laplace transform $\mathcal{F}$ defines a linear topological isomorphism between $\mathcal{E}_{\{\omega\}}(\mathbb{R}^N)'$ and the (LF)-space

$$A_{\{\omega\}} := \mathrm{ind}_K A(K), \text{ where } A(K) := \mathrm{proj}_m A(K, \frac{1}{m}),$$

and

$$A(K, \frac{1}{m}) = \{f \in H(\mathbb{C}^N) : \sup_{z \in \mathbb{C}^N} |f(z)| \exp(-K|\mathrm{Im}\, z| - \frac{1}{m}\omega(|z|)) < \infty\}.$$

- The convolution operators $T_\mu, \mu \in \mathcal{E}_{\{\omega\}}(\mathbb{R}^N)'$ are defined in a similar way as in the Beurling case.

- $F \in A_{\{\omega\}}$ is called $\{\omega\}$-slowly decreasing, if for each $m \in \mathbb{N}$ there exists $R > 0$ such that for each $x \in \mathbb{R}^N$ with $|x| \geq R$ there exists $\xi \in \mathbb{C}^N$ satisfying $|x - \xi| \leq \omega(x)/m$ such that

$$|F(\xi)| \geq \exp(-\omega(\xi)/m).$$

Proposition 6.

Let ω be a weight function and let $F \in A_{\{\omega\}}$ be given. Then the following assertions are equivalent:

(a) F is $\{\omega\}$-slowly decreasing.

(b) There exists a weight function σ satisfying $\sigma = o(\omega)$ such that $F \in A_{(\sigma)}$ and such that F is (σ)-slowly decreasing.

(c) The multiplication operator $M_F : A_{\{\omega\}} \to A_{\{\omega\}}$, $M_F(g) := Fg$, has closed range.

(d) $M_F^{-1} : FA_{\{\omega\}} \to A_{\{\omega\}}$ is sequentially continuous.

Theorem 7

Let ω be a weight function and let $\mu \in \mathcal{E}_{\{\omega\}}(\mathbb{R})'$, $\mu \neq 0$, be given. Then the following assertions are equivalent:

(1) $T_\mu : \mathcal{E}_{\{\omega\}}(\mathbb{R}) \to \mathcal{E}_{\{\omega\}}(\mathbb{R})$ is surjective.

(2) The following two conditions are satisfied:

  (a) $\hat{\mu}$ is $\{\omega\}$-slowly decreasing.

  (b) There exists $\delta > 0$ such that each limit point of the set $\{|\operatorname{Im} a|/\omega(a) : a \in V(\hat{\mu}),\ \omega(a) \neq 0\}$ is contained in $\{0\} \cup [\delta, \infty[$.

- This result for for non quasianalytic classes was proved by **Braun**, **Meise** and **Vogt** in 1990.
- For quasianalytic ultradifferential operators by **Meyer** in 1997.
- Our proof depends on recent results on (LF)-spaces due to **Vogt** and to **Wengenroth**.

Concerning the existence of continuous linear right inverses, we only know a necessary condition. A full characterization in the non quasianalytic case was obtained by **Meise** and **Vogt** in 1987.

Proposition 8

Let ω be a quasianalytic weight function which satisfies the condition (α_1) and let $\mu \in \mathcal{E}'_{\{\omega\}}(\mathbb{R})$, $\mu \neq 0$ be given. If $T_\mu : \mathcal{E}_{\{\omega\}}(\mathbb{R}) \to \mathcal{E}_{\{\omega\}}(\mathbb{R})$ admits a continuous linear right inverse, then the following assertions hold:

(a) T_μ is surjective and

(b) $\lim\limits_{\substack{a \in V(\hat{\mu}) \\ |a| \to \infty}} \dfrac{|\operatorname{Im} a|}{\omega(a)} = 0.$